Is energy conserved in general relativity ?
— The lates collaboration with Tetsuya —

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Non-perturbative Analysis of Quantum Field Theory
and its Application

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Congratulations, Tetsuya, on your 60th year (還暦) !

Tetsuya is 2 years younger than I. I became a graduate student@Hongo, in1982, while Tetsuya became a graduate student@Komaba, in 1984, and I have known him since then. Collaborations with Tetsuya started in 1995 on lattice QCD:

"Manifestation of Sea Quark Effects in the Strong Coupling Constant in Lattice QCD" Phys. Rev. Lett. 74 (1995) 22-25.

We together wrote 40 papers, 3 FLAG reviews, and 57 proceedings so far.

Today I will talk on the latest collaboration with Tetsuya.

References:

S. Aoki, T. Onogi and S. Yokoyama, “Conserved charge in general relativity”, Int. J. Mod. Phys. A36 (2021) 2150098, arXiv:2005.13233[gr-qc].

S. Aoki, T. Onogi and S. Yokoyama, “Charge conservation, Entropy Current, and Gravitation”, Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

S. Aoki and T. Onogi, “Conserved non-Noether charge in general relativity: Physical definition vs. Noether’s 2nd theorem”, to appear in IJMPA, arXiv:2201.09557[hep-th], https://doi.org/10.1142/S0217751X22501299.

S. Aoki, “Noether’s 1st theorem with local symmetries”, arXiv:2206.00283[hep-th].

Shuichi Yokoyama

A simple question I would like ask you.

Is energy conserved in general relativity (GR) ?

Please consider your answer to this question. Yes or No ? Why ?

In a flat spacetime, energy is a conserved charge of time translational symmetry.

Time translational symmetry not only defines “energy” but also leads to its conservation. (Noether’s 1st theorem)

Does Noether’s 1st theorem lead to a conserved energy in GR ?

$$\text{time translation} \in \text{general coordinate transformation (gauge symmetry)}$$

Another perspective on this problem from the Einstein equation

gravity matter

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa G_d T_{\mu\nu} \qquad \kappa = 4\pi G_N$$

energy momentum tensor (EMT)

Bianchi identity for $R_{\mu\nu}$

⟶ covariant conservation

$$\nabla_\mu (\sqrt{-g} T^\mu{}_\nu) = \partial_\mu (\sqrt{-g} T^\mu{}_\nu) + \Gamma^\mu_{\mu\alpha} (\sqrt{-g} T^\alpha{}_\nu) - \Gamma^\alpha_{\mu\nu} (\sqrt{-g} T^\mu{}_\alpha) = 0$$

but what we need for a conserved energy is $\partial_\mu (\sqrt{-g} T^\mu{}_\nu) = 0$

2nd and 3rd terms are obstructions.

I. Noether's Theorem for "gauge" theory

S. Aoki and T. Onogi, "Conserved non-Noether charge in general relativity: Physical definition vs. Noether's 2nd theorem", to appear in IJMPA, arXiv:2201.09557[hep-th], https://doi.org/10.1142/S0217751X22501299.

1. Naive derivation of conserved current

$$S_\Omega = \int_\Omega d^d x \, L(\phi, \phi_{,\mu}) \qquad \phi_{,\mu} := \partial_\mu \phi$$

Infinitesimal local transformation $\delta\phi := \theta(x) F_0(\phi) + \theta_{,\mu}(x) F_1^\mu(\phi) + \cdots$

$$\delta S_\Omega = \int_\Omega d^d x \left[\frac{\partial L}{\partial \phi} - \partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}} \right) \right] \delta\phi + \int_\Omega d^d x \, \partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}} \delta\phi \right) = 0$$

invariance

EoM (Equation of Motion) $:= E_\phi$ current $:= N^\mu[\theta]$

a region Ω is arbitrary

$$\partial_\mu N^\mu[\theta] = -E_\phi \delta\phi \simeq 0$$

on-shell conserved current

$\theta(x) = \theta_0$: constant

$$N_0^\mu := \frac{N^\mu[\theta_0]}{\theta_0} = \frac{\partial L}{\partial \phi_{,\mu}} F_0(\phi)$$

conserved current for a global symmetry ?

This seems good, but there exist infinitely many conserved currents not only for a constant θ_0 but also for an arbitrary function $\theta(x)$.

2. Maximal use of local symmetry

$$\delta\phi := \theta(x)F_0(\phi) + \theta_{,\mu}(x)F_1^{\mu}(\phi) + \cdots$$

$$\delta S_\Omega = \int_\Omega d^d x\, \underbrace{\left[E_\phi F_0 - \partial_\mu(E_\phi F_1^\mu) + \cdots\right]}_{=0}\theta + \int_\Omega d^d x\, \underbrace{\partial_\mu\left(N^\mu[\theta] + E_\phi F_1^\mu \theta + \cdots\right)}_{=0} = 0$$

1. Choose $\theta(x) = \theta_{,\mu}(x) = \cdots = 0$ at a boundary of an arbitrary Ω.

$$\longrightarrow \quad E_\phi F_0 - \partial_\mu(E_\phi F_1^\mu) + \cdots = 0$$

off-shell constraint of EOM

2. General θ and an arbitrary choice of Ω.

$$\longrightarrow \quad \partial_\mu K^\mu[\theta] = 0.$$

off-shell conservation

$$K^\mu[\theta] := N^\mu[\theta] + E_\phi F_1^\mu \theta + \cdots \quad = \underbrace{\left(\frac{\partial L}{\partial \phi_{,\mu}} F_0 + E_\phi F_1^\mu + \cdots\right)}_{:=A^\mu}\theta + \underbrace{\left(\frac{\partial L}{\partial \phi_{,\mu}} F_1^\nu + \cdots\right)}_{:=B^{\mu\nu}}\theta_\nu + \cdots$$

3. θ is an arbitrary function.

$$\longrightarrow \quad \partial_\mu A^\mu = 0 \qquad A^\mu + \partial_\nu B^{\nu\mu} = 0 \qquad B^{\mu\nu} + B^{\nu\mu} = 0$$

another off-shell conservation

Non-dynamical (off-shell) conservation $\partial_\mu K^\mu[\theta] = 0$ $\partial_\mu A^\mu = 0$

generic consequences of Noether's 2nd Theorem for gauge theory

$$N^\mu[\theta] = K^\mu[\theta] - E_\phi F_1^\mu \theta \simeq K^\mu[\theta]$$

$$N_0^\mu = A^\mu - E_\phi F_1^\mu \simeq A^\mu$$

Conserved currents from Noether's 1st theorem for gauge theory are equal to off-shell conserved currents form Noether's 2nd theorem up to EoM.

3. Consequences in GR

Gravity Action $$S_{G,\Omega} = \int_\Omega d^d x \, L_G$$ $$L_G = \frac{1}{2\kappa}\sqrt{-g}(R - 2\Lambda)$$

(infinitesimal) general coordinate transformations $\delta_\xi x^\mu = \xi^\mu$

$$\delta_\xi g_{\mu\nu}(x) = -\xi^\alpha{}_{,\mu}(x) g_{\alpha\nu}(x) - \xi^\alpha{}_{,\nu}(x) g_{\mu\alpha}(x)$$

$$\delta_\xi S_{G,\Omega} = 0 \longrightarrow \partial_\mu K^\mu_G[\xi] = 0$$

Noether's 2nd Theorem

off-shell conserved currents

$$\partial_\mu K^\mu_G[\xi] = 0$$

$$K^\mu_G[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right] = A^\mu{}_\nu\xi^\nu + B^\mu{}_\nu{}^\alpha\xi^\nu{}_{,\alpha} + C^\mu{}_\nu{}^{\alpha\beta}\xi^\nu{}_{,\alpha\beta}$$

off-shell conserved current density (covariant)

conserved for an arbitrary vector ξ

Komar energy, ADM energy, Wald entropy, asymptotic charges

referred as Quasi-local (energy)

$$\partial_\mu A^\mu{}_\nu = 0$$

off-shell conserved current density (non-covariant)

Einstein's pseudo-tensor

EoM

$$A^\mu{}_\nu = \frac{\sqrt{-g}}{2\kappa}\left[2R^\mu{}_\nu + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right] \overset{g_{\mu\nu}}{\approx} \sqrt{-g}\,(T^\mu{}_\nu + t^\mu{}_\nu)$$

$$t^\mu{}_\nu := \frac{1}{2\kappa}\left[R^\mu{}_\nu + \frac{\delta^\mu_\nu}{2}(R-2\Lambda) + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right]$$

Matter Action

$$S_{\phi,\Omega} = \int_\Omega d^d x \, L_\phi(g, \phi, \phi_{,\mu})$$

$$\delta_\xi S_{\phi,\Omega} = 0 \longrightarrow \partial_\mu K^\mu_\phi[\xi] = 0$$

$$K^\mu_{\text{scalar}}[\xi] = 0$$

$$K^\mu_{\text{vector}}[\xi] = \partial_\nu[\sqrt{-g} F^{\mu\nu} A_\alpha \xi^\alpha]$$

“Energy” defined from off-shell conserved currents ?

$$\partial_\mu K^\mu_G[\xi] = 0$$ identity

c.f. Conserved current from Noether’s 1st theorem $N^\mu(x)$

One can freely add a trivially conserved term as

$$J^\mu(x) = N^\mu(x) + \partial_\nu f^{\mu\nu}(x), \ f^{\nu\mu}(x) = -f^{\mu\nu}(x)$$ arbitrary anti-symmetric function

$K^\mu_G[\xi]$ or $A^\mu{}_\nu$ corresponds to $\partial_\nu f^{\mu\nu}$

Thus quasi-local energy or Einstein’s (pseudo-tensor) energy cannot be a physical energy.

E. Noether, Gott. Nacho. **1918**(1918)235-257 [arXiv:physics/0503066[physics]].

Hilbert enunciates his assertion to the effect that the failure of proper laws of conservation of energy is a characteristic feature of the “general theory of relativity.” In order for this assertion to hold good literally, therefore, the term “general relativity” should be taken in a broader sense than usual, and extended also to the forgoing groups depending on n arbitrary functions.[27]

II. Matter energy in GR and its (non) conservation

S. Aoki, "Noether's 1st theorem with local symmetries", arXiv:2206.00283[hep-th].

Since quasi-local energy or Einstein's energy is not physical,
we should consider an alternative.

While EOM is derived from the total action, a matter action alone is invariant under general coordinate transformations.

$$\delta_\xi S_{\phi,\Omega} = 0 \qquad S_{\phi,\Omega} = \int_\Omega d^d x \, L_\phi(g, \phi, \phi_{,\mu})$$

$$\delta_\xi g_{\mu\nu}(x) = -\xi^\alpha{}_{,\mu}(x) g_{\alpha\nu}(x) - \xi^\alpha{}_{,\nu}(x) g_{\mu\alpha}(x) \qquad \delta_\xi \phi := \xi^\mu(x) F_\mu(\phi) + \xi^\mu{}_{,\nu}(x) F_\mu{}^\nu(\phi) + \cdots$$

Since transformations do not commute with derivatives, we define

$$\bar\delta_\xi \Phi := \delta_\xi \Phi - \Phi_{,\beta} \delta_\xi x^\beta \longrightarrow \bar\delta_\xi \Phi_{,\alpha} = \partial_\alpha(\bar\delta_\xi \Phi)$$ commute with derivatives

$$\bar\delta_\xi g_{\mu\nu} = -(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \qquad \bar\delta_\xi \phi = \xi^\mu \underbrace{[F_\mu(\phi) - \phi_{,\mu}]}_{=\bar F_\mu(\phi)} + \xi^\mu{}_{,\nu} F_\mu{}^\nu(\phi) + \cdots$$

1. Noether's theorem for a matter action in GR

$\delta_\xi (d^d x)$

$$\delta_\xi S_{\phi,\Omega} = \int_\Omega d^d x \left(\bar{\delta}_\xi L_\phi + \underbrace{L_{\phi,\mu} \delta_\xi x^\mu + \partial_\mu (\delta_\xi x^\mu) L_\phi}_{= \partial_\mu (L_\phi \xi^\mu)} \right) = 0$$

$$\bar{\delta}_\xi L_\phi = \frac{\partial L_\phi}{\partial g_{\mu\nu}} \bar{\delta}_\xi g_{\mu\nu} + \frac{\partial L_\phi}{\partial \phi} \bar{\delta}_\xi \phi + \frac{\partial L_\phi}{\partial \phi_{,\mu}} \partial_\mu \bar{\delta}_\xi \phi = \frac{\partial L_\phi}{\partial g_{\mu\nu}} \bar{\delta}_\xi g_{\mu\nu} + E_\phi \bar{\delta}_\xi \phi + \partial_\mu \left(\frac{\partial L_\phi}{\partial \phi_{,\mu}} \bar{\delta}_\xi \phi \right)$$

$$\longrightarrow \quad \frac{\partial L_\phi}{\partial g_{\mu\nu}} \bar{\delta}_\xi g_{\mu\nu} + E_\Phi \bar{\delta}_\xi \phi + \partial_\mu N^\mu_\phi[\xi] = 0 \qquad N^\mu_\phi[\xi] := L_\phi \xi^\mu + \frac{\partial L_\phi}{\partial \phi_{,\mu}} \bar{\delta}_\xi \phi$$

$$\frac{\partial L_\phi}{\partial g_{\mu\nu}} = \sqrt{-g} T^{\mu\nu}$$ EMT

2nd theorem $\longrightarrow$ $K^\mu_\phi[\xi] := N^\mu_\phi[\xi] - 2\sqrt{-g} T^{\mu\nu}_\phi \xi_\nu + E_\phi (\xi^\nu F_\nu{}^\mu + \cdots)$

redefinition of current

$$J^\mu_\phi[\xi] := N^\mu_\phi[\xi] - \underline{K^\mu_\phi[\xi]} = 2\sqrt{-g} T^{\mu\nu}_\phi \xi_\nu + E_\phi (\xi^\nu F_\nu{}^\mu + \cdots) \approx 2\sqrt{-g} T^{\mu\nu}_\phi \xi_\nu$$

trivial part

2. Our proposal for matter energy current

We keep the following property for a matter energy.

A matter energy is associated with global time translation.

matter energy current $J^\mu_\phi[\xi_T]$ $\xi^\mu_T = -\frac{\delta^\mu_0}{2}$

EoM

$$\partial_\mu J^\mu_\phi[\xi_T] = \partial_\mu N^\mu_\phi[\xi_T] - \underbrace{\partial_\mu K^\mu_\phi[\xi_T]}_{=0} = -\sqrt{-g}T^{\mu\nu}_\phi \bar{\delta}_{\xi_T} g_{\mu\nu} + E_\Phi \bar{\delta}_{\xi_T}\phi \approx \sqrt{-g}T^{\mu\nu}_\phi (\nabla_\mu(\xi_T)_\nu + \nabla_\nu(\xi_T)_\mu)$$

EMT

This current is not conserved in general curved spacetime.

total energy $$E := \int_{x^0:\text{fix}} [d^{d-1}x]_\mu \, J^\mu_\phi[\xi_T] = -\int_{x^0:\text{fix}} [d^{d-1}x]_\mu \sqrt{-g}\, T^\mu_\phi{}_0$$

same as energy in flat spacetime

$E_{\text{vacuum}} = 0$ by definition, since vacuum has $T^{\mu\nu} = 0$.

matter energy (non) conservation

condition $\partial_\mu J^\mu_\phi[\xi_T] \approx \sqrt{-g}T^{\mu\nu}_\phi(\nabla_\mu(\xi_T)_\nu + \nabla_\nu(\xi_T)_\mu)$ $\quad J^\mu_\phi[\xi_T] \approx -\sqrt{-g}T^\mu_\phi{}_0$

1. If a metric $g_{\mu\nu}$ doesn't depend on x^0, ξ^μ_T is a Killing vector and satisfies

$$\nabla_\mu(\xi_T)_\nu + \nabla_\nu(\xi_T)_\mu = 0 \longrightarrow \partial_\mu J^\mu_\phi[\xi_T] \approx 0$$

conserved matter current

$$E = -\int_{x^0:\text{fix}} [d^{d-1}x]_\mu \sqrt{-g}\, T^\mu_\phi{}_0$$

conserved energy

2. Even if ξ^μ_T is NOT a Killing vector but $T^\mu_\phi{}_\nu \Gamma^\nu_{\mu 0} = 0$ is satisfied,

$$(T_\phi)^\mu{}_\nu \nabla_\mu \xi^\nu_T = -(T_\phi)^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0 \longrightarrow \partial_\mu J^\mu_\phi[\xi_T] \approx 0$$

matter energy is conserved

3. Matter energy E is not conserved for a generic metric $g_{\mu\nu}$.

3. Our answer to the question in the title

1. A matter energy can be defined in a curved spacetime.

2. A matter energy is not conserved in a general curved space time.

Next question: Does a conserved matter charge other than energy exist ?

III. Conserved matter charge in GR

S. Aoki, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

S. Aoki, "Noether's 1st theorem with local symmetries", arXiv:2206.00283[hep-th].

$$\partial_\mu J^\mu_\phi[\xi] \approx \sqrt{-g} T^{\mu\nu}_\phi (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)$$

If a special vector ζ^μ satisfies $T^{\mu\nu}_\phi (\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$

$\longrightarrow$ $\partial_\mu J^\mu_\phi[\zeta] \approx 0$ conserved matter current

This is a conserved Noether current for a global symmetry generated by $\theta_0 \times \zeta^\mu(x)$ for a given $g_{\mu\nu}$ with a constant θ_0.

$g_{\mu\nu}$ is a background field for a matter. EOM for $g_{\mu\nu}$ is not assumed.

$$T_\phi^{\mu\nu}(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$$

We take $\zeta^\mu(x) = \beta(x)\xi_T^\mu$, then the condition becomes

$$T_\phi{}^0{}_0(x)\partial_0\beta(x) + T_\phi{}^k{}_0(x)\partial_k\beta(x) + T_\phi{}^\mu{}_\nu(x)\Gamma^\nu_{\mu 0}(x)\beta(x) = 0$$

1st order linear PDE

If an initial value $\beta(x^0, {}^\forall\vec{x})$ is given at x^0, $\beta(x)$ for other x^0 is easily obtained.

If $\beta(x)$ is constant, a conserved charge is a matter energy.

There exists a conserved matter charge more general than energy in GR.

A solution is known as a Kodama vector for a spherically symmetric system. Kodama'80

condition $$T_\phi^{\mu\nu}(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$$

conserved current density $$J_\phi^\alpha[\zeta] \approx \sqrt{-g}\, T_\phi^{\mu\nu} \zeta_\nu$$

Condition and conserved current have been proposed to define a conserved charge in general relativity from a different point of view in

S. Aoki, T. Onogi and S. Yokoyama, "Conserved charge in general relativity", Int. J. Mod. Phys. A36 (2021) 2150098, arXiv:2005.13233[gr-qc].

S. Aoki, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

What is a physical meaning of this matter conserved charge ?

Homogeneous and isotropic expansing Universe

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j$$ Freedman-Lemaitre-Robertson-Walker metric

EMT (perfect fluid) $T^0{}_0 = -\rho(x^0),\ T^i{}_j = P(x^0)\delta^i_j,\ T^0{}_j = T^i{}_0 = 0$

covariant conservation $\nabla_\mu T^\mu{}_\nu = 0 \longrightarrow \dot{\rho} + (d-1)(\rho + P)\frac{\dot{a}}{a} = 0$

energy $E(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0 = V_{d-1}a^{d-1}\rho, \quad V_{d-1} := \int d^{d-1}x\,\sqrt{\tilde{g}}.$

$$\longrightarrow \quad \frac{\dot{E}}{E} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} \neq 0$$ not conserved if $P \neq 0$

condition $T^{\mu}{}_{\nu} \nabla_{\mu} \zeta^{\nu} = 0$

$$\zeta^{\mu}(x^0) = -\beta(x^0)\delta^{\mu}_0 \longrightarrow \rho\dot{\beta} - (d-1)\beta\frac{\dot{a}}{a}P = 0$$

charge

$$S(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0\beta = V_{d-1}a^{d-1}\rho\beta \quad = E(x^0)\beta(x^0)$$

$$\longrightarrow \frac{\dot{S}}{S} = \frac{\dot{E}}{E} + \frac{\dot{\beta}}{\beta} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} + (d-1)\frac{\dot{a}}{a}\frac{P}{\rho} = 0$$

conserved !

Physical interpretation of the matter charge

volume density

$$v(x^0) = a(x^0)^{d-1}$$

energy density

$$e(x^0) = \rho(x^0) v(x^0)$$

charge density

$$s(x^0) = e(x^0)\beta(x^0)$$

$$\frac{ds}{dx^0} = \frac{de}{dx^0}\beta + e\frac{d\beta}{dx^0} = \left(\frac{de}{dx^0} + P\frac{dv}{dx^0}\right)\beta$$

⟺

$$Tds = de + Pdv$$

1st law of thermodynamics

$$\frac{d\beta}{dx^0} = (d-1)\frac{\dot{a}}{a}\frac{P\beta}{\rho} = P\frac{dv}{dx^0}\frac{\beta}{e}$$

⇒ S entropy

$$\beta \propto \frac{1}{T}$$ inverse temperature

Entropy of the Universe is conserved during its expansion.

S. Aoki, T. Onogi and S. Yokoyama, "Charge conservation, Entropy Current, and Gravitation", Int. J. Mod. Phys. A36 (2021)2150201, arXiv:2010.07660[gr-qc].

"Inflation"

S. Aoki and K. Kawana (in preparation)

$$P(x^0) = -\rho(x^0) \longrightarrow \rho(x^0) = \rho_I, \quad H(x^0) := \frac{\dot{a}(x^0)}{a(x^0)} = H_I \longrightarrow a(x^0) = a_0 e^{H_I x^0}$$

de Sitter-like

exponential growth

$$\longrightarrow \beta(x^0) = \beta_0 e^{-(d-1)H_I x^0}$$

$$S = \frac{A_H}{4G_N}$$

Bekenstein-Hawking entropy !

A_H area of de Sitter horizon

$$\frac{d-2}{2}\beta_0 = \frac{2\pi}{H_I} := \frac{1}{T_H}$$

initial condition

T_H de Sitter temperature

our interpretation seems reasonable

IV. Conclusion and discussion

Is energy conserved in GR ?

1. A matter energy can be defined in a curved spacetime as a generalization of energy in a flat spacetime.

2. A matter energy is not conserved in a general curved spacetime.

3. GR does not provide a conserved total energy in a general curved spacetime.

4. Instead, there always exists a conserved matter charge more general than energy in a curved spacetime, which is regarded as entropy for special cases.

Our method can be applied to gauge theories.

A. Electric charge is a matter U(1) Noether charge w/o gauge fixing.

B. Non-abelian matter gauge charge can be defined.

Ref. Sinya Aoki, "Noether's 1st theorem with local symmetries", arXiv:2206.00283[hep-th].

Further confirmations

1. Colliding gravitational waves and singularities — S. Aoki (to appear)

2. Binary star and energy — S. Aoki, T. Onogi and T. Yamaoka (work in progress)

3. Expanding universe and entropy conservation — S. Aoki and K. Kawana (in preparation)

and more.

Congratulations again ！

Tetsuya, I wish you a happy, healthy and fruitful life even after 60th.

Let us start a new collaboration, the Kanreki (還暦) collaboration, from now on !

Thank you !

Backup

Examples of conserved energy

S. Aoki, T. Onogi and S. Yokoyama, Int. J. Mod. Phys. A36 (2021) 2150098.

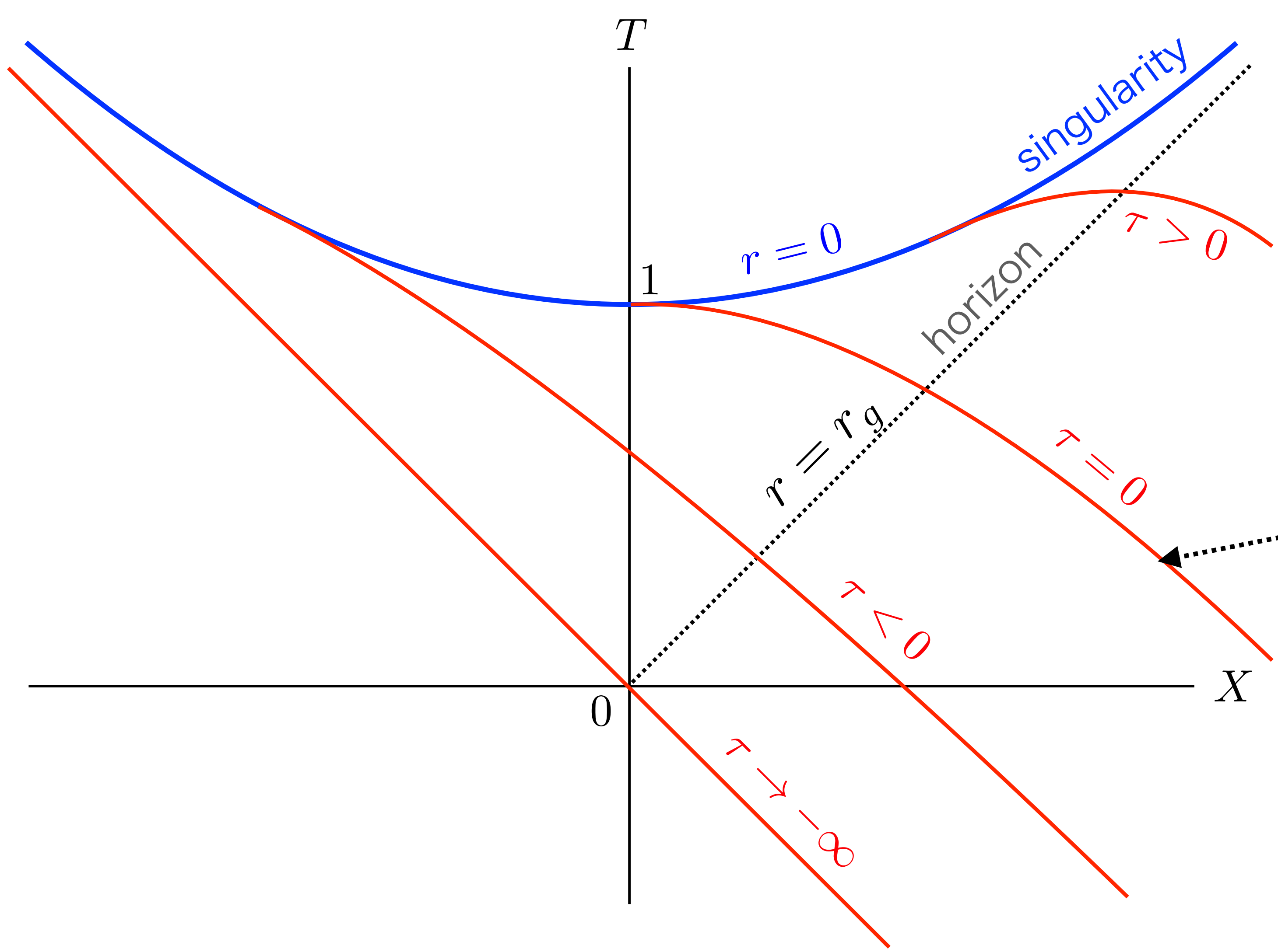
Schwarzschild black hole energy

$$ds^2 = -(1+u)d\tau^2 - 2ud\tau dr + (1-u)dr^2 + r^2 d\Omega_{d-2}^2$$

Eddington-Finkelstein coordinates

$$u(r) := \delta u(r) - \frac{2\Lambda r^2}{(d-2)(d-1)} \qquad \delta u(r) := -\frac{2GM\theta(r)}{r^{d-3}}$$

$\theta(r)$ with $\theta(0) = 0$ handles singularity at $r = 0$



stationary Killing vector

$$\xi^\mu = -\delta^\mu_\tau$$

τ constant surface

Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu}$ ⟶ $T_{\mu\nu} = 0$ at $r \neq 0$

If we calculate carefully, we obtain

$$T^{\tau}{}_{\tau} = \frac{d-2}{16\pi G} \frac{\partial_r (r^{d-3} \delta u)}{r^{d-2}} = -\frac{(d-2)M}{8\pi} \frac{\delta(r)}{r^{d-2}} = T^{r}{}_{r} \qquad r^{d-3}\delta u(r) = -2GM\theta(r)$$

$$T^{i}{}_{i} = \frac{1}{16\pi G} \frac{\partial_r^2 (r^{d-3} \delta u)}{r^{d-3}} = -\frac{1}{8\pi} \frac{\partial_r \delta(r)}{r^{d-3}}$$

⟶ black hole is not a vacuum solution to Einstein equation.

cf. Coulomb potential by a point charge is NOT a vacuum solution to Maxwell eq.

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \quad r \neq 0 \longrightarrow \nabla^2 \left(\frac{1}{r} \right) \propto \delta(x)$$

Einstein/Maxwell equations are distributional equations.

energy of black hole

$$E_{\rm BH} = -\int d^{d-1}x \sqrt{-g} T^{\tau}{}_{\tau} = \frac{(d-2)\Omega_{d-2}}{8\pi} \int_0^\infty dr \partial_r (M\theta(r)) = \frac{(d-2)\Omega_{d-2}}{8\pi} M$$

⟶ $E_{\rm BH} = M$ at $d = 4$

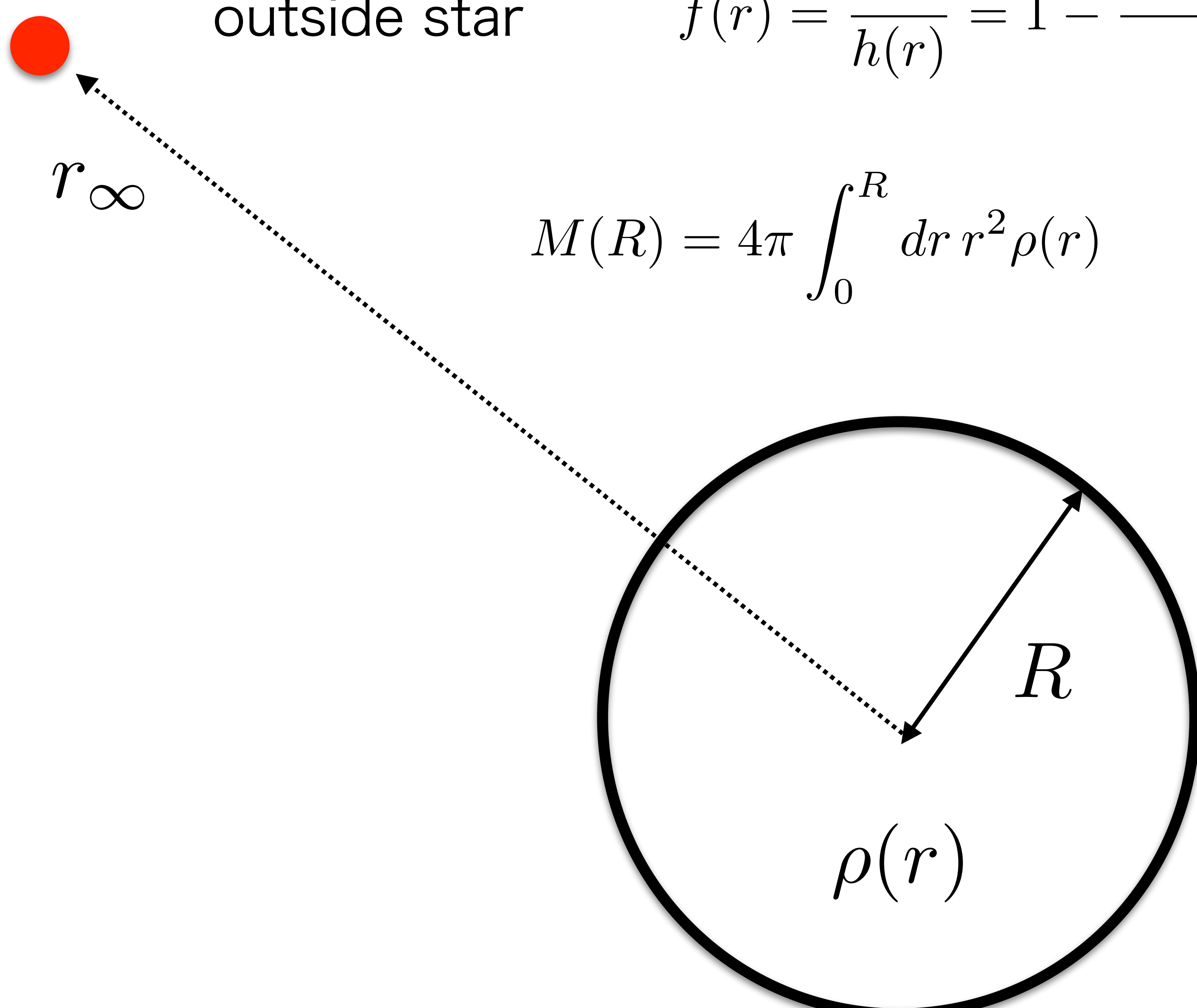
Energy of compact star

stationary spherically symmetric $$ds^2 = -f(r)(dx^0)^2 + h(r)dr^2 + r^2\tilde{g}_{ij}dx^i dx^j$$

with perfect fluid EMT $$T^0{}_0 = -\rho(r), \quad T^r{}_r = P(r), \quad T^i_j = \delta^i_j P(r)$$

outside star $$f(r) = \frac{1}{h(r)} = 1 - \frac{2GM(R)}{r}$$ Schwarzschild metric

$$M(R) = 4\pi \int_0^R dr\, r^2 \rho(r)$$ ADM mass



our energy $$E = -\int d^2x \int_0^\infty dr\, \sqrt{-g}\, T^0{}_0 = 4\pi \int_0^R dr\, \sqrt{f(r)h(r)} r^2 \rho(r)$$

difference $$E = M(R) - 4\pi G_4 \int_0^R dr\, \sqrt{f(r)h^3(r)}\, r M(r)\, (\rho(r) + P(r))$$

observed gravitational mass

correction due to the internal structure of the star $:= \Delta E$

Newtonian limit $$\Delta E \simeq \frac{G}{2} \int d^3x\, \rho(\mathbf{x})\phi(\mathbf{x}) < 0$$ Newton potential $$\phi(\mathbf{x}) = -\int d^3y \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

ADM mass $$M(R) \simeq E_0 + \frac{G}{2} \int d^3x\, \rho(\mathbf{x})\phi(\mathbf{x})$$ E_0 : energy without gravity

our energy $$E \simeq E_0 + G \int d^3x\, \rho(\mathbf{x})\phi(\mathbf{x})$$ matter energy in Newton potential

consistent with our "derivation"

ADM mass $$M(R) \simeq E + \frac{G}{2} \int d^3x \nabla\phi(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}) = E_0 - \frac{G}{2} \int d^3x\, d^3y\, \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

energy of gravitational field ? ⟶ a factor "1/2"