# A mathematical formulation of index theorem on a lattice

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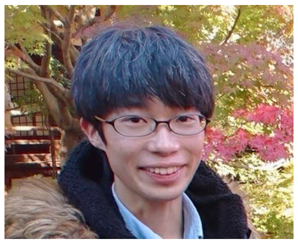





# Physicist-friendly index theorem project

- Physicist-friendly Atiyah-Patodi-Singer (APS) index on a flat space [F, Onogi, Yamaguchi 2017]
- Mathematical proof for the physicist-friendly APS index on general curved manifold [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019]
- Lattice version [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]
- Mod-two APS index [F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]

# What is  the "physicist-friendly" formulation of the index?

Original definition of index

= number of chiral zero modes of massless Dirac operator:

$$\mathrm{Tr}\gamma_5^{\mathrm{reg.}} = n_+ - n_-$$

It is O.K. on a closed manifold but with boundary,

a nonlocal (= physicist-unfriendly) boundary condition is needed.

# What is the "physicist-friendly" formulation of the index?

The physicist-friendly formulation in terms of the eta invariant

$$\frac{1}{2}\eta(\gamma_5(D+m))$$

$$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

is

* Defined by massive Dirac operator :

Chiral symmetry is not required.

* United on a closed manifold with a cap of the boundary

(nonlocal) boundary condition is not required.

Physicist-friendly = Lattice-friendly.

Nielsen-Ninomiya no-go theorem[1981]

To avoid fermion doubling problem, we have to break the chiral symmetry,

$$\gamma_5 D + D\gamma_5 \neq 0$$

With the physicist-friendly formulation, $\frac{1}{2}\eta(\gamma_5(D+m))$

* Defined by massive Dirac operator :

Chiral symmetry is not required.

-> Lattice version looks easy !

# Our goals

We try to formulate the Atiyah-Singer index on a flat even-dimensional torus using a square lattice

= this talk.

[Cf. Kubota2020, Yamashita 2020]

Outlook:

Extension to the APS index

Mod-two version

Curved space version using domain-wall fermion [N. Kan's talk]

etc.

# Contents

# Eigenvalues of $H(m) = \gamma_5(D+m)$

For $D\phi = 0$, $H(m)\phi = \gamma_5 m\phi = \underbrace{\pm}_{\text{chirality}} m\phi.$

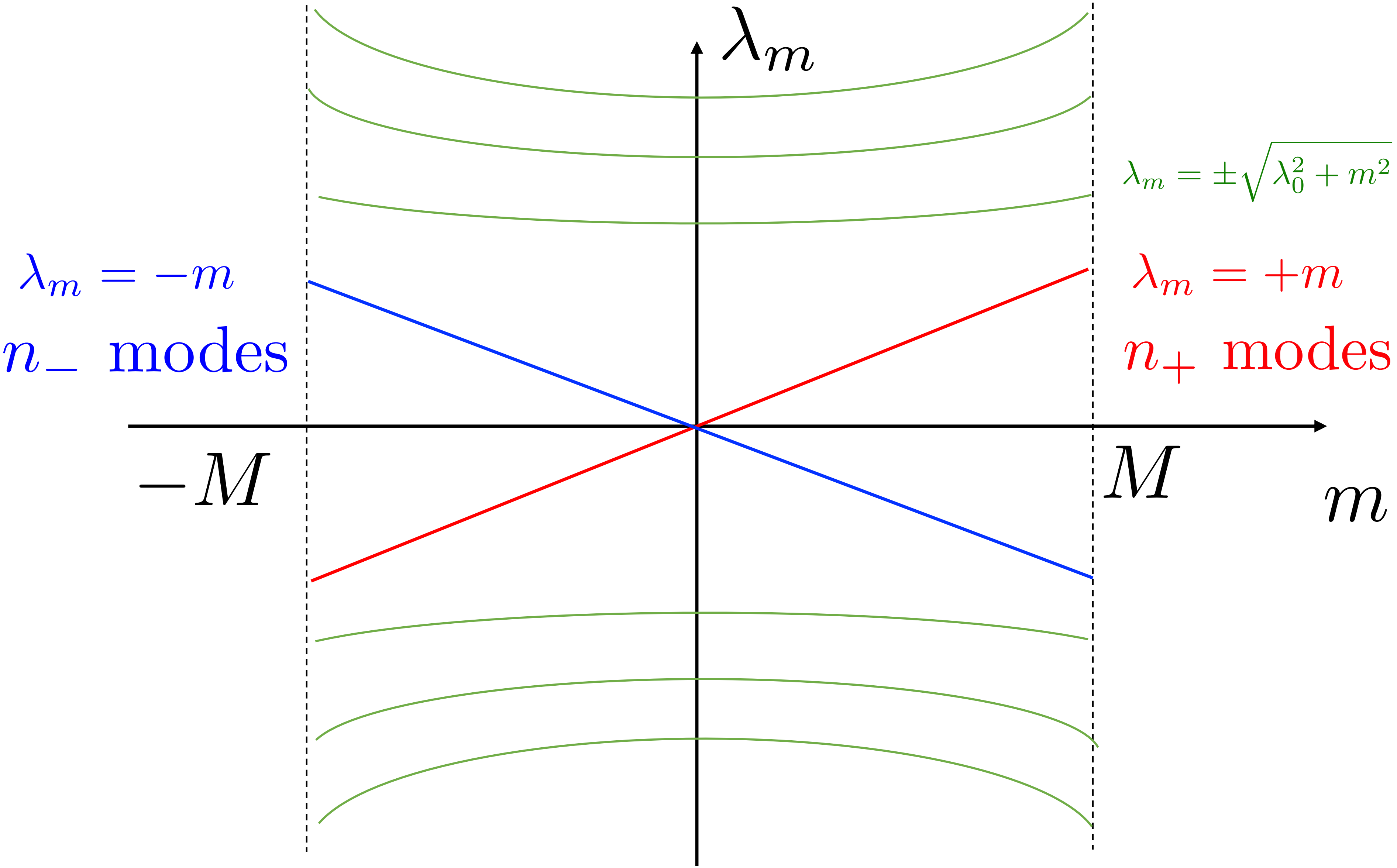
For $D\phi \neq 0$, $\{H(m), D\} = 0.$

The eigenvalues are paired:
$$H(m)\phi_{\lambda_m} = \lambda_m \phi_{\lambda_m}$$
$$H(m)D\phi_{\lambda_m} = -\lambda_m D\phi_{\lambda_m}$$

As $H(m)^2 = -D^2 + m^2$, we can write them

$$\lambda_m = \pm\sqrt{\lambda_0^2 + m^2}$$

# Spectrum of $H(m) = \gamma_5(D+m)$



# Spectral flow = AS index = η invariant

$n_+$ = # of crossing eigenvalues from - to +

$n_-$ = # of crossing eigenvalues from + to -

$n_+ - n_-$ = spectral flow of $H(m) \quad m \in [-M, M]$

Equivalent to the eta invariant: $$\eta(H) = \sum_{\lambda \geq 0}^{reg} - \sum_{\lambda < 0}^{reg}$$

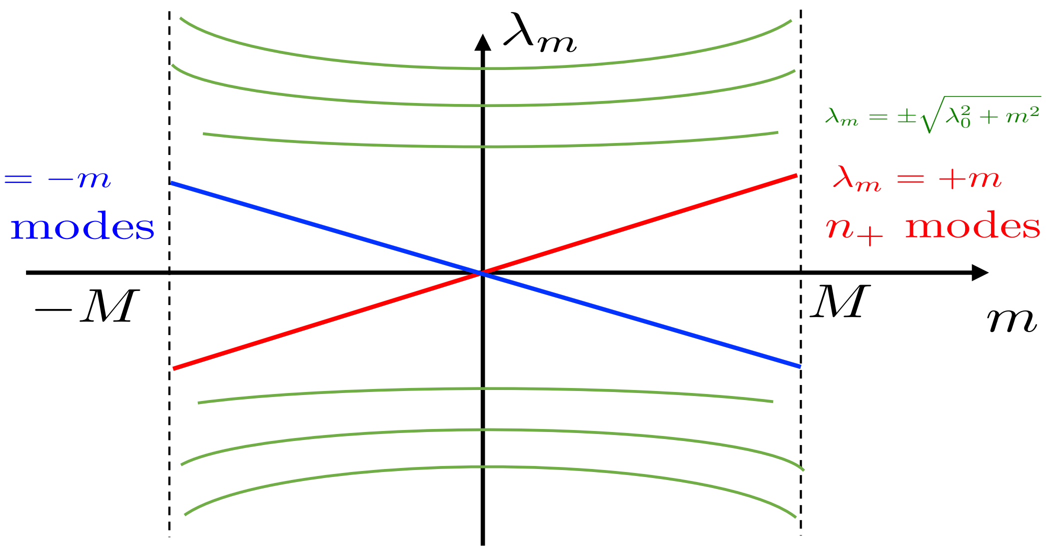
Whenever an eigenvalue crosses zero,

$\eta(H(m))$ jumps by two.

The physicist-friendly AS index $$\frac{1}{2}\eta(H(M)) - \frac{1}{2}\eta(H(-M)) = n_+ - n_-.$$

Pauli-Villars subtraction

# Suspension isomorphism in K theory




Massless= Couting by points

Massive= Counting by lines

$$K^0(pt) \simeq K^1(I, \partial I)$$

point

line

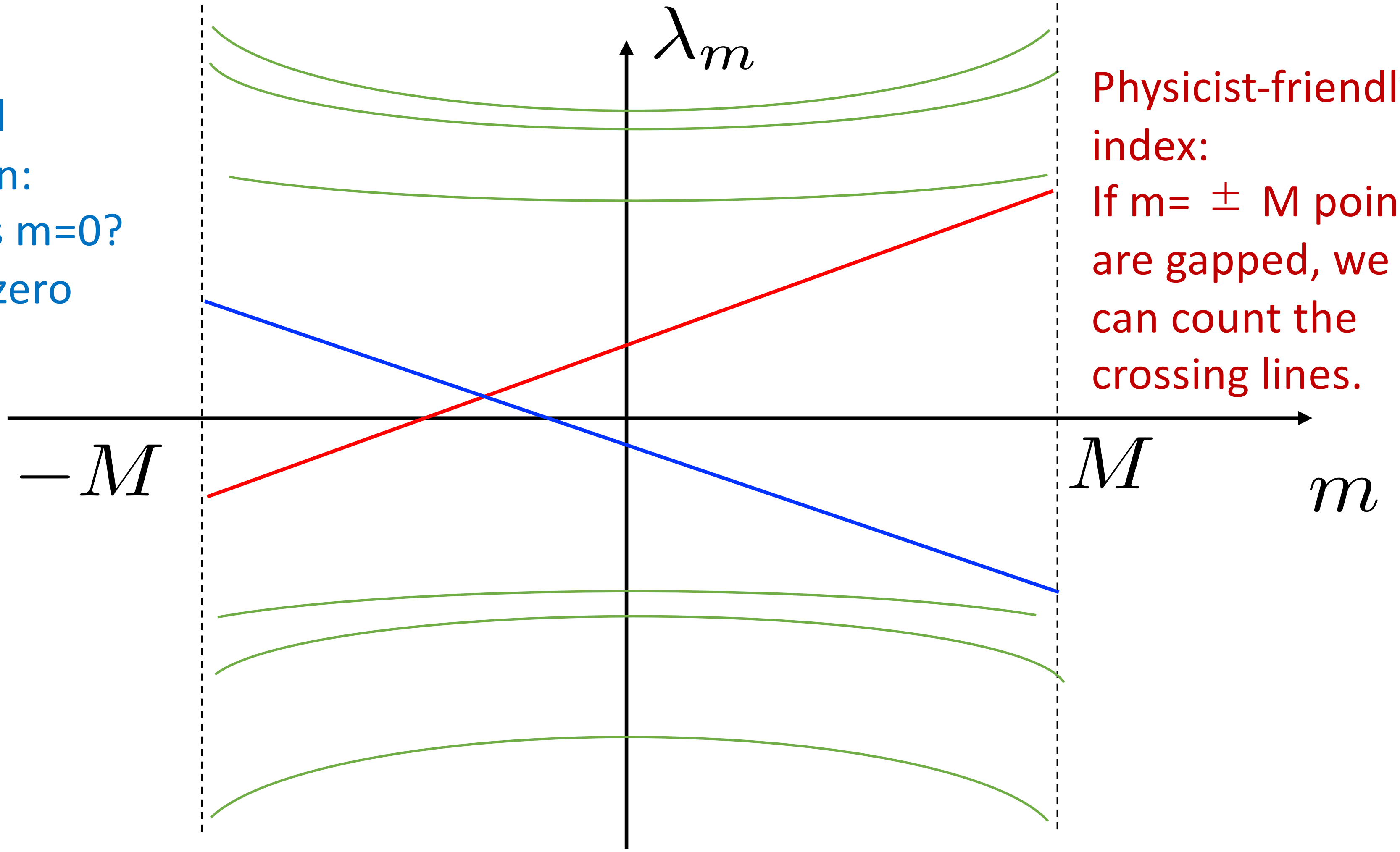
With chirality operator

Without chirality operator

⇨ The two definitions of the index agree.

With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) is possible.

Standard definition:
Where is m=0?
What is zero modes?




Physicist-friendly index:
If m= $\pm$ M points are gapped, we can count the crossing lines.

# Contents

# Dirac operator in continuum theory

E : Complex vector bundle

Base manifold M: 2n-dimensional flat torus $T^{2n}$

Fiber F : vector space of rank r with a Hermitian metric

Connection : Parallel transport with gauge field $A_i$

D : Dirac operator on sections of E

$$D = \gamma_i(\partial_i + A_i)$$

Chirality ($Z_2$ grading) operator: $\gamma = i^n \prod_i \gamma_i$

$$\{\gamma, D\} = 0, \{\gamma, \gamma_i\} = 0.$$

# Wilson Dirac operator on a lattice

Base manifold regularizing 2n-dimensional $T^{2n}$ is regularized by a square lattice with lattice spacing $a$

But the fiber is kept continuous.

We denote the bundle by $E^a$ and

Link variables : $U_k(\boldsymbol{x}) = P\exp\left[i\int_0^a A_k(\boldsymbol{x}')dl\right]$,

$$D_W = \sum_i \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{a}{2}\nabla_i^f \nabla_i^b\right]$$

$$a\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

Wilson term

$$a\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x}-\boldsymbol{e}_i)\psi(\boldsymbol{x}-\boldsymbol{e}_i)$$

# Definition of $K^1(I, \partial I)$

Let us consider a Hilbert bundle with

Base space $I$ = range of mass [-M, M]

boundary $\partial I$ = $\pm$ M points

Fiber space $H$ = Hilbert space to which D acts

$D_m$ : one-parameter family labeled by m.

We assume that $D_{\pm M}$ has no zero mode.

The group element is given by a equivalence class of

$$\{(H, D_m)\}$$

having the same spectral flow.

# Definition of $K^1(I, \partial I)$

Group operation:

$$\{(H^1, D^1_m)\} \pm \{(H^2, D^2_m)\} = \{(H^1 \oplus H^2, \begin{pmatrix} D^1_m & \\ & \pm D^2_m \end{pmatrix})\}$$

Identity element: $\{(H, D_m)\}|_{\mathrm{Spec.flow}=0}$

We compare $\{(H^{\mathrm{cont.}}, \gamma(D+m))\}$

and $\{(H^{\mathrm{lat.}}, \gamma(D_W+m))\}$ (finite-dimensional)

Taking their difference, and confirm if

the lattice-continuum combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D+m) & f_a \\ f_a^* & -\gamma(D_W+m) \end{pmatrix}$$

has Spectral flow =0 $\quad f_a^* \; f_a$ are mixing perturbation with 4 nice properties (see our paper for the details)

# Contents

✓ 1. Introduction

We try lattice formulation of the physicist-friendly AS index.

✓ 2. Spectral flow (= the key point in this work)

Massive = line counting is easier than massless=point counting.

✓ 3. Mathematical preparation

Our goal is to show the summed $K^1$ (lattice+continuum) group element has zero spec.flow.

4. Main theorem and its proof

5. Comparison with previous works

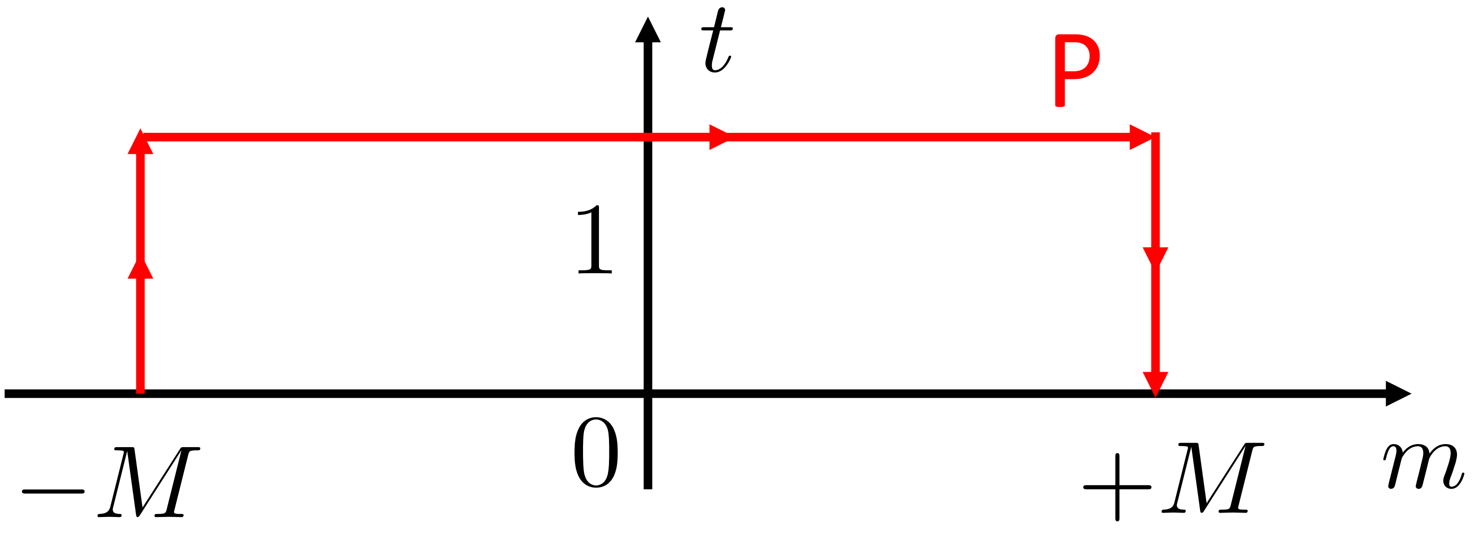
6. Summary and discussion

# Main theorem

Consider a continuum-lattice combined Dirac operator

$$\hat{D} = \begin{pmatrix} \gamma(D+m) & tf_a \\ tf_a^* & -\gamma(D_W+m) \end{pmatrix}$$

on the path P :



# Main theorem

There exists a finite lattice spacing $a_0$ such that
for any $a < a_0$

$$\hat{D} = \begin{pmatrix} \gamma(D+m) & tf_a \\ tf_a^* & -\gamma(D_W+m) \end{pmatrix}$$

is invertible (having no zero mode) in the path P [which is a sufficient condition for Spec.flow=0]

⇨ $\gamma(D+m), \;\; \gamma(D_W+m)$ have the same spec.flow

⇨ $$\frac{1}{2}\eta(\gamma(D-M))^{\mathrm{PV\ reg.}} = \frac{1}{2}\eta(\gamma(D_W-M))$$

Index is mathematically formulated on a finite lattice.

# Proof (by contradiction)

Assume $$\hat{D} = \begin{pmatrix} \gamma(D+m) & tf_a \\ tf_a^* & -\gamma(D_W+m) \end{pmatrix}$$

has zero mode(s) at arbitrarily small lattice spacing.

⇨ For a decreasing series of $\{a_j\}$

$$\begin{pmatrix} \gamma(D+m_j) & t_j f_{a_j} \\ t_j f_{a_j}^* & -\gamma(D_W^{a_j}+m_j) \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = 0$$

is kept.

# Continuum limit

Multiplying
$$\begin{pmatrix} 1 & \\ & f_{a_j} \end{pmatrix}$$

and taking the continuum limit

$$\begin{pmatrix} \gamma(D+m_\infty) & t_\infty \\ t_\infty & -\gamma(D+m_\infty) \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix} = 0$$

is obtained.

$$\hat{D}^2_\infty = D^2 + m^2_\infty + t^2_\infty$$

requires $m_\infty = t_\infty = 0.$

$u_\infty, \quad v_\infty$ are

$L^2_1$ weakly convergent

=

$L^2$ strongly convergent

(Rellich's theorem)

Contradiction with $m \in [-M, M], \quad t \in (0,1]$

# Contents

# Chiral symmetry on a lattice

Nielsen-Ninomiya theorem [1981]

$\gamma_5 D + D\gamma_5 = 0,$ requires fermion doubling.

Ginsparg-Wilson relation [1982]

$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$ $a$ :lattice spacing

can avoid NN theorem.

Overlap Dirac operator [Neuberger, 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right) \qquad H_W = \gamma_5(D_W - M). \quad M = 1/a.$$

Realizes an exact chiral symmetry !

# Atiyah-Singer index on a lattice

Moreover, Atiyah-Singer index can be given by

$$\mathrm{Tr}\gamma_5 \left( 1 - \frac{aD_{ov}}{2} \right)$$

[Hasenfraz 1998, Neuberger 1998]
which counts the chiral zero modes of Dov.

# But AS index is defined by Wilson Dirac op…

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \quad D_{ov} = \frac{1}{a}\left(1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}\right)$$

$$H_W = \gamma_5(D_W - M) \qquad M = 1/a$$

It is equivalent to the $\eta$ invariant (physicist-friendly index) !

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

Note) this fact is known even before overlap Dirac by Itoh-Iwasaki-Yoshie 1982, but its mathematical meaning was not discussed. Cf.) Adams 2001

Namely our formulation is consistent with the AS index of the overlap Dirac operator.

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✓ 3. Mathematical preparation

Our goal is to show the summed $K^1$ (lattice+continuum) group element has zero spec.flow.

✓ 4. Main theorem and its proof

The gap is proved by contradiction.

✓ 5. Comparison with previous works

Our formulation is consistent with overlap index.

6. Summary and discussion

# Summary

Atiyah-Singer index can be understood by massive fermion Dirac operator (with PV reg.):

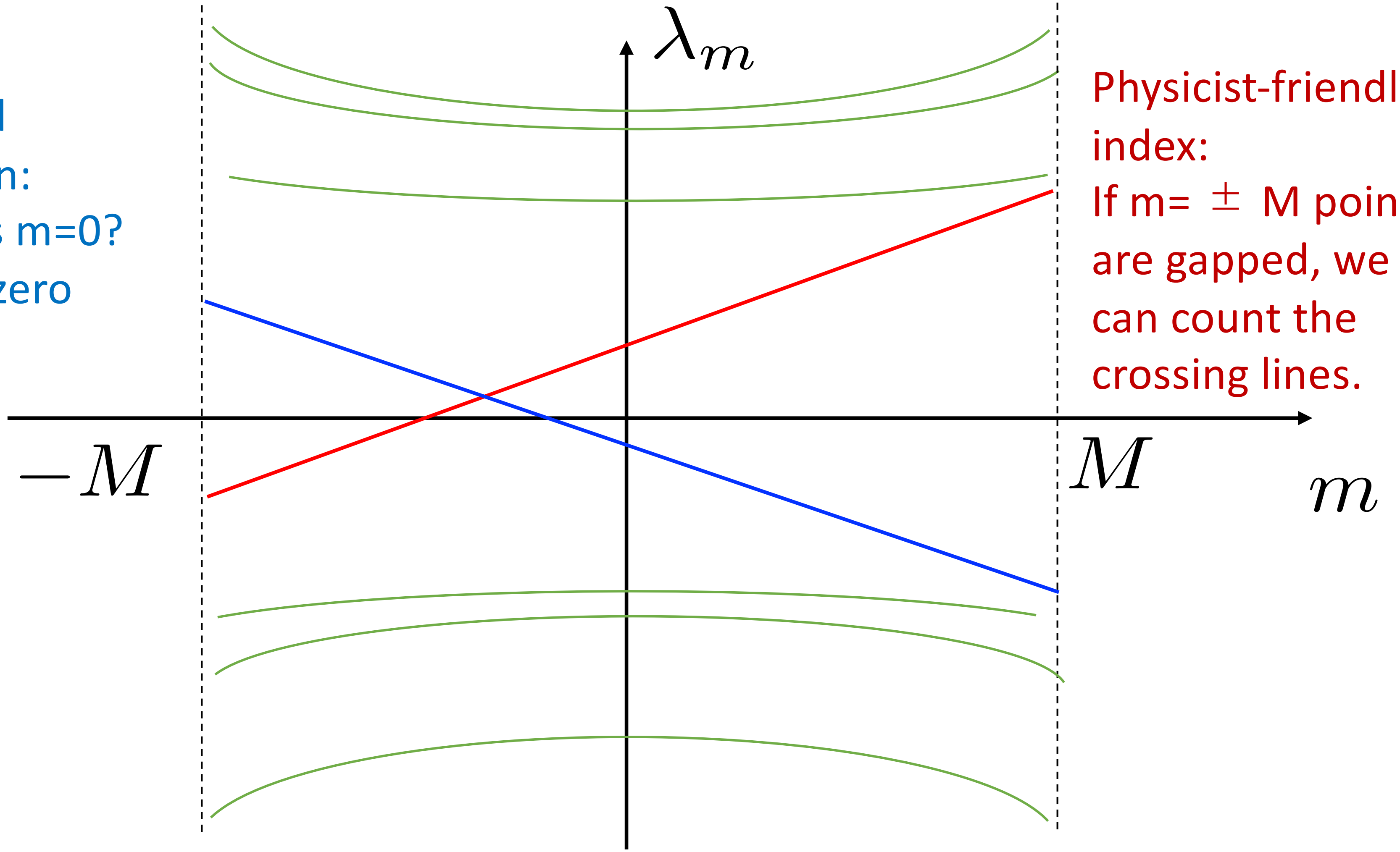
$$-\frac{1}{2}\eta(\gamma_5(D-M))+\frac{1}{2}\eta(\gamma_5(D+M))$$

We have given a mathematical proof that for a fine but finite lattice spacing, the Wilson Dirac operator has the Atiyah-Singer index which is equal to that of continuum theory.

$$-\frac{1}{2}\eta(\gamma_5(D_W-M))$$

# With chiral symmetry breaking regularization (on a lattice), counting points (massless) is difficult but counting lines (massive) is possible.

Standard definition:
Where is m=0?
What is zero modes?




Physicist-friendly index:
If m= $\pm$ M points are gapped, we can count the crossing lines.

# Outlook [index counted by lines]

mod-2 index in odd dimensions (including Witten's SU(2)anomaly ) can be formulated by massive Dirac operator.

[F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita 2020]

| | continuum | lattice | |
|---|---|---|---|
| AS | $\mathrm{Sf}(\gamma_5(D-M))$ | $\mathrm{Sf}(\gamma_5(D_W-M))$ | ◎ |
| APS | $\mathrm{Sf}(\gamma_5(D-\varepsilon M))$ | $\mathrm{Sf}(\gamma_5(D_W-\varepsilon M))$ | △ |
| mod-two AS | $\mathrm{Sf}'\begin{pmatrix} & D-M \\ -(D-M)^\dagger & \end{pmatrix}$ | $\mathrm{Sf}'\begin{pmatrix} & D_W-M \\ -(D_W-M)^\dagger & \end{pmatrix}$ | △ |
| mod-two APS | $\mathrm{Sf}'\begin{pmatrix} & D-\varepsilon M \\ -(D-\varepsilon M)^\dagger & \end{pmatrix}$ | $\mathrm{Sf}'\begin{pmatrix} & D_W-\varepsilon M \\ -(D_W-\varepsilon M)^\dagger & \end{pmatrix}$ | △ |

$\mathrm{Sf}' = \text{mod-two spectral flow : counting zero-crossing pairs from PV op.}$

# Additional comments about

$$Ind D_{\rm ov} = -\frac{1}{2}\eta(\gamma(D_W - M))$$

1. Wilson [1975] is great. Topological feature is already contained in his lattice Dirac operator.
2. Neuberger [1998], Hasenfraz[1998] are great. They defined a chiral zero mode on a lattice.
3. Mathematics is great. Index can be well-defined with finite lattice spacing.
4. But if eta invariant was emphasized from beginning…

# Backup slides

# Mathematical motivation

Previous works [Hasenfraz 1998, Neuberger 1998] employs the Dirac operator satisfying

Ginsparg-Wilson relation

$$\gamma_5 D_{\rm ov} + D_{\rm ov}\gamma_5 = aD_{\rm ov}\gamma_5 D_{\rm ov}$$

to define the index

$$\mathrm{Ind} D_{\rm ov} = n_+ - n_-$$

Furuta : "IInteresting to see that index can be defined with finite-dimensional vector space. This should be mathematically formulated." = Mathematical motivation.

Lattice version for $\frac{1}{2}\eta(\gamma_5(D+m))$ is more mathematically interesting!

# Elliptic estimate

In continuum theory, For any $\phi \in \Gamma(E)$ and i,
a constant c exists such that

$$||D_i\phi||^2 \leq c(||\phi||^2 + ||D\phi||^2)$$

When a covariant derivative is large、 D is also large.
This property is nontrivial on a lattice.

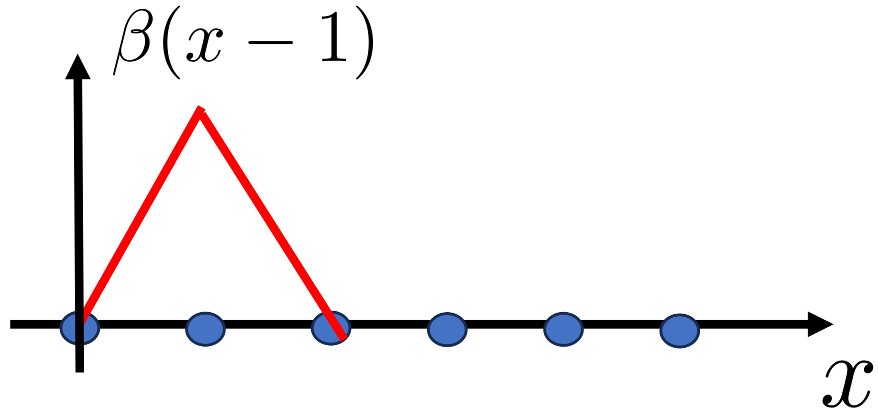
$$||\nabla_i^f\phi||^2 \leq c(||\phi||^2 + ||D_W\phi||^2)$$

Doubler modes have small Dirac eigenvalue with large wave number.

-> Wilson term is mathematically important, too!

$$f_a$$

$$f_a : H^{\mathrm{lat.}} \to H^{\mathrm{cont.}}$$




From finite-dimensional vector bundle on a discrete lattice we need to make infinite-dimensional vector bundle on continuous x :

$$f_a \phi^{\mathrm{lat.}}(x) = \sum_{l \in C_x} \beta(x-l) P(x-l) \phi^{\mathrm{lat.}}(l)$$

$C_x$ : a hyper cube containing $x$ . $l$ : lattice sites

$$P(x-l) = P \exp\left[ i \int_l^x dx'^i A_i(x') \right]$$ Wilson line.

$\beta(x-l)$ : linear partition of unity s.t.

$$\beta(0) = 1, \beta(1_\mu) = 0, \quad \sum_{l \in C_x} \beta_l(x) = 1.$$

$$f_a^*$$

$$f_a^* : H^{\text{cont.}} \to H^{\text{lat.}}$$

Is defined by

$$f_a^* \phi^{\text{cont.}}(l) = \int_{y \in C_l} dy \beta(l-y) P(l-y) \phi^{\text{cont.}}(y)$$

Note) $f_a^* f_a$ is not the identity but smeared to nearest-neighbor sites. (The gauge invariance is maintained by the Wilson lines.)

# Continuum limit of $f_a^*$ $f_a$

1. For arbitrary $\phi^{\mathrm{lat.}}$

   $\lim_{a\to 0} f_a \phi^{\mathrm{lat.}}$ weakly converges to a $\phi_0^{\mathrm{cont.}} \in L_1^2$

   where $L_1^2$ is the square-integrable subspace of $H^{\mathrm{cont.}}$

   to the first derivatives.

2. $\lim_{a\to 0} f_a \gamma(D_W + m)\phi^{\mathrm{lat.}}$ weakly converges to

   $\gamma(D+m)\phi_0^{\mathrm{cont.}} \in L^2$

3. There exists c s.t. $||f_a^* f_a \phi^{\mathrm{lat.}} - \phi^{\mathrm{lat.}}||_{L^2}^2 < ca^2 ||\phi^{\mathrm{lat.}}||_{L_1^2}^2$

4. For any $\phi^{\mathrm{cont.}} \in L_1^2$ , $\lim_{a\to 0} f_a f_a^* \phi^{\mathrm{cont.}}$

weakly converges to $\phi_0^{\mathrm{cont.}} \in L_1^2$ and

$$\lim_{a\to 0} f_a f_a^* \phi_0^{\mathrm{cont.}} = \phi_0^{\mathrm{cont.}}$$

# What are the weak convergence and strong convergence?

The sequence $v_j$ weakly converges to $v_\infty$

when for arbitrary $w$

$$\lim_{j\to\infty} \langle (v_j - v_\infty), w \rangle = 0.$$

Note) $\lim_{j\to\infty} (v_j - v_\infty)(x) \to \lim_{k\to\infty} e^{ikx}$ is weakly convergent.

Strong convergence means $\lim_{j\to\infty} ||v_j - v_\infty||^2 = 0.$

Rellich's theorem:

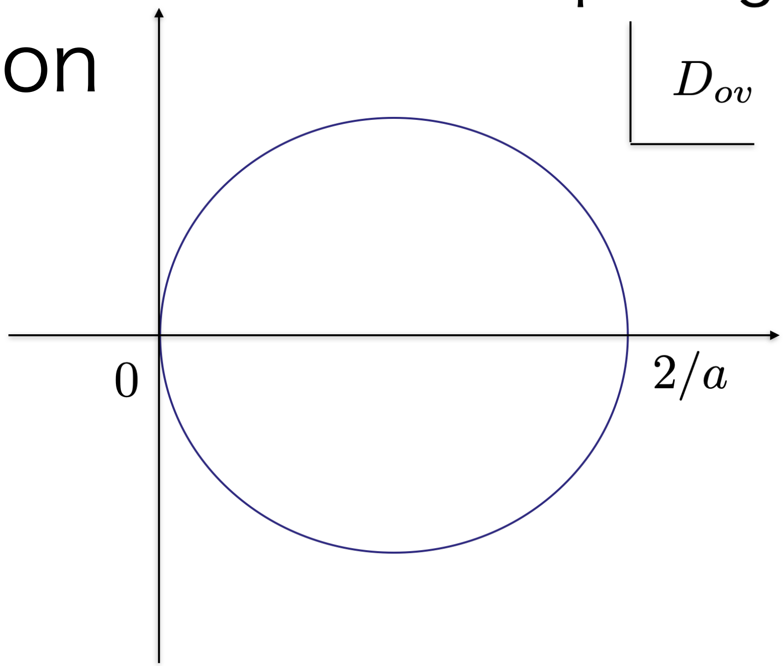
$L_1^2$ weak convergence = $L^2$ convergence

# Atiyah-Singer index on a lattice

$a$ :lattice spacing

Overlap Dirac spectrum lies on

a circle with radius 1/a




Complex eigenvalues of

$$\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$$

have ± pairs (zero contribution to the trace)
The real 2/a (doubler poles) does not contribute.

$$\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) = \mathrm{Tr}_{\mathrm{zeros}}\gamma_5.$$