Why is quantum gravity so difficult (compared to QCD)?

Hidenori Fukaya (Osaka Univ.)

Discussion based on

HF, “なぜ、量子重力は(QCD に比べて)難しいのか?

(Why is quantum gravity so difficult (compared to QCD?))”

Soryu-shi-ron Kenkyu Vol 25 (2016) No. 2,

English translation: arXiv: 1811.11577 [hep-th]

Let me introduce myself.

1978 Born in Kyoto

1986-1996 Grown up in Hamamatsu

1997- 2006 Undergraduate & Ph.D. course at Kyoto Univ.

2006-2007 Postdoc at RIKEN

2007-2008 Postdoc at Niels Bohr Institute

2009-2010 GCOE assistant prof. at Nagoya U.

2010- Assistant prof. at Osaka U.

My main subject = QCD

I am studying Quantum Chromo Dynamics using numerical simulations on a lattice

with domain-wall fermion

in JLQCD project.

It has nothing to do with gravity.

Fugaku supercomputer at RIKEN KOBE

In 2014

I was assigned to teach general relativity (GR) to B4 students.

But I realized I forgot almost everything about GR. So I started learning it again.

But no B4 student entered our lab… (it was a historically unusual bad event.) and my class did not open.

I took this occasion (= 90min free time) positively and I decided to continue learning GR beyond the textbook level.

Motivation

In particular, I wanted to understand WHY gravity and QCD are so different, while they are described in the similar way in terms of gauge invariance.

Similarities/Differences

	gravityGR	QCD
connection	$\Gamma^{\rho}_{\mu\nu}$	A_μ
curvature	$R^{\sigma}_{\mu\nu\rho}$	$F_{\mu\nu}$
fundamental d.o.f.	$g_{\mu\nu}$	A_μ
Lagrangian	$\sqrt{g}R$	$\mathrm{Tr} F_{\mu\nu}F^{\mu\nu}$
basic solutions static/stationary?	No.	Yes.
vierbein and torsion?	Yes.	No.
renormalizability	No !	Yes !

What I learned from textbooks

R. Utiyama,　一般ゲージ理論序説 (岩波書店)

M. Heller, Theoretical Foundations of Cosmology (World Scientific)
Nash and Sen, Topology and Geometry for Physicists (DOVER)
Kobayashi & Nomizu, Foundation of Differencial geometry (Wiley)

- Yang-Mills theory can be formulated by fiber bundle and is renormalizable.
- gravity can be formulated by Riemannian manifold and is non-renormalizable.
- Riemannian manifold is obtained from a "special" fiber bundle:

Affine gauge theory (1st order formalism) -> General relativity (2nd order formalism)

Progress

2015 May : in journal club at Osaka, I gave a talk

"Why is gravity so difficult compared to QCD?"

2016 Dec : I submitted a paper(essay) to Soryu-shi-ron

Kenkyu (following suggestion by Oda-Kin-san).

2018 Nov : submitted English translation to arXiv.

2023 Aug : Oda-san told me that Purnendu Karmakar is

interested in the article.

2023 today : seminar here.

What I explain today

* 1st order formalism of general relativity is based on a parallelizable principal bundle in mathematics (= difficulty for quantization).

* 2nd order formalism of general relativity (conventional GR) is obtained from 1st order formalism by fiber bundle reduction, which is known as the Higgs mechanism.

We work with a Euclidean metric

In this talk, we mainly focus on a Riemannian manifold (= a special fiber bundle) : our base manifold has a metric with Euclidean signature = Wick-rotated space time.

To apply the general relativity, we would need pseudo-Riemannian manifold., which is mathematically much more difficult.

There is (almost) no my original study

In this talk, I just gathered what are previously known in mathematics and physics.

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Contents

Fiber bundle = space x field

Fiber bundle E(F, M, G)consists of

1. E: total space,
2. M: base space, =space-time
3. F: fiber space, =fields
4. Projection π: E→M
5. (linear) structure group G: F→F =gauge group
6. Open coverings {U_i} of M and homeomorphism
7. For p on M which is on $U_i \cap U_j$ $\phi_i : U_i \times F \to \pi^{-1}(U_i)$

$^\exists t_{ij}$ s.t. $\phi_j(p, f) = \phi(p, t_{ij}(p) f)$ = transition function.

Principal bundle

When F=G we call it principal bundle P(G,M).

Space-time M and gauge group G

→ one can construct a fiber bundle P.

Moreover, the other bundles of different representations can be made from the principal bundle (associated bundles) = matter field

$$E = P \times F/G$$

Connections

Connection = to give local structure of P :

for $u \in P$, connection = decomposing the tangent space on u into two parts,

$$T_u(P) = V_u(P) \oplus H_u(P) \qquad (V_u(P)//F, \;\; H_u(P)//M)$$

which can be uniquely given by

the connection 1-form (with local coordinate = (x, g))

$$\omega = g^{-1}dg + g^{-1}Ag \qquad (A = A^a_\mu(x)T_a dx^\mu)$$

and the condition $\langle \omega, X \rangle = 0 \;\; \text{for } X \in H(P)$

Gauge transformation and curvature

Gauge transformation = changing coordinates

in fiber's direction.

Note : ω does not depend on coordinate.

$$g \to hg \qquad A' = hdh^{-1} + hAh^{-1}$$

$$\omega \to (hg)^{-1}d(hg) + (hg)^{-1}A'(hg),$$

Curvature form

$$\Omega = d\omega + \omega \wedge \omega \quad = g^{-1}(dA + A \wedge A)g \quad = g^{-1}Fg$$

Analogy (for M1 students)

M (space-time) = your head

F (fiber or field) = your hair

E (fiber bundle) = your hair style

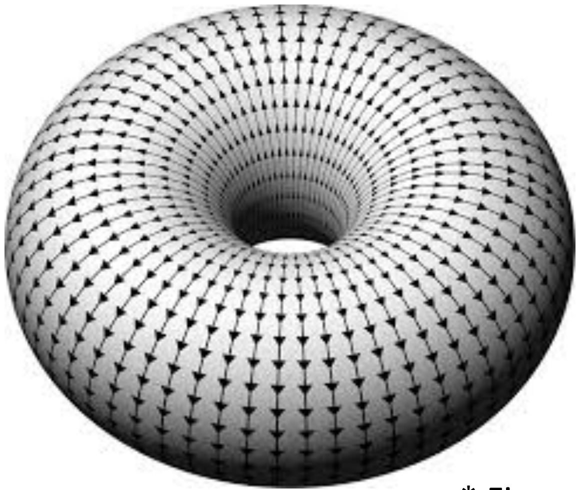
Connection = hair wax (local hair design)

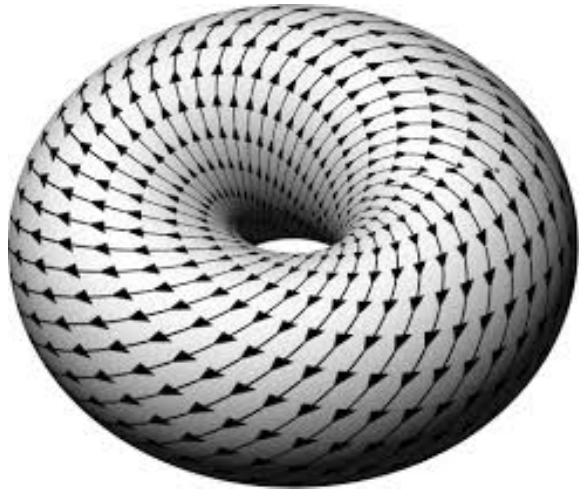
Example

Spacetime $M = S^1$ (circle)

Gauge group $G = U(1)$ (circle)

(principal) fiber bundle $P = T^2$ (donut)





* Figures from Wikipedia

QCD from fiber bundle

M = Minkowski or Euclidean 4d space

G = SU(3)

Metric on M is given : $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

→ dual of F

$$*F_{\mu\nu} = \frac{1}{2} F^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma}$$

Gauge invariant scalars

= gauge action

$$S_g = \frac{1}{4} \int_M F \wedge *F$$

* Θ term (topological):

$$S_\theta = \theta \frac{1}{4} \int_M F \wedge F$$

Quark field = vector bundle

M = Minkowski or Euclidean 4d space

F = associated vector space in fundamental rep. of SU(3)

G = SU(3) → Q = P x F / G.

Gauge inv. scalar $S_q = \int_M \bar{q} D q$

Least action principle

Action

$$S = S_g + S_q + S_\theta$$

describes how much P ~ M_4 x SU(3)

and Q ~ M_4 x $V_{su(3)}$ are curved.

Least action = flat P (and Q).

Quantization

We can quantize the theory by the path-integral = statistical mechanics of the fiber bundles

$$Z_{\mathrm{QCD}} = \int dA \int d\psi e^{-S}$$

with the renormalizable action:

$$S = S_g + S_q + S_\theta$$

Contents

✓ 1. What is fiber bundle (review again)?

Fiber bundle = united manifold of space-time x fields.

2. Why is gravity difficult?

3. Fiber bundle reduction and Higgs mechanism

4. Fiber bundle reduction in gravity

5. Summary

Frame bundle

Let us consider a principal bundle :

M : 4d space-time

G : GL(4,R) (or GL(4) in short) acts on tangent space of M.

P ~ M x GL(4,R)

We call it the frame bundle F(M)

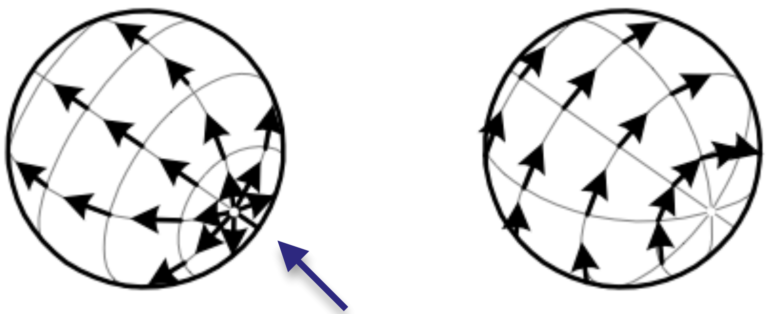
What is special?

It is parallelizable.

Parallelizable manifold

has a global tangent vector field.

Ex. 1: 2dim. sphere is not parallelizable.



not well-defined here.

figure from https://www.mathphysicsbook.com

Ex. 2: 2dim. torus is parallelizable.

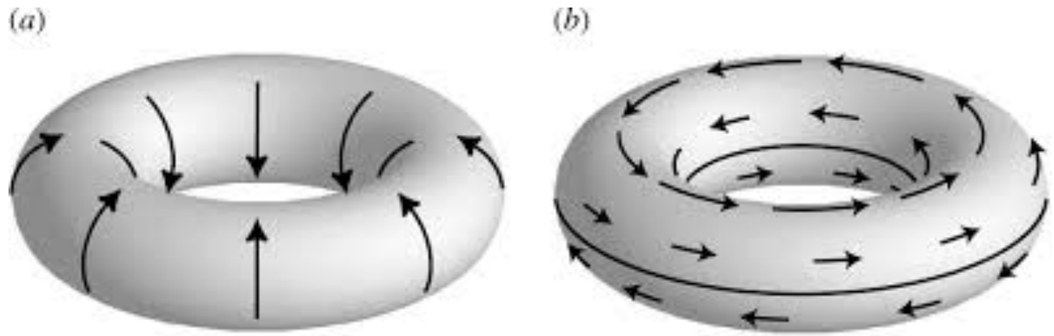


figure from Fowler and Guest 2005

Frame bundle is parallelizable (just believe me or textbooks).

F(M) = 4+4x4 = 20 dim. manifold.

connection 1-form $\omega = g^{-1}dg + g^{-1}Ag$

has only 4x4=16 dim.

In fact, we have 4 more (global) vectors.

= solder form : $\theta^a = [g^{-1}e]^a = [g^{-1}]^a_b e^b_\mu dx^\mu$

vierbein

A crucial difference = vierbein

Usual principal bundle has A_μ only.

-> actions $S_g = \frac{1}{4}\int_M F \wedge *F$ $\quad S_\theta = \theta \frac{1}{4}\int_M F \wedge F$

But the frame bundle has A_μ, e_μ, with which

$$S_{EH} = M_{\mathrm{pl}}^2 \int_M e^a \wedge e^b \wedge [F]^c_d \eta^{de} \epsilon_{abce} \qquad S_\Lambda = \Lambda M_{\mathrm{pl}}^2 \int_M e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}$$

Einstein-Hilbert term $\qquad$ cosmological constant term

are allowed (after GL(4)->SO(4) reduction).

Metric and general covariance

metric is a composite field,

$$g_{\mu\nu} \equiv e_\mu^a e_\nu^b \eta_{ab}$$

and from the metricity condition,

$$\nabla_\rho g_{\mu\nu} = 0.$$

general covariance appears as a secondary gauge invariance.

Equivalence principle

or torsionless condition

$$T = De = 0.$$

is also a secondary condition.

* with metricity condition, we have

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}.$$

* Torsion exists only in gravity.

From GL gauge theory to GR

GL(4) theory : $[A_\mu]^a_b, [e_\mu]^a$ (4x4x4+4x4= 80 d.o.f.)

⇩ fiber bundle reduction & metricity (-40)

SO(4) + general covariance: $[A^{LL}_\mu]^a_b, [e_\mu]^a$ (40)

⇩ torsionless condition (-24)

+ equivalence principle : $[e_\mu]^a$ (16)

⇩ Local Lorentz gauge fixing (-6)

general relativity : $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ (10)

(Riemannian manifold)

2nd vs. 1st. at quantum level

The two are equivalent in classical theory.

2nd order formalism [$g_{\mu\nu}$]

spin 2 field looks hopelessly norenormalizable.

1st order formalism [$A_\mu, \quad e_\mu$]

two spin 1 fields look easier.

Cf.) Maitiniyazi et al. 2023 claims that 1st order formalism is renormalizable with the irreversible vierbein.

Analogy to QCD

QCD renormalizable

⇕

Pion theory

(chiral perturbation)

non-renormalizable.

$$\pi(x) = \bar{q}\gamma_5 q$$

1st order formalism

(???)

⇕

2nd order formalism

non-renormalizable.

$$g_{\mu\nu} \equiv e_\mu^a e_\nu^b \eta_{ab}$$

Quantization of spin-1 particles

On our textbooks, we learn that

spin-1 field is renormalizable (only) IF

it is a gauge field.

A_μ = gauge field of Local Lorentz sym. May be O.K.

e_μ NO associated gauge symmetry.

Non renormalizable !

Gravity is non-renormalizable because of vierbein.

Vierbein = gauge field ?

If we can extend the gauge symmetry so that vierbein = a gauge field, then the quantum gravity may be renormalizable.

A beautiful candidate = local translation : $e_\mu = e_\mu^a P_a$

local Lorentz $\rightarrow$ local Poincare

Special relativity = global Poincare	⟺	General relativity = local Poincare

However, our life is not so easy (except for 3d).

EH action is not gauge invariant…

$$S_{4\mathrm{d}} = \int_M e \wedge e \wedge F \to \int_M \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F}$$

$$\mathcal{A}_\mu = e_\mu^a P_a + A_\mu^a J_a$$

3d is special !!! [Witten, NPB311(1988)46]

$$S_{3\mathrm{d}} = \int_M e \wedge F \to \int_M \mathcal{A} \wedge \mathcal{F}$$

Chern-Simmons theory with local Poincare [ISO(2,1)] invariance -> renormalizable !!!

Poincare gauge theory is BEYOND the fiber bundle.

Fiber bundle E(F, M, G)consists of

$$GL(4): v'_\mu = M v_\mu$$

$$P: v'_\mu = v_\mu + \underline{a_\mu}$$

1. E: total space,
2. M: base space, =space-time
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Frame bundle is parallelizable: we need vierbein(spin1).

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From GL gauge theory to

GL(4) theory : $[A_\mu]^a_b, [e_\mu]^a$ (4x4x4+4x4= 80 d.o.f.)

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(Riemannian manifold)

When fiber bundle reduced?

Kobayashi-Nomizu's textbook says

1) P(G, M) is reducible to P(H,M) if and only if E(G/H,M,G) admits a section.
2) If the section is parallel to the connection of P(G, M), the connection of P(H, M) is uniquely given.

When fiber bundle reduced?

Translation into physics language:

1) G-theory is reducible to H-theory if and only if a field in G/H takes a vev.

2) If the vev is parallel to the G-gauge field, the H-gauge field is uniquely given.

Fiber bundle reduction = Higgs mehanism !

Example : SO(3) gauge + Higgs

(SU(2)+ adjoint Higgs)

$$h_0 = (0, 0, v)^T$$

$$h(x) = g(x) h_0 \in G/H \longrightarrow$$ section of E(G/H, G, M)

$$H = SO(2) \simeq U(1)$$

1) of Kobayashi-Nomizu.

M. Honda, “Fiber bundle structure of gauge field and reduction by the Higgs mechanism”, PTP 63, No. 4 (1980), 1429

U(1) gauge field

2) of Kobayashi-Nomizu

is given by Higgs "EOM" : $D_\mu^{SO(3)} h = 0.$

* it is stronger than usual EOM: $D^{SO(3)\mu} D_\mu^{SO(3)} h = 0.$

⟶ U(1) gauge field is uniquely determined:

(upto gauge tr.) $$A_\mu^{U(1)} = \frac{h^T \left[A_\mu + h \times \partial_\mu h\right]}{h^T h}$$

For simplest case, $h(x) = (0,0,1)^T$

$$A_\mu^{U(1)} = A_\mu^3, \quad A_\mu^{1,2} = 0.$$

* W bosons are not excited in classical FT.

t'Hooft-Polyakov monopole

If Higgs has a zero point: $h(x_0) = 0$

then on a 2D sphere around x_0 ,

$h(x)$ makes a map : S^2 -> SO(3)/U(1) = S^2

labelled by an integer = monopole charge.

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VEV = section of G/H, EOM determines H-gauge field.

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(Riemannian manifold)

Q. What is the Higgs in gravity?

A. In fact, vierbein is the Higgs !

Let's assume $[D_\nu^{GL} e_\mu]^a = (\partial_\nu \delta_b^a + [A_\nu]_b^a) e_\mu^b = 0.$

Cf.) $D_\mu^{SO(3)} h = 0$ in SO(3).

Then symmetric part of gauge field is killed: $[A_\nu^S]_b^a = -(e^{-1})_b^\mu (\partial_\nu \delta_c^a + [A_\nu^A]_c^a) e_\mu^c.$

$A_\nu = A_\nu^S + A_\nu^A$

40 16 24

= SO(4) Local-Lorentz

Metric = Higg VEV.

$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}$ breaks GL(4) but is SO(4) invariant: element of GL(4)/SO(4)

= section of E(G/H, M, G)

1) and 2) of Kobayashi-Nomizu are satisfied by

$$[D^{GL}_\nu e_\mu]^a = 0.$$

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}$$

→ We obtain Local Lorentz SO(4) bundle (with vierbein).

Higgs EOM = vierbein postulate

The Higg EOM, $[D_\nu^{GL} e_\mu]^a = (\partial_\nu \delta_b^a + [A_\nu^S + A_\nu^A]_b^a) e_\mu^b = 0,$

can be expressed by $\Gamma_{\mu\nu}^\rho \equiv -[A_\nu^S]_b^a e_\mu^b [e^{-1}]_a^\rho,$

as

$$[D_\nu^{LL} e_\mu]^a \equiv (\partial_\nu \delta_b^a + [A_\nu^A]_b^a) e_\mu^b = \Gamma_{\mu\nu}^\rho e_\rho^a,$$

known as the vierbein postulate.

Metricity condition

$$\frac{\partial}{\partial x_\rho}(g_{\mu\nu}) = \frac{\partial}{\partial x_\rho}(e^a_\mu e^b_\nu \eta_{ab}) = [D^{LL}_\rho e_\mu]^a e^b_\nu \eta_{ab} + e^a_\mu [D^{LL}_\rho e_\nu]^b \eta_{ab}$$

$$= \Gamma^\lambda_{\mu\rho} g_{\lambda\nu} + \Gamma^\lambda_{\nu\rho} g_{\mu\lambda}, \longrightarrow \nabla_\rho g_{\mu\nu} = 0.$$

Then, the world lines become invariant:

general covariance dynamically appears as a consequence of the Higgs mechanism.

Equivalence principle is also dynamical

Torsion mode

$$T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$$

can have a mass ~ M_{planck}.

[See Shapiro 2002]

It is natural to assume that they are just decoupled at low energy.

Which Higgs mechanism?

In usual G->H Higgs mechanism, G-invariant action is easily constructed: $\int d^4x\sqrt{-g}\left[g_{\mu\nu}D^\mu\phi^\dagger D^\nu\phi - V(\phi^\dagger\phi)\right]$

But for gravity, we cannot use metric since it is not G=GL(4) invariant.

~~$g^{\mu\rho}g^{\nu\sigma}D_\mu e^a_\nu D_\rho e^b_\sigma \eta_{ab}$~~

We have an EOM $[D^{GL}_\nu e_\mu]^a = 0.$

But we don't know its action.

Action without metric?

Integral over 4-form

$$\int_M f = \int \sqrt{-g} f(x) dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$$

is metric independent.

$$\int_M \mathrm{Tr}\left[\sigma \wedge \sigma \wedge F\right],$$

$$\sigma_a^b = (e^{-1})_a^\mu [D_\nu e_\mu]^b dx^\nu$$

$$F = [D^{GL} A]$$

looks consistent with Higgs EOM but⋯

How Einstein-Hilbert action appears?

It is difficult to explain how

$$S_{EH} = M_{\rm pl}^2 \int_M e^a \wedge e^b \wedge [D^{LL} A^A]^c_d \eta^{de} \epsilon_{abce},$$

appears. Also, the matter is difficult:

$$T_{\mu\nu}$$

is not a tensor of the GL(4) theory.

In the literature···

The idea that vierbein = Higgs is found in the literature,

Sardanashhvily 1994, 1994, Krisch 2005, Leclerk 2006, Obukhov 2006···

but no convincing Lagrangian is given.

Contents

Summary

- Among gauge theories, gravity is special because GL(4) bundle is parallelizable: there exists vierbein.
- GL(4)→SO(4) + general covariance theory (= 1st order formalism) is obtained by fiber bundle reduction = Higgs mechanism.
- Vierbein = Higgs! $[D_\nu^{GL} e_\mu]^a = 0, \quad g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}$

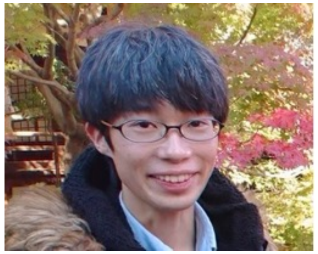
  Higgs EOM= vierbein postulate, Higgs vev = metric.
- But there appears to be no Lagrangian to describe it.

Outlook

In this talk,

Riemannian manifold = special fiber bundle

In our recent works with Shoto Aoki



we try an opposite way:

Riemannian manifold = submanifold of trivial flat space

[Nash 1956]

→ Lattice regularization of fermion in curved space

S. Aoki, HF, PTEP 2022 (2022) 6, 063B04 2203.03782

S. Aoki, HF, PTEP 2023 (2023) 3, 033B05, 2212.11583