Is N_c=2 large ?

Norikazu Yamada (KEK / SOKENDAI)

Refs. Otake and NY, JHEP 06 (2022) 044
Kitano, Matsudo, NY, Yamazaki, PLB822, 136657 (2021)
Kitano, NY, Yamazaki, JHEP 02 (2021) 073
Kitano, Suyama, NY, JHEP 09 (2017) 137

Nonperturbative Analysis of Quantum Field Theory and its Applications @Osaka University Hall, Sep. 22, 2022

4d $SU(N_c)$ and 2d CP^{N-1}

4d $SU(N_c)$ YM

$$\mathscr{L} = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - i\theta q$$

$$q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

$\sim$

2d CP^{N-1}

$$\mathscr{L} = \frac{N}{2g} \overline{D_\mu z}\, D_\mu z - i\theta q$$

$z_i \in \mathbb{C}$ $(i = 1, \cdots, N)$ with $\bar{z}_i z_i = 1$

$$D_\mu = \partial_\mu + iA_\mu \,, \quad A_\mu = i\bar{z}\partial_\mu z$$

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e.g.) asymptotically free, dynamical mass, instanton, $1/N$ expandable

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θ dependence of vacuum energy density

Vacuum energy density $f(\theta)$

$$e^{-Vf(\theta)} = \frac{Z(\theta)}{Z(0)}$$

where

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$$Q = \int d^4x\, q(x)$$

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- In both theories,
  $Q \in \mathbb{Z} \Rightarrow f(\theta) = f(\theta + 2\pi)$
  S is CP even $\Rightarrow f(\theta) = f(-\theta)$

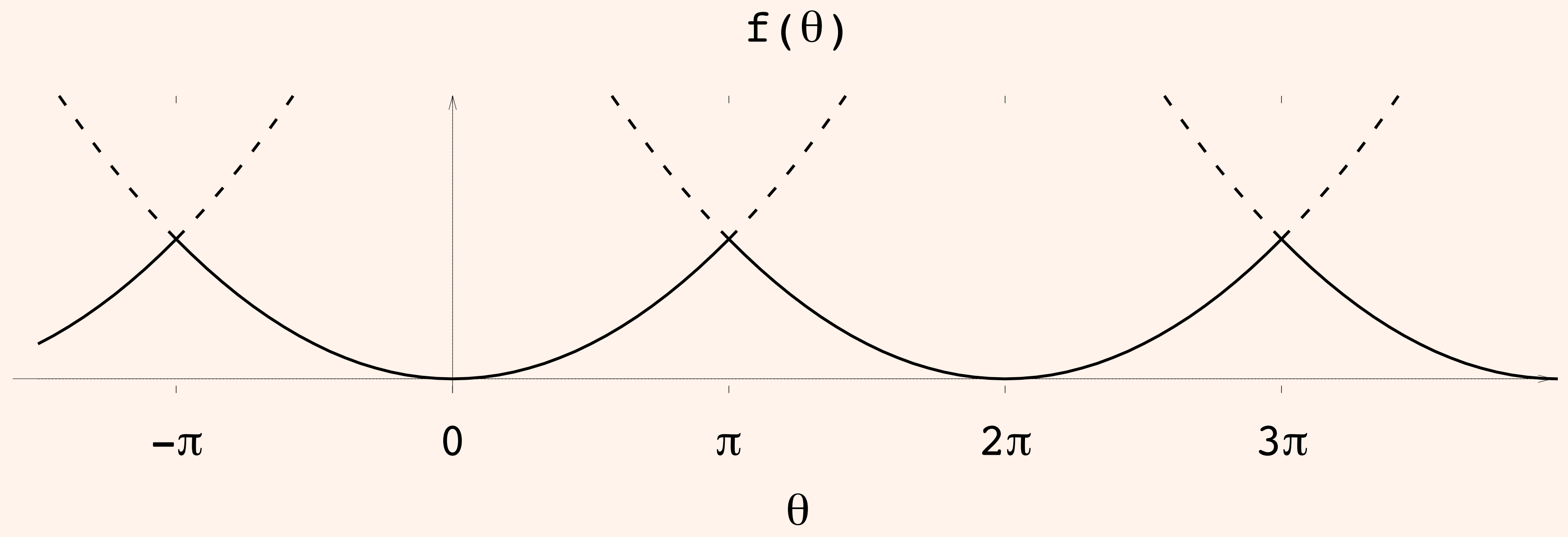
⬇

$$f(\pi - \theta') = f(\pi + \theta')$$

Expected θ dependence: large-N_c vs instanton

Large N_c [Witten (1980, 1998)]

$$\Rightarrow f(\theta) = \chi/2 \min_{k\in\mathbb{Z}}(\theta + 2\pi k)^2 + O(1/N_c^2)$$



- many branches
- spontaneous CPV (1st order PT) at $\theta = \pm(2n+1)\pi$
- order parameter $df(\theta)/d\theta|_{\theta=\pi} = -i\langle q(x)\rangle_{\theta=\pi}$

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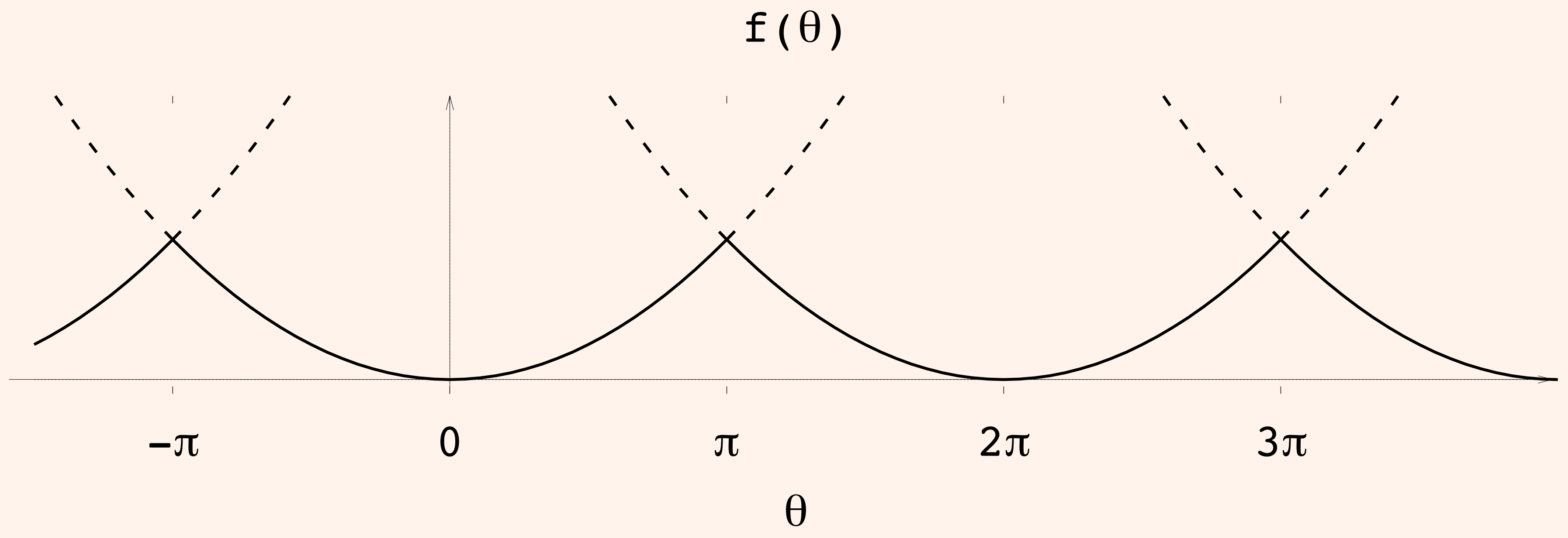

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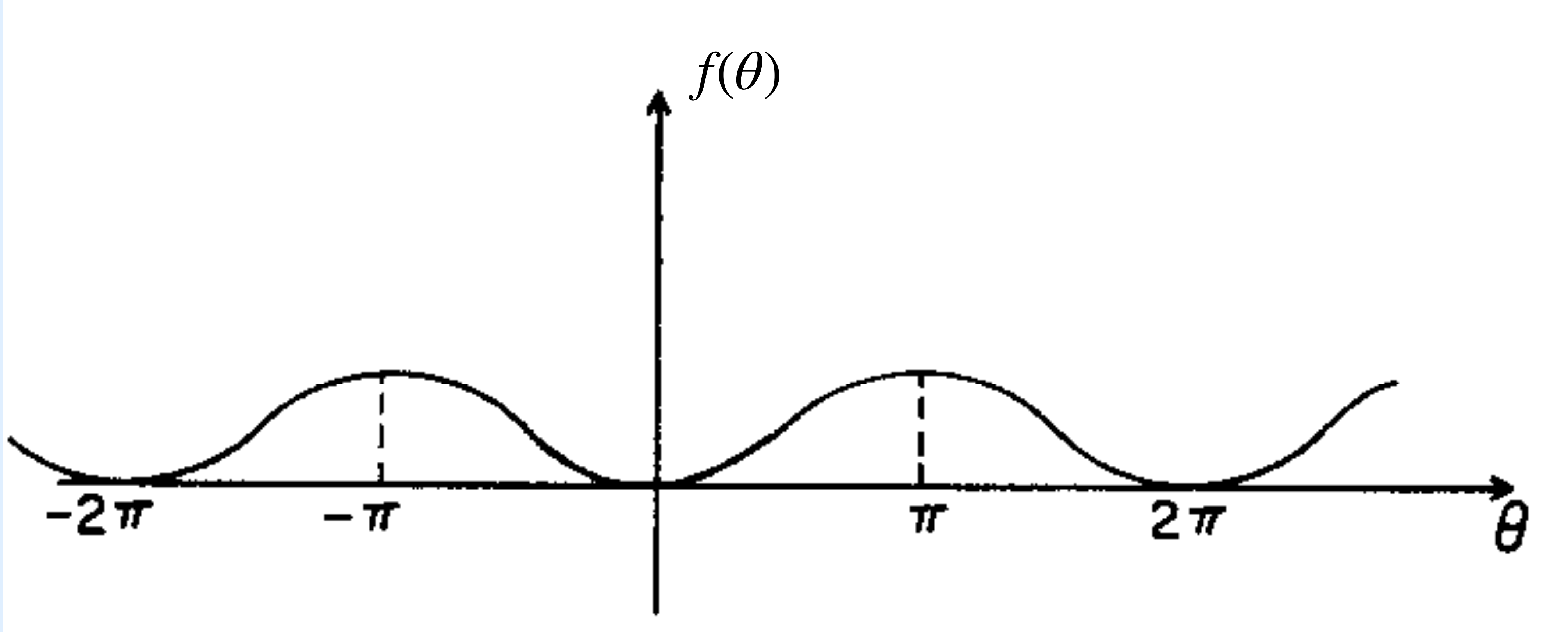
Dilute instanton gas approximation (DIGA)

$$\Rightarrow f(\theta) = \chi(1 - \cos\theta)$$



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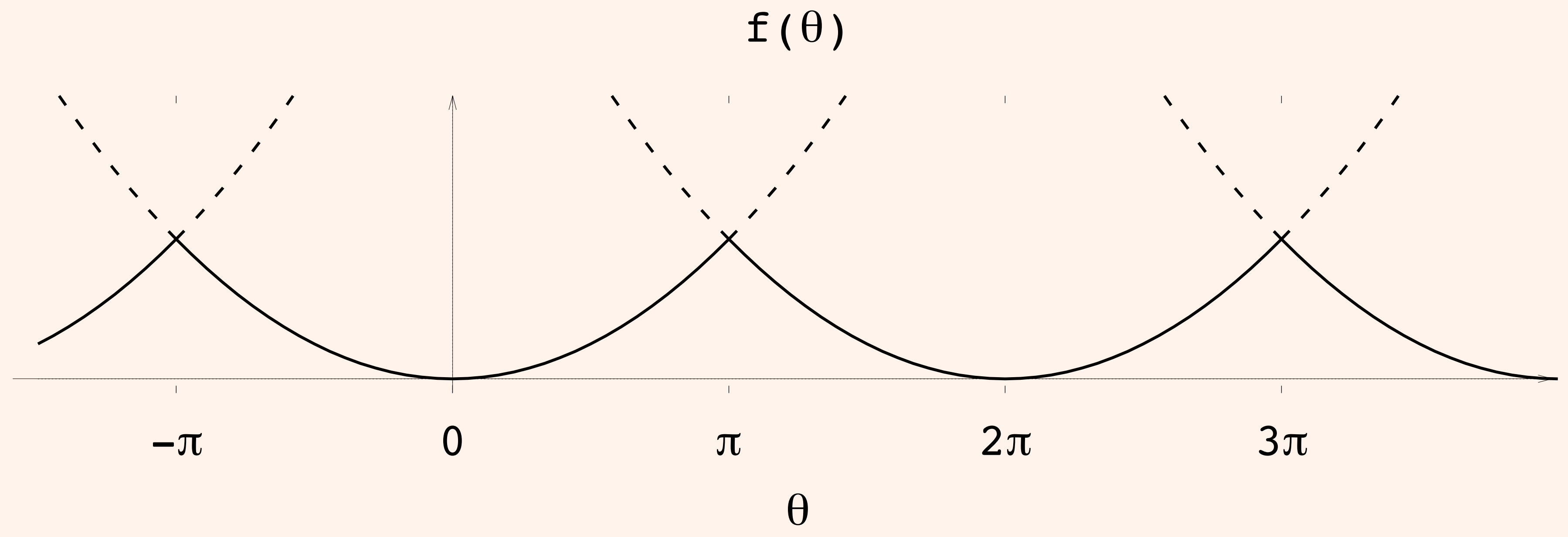

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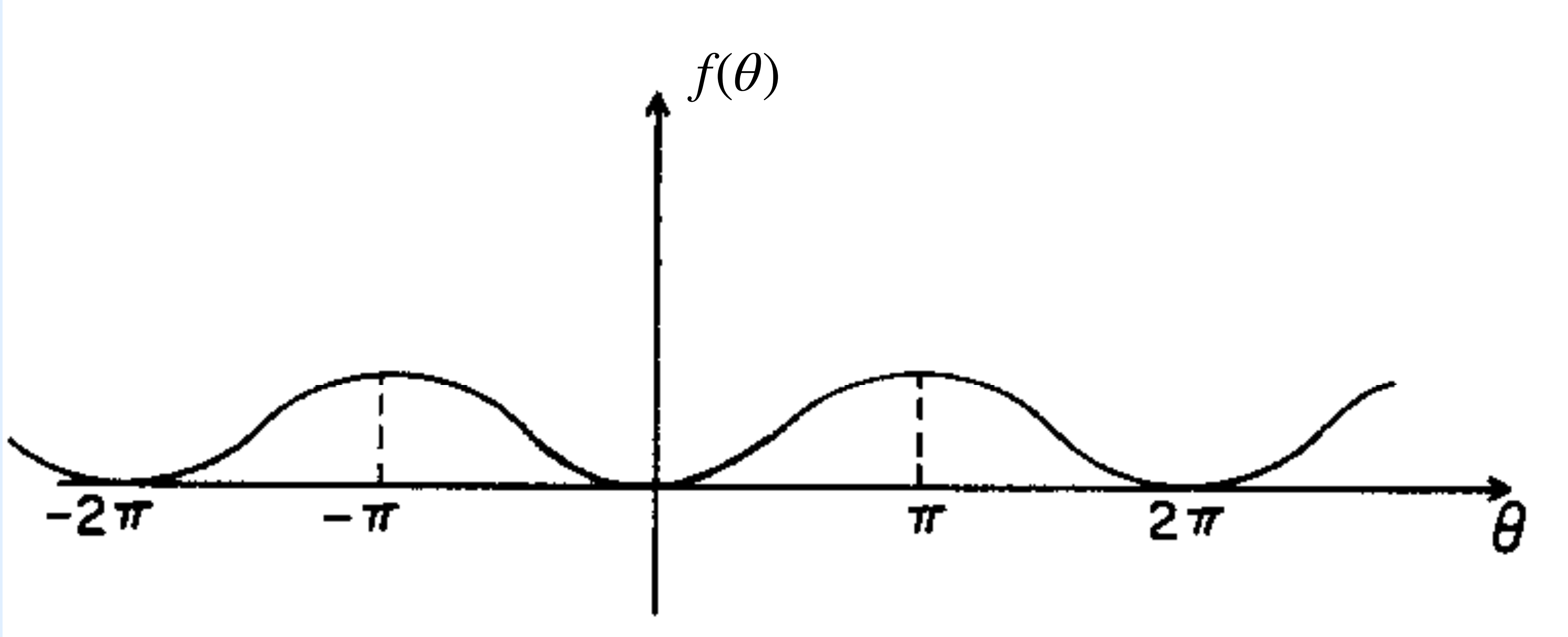

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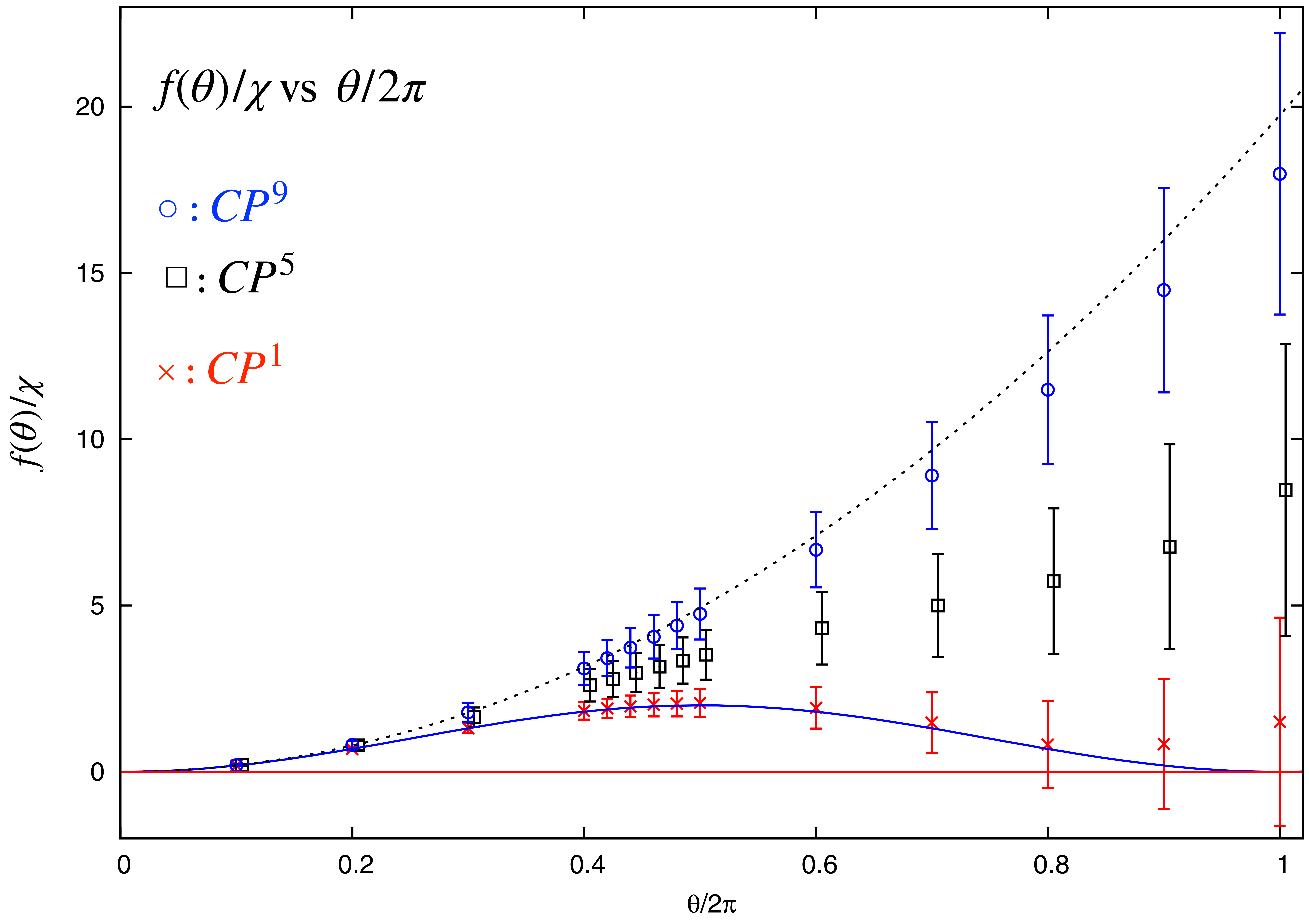
Significant difference in $d f(\theta)/d\theta$ around $\theta = \pi$

Lattice calculation: $f(\theta)$ in 2d CP^{N-1}

[Keith-Hynes and Thacker (2008)]



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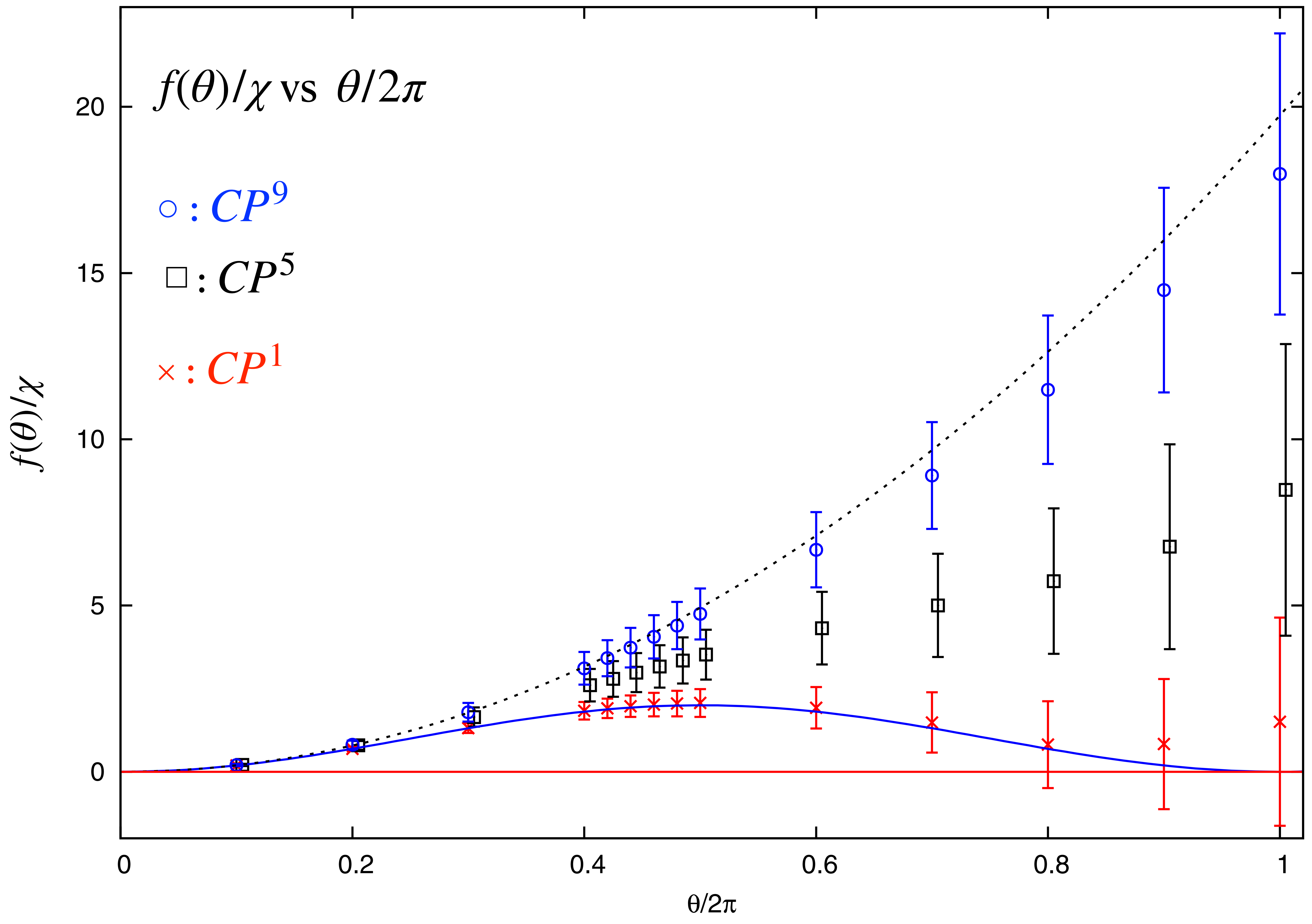

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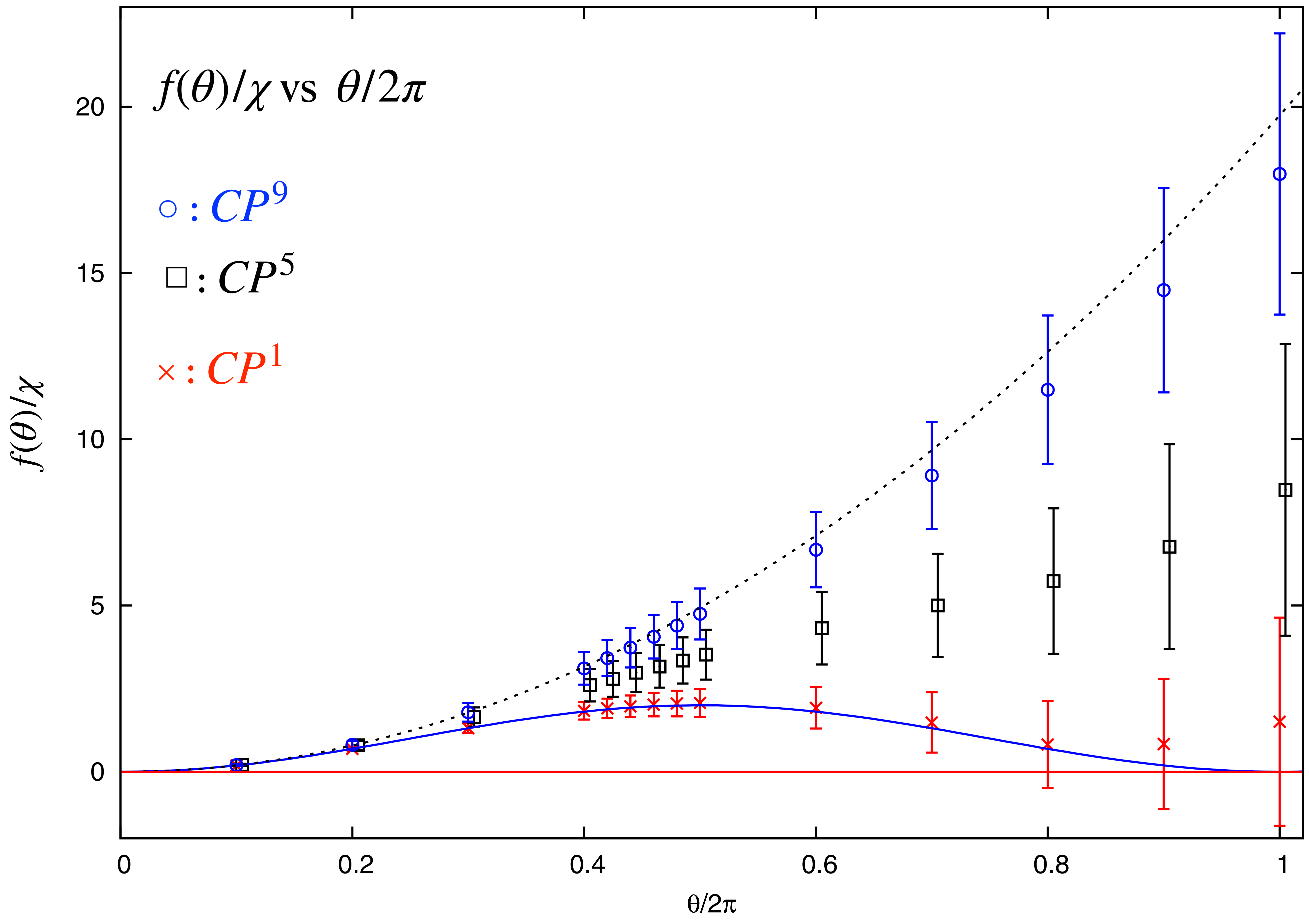

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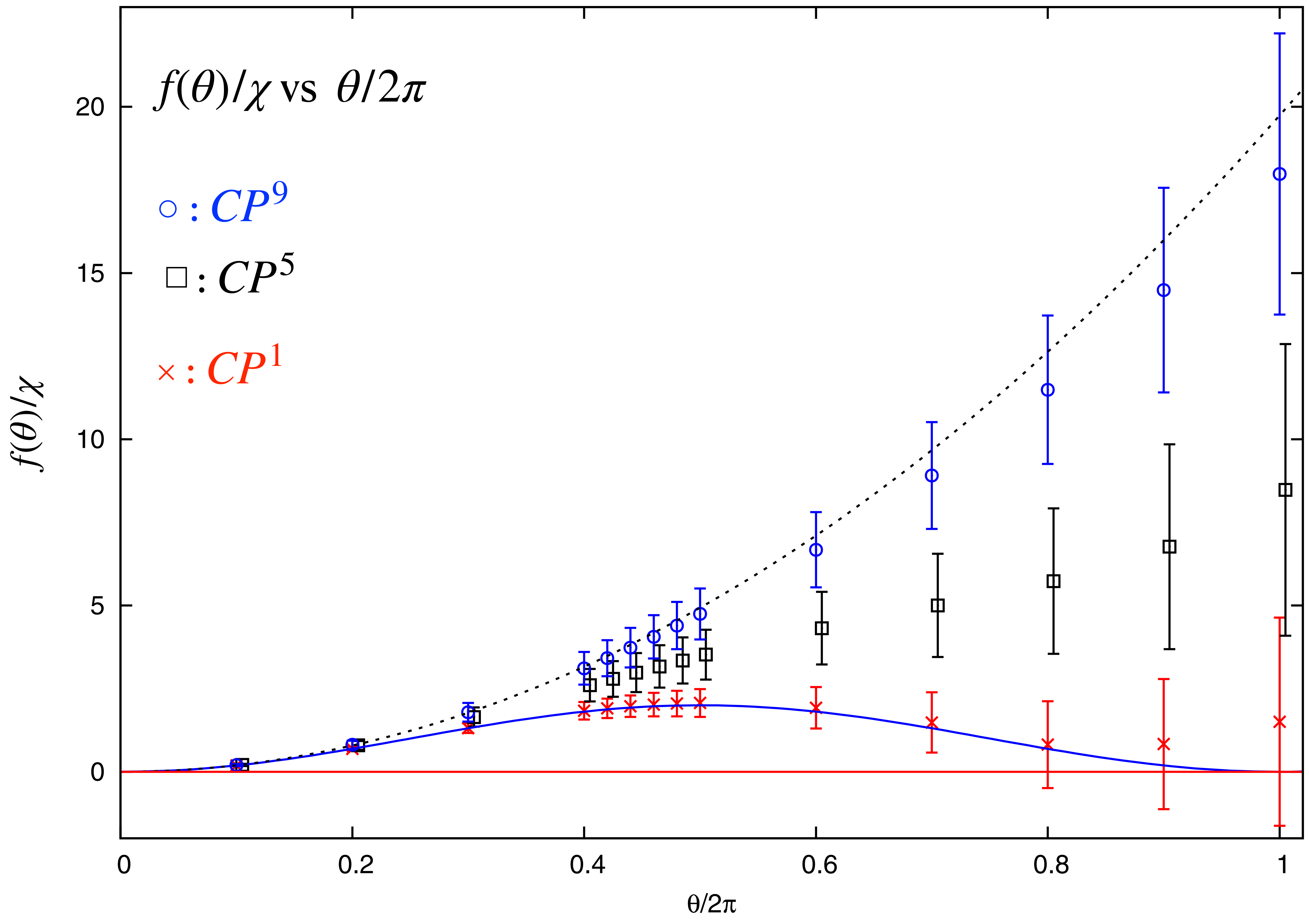

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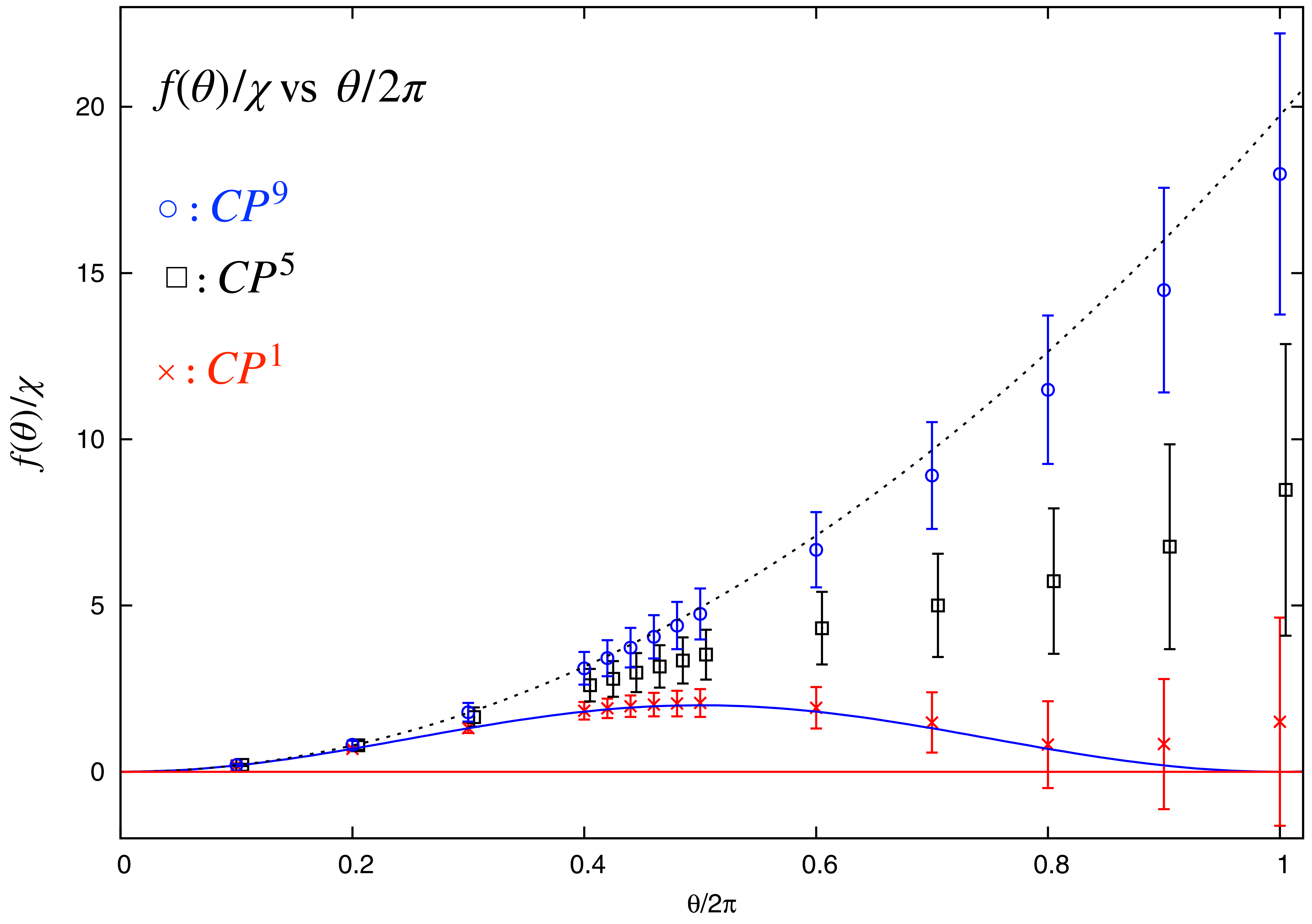

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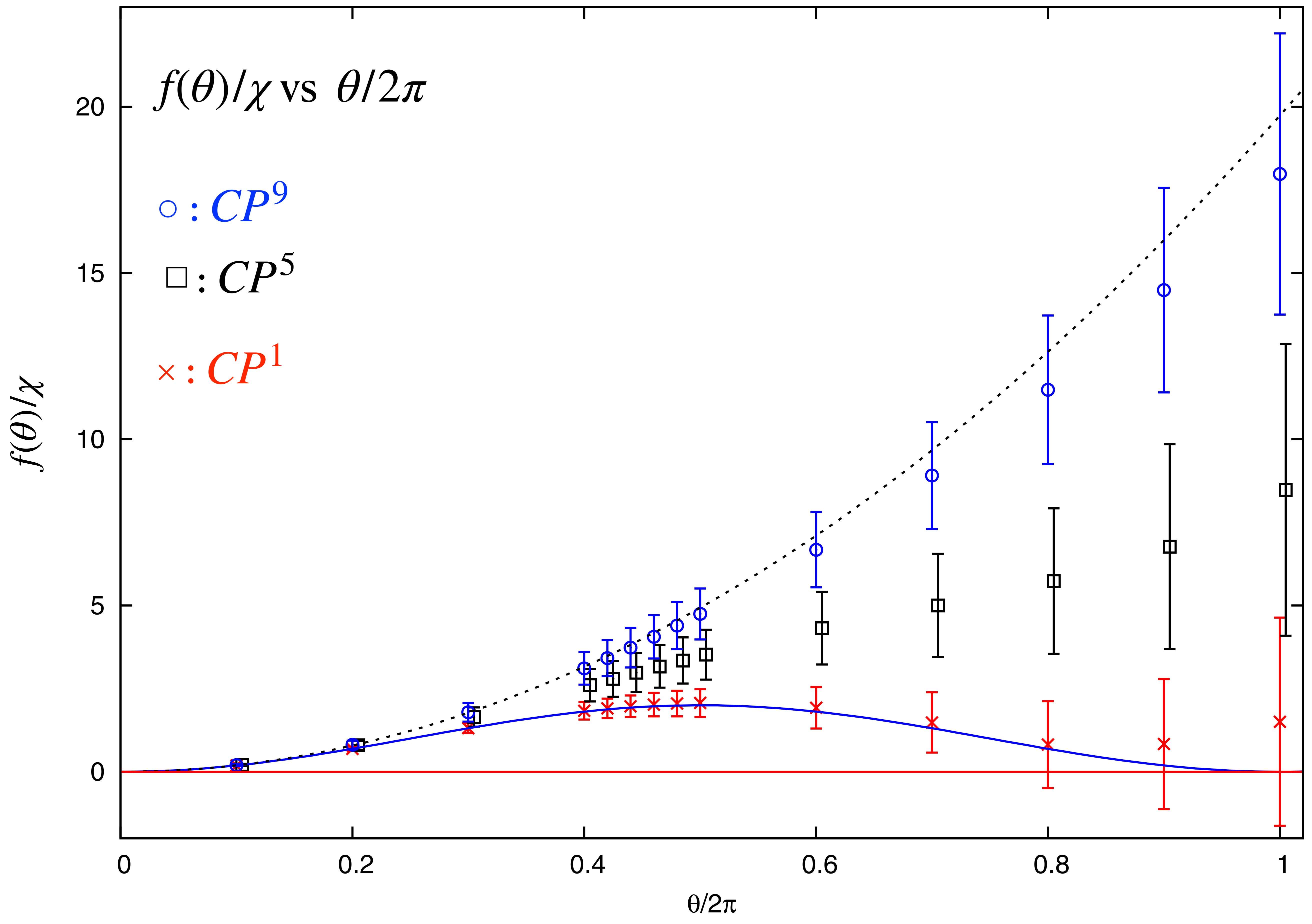

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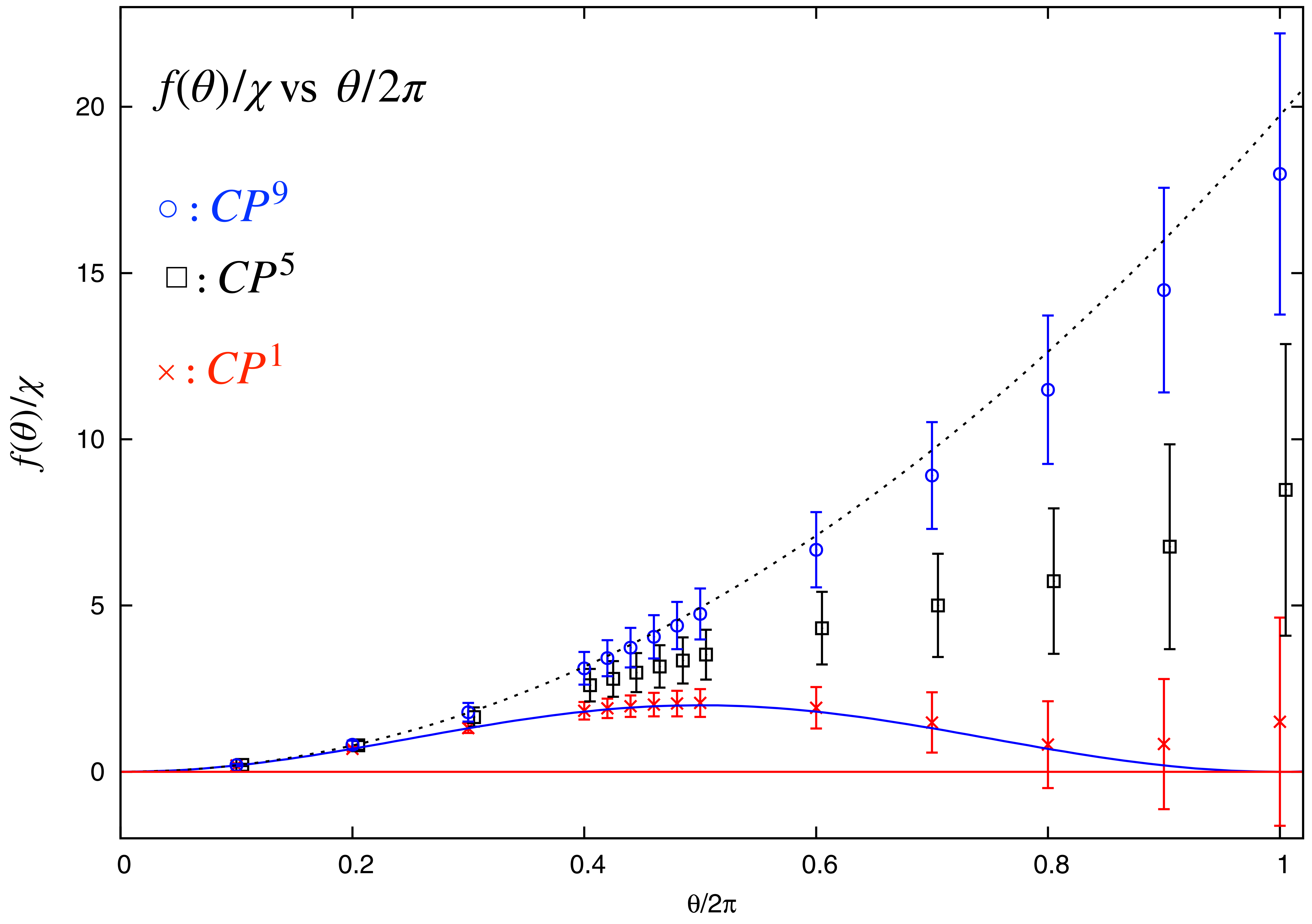

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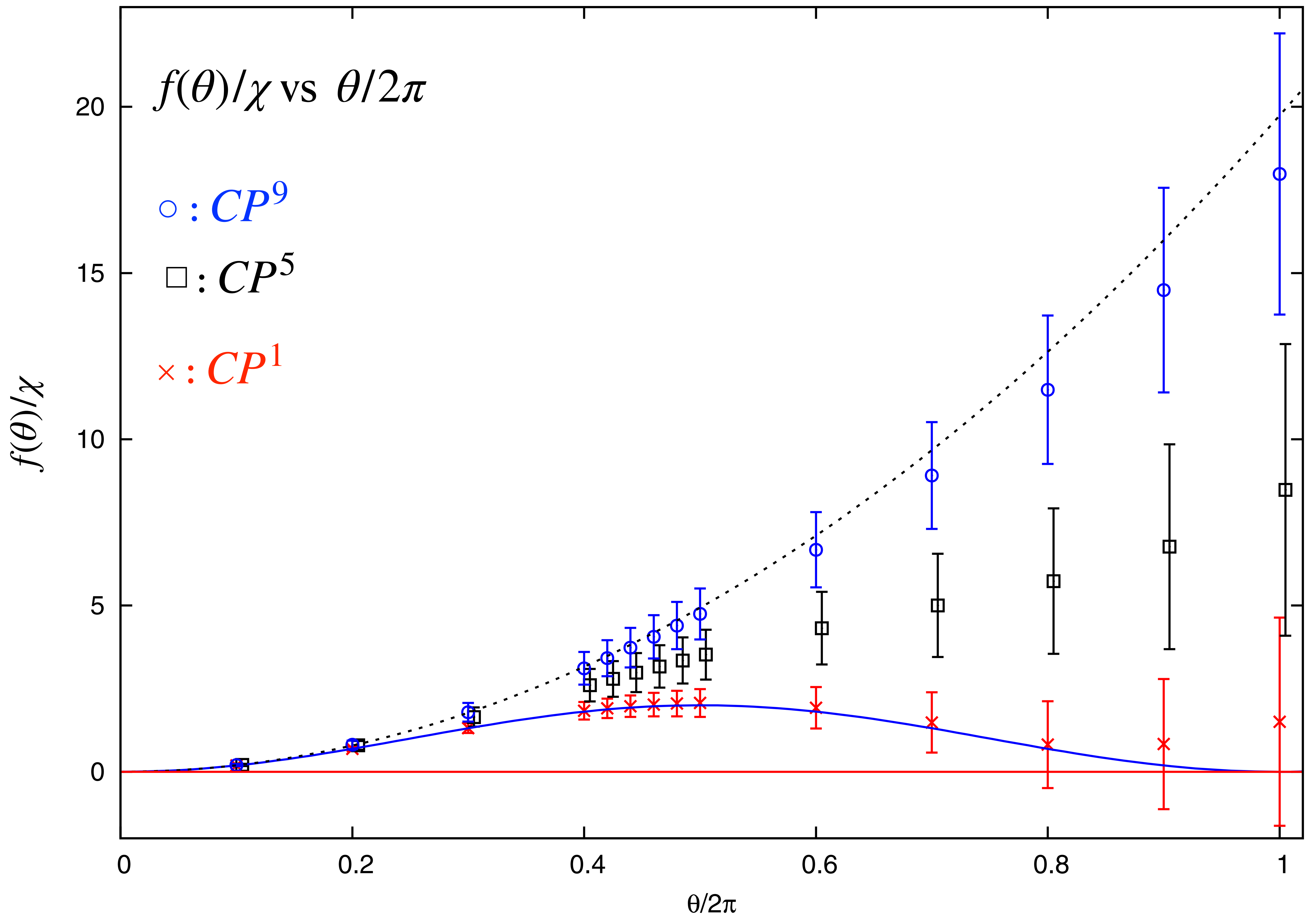

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CP^1 (*i.e.* $N = 2$) is **not large-N like,**

and gapless and no CPV at $\theta = \pi$.

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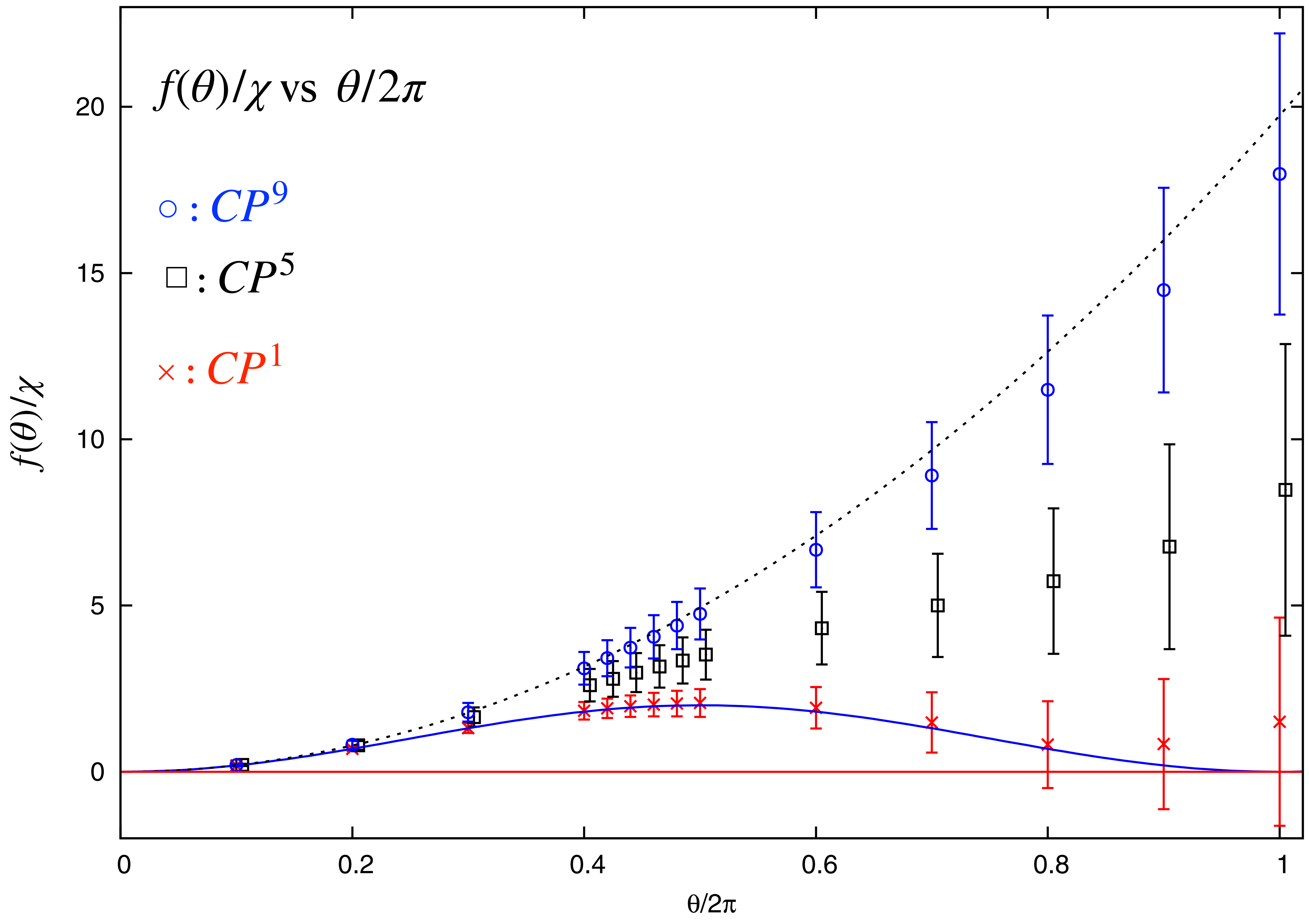

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CP^1 (*i.e.* $N=2$) is not large-N like,

and gapless and no CPV at $\theta=\pi$.

$\Longleftrightarrow$ Haldane conjecture

Lattice calculations of $f(\theta)$ in 4d $SU(N)$

$$\mathscr{L}_\theta = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - i\theta q$$

- Direct simulation ❌ $\Longrightarrow$ sign problem
- Taylor expansion around $\theta = 0$

$$f(\theta) = \frac{\chi}{2}\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \cdots)$$

- Determine each coefficient on the lattice by

$$\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V}$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle^2_{\theta=0}}{12\langle Q^2 \rangle_{\theta=0}}$$

$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15\langle Q^2 \rangle_{\theta=0}\langle Q^4 \rangle_{\theta=0} + 30\langle Q^2 \rangle^3_{\theta=0}}{360\langle Q^2 \rangle_{\theta=0}}$$

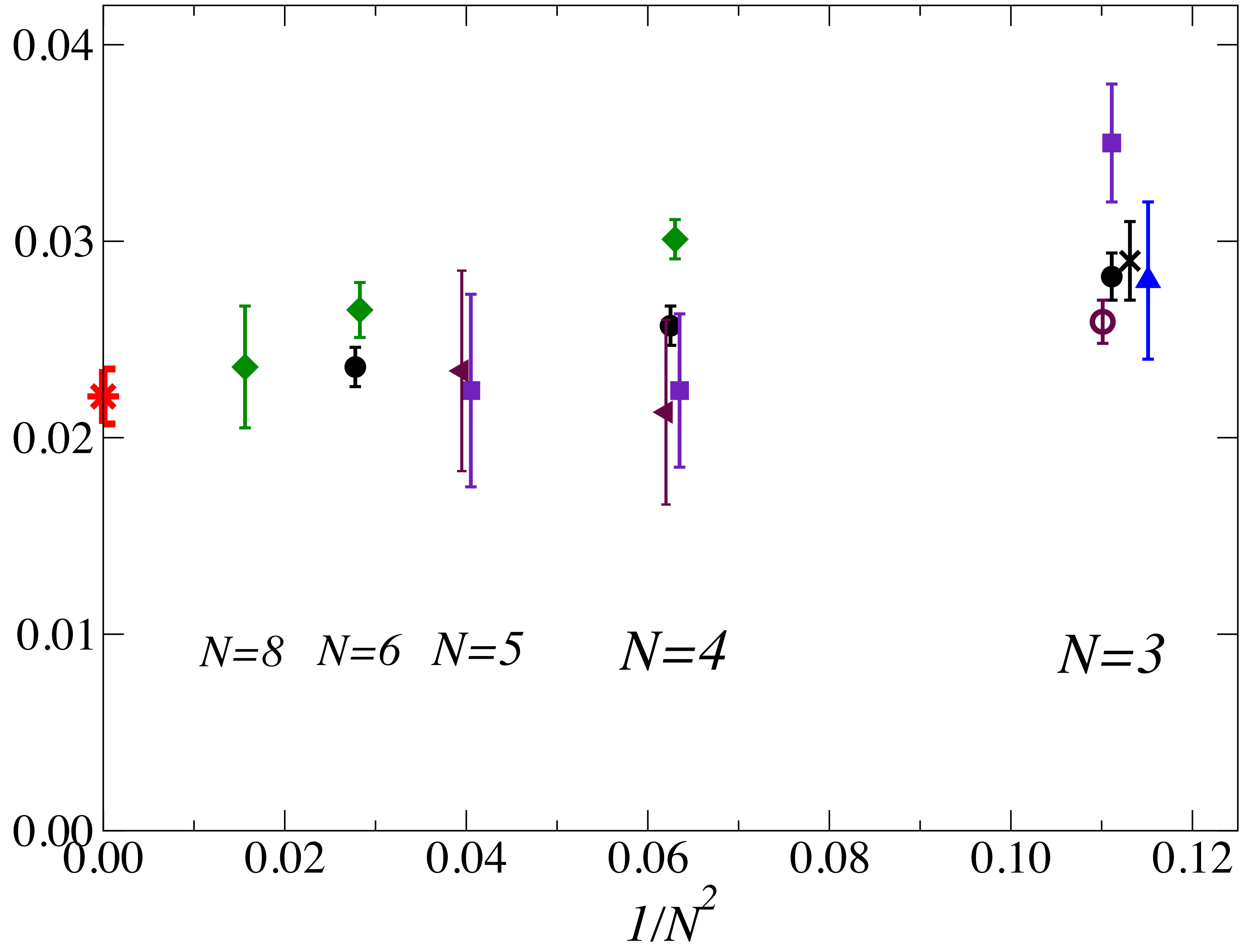
$$\vdots$$

If DIGA works, $f(\theta) = \chi(1 - \cos\theta) \;\Rightarrow b_2 = -\frac{1}{12}, \;\cdots$

First two coefficients for $N_c \geq 3$

[Review by Vicari and Panagopoulos (2018)]

$$\chi/\sigma^2 = C_\infty + \frac{c_2}{N_c^2} + O(1/N_c^4)$$



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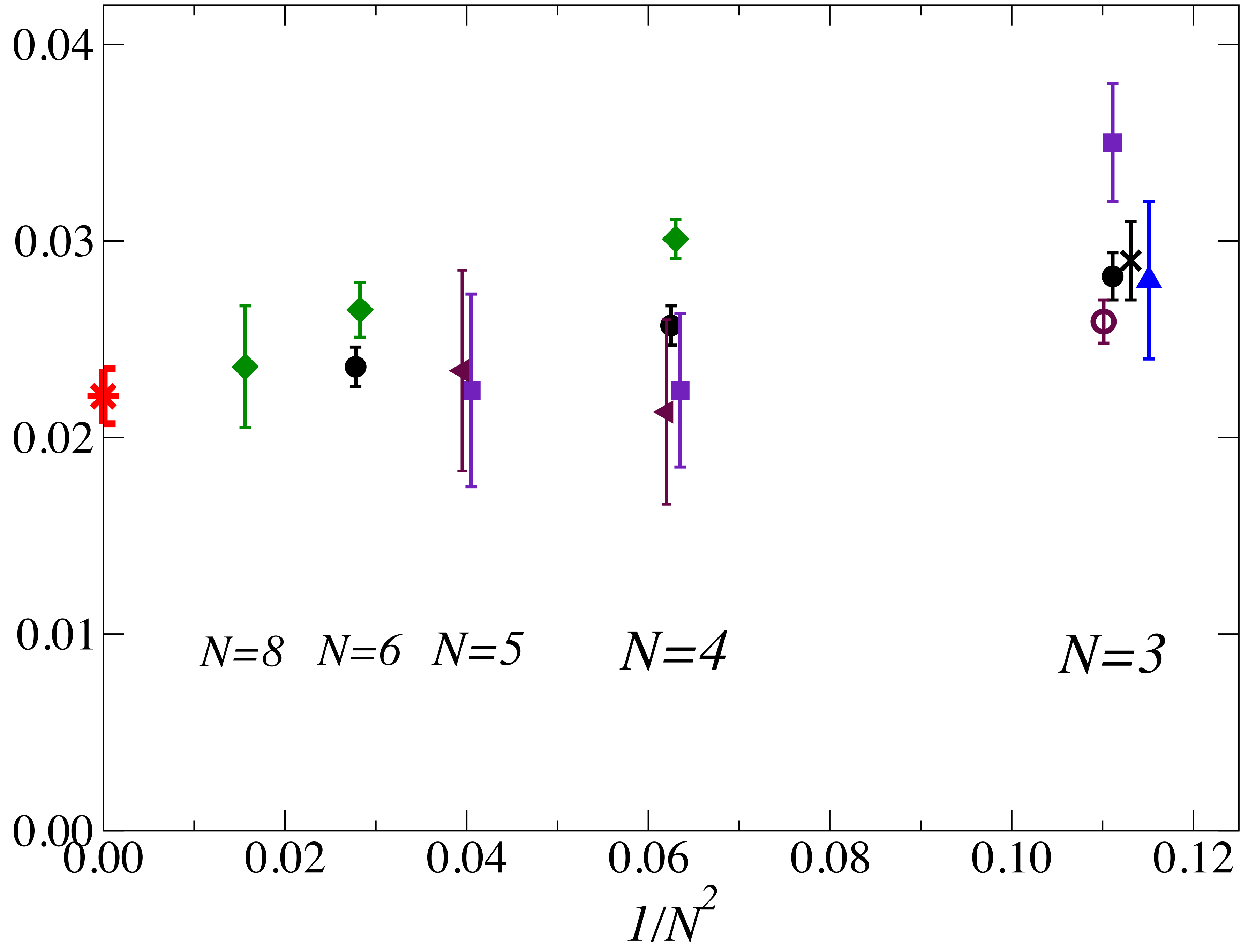

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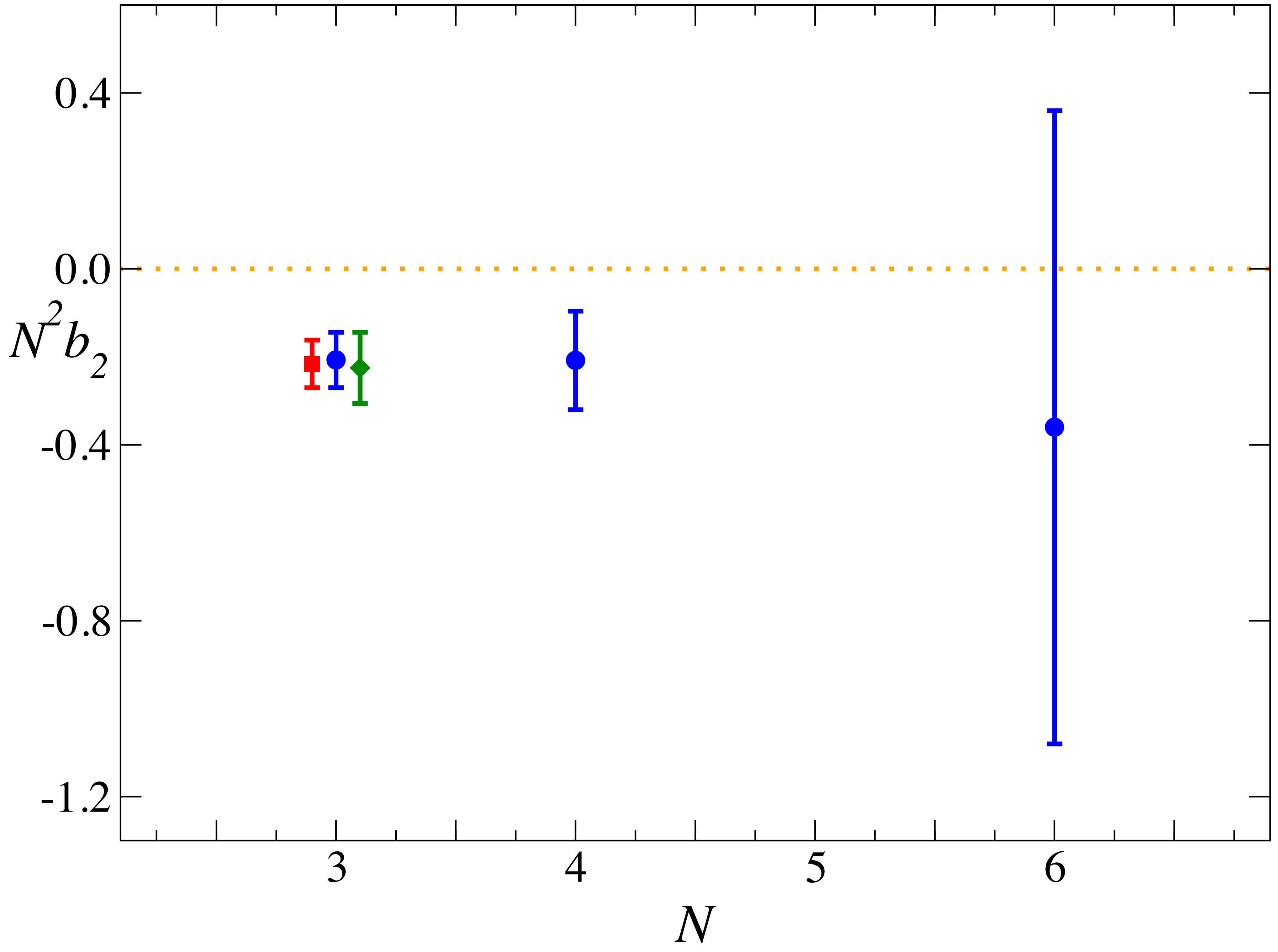

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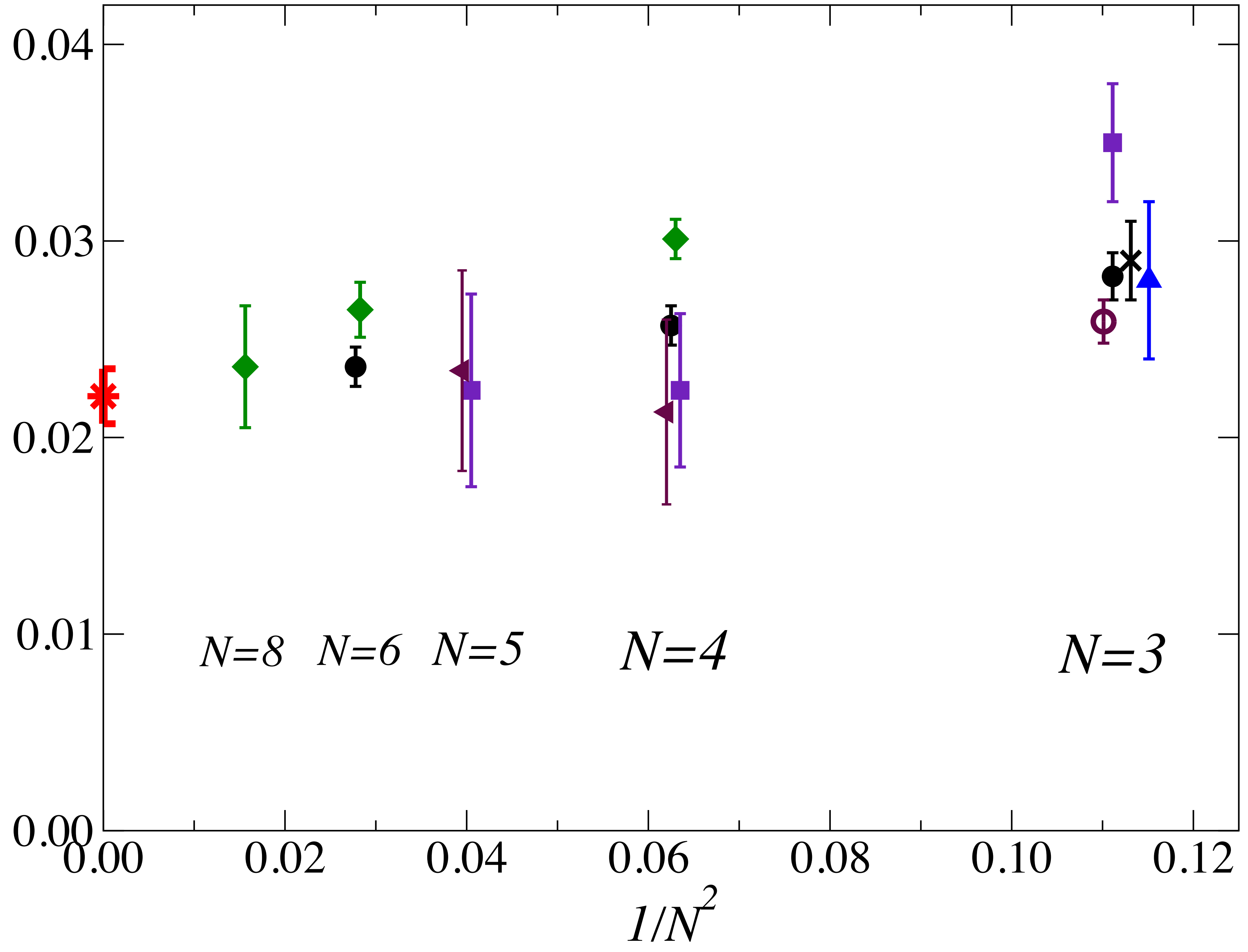

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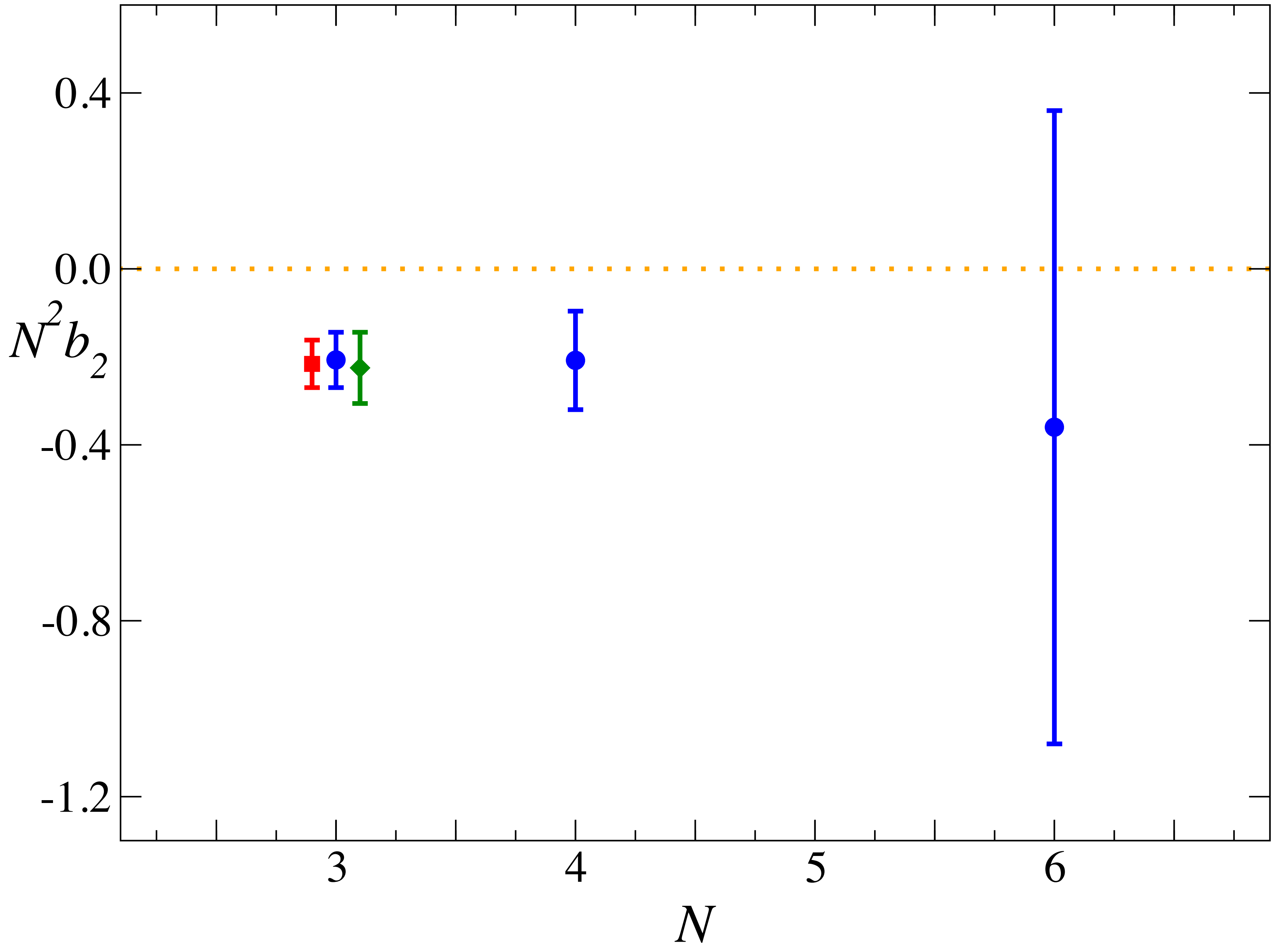
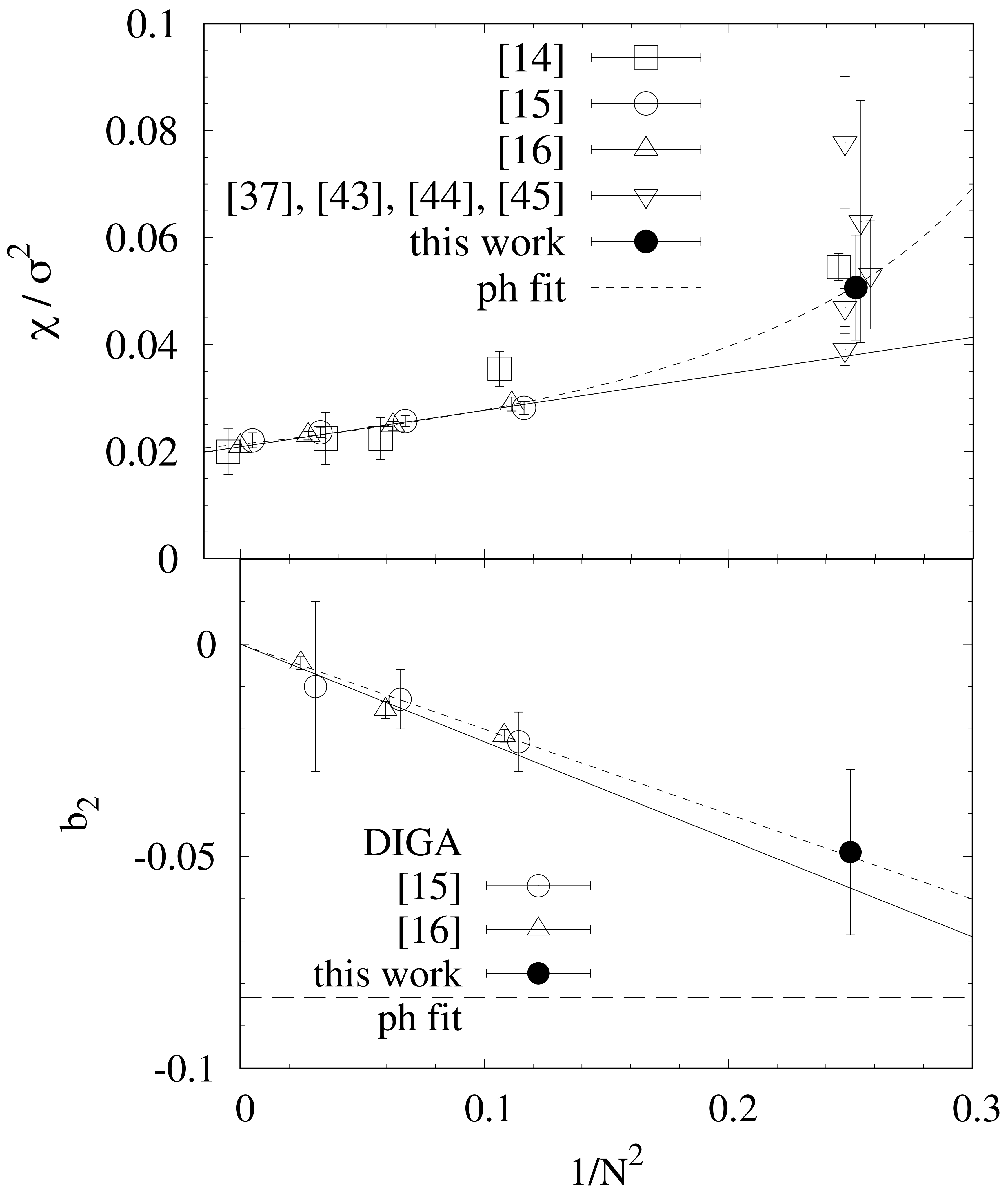


These behavior looks smooth ⇒ Nothing special happens down to $N_c = 3$

How about $N_c = 2$?

χ and b_2 in $SU(2)$ at $T = 0$

[Kitano, NY, Yamazaki (2021)]



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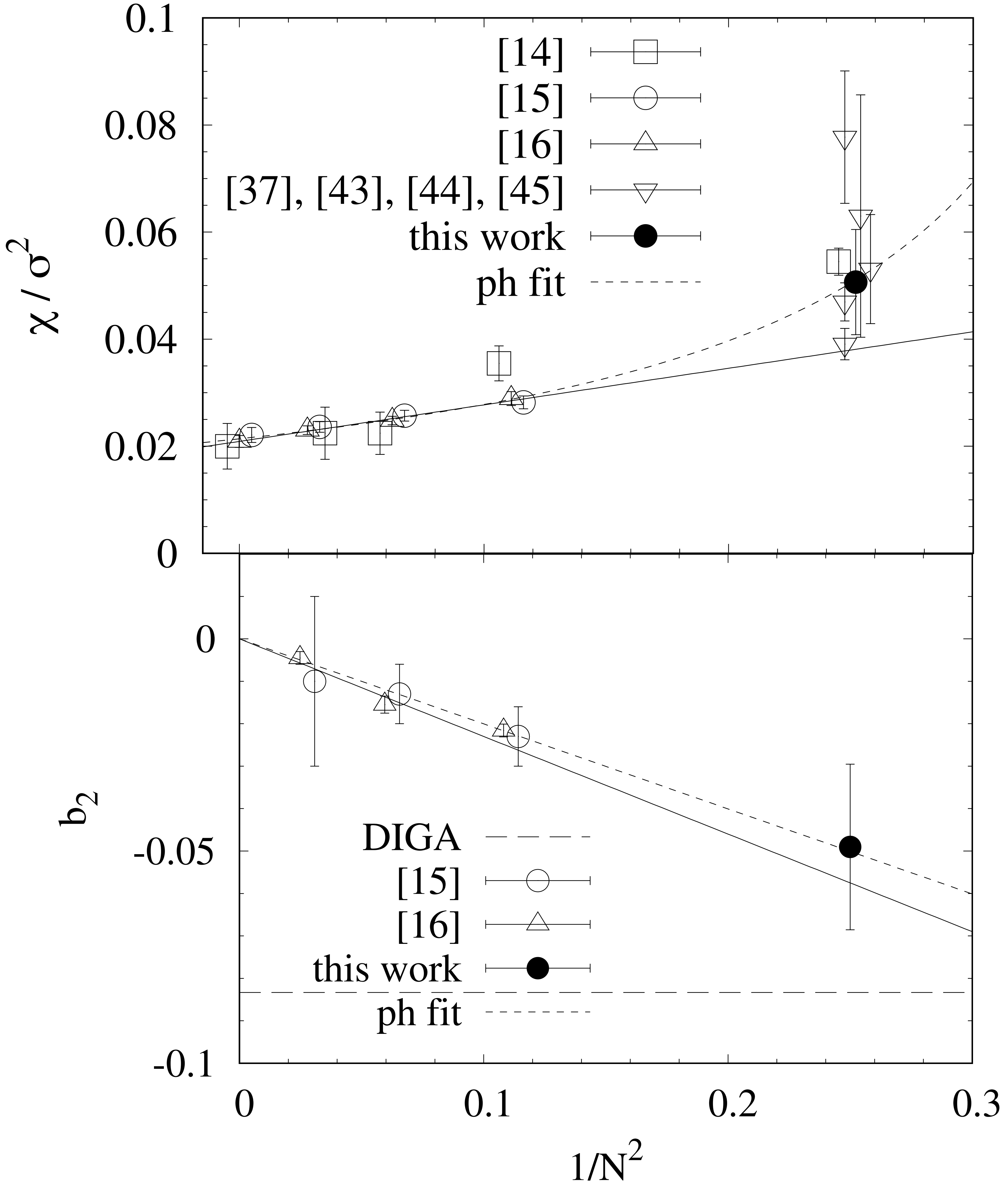

- Still smoothly connected down to $N_c = 2$

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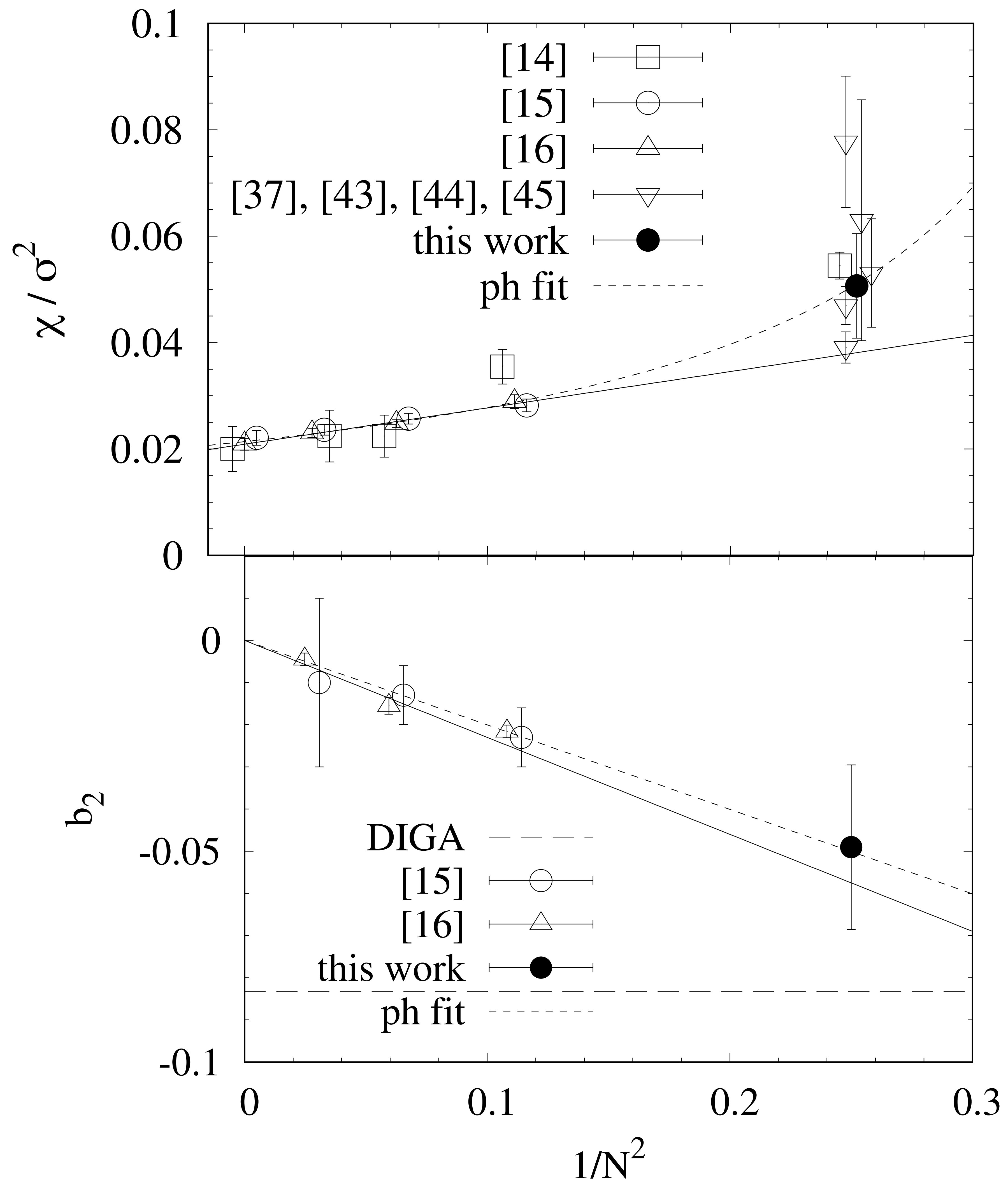

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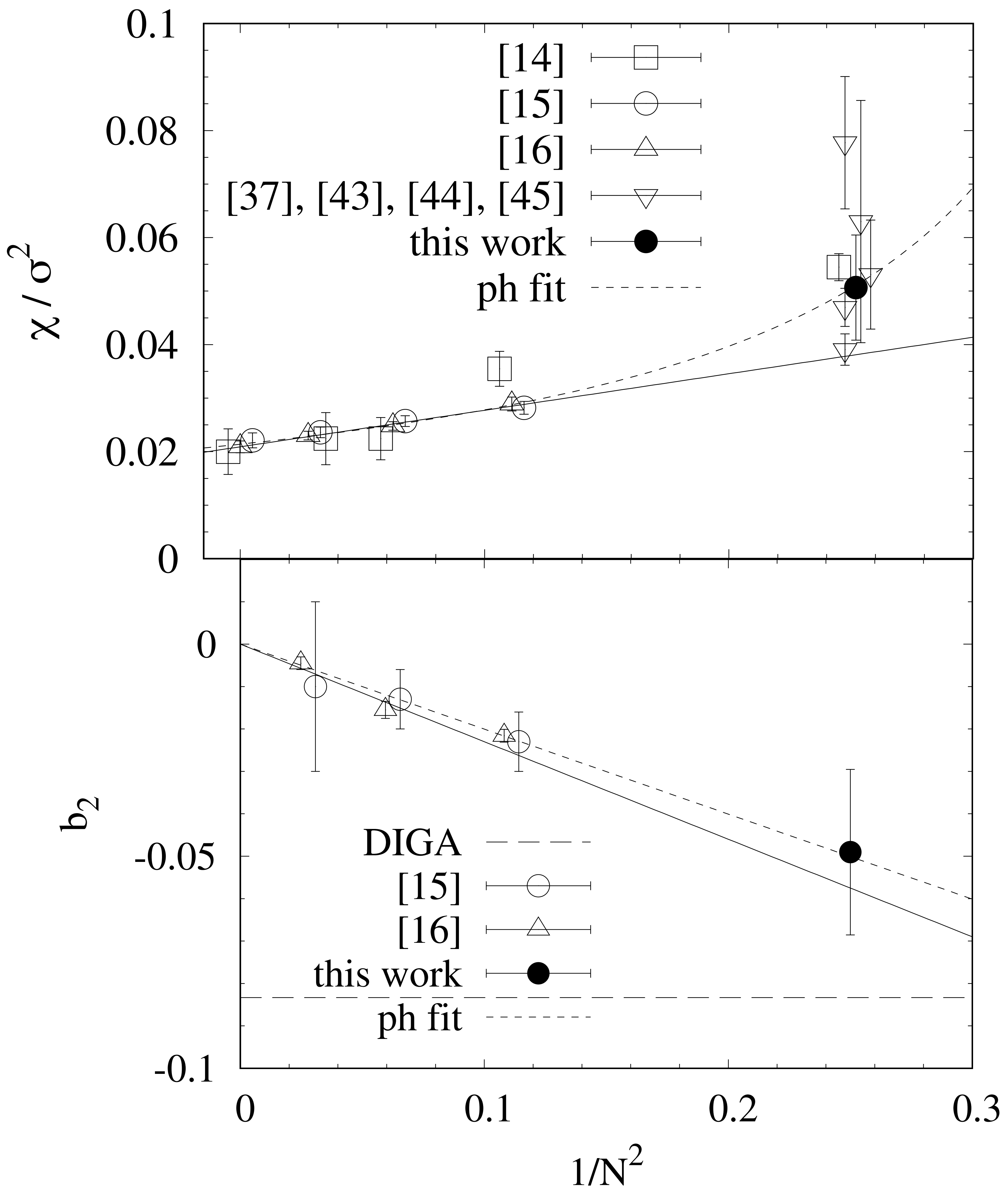

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Speculation:
$SU(2)$ YM belongs to large N_c class and
CPV takes places at $\theta = \pi$.

Conjectured θ-T phase diagram

- θ-vacuum undergoes CPV at $\theta = \pi$ when $N_c \gg 1$.
- Above T_c, instanton calc. $\Rightarrow$ no CPV at $\theta = \pi$.
  [Frison, Kitano, Matsufuru, Mori and NY(2016)]
- "For general N_c , CP has to be broken at $\theta = \pi$ if the vacuum is in the confining phase."
  [Gaiotto, et al.(2017)], [Kitano, Suyama, NY(2017)]
- $T_c(\theta)$ is available for SU(3) around $\theta = 0$.
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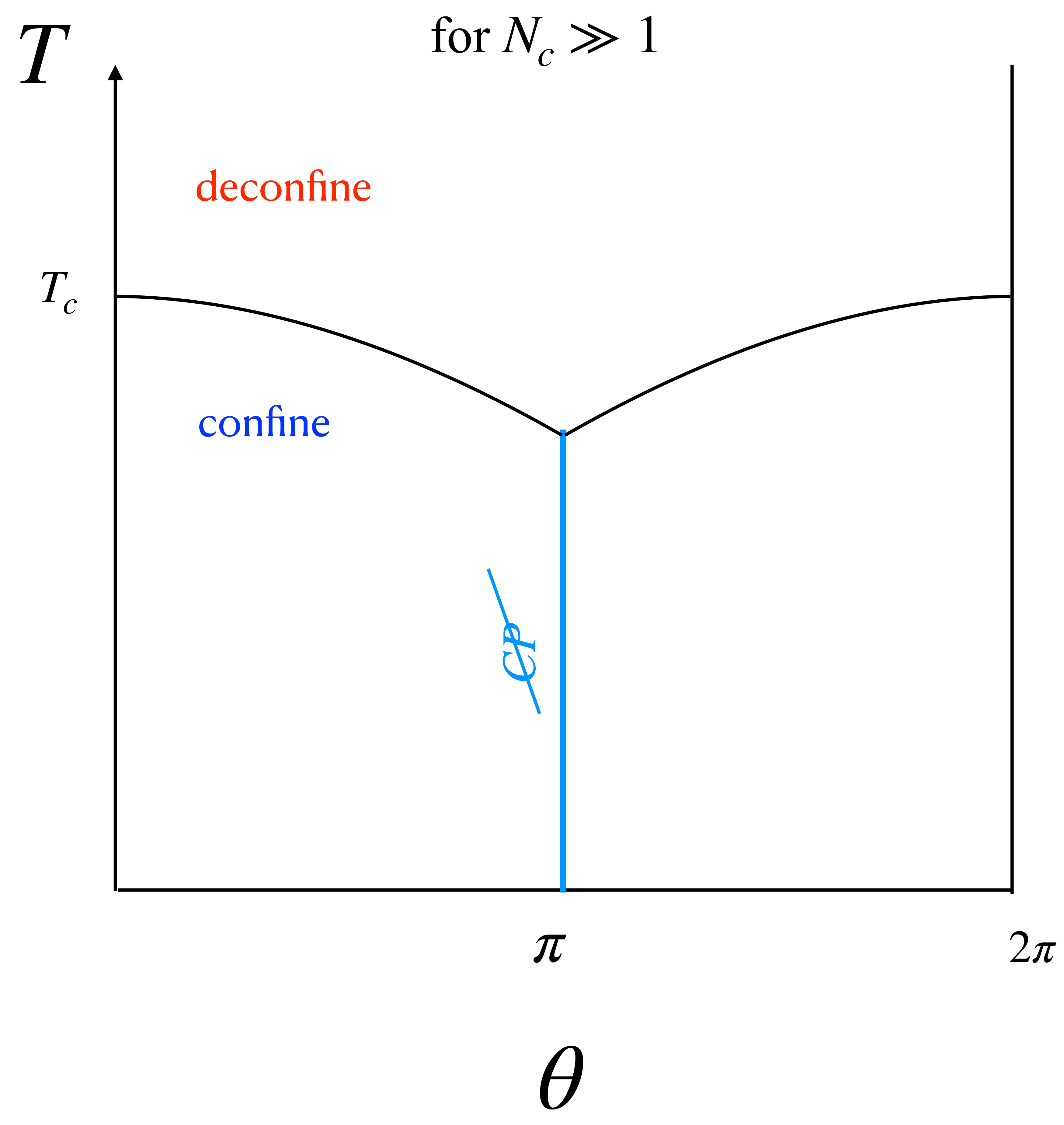

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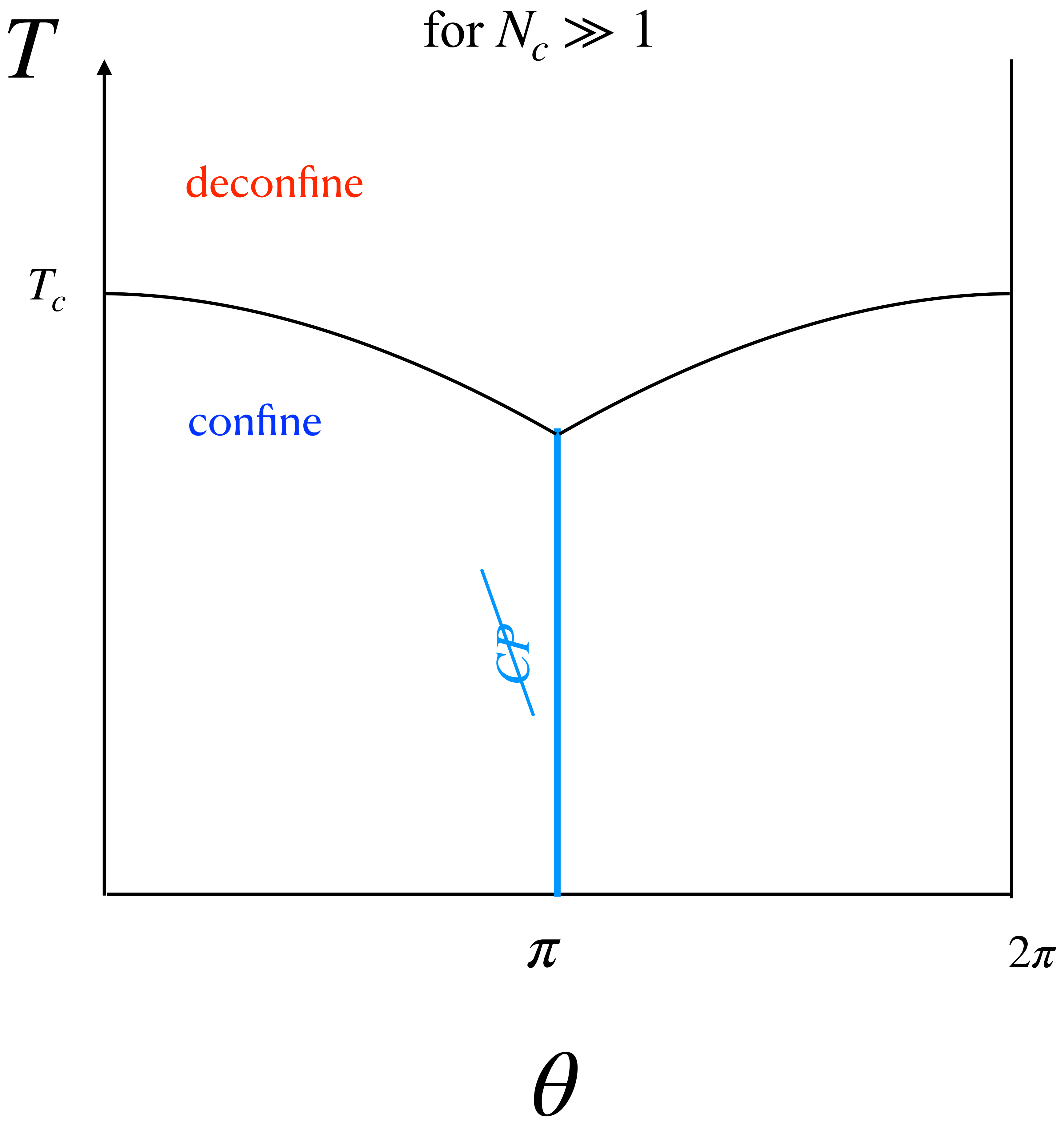
We want to check **whether $SU(2)$ is large N_c-like.**

$\Rightarrow$ How to avoid the sign problem ?

4d $SU(N_c)$ YM



Sub-volume method

[Kitano,Matsudo,NY,Yamazaki(2021)]

c.f. [Keith-Hynes and Thacker (2008)]

does not rely on any expansions

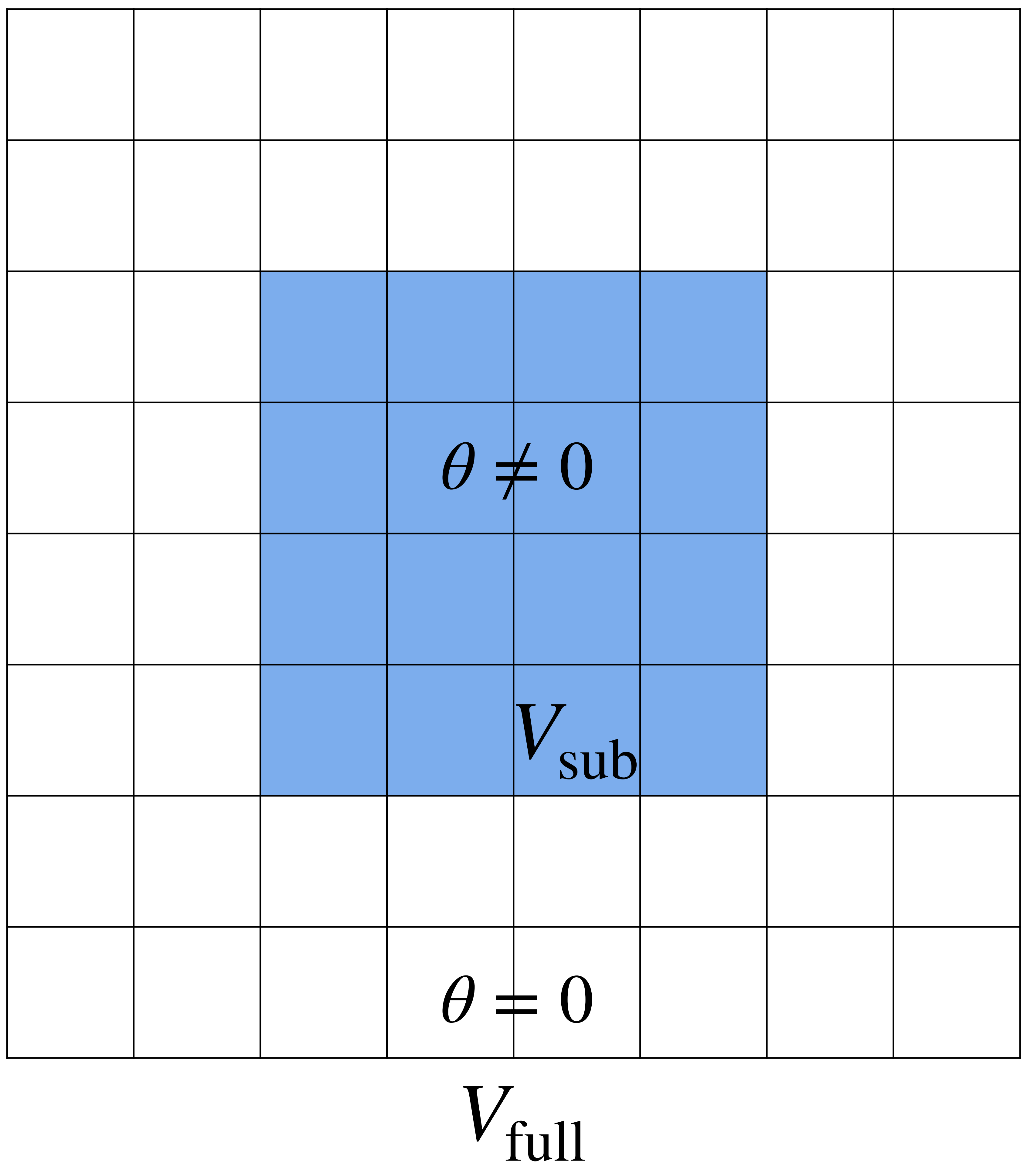
Replace Q with $Q_{\rm sub} = \sum_{x\in V_{\rm sub}} q(x) \notin \mathbb{Z}$

where $V_{\rm sub} = l^4$ is a sub-volume.

$$e^{-V_{\rm sub} f_{\rm sub}(\theta)} = \frac{Z_{\rm sub}(\theta)}{Z(0)} = \frac{1}{Z(0)} \int \mathscr{D}U\, e^{-S+i\theta Q_{\rm sub}} = \langle e^{i\theta Q_{\rm sub}} \rangle$$

$$f_{\rm sub}(\theta) = -\frac{1}{V_{\rm sub}} \ln \langle \cos(\theta Q_{\rm sub}) \rangle$$

$$f(\theta) = \lim_{V_{\rm sub}\to\infty} f_{\rm sub}(\theta) = \lim_{l\to\infty} \left\{ f(\theta) + \frac{s(\theta)}{l} + O(1/l^2) \right\}$$



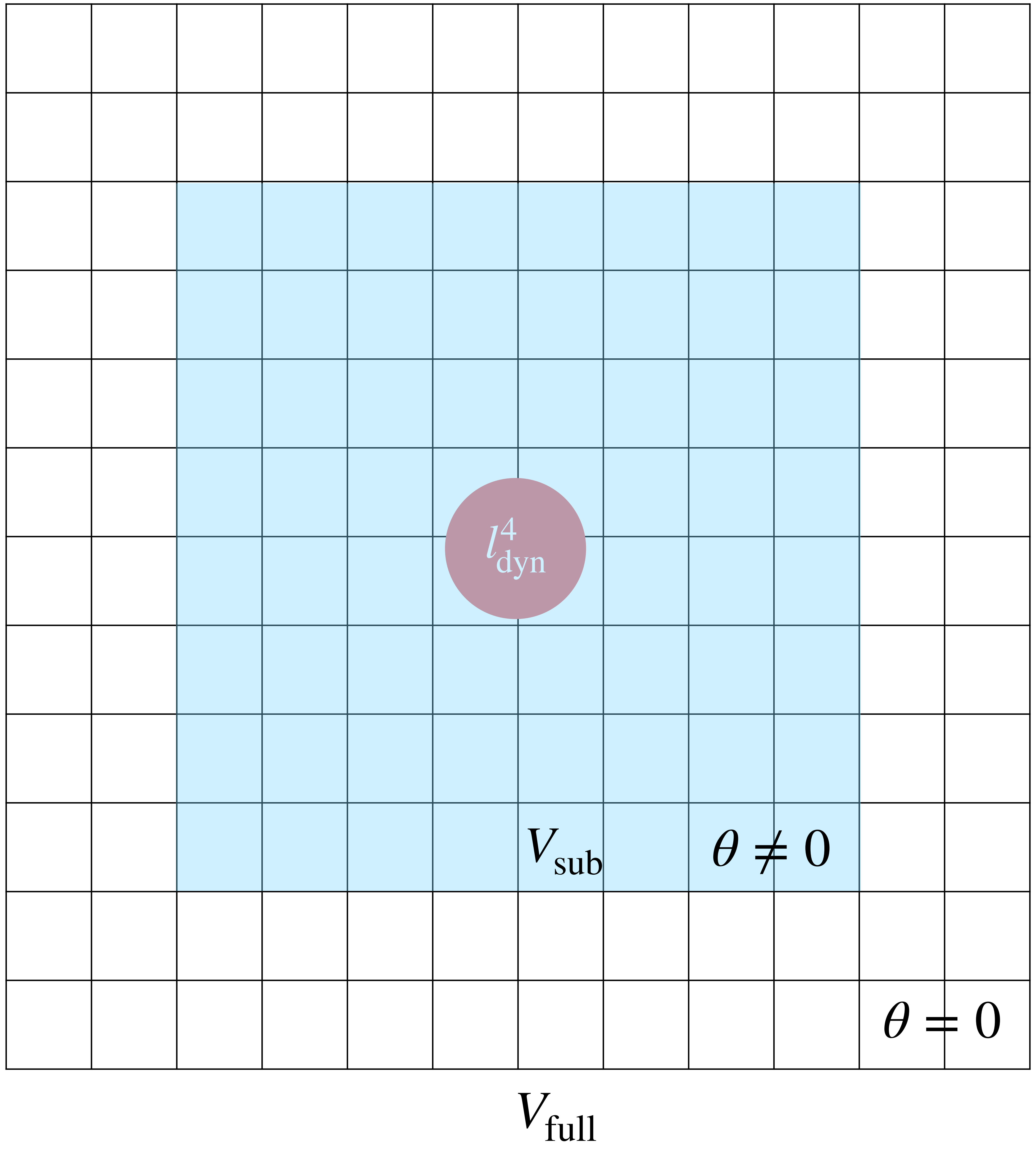
Some remarks on the sub-volume method

What is the suitable range for V_{sub} ?

- $V_{\text{sub}} \gg l_{\text{dyn}}^4$ (l_{dyn}: dynamical length scale)

- As long as $V_{\text{sub}} \gg l_{\text{dyn}}^4$, $f_{\text{sub}}(\theta)$ is expected to show a scaling behavior, $f_{\text{sub}}(\theta) = f(\theta) + \dfrac{s(\theta)}{l} + O(1/l^2)$.

- As $V_{\text{sub}} \to V_{\text{full}}$, finite size effects may appear. $\Rightarrow V_{\text{sub}} \ll V_{\text{full}}$.



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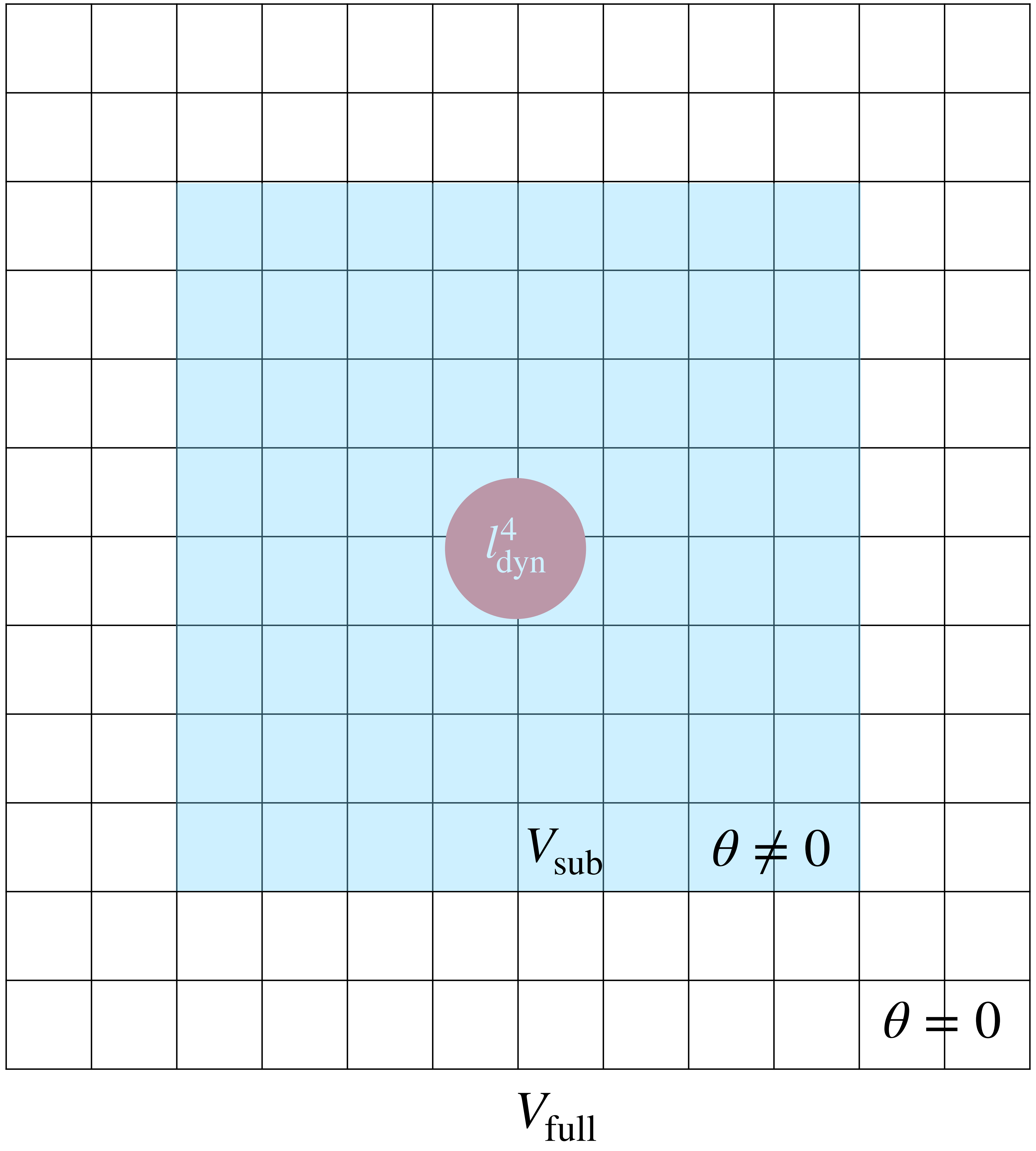
$$l_{\rm dyn}^4 \ll V_{\rm sub} \ll V_{\rm full}$$

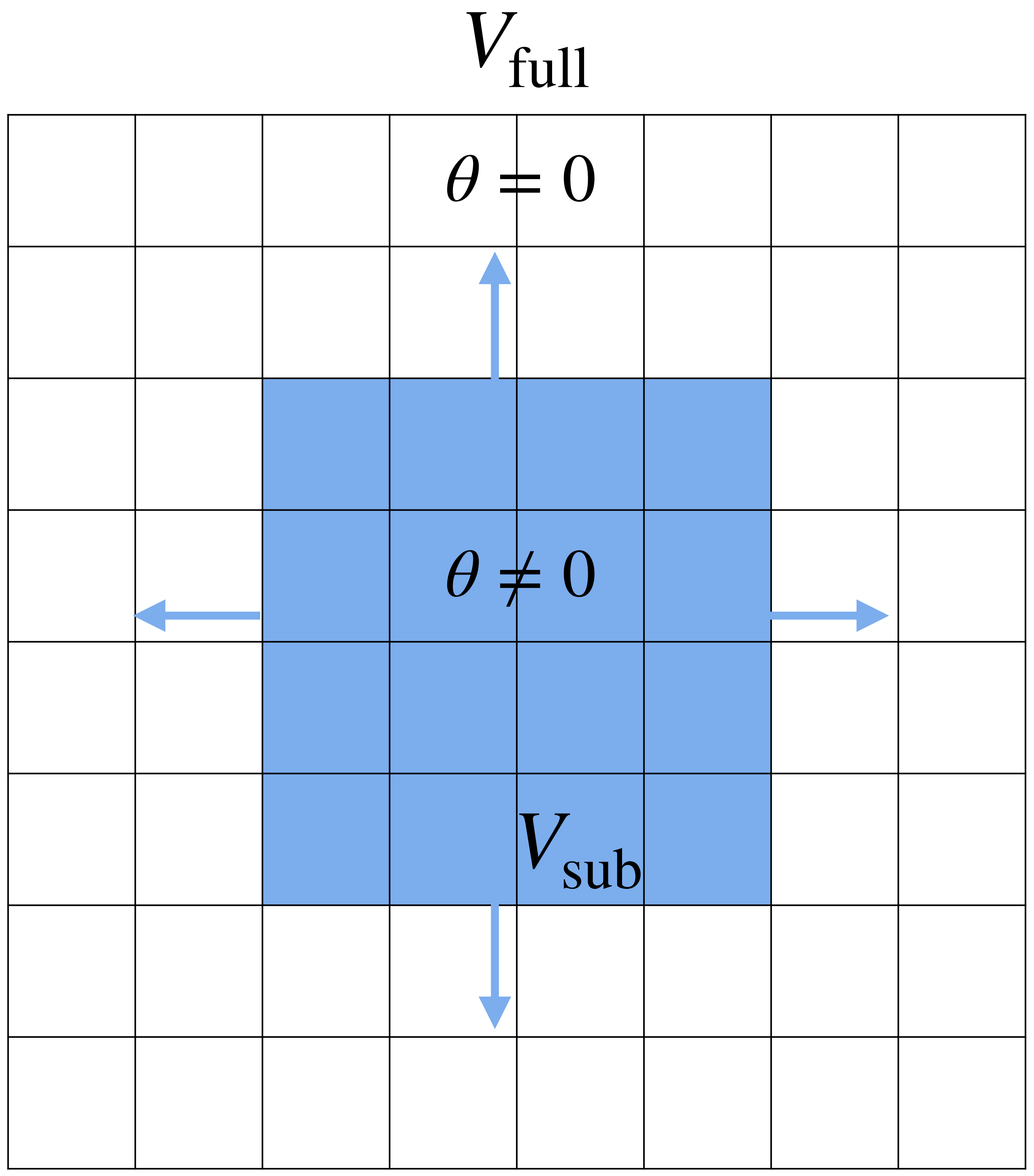


Lattice parameters and observables

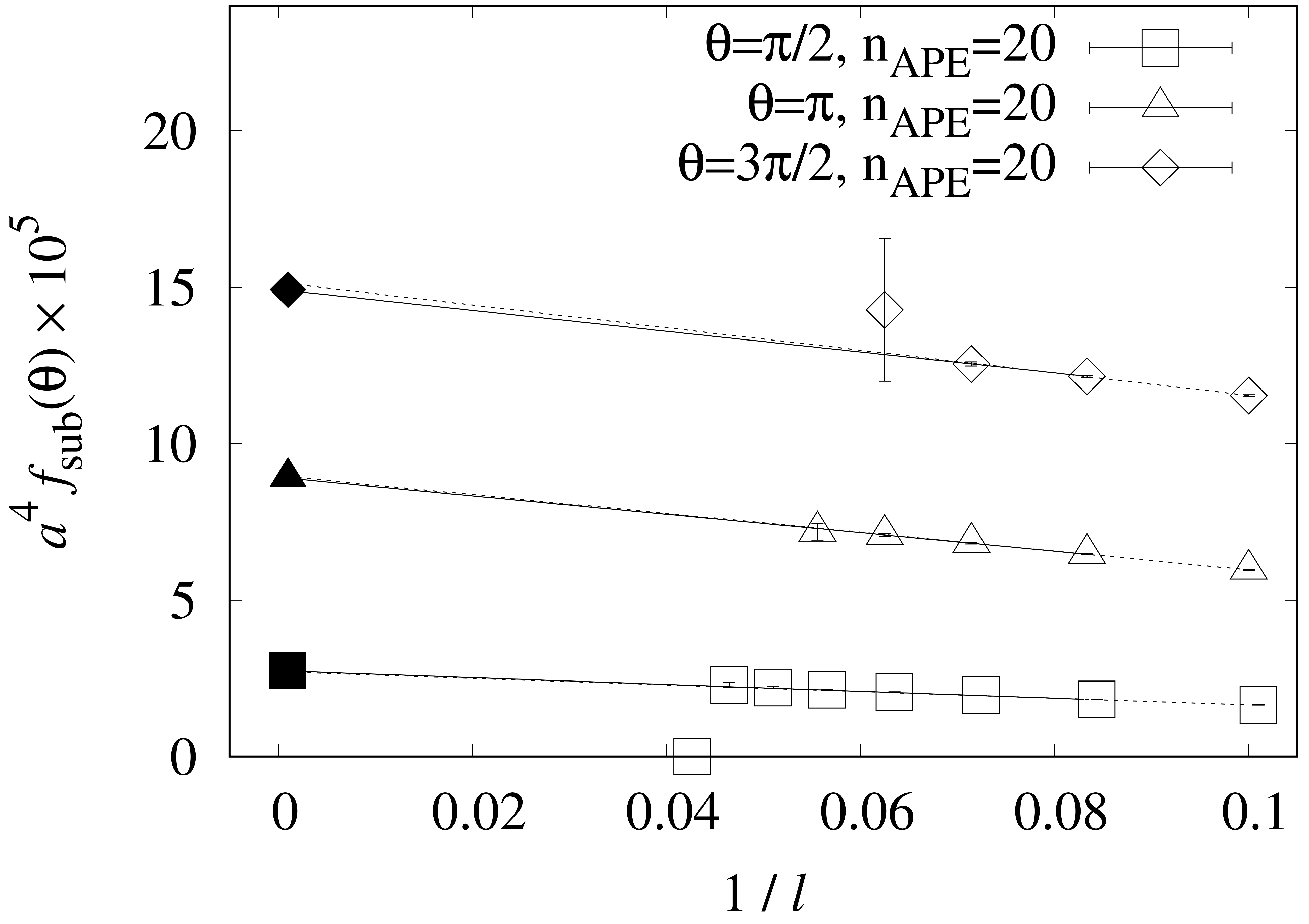
- $SU(2)$ YM by Symanzik improved gauge action
- $\beta = \dfrac{4}{g^2} = 1.975$ [cf. $1/(aT_c) = 9.50$]
- $V_{\text{full}} = 24^3 \times \{48,\ 8,\ 6\}$ ($T = 0,\ 1.2T_c,\ 1.6T_c$)
- Periodic boundary conditions
- # of configs = { 68000 , 5000 , 5000 }
- $V_{\text{sub}} = l^4$ for T=0 and $V_{\text{sub}} = l^3 \times N_T$ for finite T
- After applying APE smearing, we estimate

✓ $f(\theta) = -\lim_{V_{\text{sub}}\to\infty} \dfrac{1}{V_{\text{sub}}} \ln\langle \cos(\theta Q_{\text{sub}}) \rangle$

✓ $\dfrac{df(\theta)}{d\theta} = \lim_{V_{\text{sub}}\to\infty} \dfrac{1}{V_{\text{sub}}} \dfrac{\langle Q_{\text{sub}} \sin(\theta Q_{\text{sub}}) \rangle}{\langle \cos(\theta Q_{\text{sub}}) \rangle}$



$l \to \infty$ limit at $T = 0$

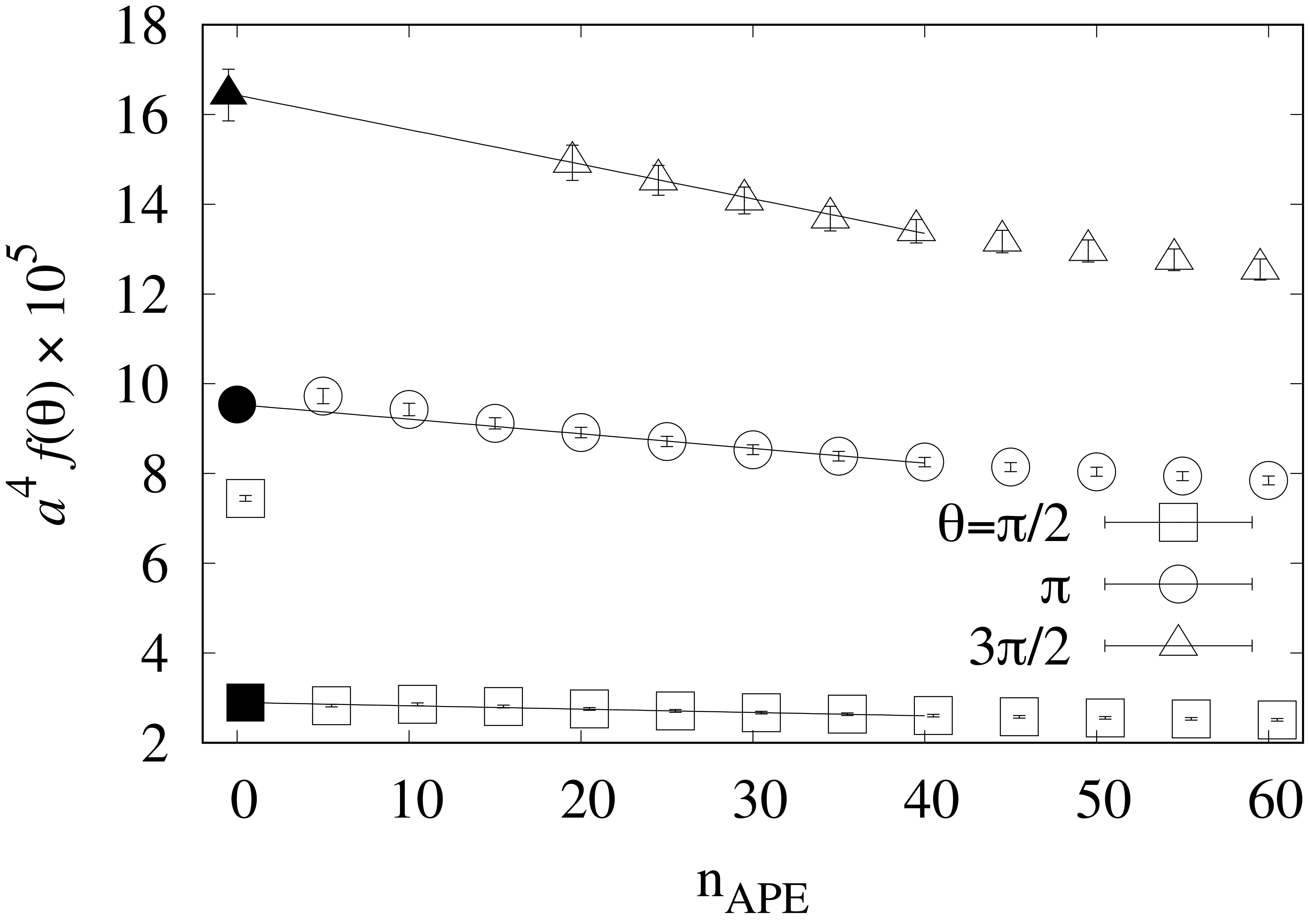


- $V_{\mathrm{sub}} = l^4$ with $l \in \{10,\ 12,\ \cdots,\ 20\}$
- Linear extrapolation with

$$f_{\mathrm{sub}}(\theta) = f(\theta) + \frac{as(\theta)}{l}$$

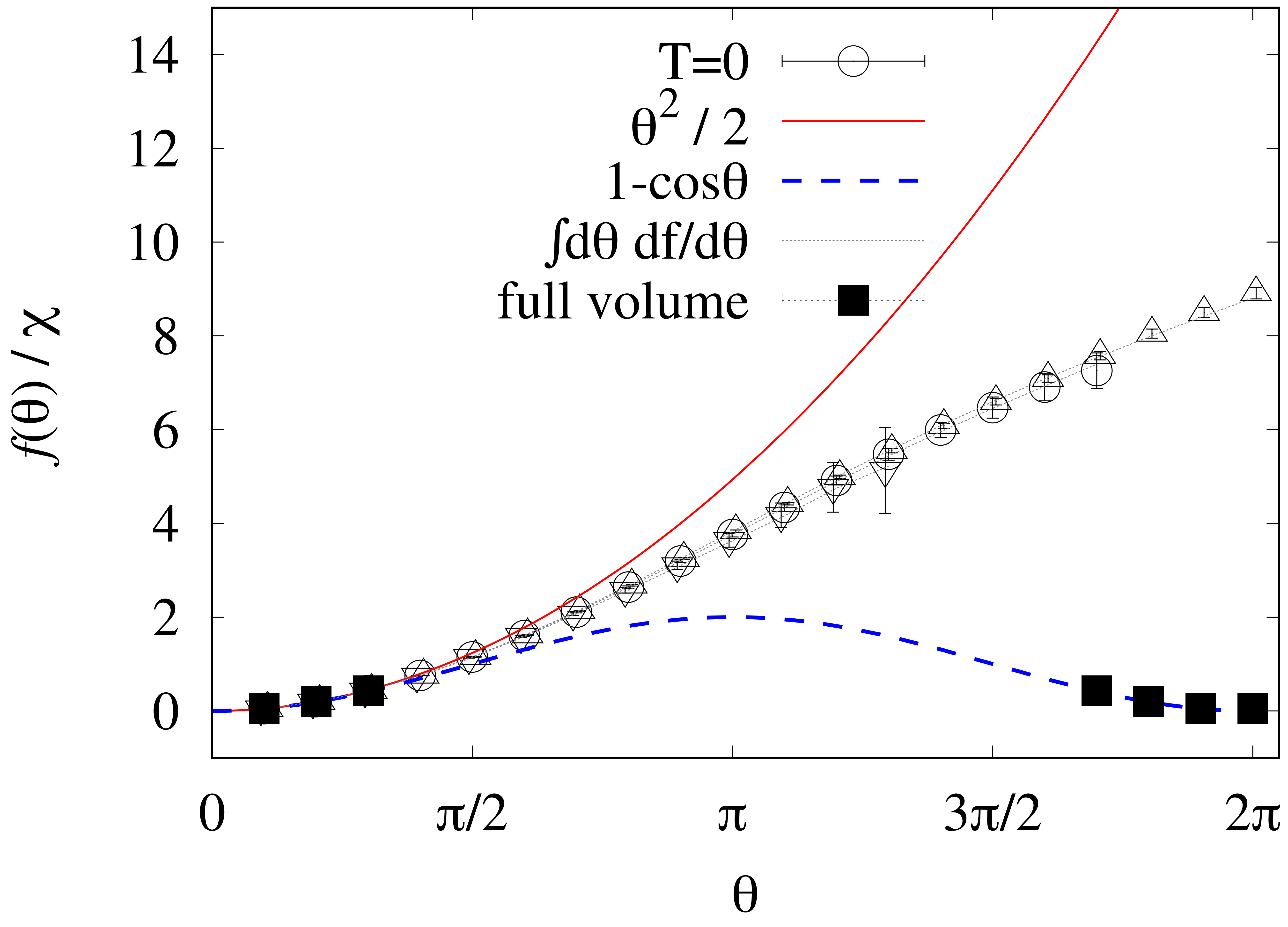
works well.

$n_{\mathrm{APE}} \to 0$ limit at $T = 0$



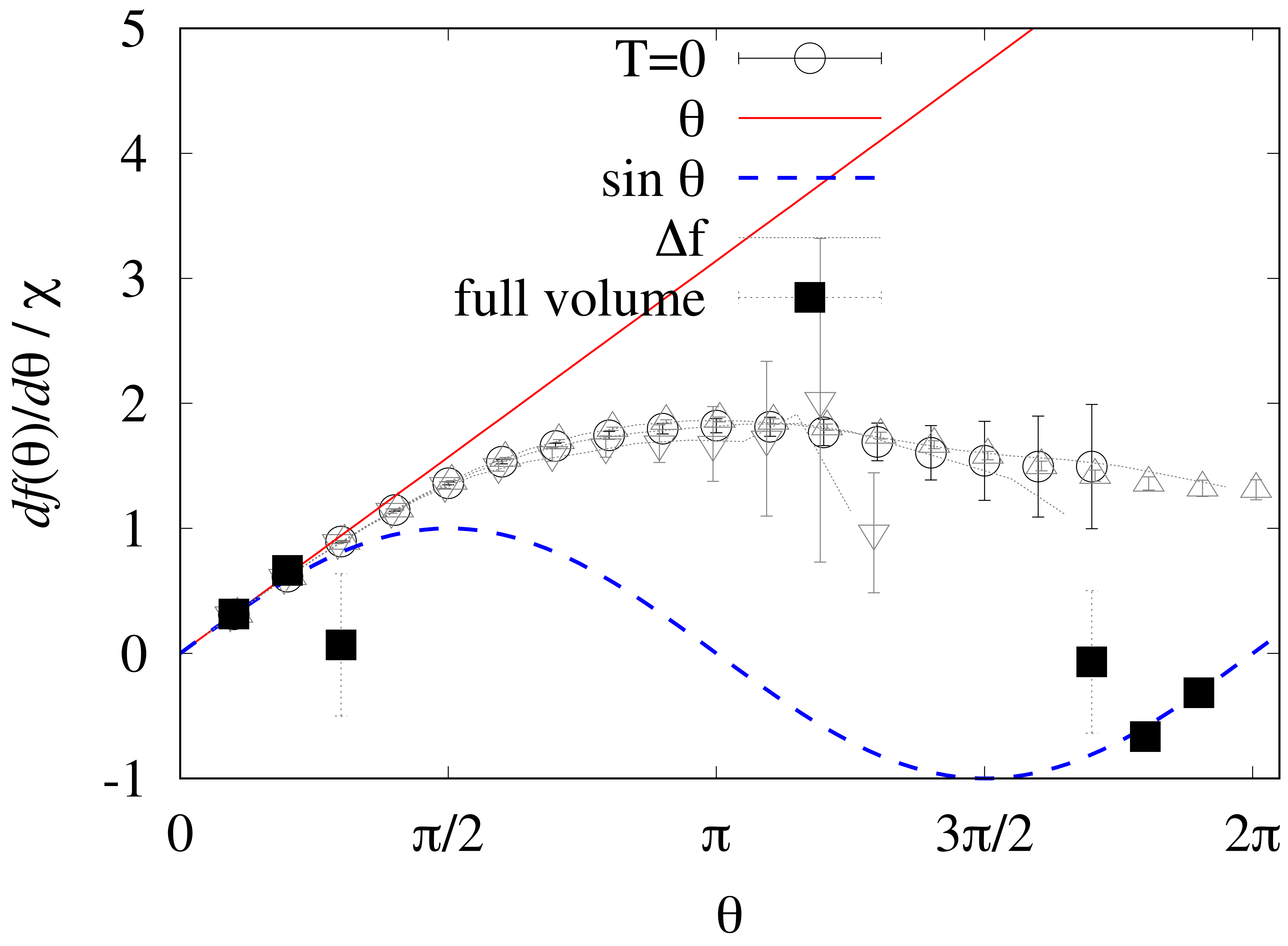
- Fit range $n_{\mathrm{APE}} = [20, 40]$ determined in [Kitano, NY, Yamazaki (2021)].
- Linear fit works well.
- Monotonic function $f(\pi) < f(3\pi/2)$

θ dependence of $f(\theta)$ at $T = 0$



- Succeed to calculate up to $\theta \sim 3\pi/2$
- Monotonically increasing function
- Inconsistent with DIGA
- Numerical consistency with $\int d\theta \, \frac{df}{d\theta}$

$df(\theta)/d\theta$ at $T = 0$

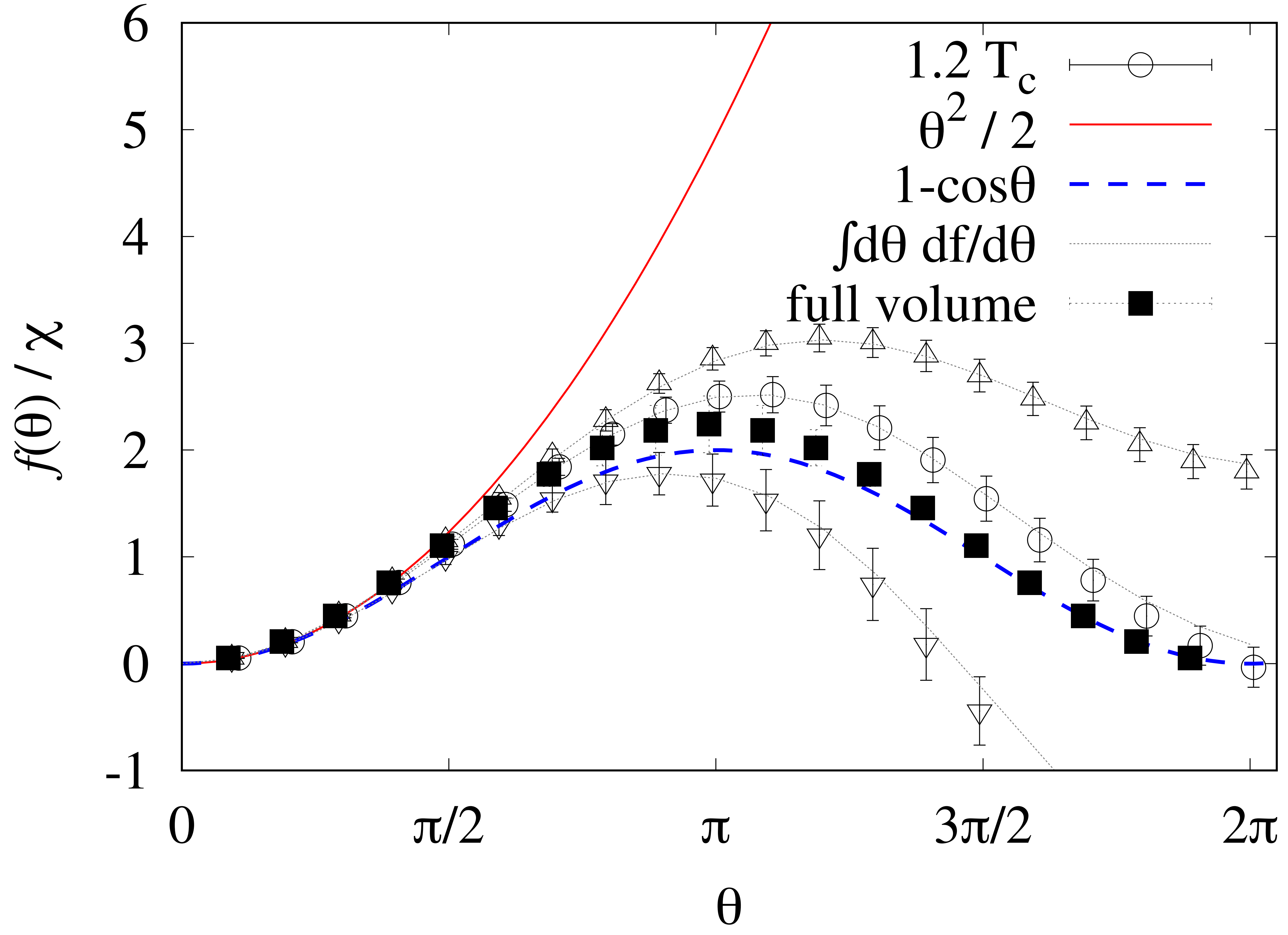


- Order parameter is non-zero

$$df(\theta)/d\theta\Big|_{\theta=\pi} = -i\langle\, q(x)\,\rangle_{\theta=\pi} \neq 0$$

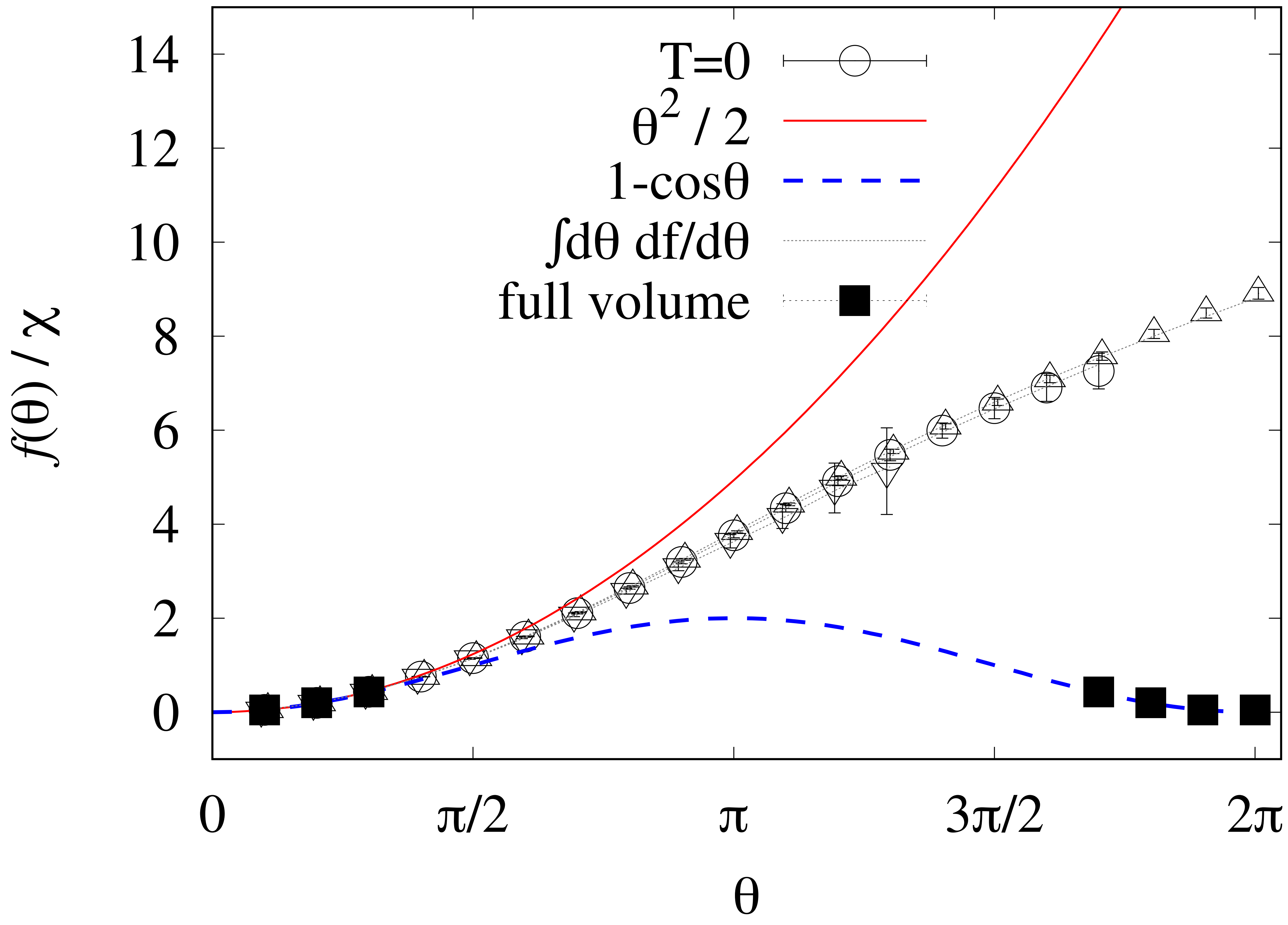
⇒ spontaneous CPV at $\theta = \pi$

θ dependence of $f(\theta)$ at $T = 1.2T_c$

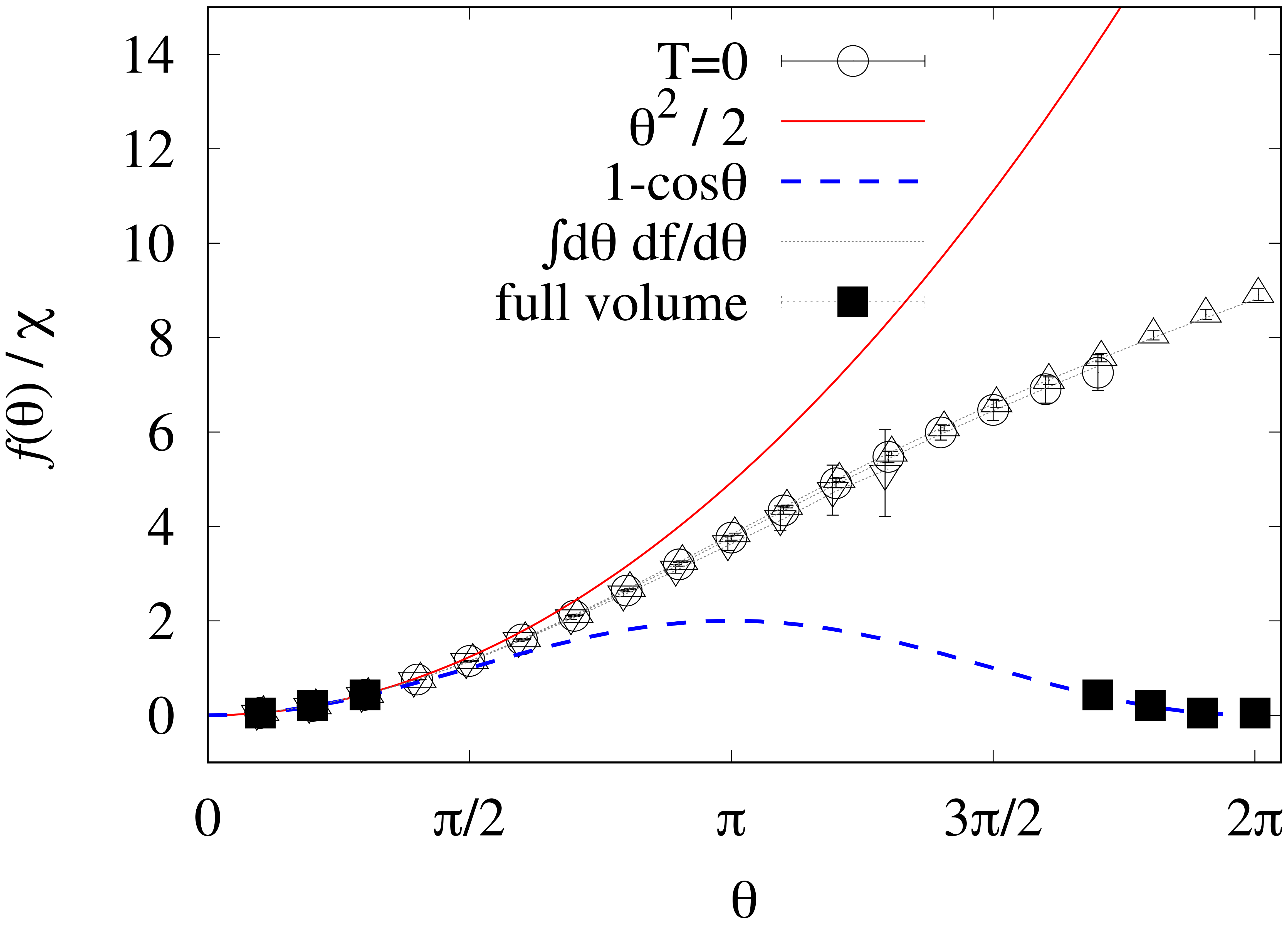


- Large uncertainty due to ambiguity of the scaling region
- Within large uncertainty, consistent with the DIGA
- Similar results at $T = 1.6\,T_c$

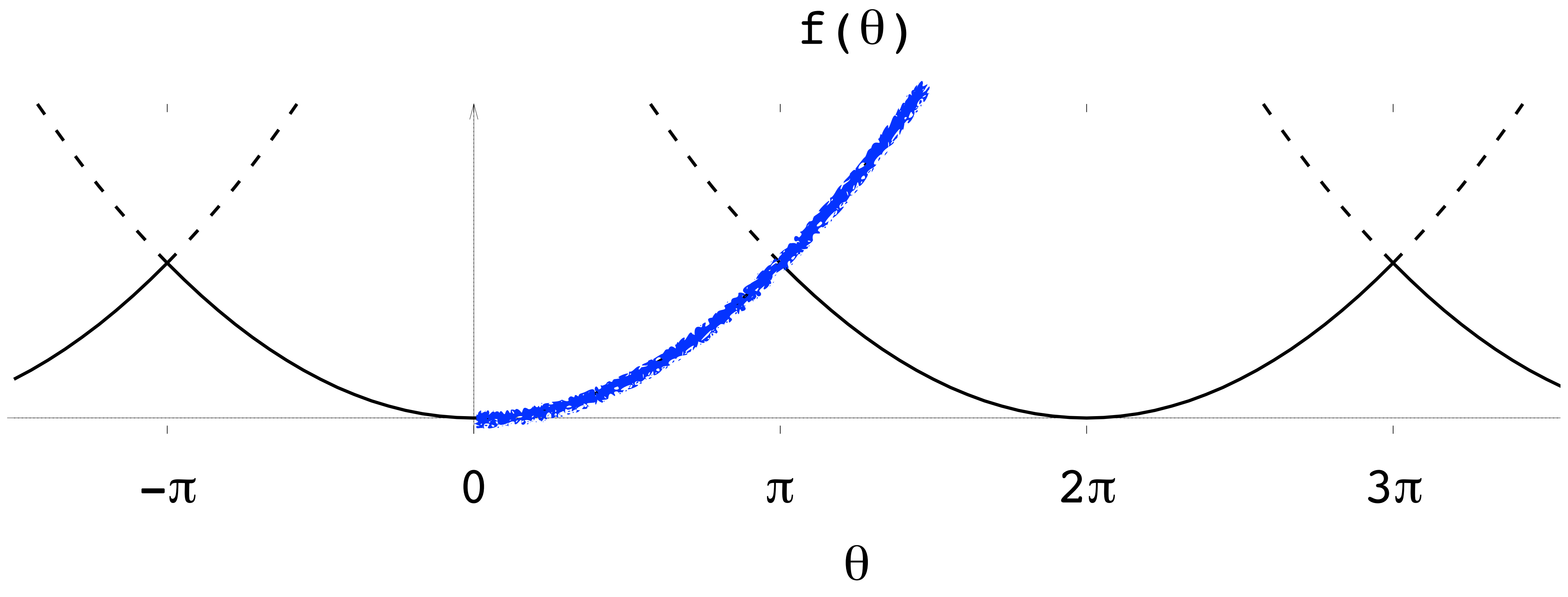
$f(\pi - \theta) = f(\pi + \theta)\,?$



$f(\pi - \theta) = f(\pi + \theta)$?



Interpretation:
Sub-volume method sticks to the original branch even after passing through the transition point.



$f(\pi - \theta) = f(\pi + \theta)$? (Cont'd)

- Similar experience in the the static potential on the dynamical configs, where"string breaking" is expected to occur at large separation but …

  $\Rightarrow$ Nothing but overlap problem

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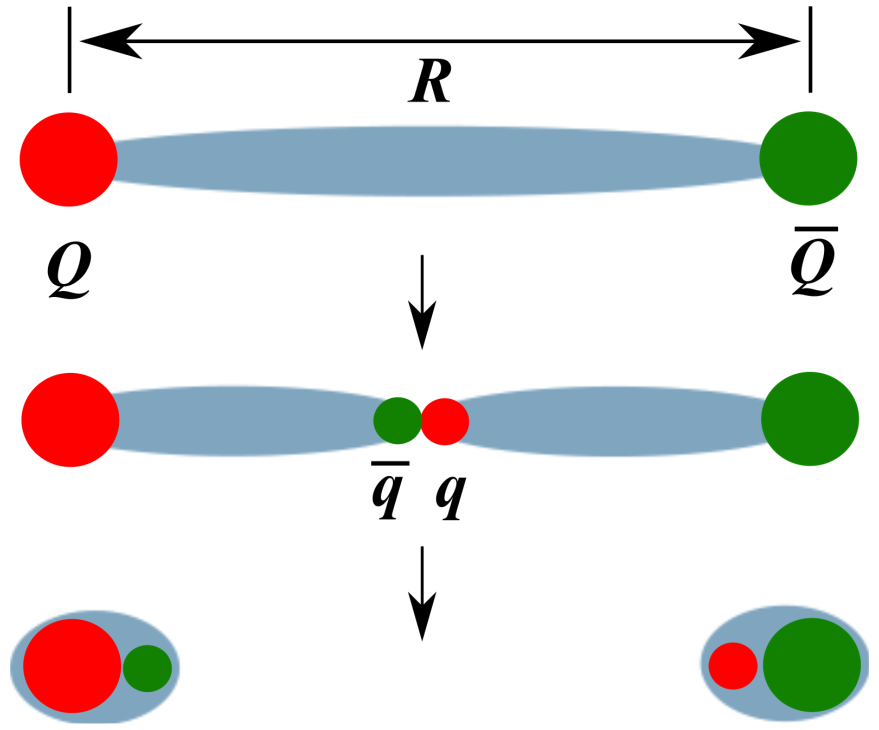

Chernodub (2010)

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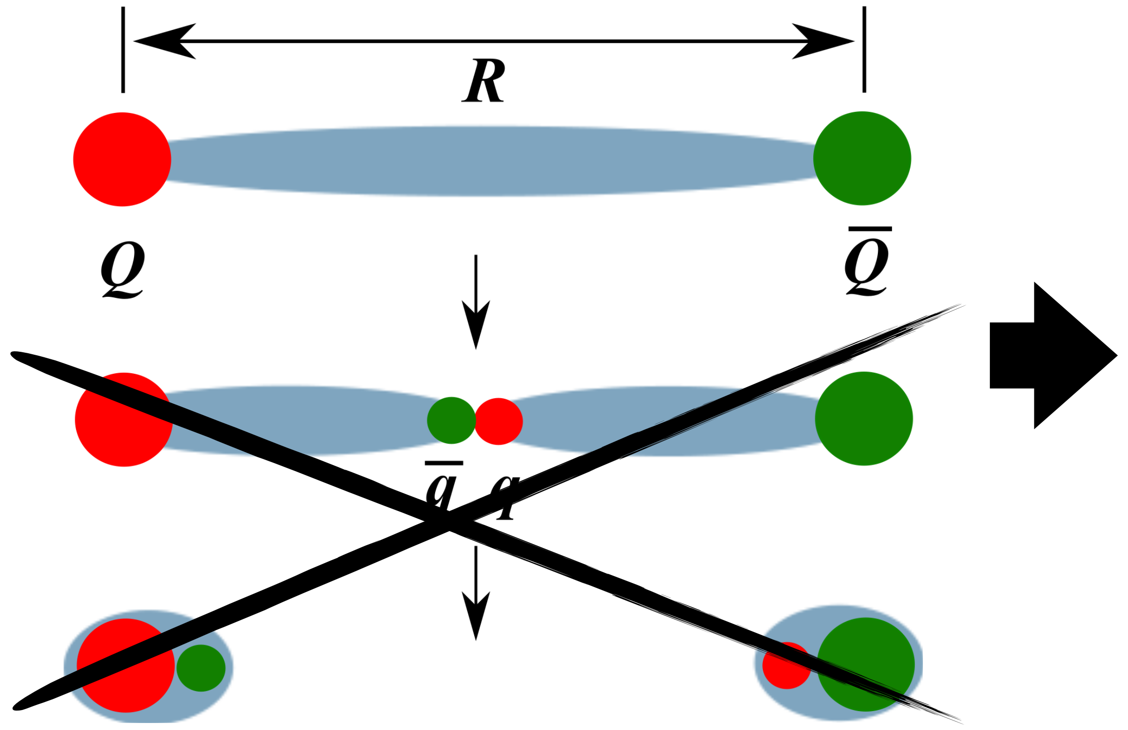

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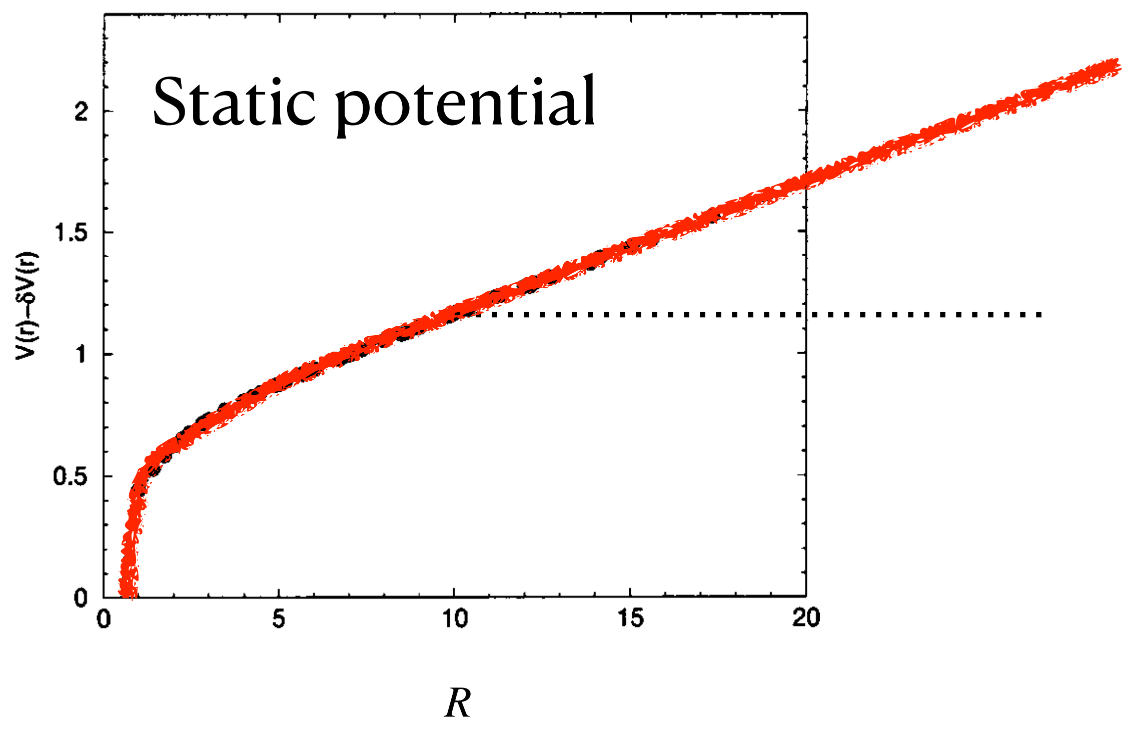

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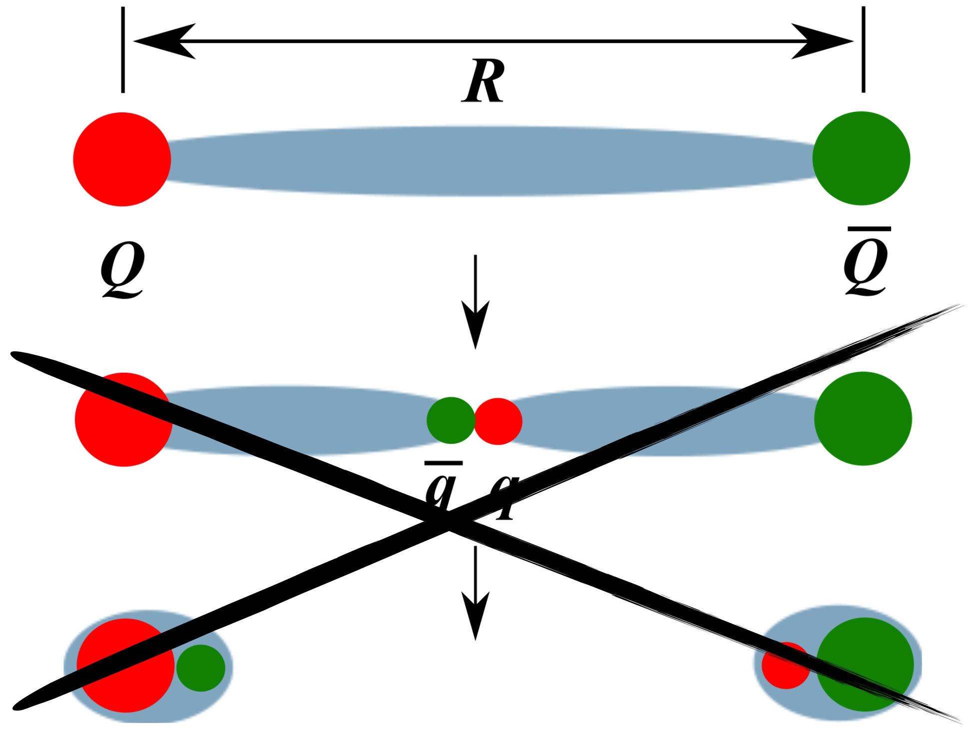

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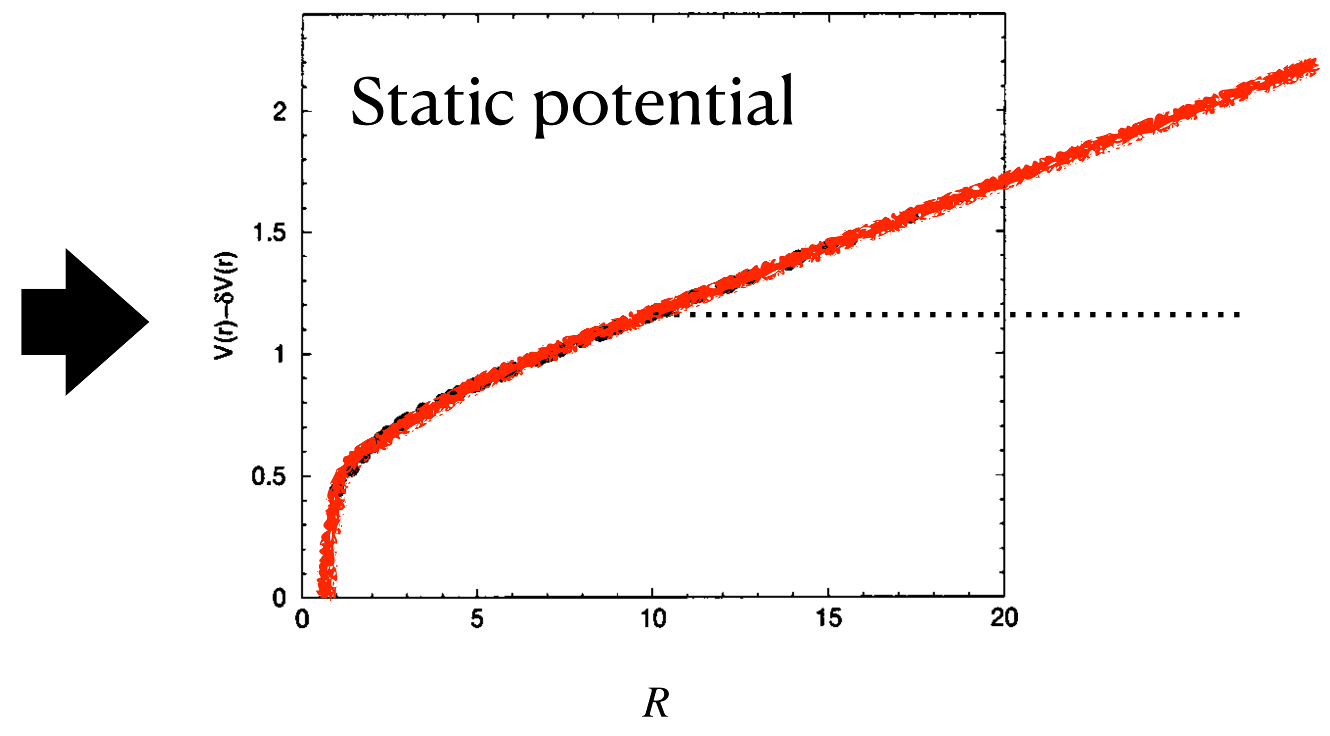

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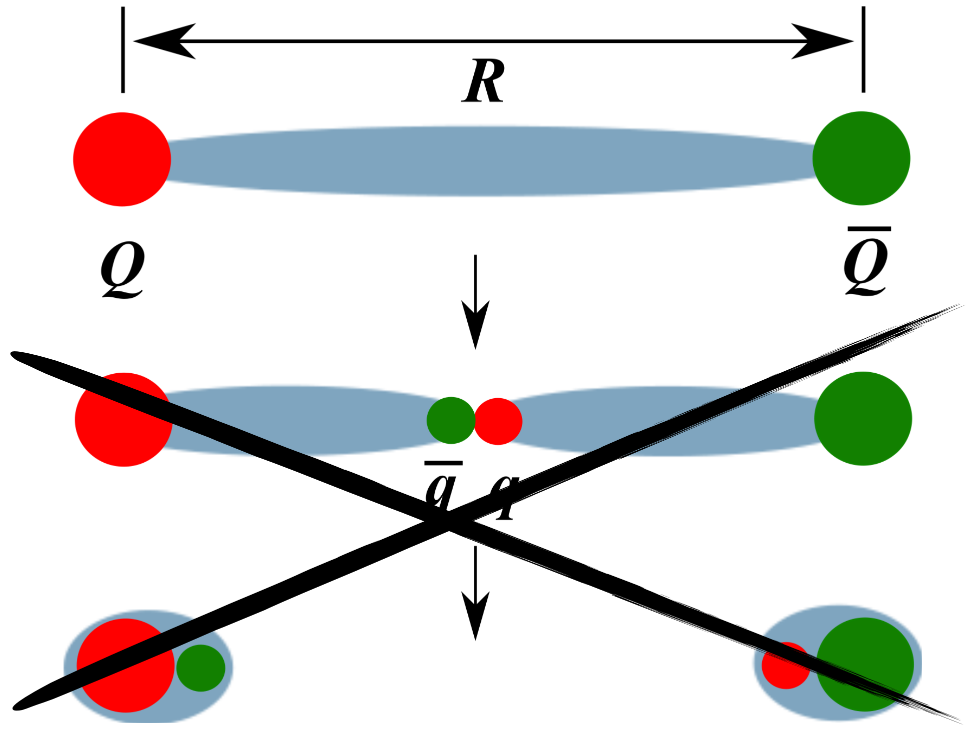

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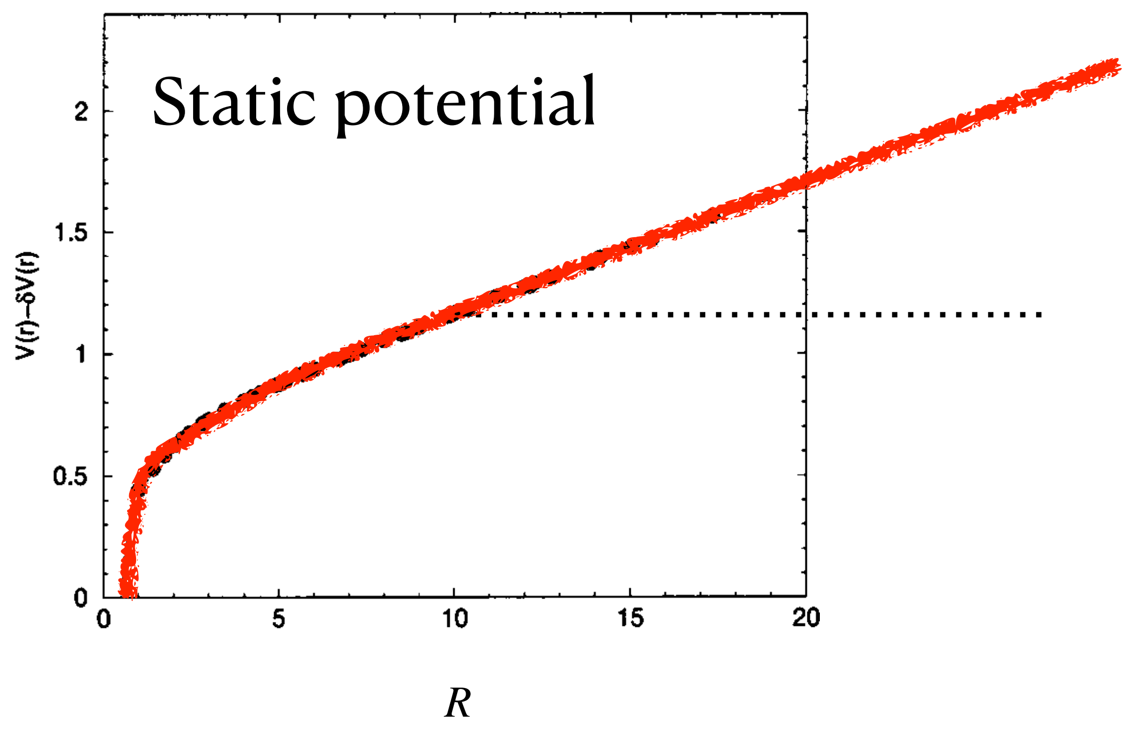

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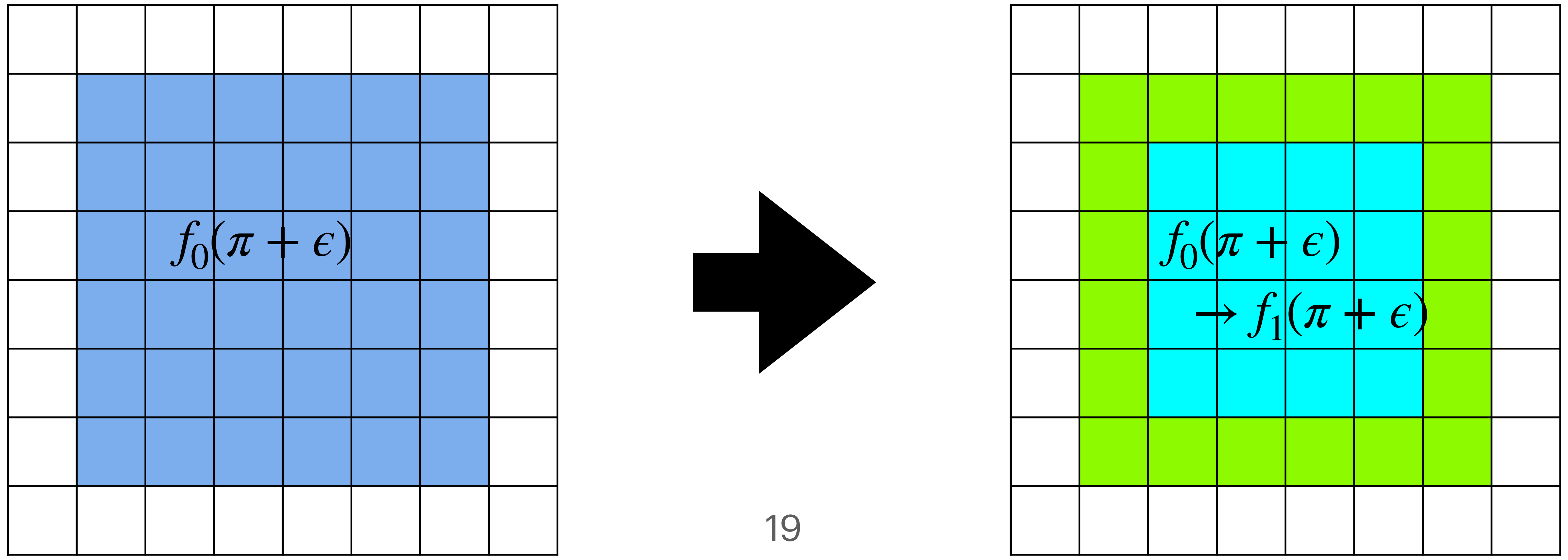

4d SU(N) YM has a topological object called "glue bag". [Luscher (1978)].

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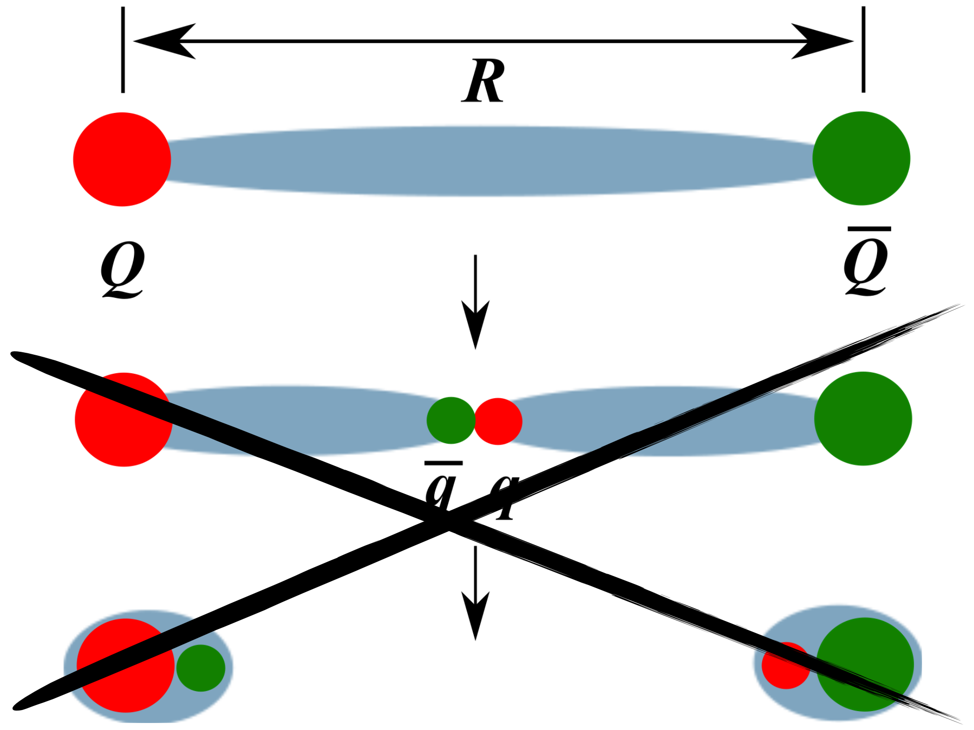

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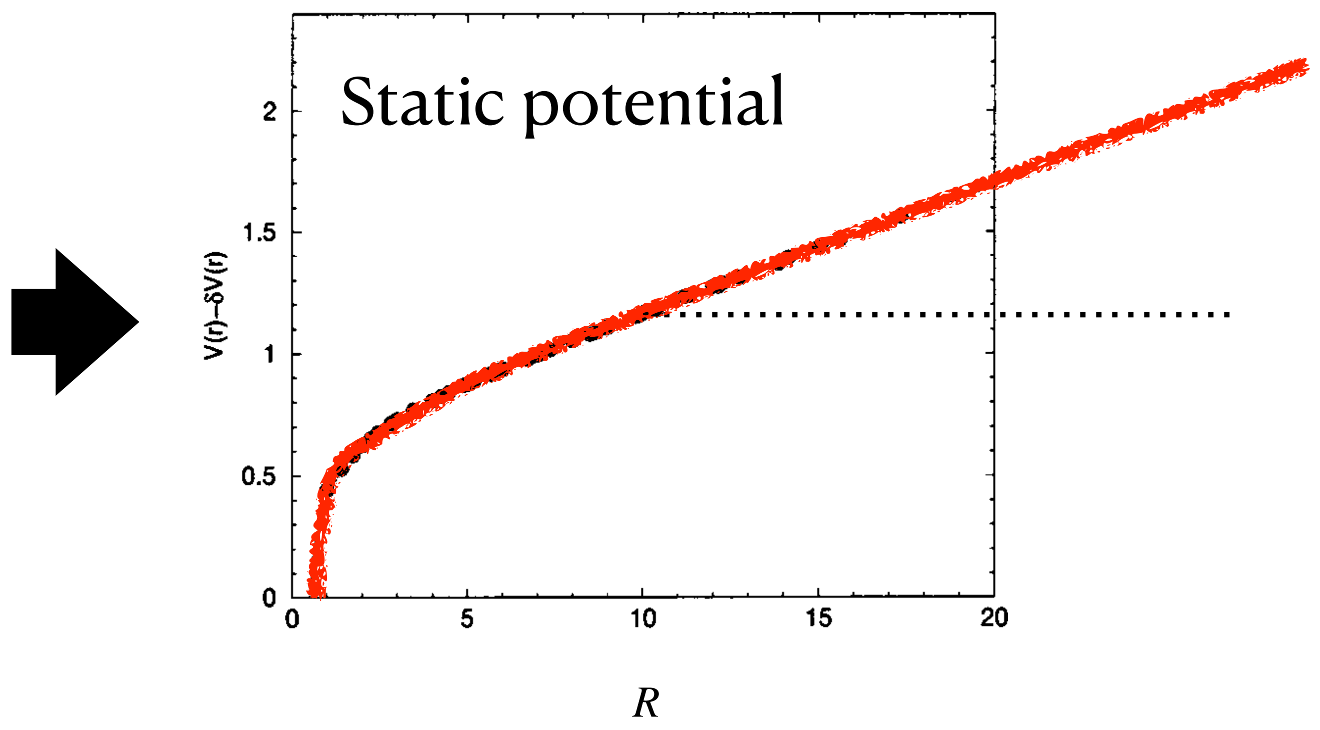

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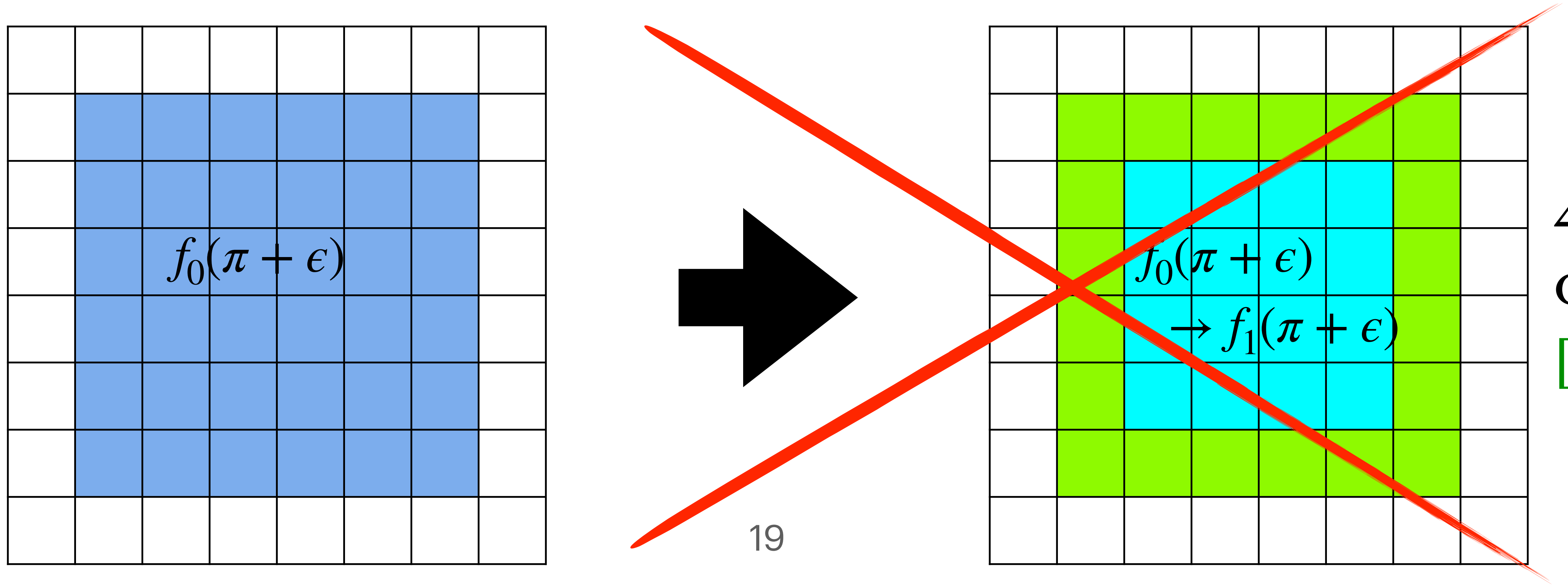

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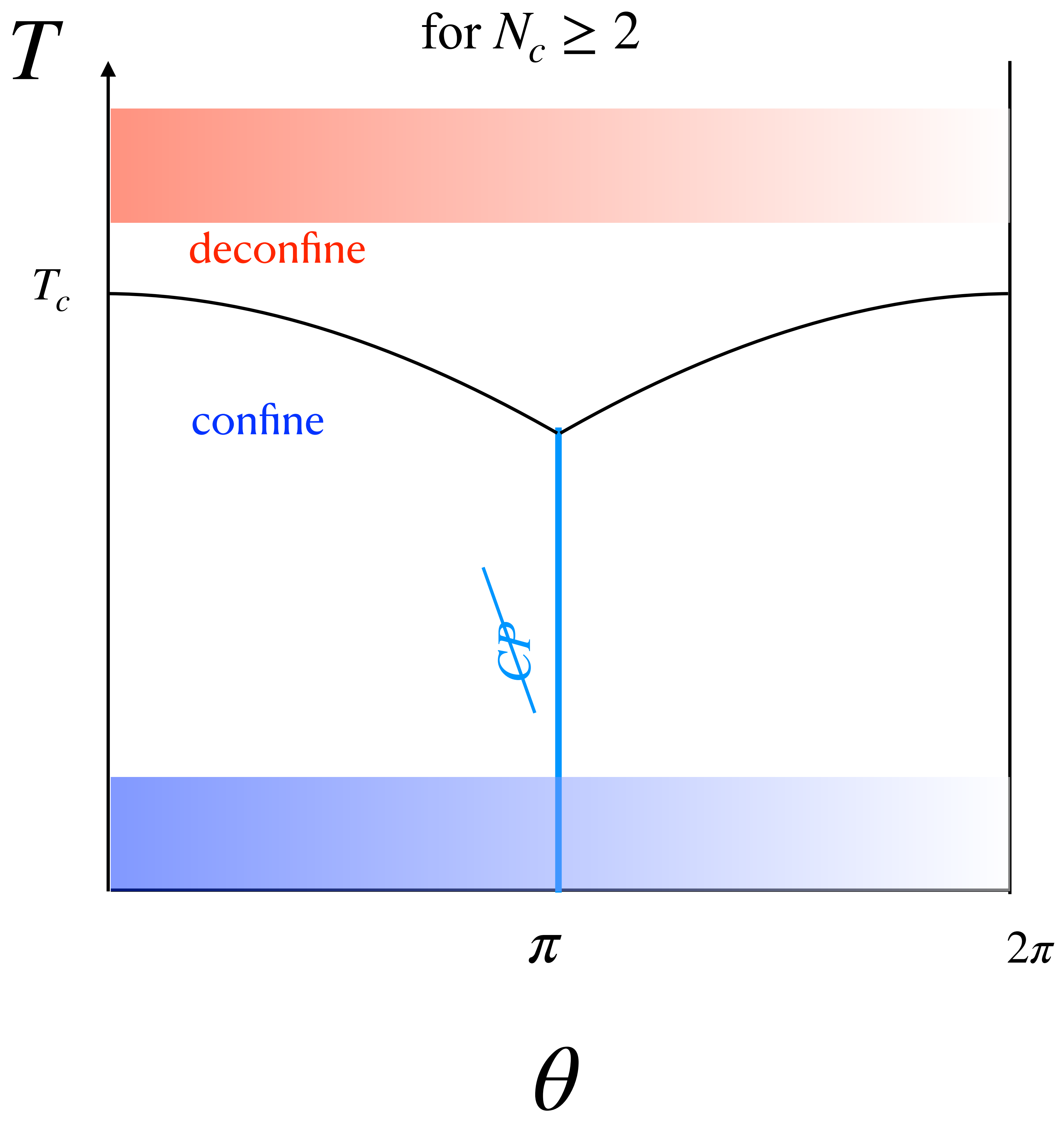

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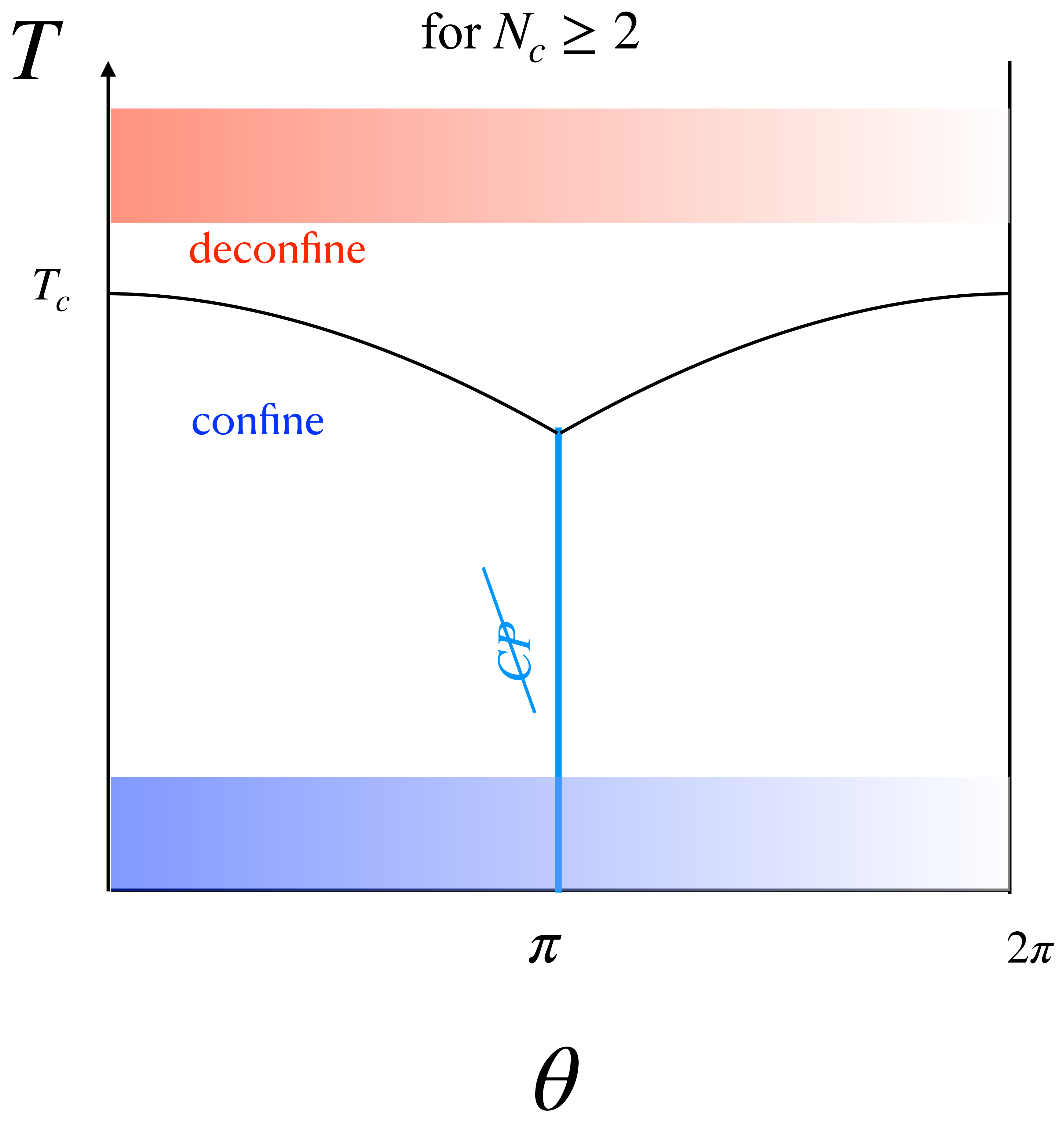

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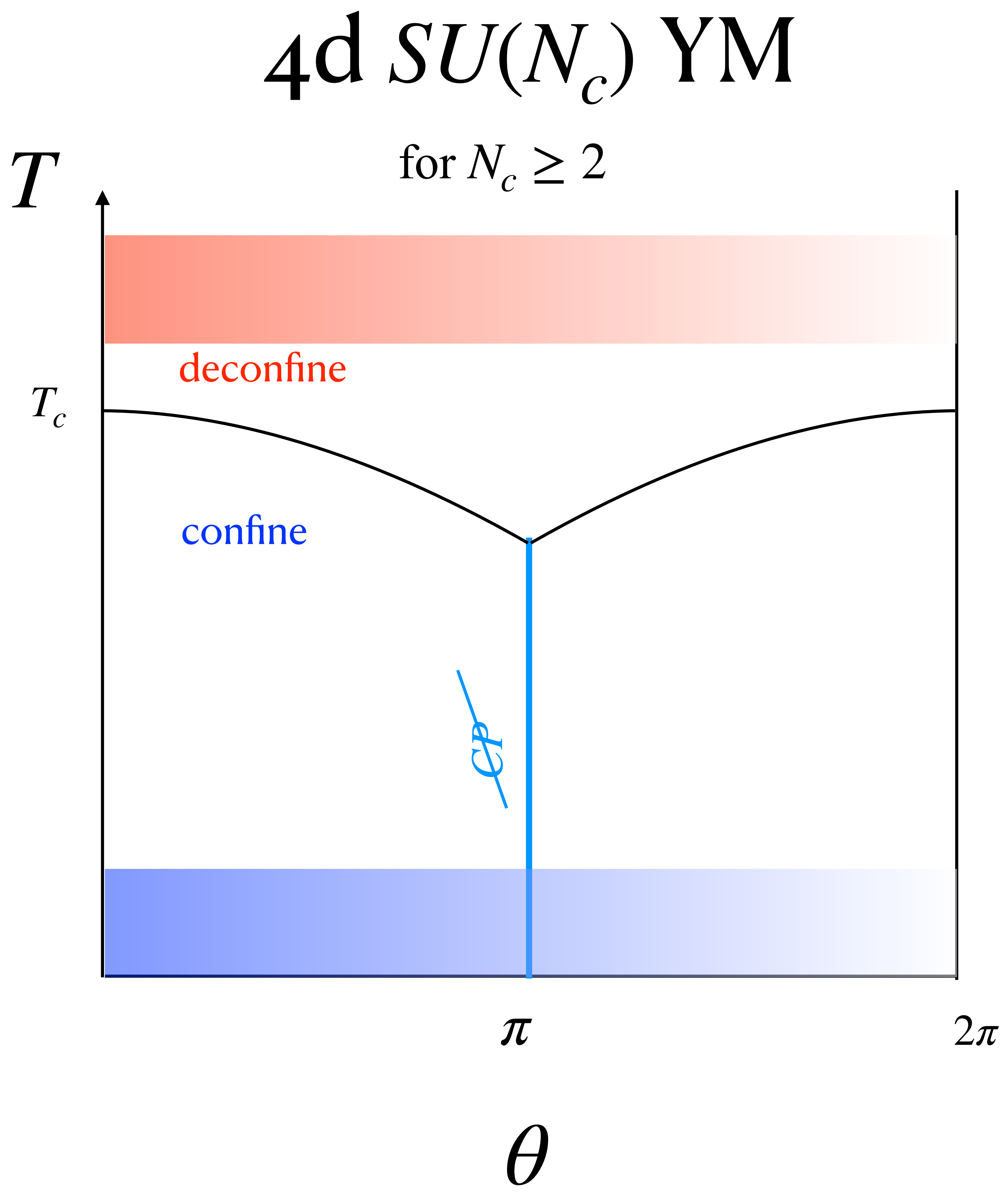
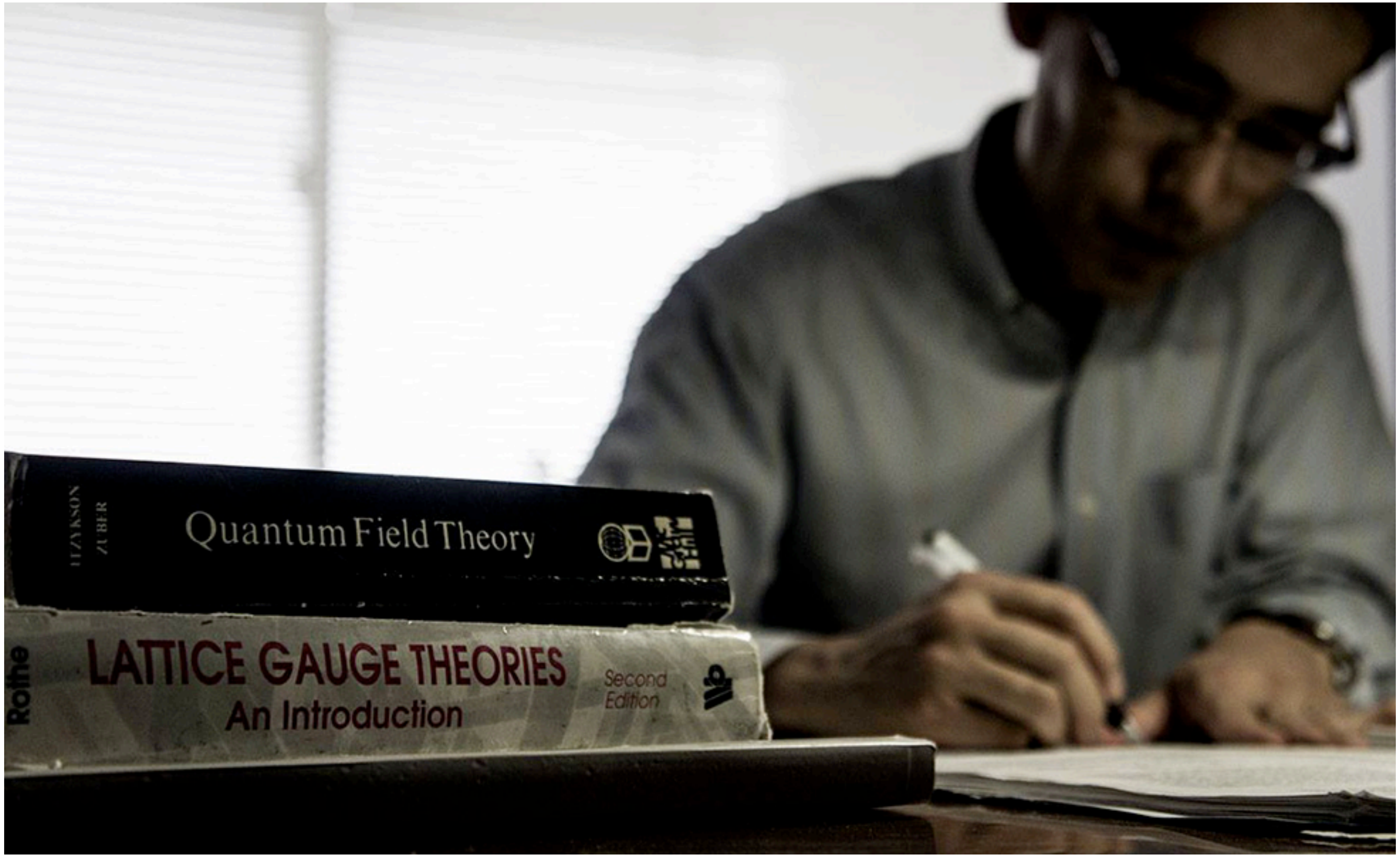
  $\Rightarrow N_c = 2$ is large.





When I was a master course student, he kindly invited Ishikawa-san and me to the Lattice QCD and started to read Rothe's textbook. LQCD is still my favorite field.

I really appreciate your guidance.