Magnetic monopole and domain-wall fermion Dirac operator

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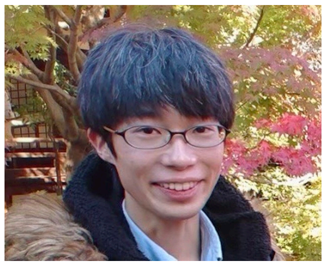
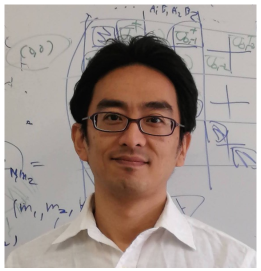
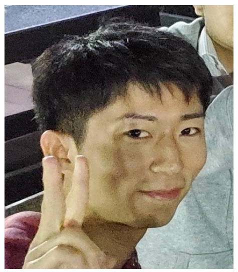
S. Aoki, HF, N. Kan, M. Koshino and Y. Matsuki (Osaka U.) arXiv:2304.13954



My research interests

1. Particle physics : numerical simulation of Quantum Chromo-Dynamics [JLQCD collaboration 2006-]
2. Index theorems [with Furuta, Matsuo, Onogi, Yamaguchi, Yamashita 2019-]
3. Topological insulator [with Aoki, Kan, Koshino, Matsuki, 2023-]

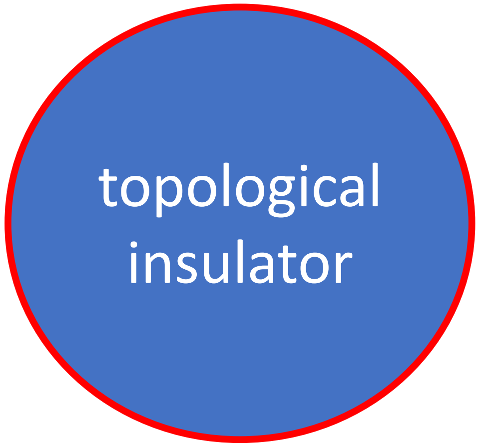
A common tool =

Domain-wall fermion

Dirac operator

This talk

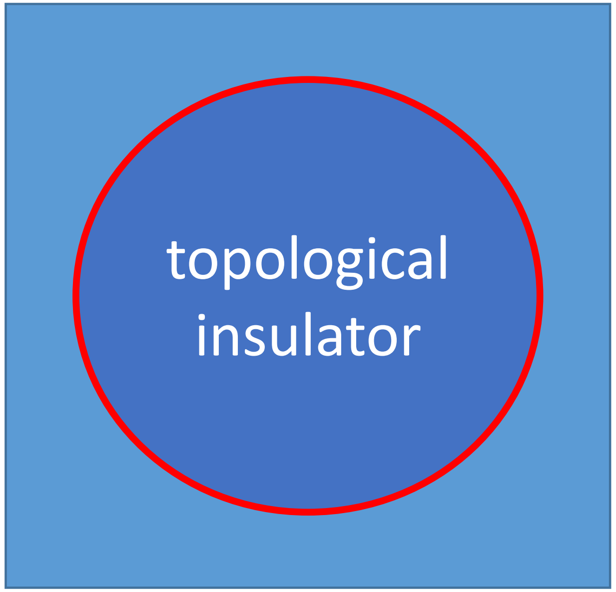
What is domain-wall fermion?



Standard approach = manifold w/ boundary:
fermion field is confined in
a disk with a boundary S^1.

Domain-wall fermion:
fermion field is defined everywhere in R^2
having two domains (classified by the mass term)
domain $r < 1$ -> topological insulator
domain $r > 1$ -> normal insulator with the
gap too big to excite electrons

Outside is important.

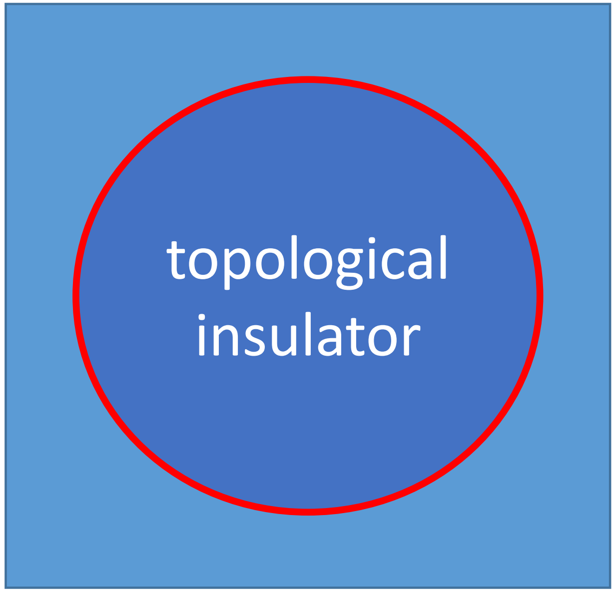


Existence of the edge-modes depends on "outside" of the topological insulator.

When the outside is normal insulator, edge-localized gapless modes appear.

When the outside is topological insulator (with the same Chern number)
the edge modes are absent.

Outside is important.



Standard approach:

If you ignore “outside”, we have to impose the boundary condition by hand.

Domain-wall fermion:

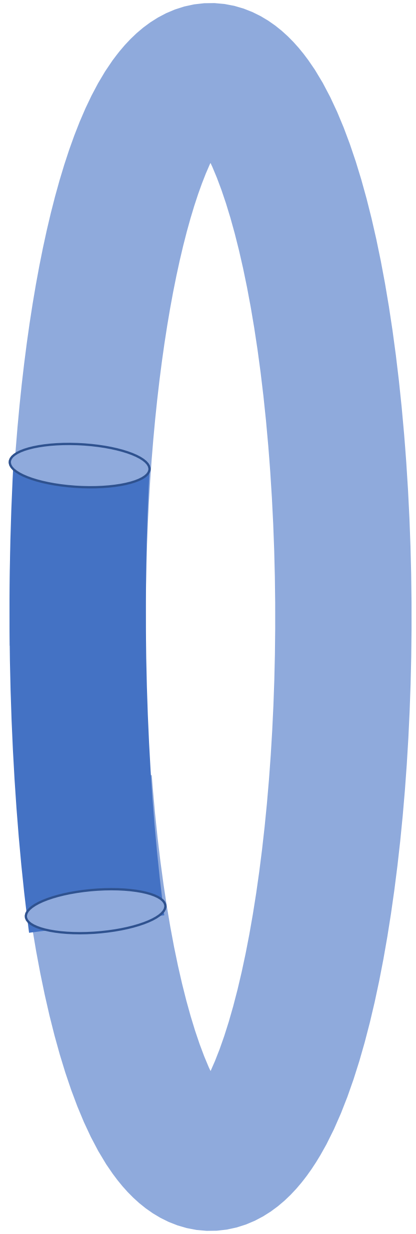
Boundary condition is automatically and uniquely determined by the “outside”.

Our previous work on the Atiyah-Patodi-Singer(APS) index

[F,Furuta,Matsuo, Onogi, Yamaguchi, Yamashita 2019]

APS index of massless Dirac op. on a manifold w/ nonlocal boundary (APS) condition by hand

= eta invariant of massive domain-wall Dirac operator on an extended closed manifold



In this work

We start from an (imaginary) infinitely
large 3-dimensional topological insulator having a single domain ($\mathbb{R}^3$ or S^3) only.

But non-trivial domain(s) will appear later
because of the monopole (and outside is important !)

What is a magnetic monopole.

In our world, every magnetic object is paired N and S poles.

Figure from sozai-library.com

But magnetic monopoles naturally appear in grand-unified theory (or low energy limit of string theory).

They are stable but extremely sparse because they are created only before the inflation of the universe.

What will happen to monopole in topological insulator?

Witten effect [1979]: A magnetic monopole inside topological insulators will become a dyon with 1 / 2 of unit electric charge.

[Cf. Recent works: Hidaka et al. 2020,2021, Fukuda-Yonekura 2021,,,]

Effective theory description [Witten 1979]

θ term modifies

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$= \boldsymbol{E} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{B} \qquad = \boldsymbol{E} \cdot \boldsymbol{B}$$

Maxwell equation :

$$\partial_\mu F^{\mu\nu} = -\frac{\theta}{8\pi^2} \partial_\mu \tilde{F}^{\mu\nu}$$

Divergences of E and B are related.

$$q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2} \int d^3x \nabla \cdot \boldsymbol{B} = -\frac{\theta n}{2\pi}$$

n : magnetic charge

Topological insulator is in the θ=π vacuum

θ term breaks Time(T) reversal symmetry (or CP) except for

θ=0: normal insulator

θ=π: topological insulator

(T sym. is protected in nontrivial way.)

$$q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2} \int d^3x \nabla \cdot \boldsymbol{B} = -\frac{\theta n}{2\pi}$$

A monopole with n=1 will have -1/2 of unit electric charge in topological insulator.

Previous studies

Effective theory is simple but tells nothing about the origin of the electric charge, although electrons are obviously the unique candidate.

Previous works [Yamagishi 1983, Zhao & Shen 2012, Khalilov 2012, Lee, Furusaki & Yang, 2019…] reported a bound state solution around the monopole of the Dirac equation but none (as far as we know) completely answered to the following questions:

1. What is the difference between normal/topological insulators?
2. How does monopole capture electron states?
3. Why is a chiral boundary condition by hand needed?
4. Why is the electric charge 1/2 ?

Our study

S. Aoki, HF, N. Kan, M. Koshino and Y. Matsuki (Osaka U.) arXiv:2304.13954

We carefully study the short-distance behavior of the Dirac equation of the electron fields in the presence of a monopole,

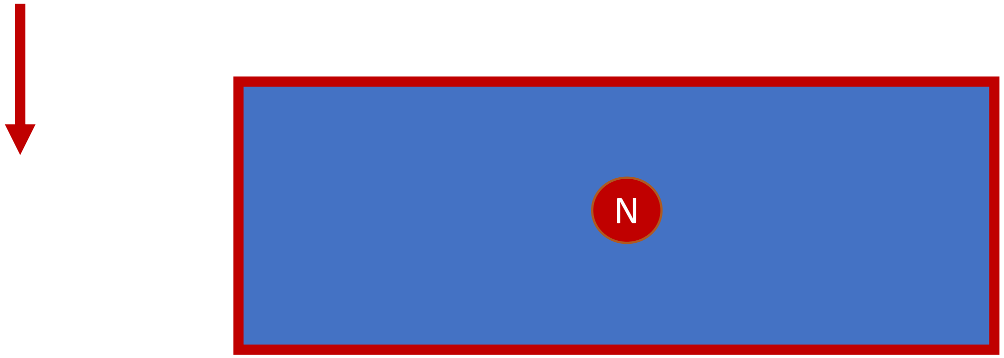
1) smearing the point-like singularity of the magnetic fields
2) adding the Wilson term (LO correction from Pauli-Villars regulator) both in continuum and on a lattice.

-> We obtain a bound-state solution without any singularity where topological/normal insulators are manifestly distinguished.

The system is described by domain-wall fermion Dirac operator.

Take-home message :

Outside is important.



Topological insulator ~~= manifold with boundary~~.

= a domain of electron system.

Contents

Dirac operator (mathematical set up)

We consider a principal U(1) bundle on $X = \mathbb{R}^3$ or S^3
with a connection 1-form $A_\mu(x)$

= electro-magnetic field,

and an associated complex vector bundle E:

$\psi \in \Gamma(E)$ = electron field.

Dirac operator (or Hamiltonian in physics)

$$H: C^\infty(X; E) \to C^\infty(X; E)$$

Dirac operator (examples)

$$H: C^\infty(X;E) \to C^\infty(X;E)$$

Massless Dirac operator on a 3-dim. manifold X

$$H = \Gamma^i(\partial_i - iA_i)$$

Covariant derivative

$$[\Gamma_i, \Gamma_j] = 2g_{ij}$$ Metric on X

Massive Dirac operator: $H = \Gamma^i(\partial_i - iA_i) + m\Gamma_0$

Constant $\quad$ Z_2 grading operator

Domain-wall Dirac operator $H = \Gamma^i(\partial_i - iA_i) + m(x)\Gamma_0$

Position dependent mass

U(1) gauge connection by a magnetic monopole

$$A_1 = \frac{-q_m y}{2r(r+z)}, \quad A_2 = \frac{q_m x}{2r(r+z)}, \quad A_3 = 0,$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Curvature

$$\partial_i A_j - \partial_j A_i = \frac{q_m}{2}\epsilon_{ijk}\frac{x_k}{r^3} - \pi q_m \delta(x)\delta(y)[1 - \mathrm{sgn}(z)]\epsilon_{ij3}$$

Dirac string extending to $z = -\infty$

Dirac's quantization condition: $q_m = n \in \mathbb{Z}$

-> the Dirac string has no physical effect.

Orbital angular momentum

Due to the presence of monopole, the orbital angular momentum is not the normal one:

$$L_i = -i\epsilon_{ijk}x_j(\partial_k - iA_k) - n\frac{x_i}{2r},$$

which satisfies

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

Dirac Hamiltonian (in 3 spatial dimensions)

$$H = \gamma_0 \left(\gamma^i(\partial_i - iA_i) + m\right) = \begin{pmatrix} m & \sigma^i (\partial_i - iA_i) \\ -\sigma^i (\partial_i - iA_i) & -m \end{pmatrix},$$

Constant mass term

$$\gamma_0 = \sigma_3 \otimes 1 \quad \gamma_i = \sigma_1 \otimes \sigma_i \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

Note: electric field=0, magnetic field is static.

$$A_0 = 0, \quad \partial_t A_i = 0.$$

"chirality" Z_2 grading operator (not usual $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$):

$$\bar{\gamma} = -i\gamma_1\gamma_2\gamma_3 = \sigma_1 \otimes 1 \qquad \{H, \bar{\gamma}\} = 0.$$

Total angular momentum and sphere operator

$$J_i = L_i 1 \otimes 1 + \frac{1}{2} 1 \otimes \sigma_i, \quad [J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, H] = 0$$

"Sphere" operator [physical meaning will be discussed later.]

$$D^{S^2} = \left[\sigma^i \left(L_i + \frac{n}{2}\frac{x_i}{r}\right) + 1\right]$$

$$[H, \sigma_3 \otimes D^{S^2}] = 0$$

Angular degrees of freedom are completed by the eigenstates of

$$J^2, \quad J_3, \quad D^{S^2}$$

Useful formulas

$$\sigma_r = \frac{\sigma^j x_j}{r} \quad \left\{D^{S^2}, \sigma_r\right\} = 0,$$

$$[D^{S^2}]^2 = \left(j + \frac{1}{2}\right)^2 - \frac{n^2}{4},$$

$$\sigma^i(\partial_i - iA_i) = \sigma_r \left(\frac{\partial}{\partial r} - \frac{1}{r}\left(D^{S^2} - 1\right)\right)$$

$$D^{S^2}\chi_{j,j_3,\pm}(\theta,\phi) = \pm\sqrt{\left(j + \frac{1}{2}\right)^2 - \frac{n^2}{4}}\chi_{j,j_3,\pm}(\theta,\phi) \quad \text{for } j > |\tfrac{n}{2}| - 1/2$$

$$D^{S^2}\chi_{j,j_3,0}(\theta,\phi) = 0 \quad \text{for } j = |\tfrac{n}{2}| - 1/2 \quad \sigma_r\chi_{j,j_3,0}(\theta,\phi) = \mathrm{sign}(n)\chi_{j,j_3,0}(\theta,\phi)$$

Radial direction

$$\psi^{E}_{j,j_3,\pm}(r,\theta,\phi) = \frac{1}{\sqrt{r}} \left(\begin{array}{c} f(r)\chi_{j,j_3,\pm}(\theta,\phi) \\ g(r)\sigma_r\chi_{j,j_3,\pm}(\theta,\phi) \end{array} \right)$$

$$\text{for } j > |\tfrac{n}{2}| - 1/2$$

$$\psi^{E}_{j,j_3,0}(r,\theta,\phi) = \frac{1}{r} \left(\begin{array}{c} f(r)\chi_{j,j_3,0}(\theta,\phi) \\ g(r)\chi_{j,j_3,0}(\theta,\phi) \end{array} \right)$$

$$\text{for } j = |\tfrac{n}{2}| - 1/2$$

The Dirac equation $H\psi = E\psi.$
is reduced to the one for f(r) and g(r).

Radial direction

$$\psi^{E}_{j,j_3,\pm}(r,\theta,\phi)=\frac{1}{\sqrt{r}}\left(\begin{array}{c} f(r)\chi_{j,j_3,\pm}(\theta,\phi)\\ g(r)\sigma_r\chi_{j,j_3,\pm}(\theta,\phi)\end{array}\right)$$

No bound state.

$$\text{for } j>|\tfrac{n}{2}|-1/2$$

$$\psi^{E}_{j,j_3,0}(r,\theta,\phi)=\frac{1}{r}\left(\begin{array}{c} f(r)\chi_{j,j_3,0}(\theta,\phi)\\ g(r)\chi_{j,j_3,0}(\theta,\phi)\end{array}\right)$$

$$\text{for } j=|\tfrac{n}{2}|-1/2$$

The Dirac equation $H\psi=E\psi$.
is reduced to the one for f(r) and g(r).

Zero mode solution localized at monopole

$$\begin{pmatrix} m & \mathrm{sign}(n)\partial_r \\ -\mathrm{sign}(n)\partial_r & -m \end{pmatrix} \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = E \begin{pmatrix} f(r) \\ g(r) \end{pmatrix}.$$

We can solve this equation to obtain E = 0 : $H\psi^{E=0}_{j,j_3,0} = 0$

$$\psi^{E=0}_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \chi_{j,j_3,0}(\theta,\phi),$$

This bound state may be the origin of electric charge that magnetic monopole acquires. But…

Still insufficient...

$$\psi^{E=0}_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \chi_{j,j_3,0}(\theta,\phi),$$

1. The solution is a chiral eigenstate of $\sigma_1 \otimes \sigma_r = \mathrm{sign}(m)$
   But what is the origin?
2. Both sign of m allowed until we impose the chiral boundary condition by hand. But why ?
3. How 1/2 unit electric charge derived? Yamagishi 1983 summed up all the states in Dirac sea but with an awkward regularization breaking the charge conservation.
4. Divergence at r=0 O.K.?

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6. Summary and discussion

Adding the Wilson term

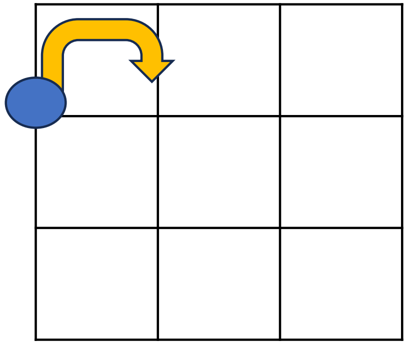
$$\gamma_0 \left[\gamma^i D_i + m + \frac{1}{M_{\rm PV}} D_i^\dagger D^i \right] \psi = E\psi, \qquad \begin{array}{l} \gamma_0 = \sigma_3 \otimes 1 \\ \gamma_i = \sigma_1 \otimes \sigma_i \end{array}$$

This second derivative term may not be welcome in mathematics, loosing symmetry, making the problem more complicated…

Higher derivatives naturally appear in physics

In the original lattice system electrons are hopping atoms in crystals

with finite spacing a



Differential operator = approximation of difference operator

$$\nabla_i^f \psi(\boldsymbol{x}) = \frac{\psi(\boldsymbol{x} + a\boldsymbol{e}_i) - \psi(\boldsymbol{x})}{a} = \left[\partial_i + \frac{a}{2}\partial_i^2 + \cdots\right]\psi(\boldsymbol{x})$$

Wilson Dirac operator on a lattice

$$H_W\psi = \gamma^0 \left(\sum_{i=1}^{3} \left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{1}{2}\nabla_i^f \nabla_i^b \right] + m \right) \psi = E\psi$$

$$\gamma_0 = \sigma_3 \otimes 1$$
$$\gamma_i = \sigma_1 \otimes \sigma_i$$

Wilson term is crutial in removing unphysically oscillating modes.

Here

$$\nabla_i^f \psi(\boldsymbol{x}) = U_i(\boldsymbol{x})\psi(\boldsymbol{x} + \boldsymbol{e}_i) - \psi(\boldsymbol{x})$$

$$\nabla_i^b \psi(\boldsymbol{x}) = \psi(\boldsymbol{x}) - U_i^\dagger(\boldsymbol{x} - \boldsymbol{e}_i)\psi(\boldsymbol{x} - \boldsymbol{e}_i)$$

$$U_j(\boldsymbol{x}) = \exp\left[i \int_0^1 A_j(\boldsymbol{x}')dl \right],$$

Wilson term makes sign of the mass well-defined

Against Wilson term(M_{PV}>0) the sign of m is well-defined.

$$\gamma_0\left[\gamma^i D_i + m + \frac{1}{M_{\rm PV}} D_i^\dagger D^i\right]\psi = E\psi, \quad \gamma^0\left(\sum_{i=1}^{3}\left[\gamma^i \frac{\nabla_i^f + \nabla_i^b}{2} - \frac{1}{2}\nabla_i^f \nabla_i^b\right] + m\right)\psi = E\psi$$

For negative $mM_{\rm PV}$, the Chern number in momentum space is nontrivial (topological phase) [Hatugai et al.]

Moreover, in the path-integral of electron fields (w/ anomaly),

$$Z = \det\left[\frac{D+m}{D+M_{\rm PV}}\right] = \det\left[\frac{D+|m|}{D+M_{\rm PV}}\right] \exp\left(\theta \int d^4x F\tilde{F}\right)$$

$$m > 0 \rightarrow \theta = 0 : \ \text{normal phase}$$
$$m < 0 \rightarrow \theta = \pi : \ \text{topological phase}$$

The Wilson matters in the continuum limit?

Cut-off of electron field: $1/M_{\rm PV} \sim a \sim 10^{-10} m.$

Size of ('tHooft-Polyakov) monopole: $r_1 \sim 10^{-30} m.$

It is nontrivial to study the 20-digit difference.

Additive mass correction from Wilson term

The correction term is a positive operator.

$$\gamma_0 \left[\gamma^i D_i + m + \frac{1}{M_{\rm PV}} D_i^\dagger D^i \right] \psi = E\psi.$$

When the magnetic field is dense in a region $r < r_1$

the correcction $\sim \frac{1}{M_{\rm PV} r_1^2} \sim 10^{+50}(m^{-1})$ $\quad r_1 \sim 10^{-30} m.$

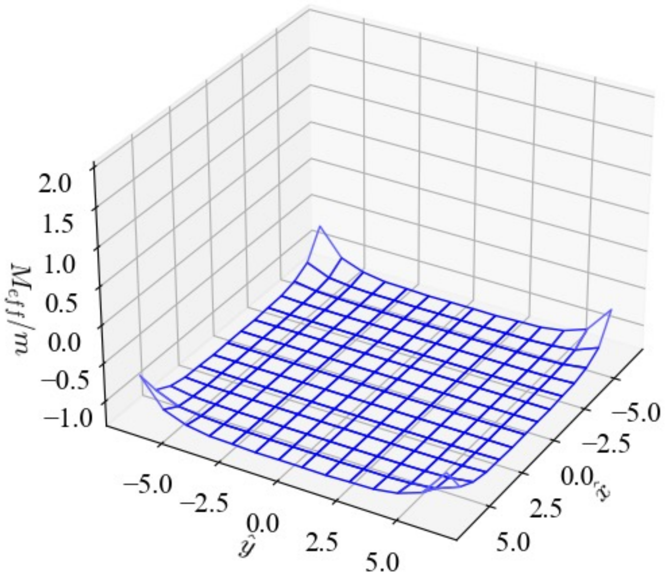
can change the sign of the mass, $\quad 1/M_{\rm PV} \sim a \sim 10^{-10} m.$

creating a domain-wall near the monopole.

[This never happens in normal insulator with m>0.]

Additive mass correction from Wilson term

Numerical evaluation of $m_{\rm eff} = \left\langle m + \frac{1}{M_{\rm PV}} D_i^\dagger D^i \right\rangle$



Local effective mass /|m| at the z=0 slice

- Bra-ket taken for the nearest-zero mode state:

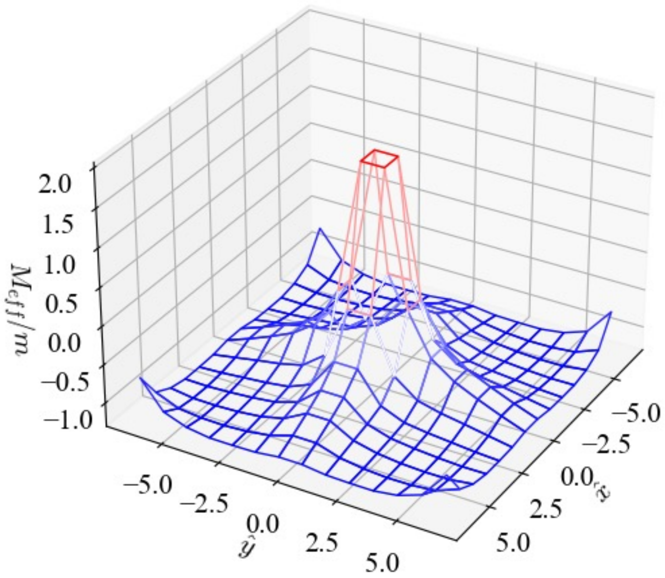
$$= \frac{\phi_0(\boldsymbol{x})^\dagger \left[-\sum_{i=1,2,3} \frac{1}{2} \nabla_i^f \nabla_i^b + m \right] \phi_0(\boldsymbol{x})}{\phi_0(\boldsymbol{x})^\dagger \phi_0(\boldsymbol{x})}$$

* Lattice setup details will be given later.

m= -0.43, n=0: When monopole is absent, the mass is negative everywhere.

Additive mass correction from Wilson term

Numerical evaluation of $m_{\text{eff}} = \left\langle m + \frac{1}{M_{\text{PV}}} D_i^\dagger D^i \right\rangle$



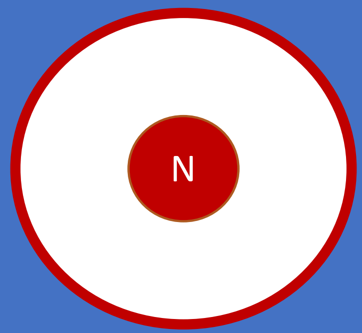
Local effective mass /|m| at the z=0 slice

m= -0.43, n=1:

the monopole makes a positive mass region around it.
= a small but finite-sized domain-wall is created!

Inside topological insulator

A magnetic monopole locally gives a positive mass shift to make its neighbor (atomic scale) a normal insulator.



This system is essentially a domain-wall fermion.

The chiral edge-localized gapless modes appear on the domain-wall between the normal and topological insulators.

Edge modes on the domain-wall

satisfying

$$\left[\sigma_1 \otimes \sigma_r \partial_r + m_{\text{eff}}(r)\right] \psi = 0,$$

$$\psi \sim \exp\left[-s \int_{r_1}^{r} dr' m_{\text{eff}}(r')\right]$$

s : eigenvalue of $\sigma_1 \otimes \sigma_r$

If s=-1 and $m_{\text{eff}}(r \gg r_1) \to m < 0, \quad m_{\text{eff}}(r < r_1) > 0$

this mode is normalizable.

Cf) We refer to studies on the domain-wall fermion [Jackiw-Rebbi 1976, Callan-Harvery 1985, Kaplan 1992]

Recall Yamagishi 1983

$$\psi^{E=0}_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \chi_{j,j_3,0}(\theta,\phi),$$

$$\sigma_r \chi_{j,j_3,0}(\theta,\phi) = \mathrm{sign}(n)\chi_{j,j_3,0}(\theta,\phi)$$

For m<0, it is an eigenstate of $\sigma_1 \otimes \sigma_r = -1$

For m>0, the solution with $\sigma_1 \otimes \sigma_r = -1$ is not normalizable.

Exact solution with Wilson term

$$\gamma_0 \left[\gamma^i D_i + m + \frac{1}{M_{\rm PV}} D_i^\dagger D^i\right]\psi = E\psi,$$

is analytically solvable (with a hard work) [See our paper for details]

In particular, the $r_1 \to 0$ limit gives a simple form with E=0.

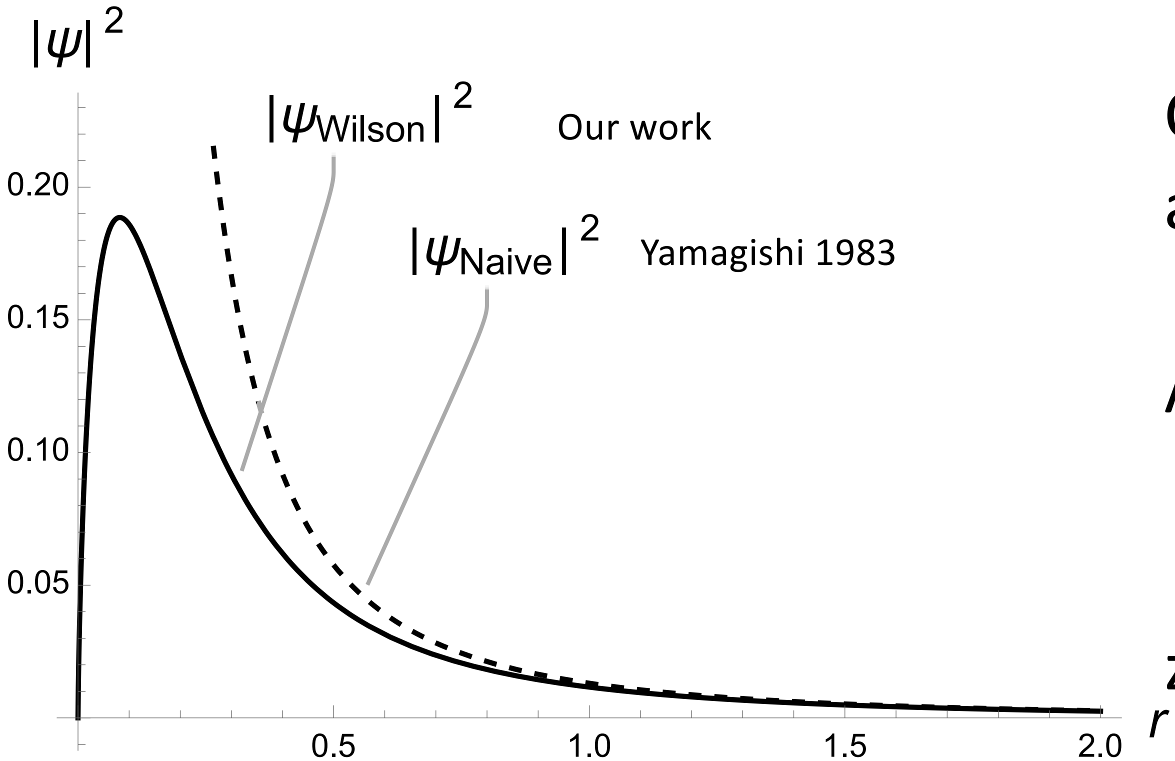
$$\psi^{\rm mono}_{j,j_3} = \frac{B e^{-\frac{M_{\rm PV} r}{2}}}{\sqrt{r}} I_\nu(\kappa r) \left(\begin{array}{c} 1 \\ -s \end{array}\right) \otimes \chi_{j,j_3,0}(\theta,\phi)$$

$$s = {\rm sign}(n)$$

$$\nu = (\sqrt{2|n|+1})/2$$

$$\kappa = \frac{M_{\rm PV}}{2}\sqrt{1+4m/M_{\rm PV}}$$

The edge mode on the domain-wall



Plot with $n = 1, \quad m = 0.1, \quad M_{\text{PV}} = 10.$

Finite everywhere.

Converges to Yamagishi 1983 at long distance.

A peak located at $r = \dfrac{|n|}{2M_{\text{PV}}}$

≒ domain-wall radius

zero at r=0

(Wilson term -> ∞)

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✓ 3. Regularized Dirac eq. and fundamental reason for electron bound state

Dense magnetic field forms a finite-sized domain-wall whose chiral edge modes are identified as the origin of the electric charge.

4. Atiyah-Singer index theorem responsible for 1/2 electric charge

5. Reinterpretation of Effective Maxwell theory and continuous magnetic charge

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What is the “sphere” operator

Our solution $$\psi^{\rm mono}_{j,j_3} = \frac{Be^{-\frac{M_{\rm PV} r}{2}}}{\sqrt{r}} I_\nu(\kappa r) \left(\begin{array}{c} 1 \\ -s \end{array} \right) \otimes \chi_{j,j_3,0}(\theta,\phi)$$

is also a zero mode of $$D^{S^2} = \left[\sigma^i \left(L_i + \frac{n}{2}\frac{x_i}{r}\right) + 1\right]$$

$$D^{S^2}\chi_{j,j_3,0}(\theta,\phi) = 0 \quad \text{for } j = |\tfrac{n}{2}| - 1/2$$

In fact, this operator is identified as the effective Dirac operator on the two-dimensional spherical domain-wall.

Two-dimensional Dirac operator

With a Local Lorentz transformation $R(\theta,\phi)=\exp(i\theta\sigma_2/2)\exp(i\phi(\sigma_3+1)/2)$

$$\frac{1}{r_2}D^{S^2\prime}:=\frac{1}{r_2}R(\theta,\phi)\left[\sigma^i\left(L_i+\frac{n}{2}\frac{x_i}{r}\right)+1\right]R(\theta,\phi)^{-1}$$

r_2: domain-wall radius.

$$=-\frac{1}{r_2}\sigma_3\left[\sigma_1\frac{\partial}{\partial\theta}+\sigma_2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}+i\hat{A}_\phi+i\hat{A}^s_\phi\right)\right],$$

$$\hat{A}_\phi=\frac{n}{2}\frac{\sin\theta}{1+\cos\theta}\qquad \hat{A}^s_\phi=\frac{1}{2\sin\theta}-\frac{\cos\theta}{2\sin\theta}\sigma_3$$

U(1) gauge connection by the monopole

Gravitational potential (Spin^c connection) induced by strongly curved domain-wall.

Note : Einstein's equivalence principle indicates constraint force to domain-wall= gravity. [Aoki-Fukaya 2022 Mar, 2022 Dec]

Gravity vs. magnetic field

$$\frac{1}{r_2} D^{S^2\prime} = -\frac{1}{r_2}\sigma_3 \left[\sigma_1 \frac{\partial}{\partial\theta} + \sigma_2 \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + i\hat{A}_\phi + i\hat{A}^s_\phi\right)\right],$$

r_2 : domain-wall radius.

Aoki Fukaya 2022 Mar, 2022Dec : With pure gravity,

the Dirac specturm is gapped: $E_1 \sim 1/r_2.$

The Monopole magnetic field cancels the gravitational effect to

have a zero mode: $D^{S^2}\chi_{j,j_3,0}(\theta,\phi) = 0 \quad \text{for } j = |\frac{n}{2}| - 1/2$

Moreover, D^{S^2} and $\sigma_r = \sigma^j x_j/r$ anticommute:

This zero mode is a chiral zero mode on the two-dimensional sphere.

Cf.) Localization of the zero modes [Furuta, "Index Theorem 1"]

Atiyah-Singer index on the S^2 domain-wall

$$D^{S^2}\chi_{j,j_3,0}(\theta,\phi)=0 \quad \text{for } j=|\tfrac{n}{2}|-1/2$$

The degeneracy is $2j+1 = |n|$.

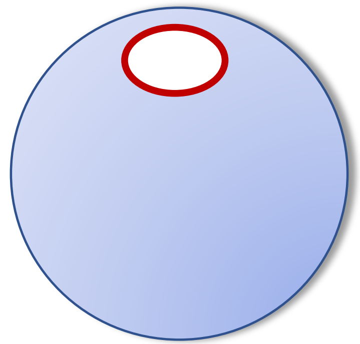
The 2-dim chirality is $\sigma_r\chi_{j,j_3,0}(\theta,\phi)=\text{sign}(n)\chi_{j,j_3,0}(\theta,\phi)$

→ $$\text{Ind}D^{S^2}=n.$$

Geometrical index is $$\frac{1}{4\pi}\int_{S^2} d^2x\epsilon^{\mu\nu}F_{\mu\nu}=\frac{1}{2\pi}\int_{S^2} d^2x\boldsymbol{B}\cdot\boldsymbol{n}=n.$$

Namely, the zero modes on the domain-wall around the monopole are topologically protected by the Atiyah-Singer index theorem.

Infra-red regularization for topological insulators



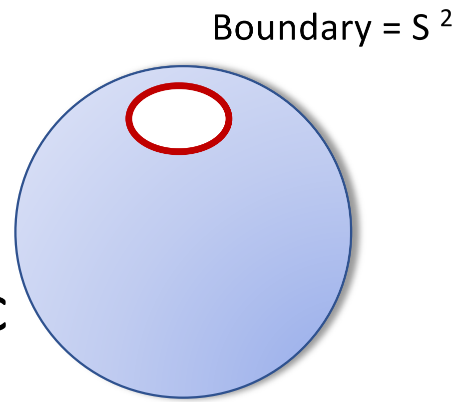
In particle physics, we often employ one-point compactification of the infinity.

This is not allowed : Atiyah-Singer index is a cobordism invariant:
It must be zero when it is located at a boundary of some manifold.

$$\int_{\partial M} F = \int_M dF = 0.$$

$$M = S^2$$

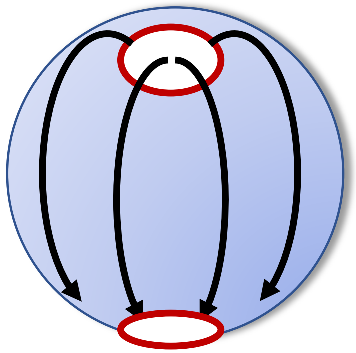
We need "outside".



We need a "sink" of the magnetic flux or another monopole with the opposite magnetic charge, which creates another domain-wall.

= Another domain-wall must be located at r<∞

= We need outside (= lab) of topological insulators



Another domain-wall is needed, which has the same Atiyah-Singer index.

Moreover, the two domain-walls share the same Atiyah-Singer index.

$$0 = \int_M dF = \int_{\partial M_1} F + \int_{\partial M_2} F$$

Outer domain-wall

$$\epsilon(r - r_0) = \pm 1 \ \text{ for } \ r \gtrless r_0$$

$$\gamma_0 \left[\gamma^i D_i + m_0 \epsilon(r - r_0) + \frac{1}{M_{\rm PV}} D_i^\dagger D^i \right] \psi = E\psi,$$

is analytically solvable (with a hard work) [See our paper for details]

In the large $r_0 m_0$ limit, we obtain

$$\psi^{\rm DW}_{j,j_3} = \begin{cases} \frac{\exp\left(\frac{M_{\rm PV} r}{2}\right)}{\sqrt{r}} \left(e^{\kappa_- r_0} B' K_\nu(\kappa_- r) + e^{-\kappa_- r_0} C' I_\nu(\kappa_- r)\right) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi), & (r < r_0), \\ \frac{D' \exp\left(\kappa_+ r_0 + \frac{M_{\rm PV} r}{2}\right)}{\sqrt{r}} K_\nu(\kappa_+ r) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi), & (r > r_0), \end{cases}$$

$$\kappa_\pm = \frac{M_{\rm PV}}{2}\sqrt{1 \pm 4|m|/M_{\rm PV}}$$

$$\frac{C'}{B'} = \pi \frac{\kappa_- - \kappa_+}{\kappa_- + \kappa_+},$$

$$\frac{D'}{B'} = \frac{2\sqrt{\kappa_- \kappa_+}}{\kappa_- + \kappa_+}$$

1) E = 0.
2) $\sigma_1 \otimes \sigma_r = +1$ (opposite to inner DW solution) with degeneracy 2j+1=|n|.
3) Also an eigenstate of $\bar{\gamma} = \sigma_1 \otimes 1 = s = \text{sign}(n)$

Mixing of the zeromodes on two domain-walls

At finite r0, the two zero mode mix and are split.

$$\psi = \alpha\psi^{\mathrm{mono}}_{j,j_3} + \beta\psi^{\mathrm{DW}}_{j,j_3}$$

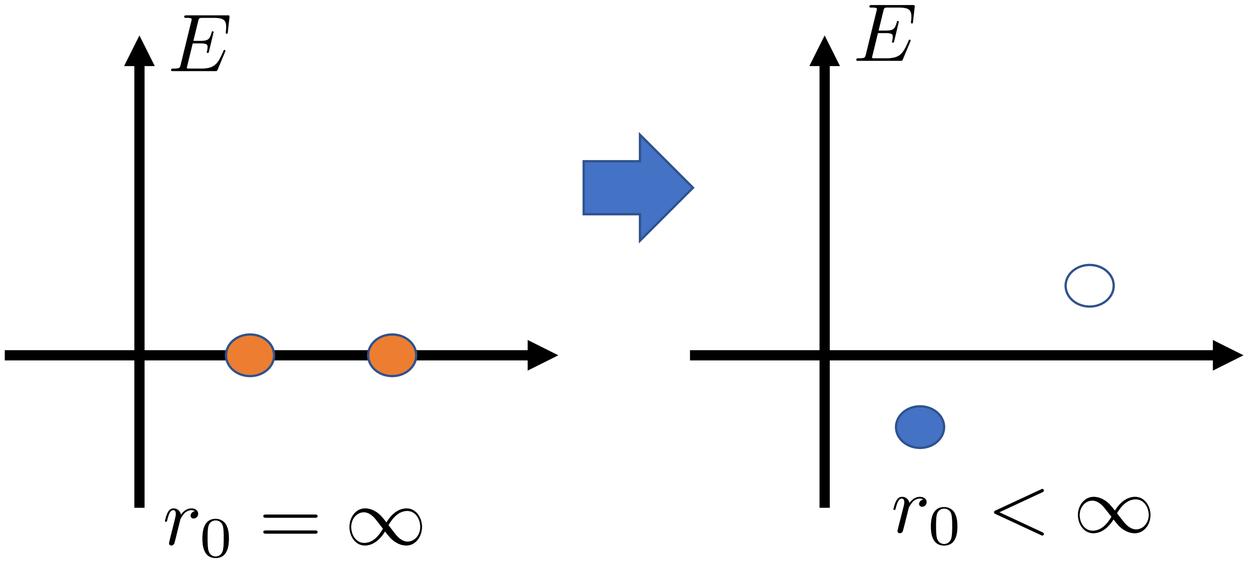
$$(\psi^{\mathrm{mono}}_{j,j_3})^\dagger H\psi^{\mathrm{mono}}_{j,j_3} = (\psi^{\mathrm{DW}}_{j,j_3})^\dagger H\psi^{\mathrm{DW}}_{j,j_3} = 0 \longleftarrow \{\bar{\gamma}, H\} = 0$$

$$(\psi^{\mathrm{mono}}_{j,j_3})^\dagger H\psi^{\mathrm{DW}}_{j,j_3} = (\psi^{\mathrm{DW}}_{j,j_3})^\dagger H\psi^{\mathrm{mono}}_{j,j_3} =: \Delta \sim \exp(-|m|r_0)$$

→ $E = \pm\Delta \quad \alpha = \pm\beta$

For however small Δ the mixing is maximal: 50% vs. 50%

Why-1/2 electric charge?



Zeromodes are degenerage

split to $E = \pm\Delta$

$$\psi_{\mp\Delta} \sim \frac{1}{\sqrt{2}}\left[\psi^{\rm mono}_{j,j_3} \pm \psi^{\rm DW}_{j,j_3}\right]$$

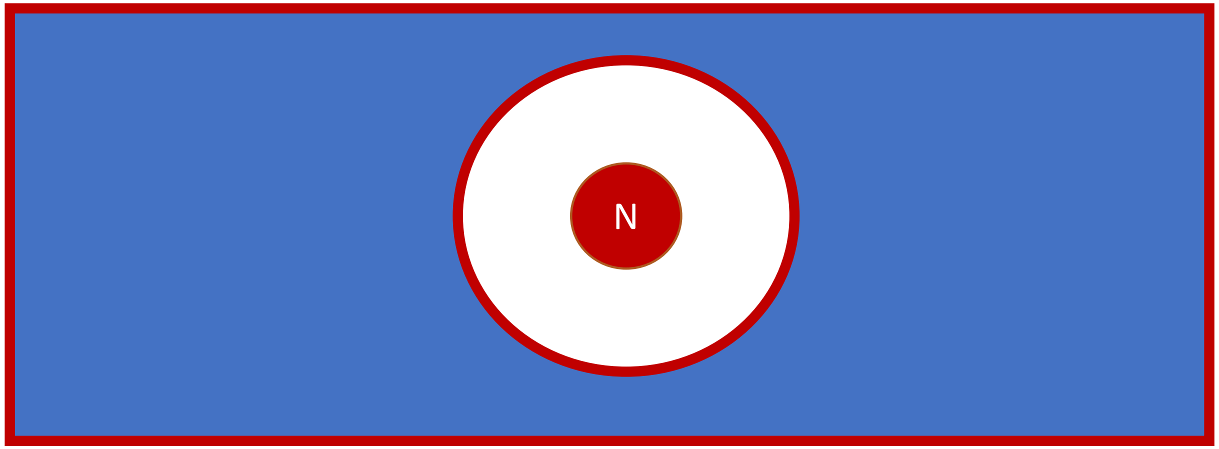
At Half filliing (Fermi energy =0) only lower state is occupied.

Only ½ of the amplitude is located around the monopole.

The electric charge (expectation value) of the monopole is -1/2!

Why 1/ 2 electric charge?

A magnetic monopole locally gives a positive mass shift to turn its neighbor into a normal insulator.



The created domain-wall is cobordant to the surface so that every zero mode is paired.

The paired zero modes maximally mix via tunneling effect.

At the half-filling, 50% amplitude around the monopole is responsible for the half electric charge.

Numerical analysis

L=24,32,48 lattices

Domain-wall radius r0=(3/8)L

Monopole put at (L/2,L/2,L/2)

Anti-monopole at (L/2,L/2,1/2)

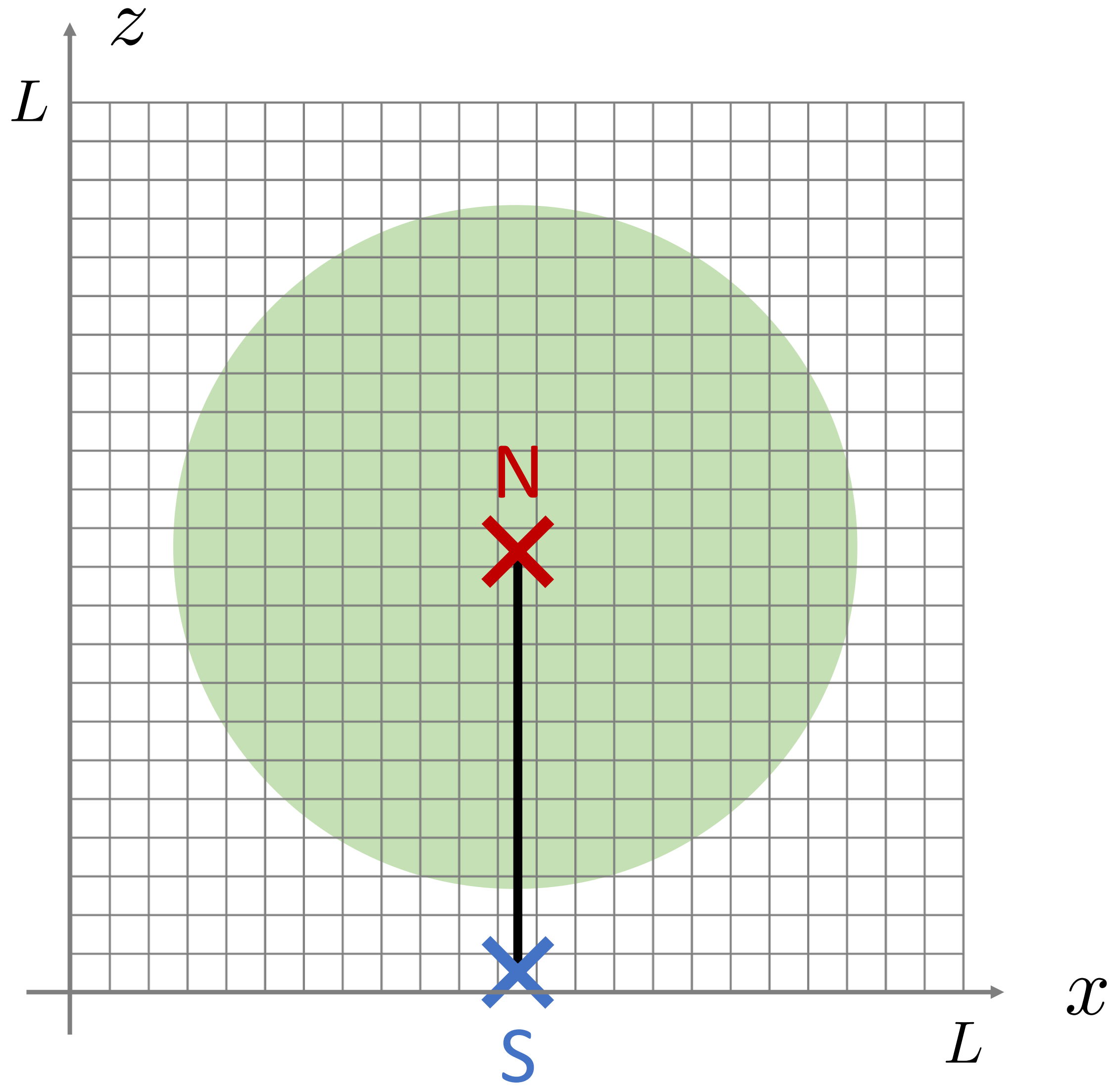
m(r<r0) = -14/(L+1)

m(r>r0) = +14/(L+1)

Open boundary condition at

x_i = 0 or L (m=∞ outside)

Monopole charge n=0,1,-2



Near-zero eigenvalues/functions

Circle: lattice results

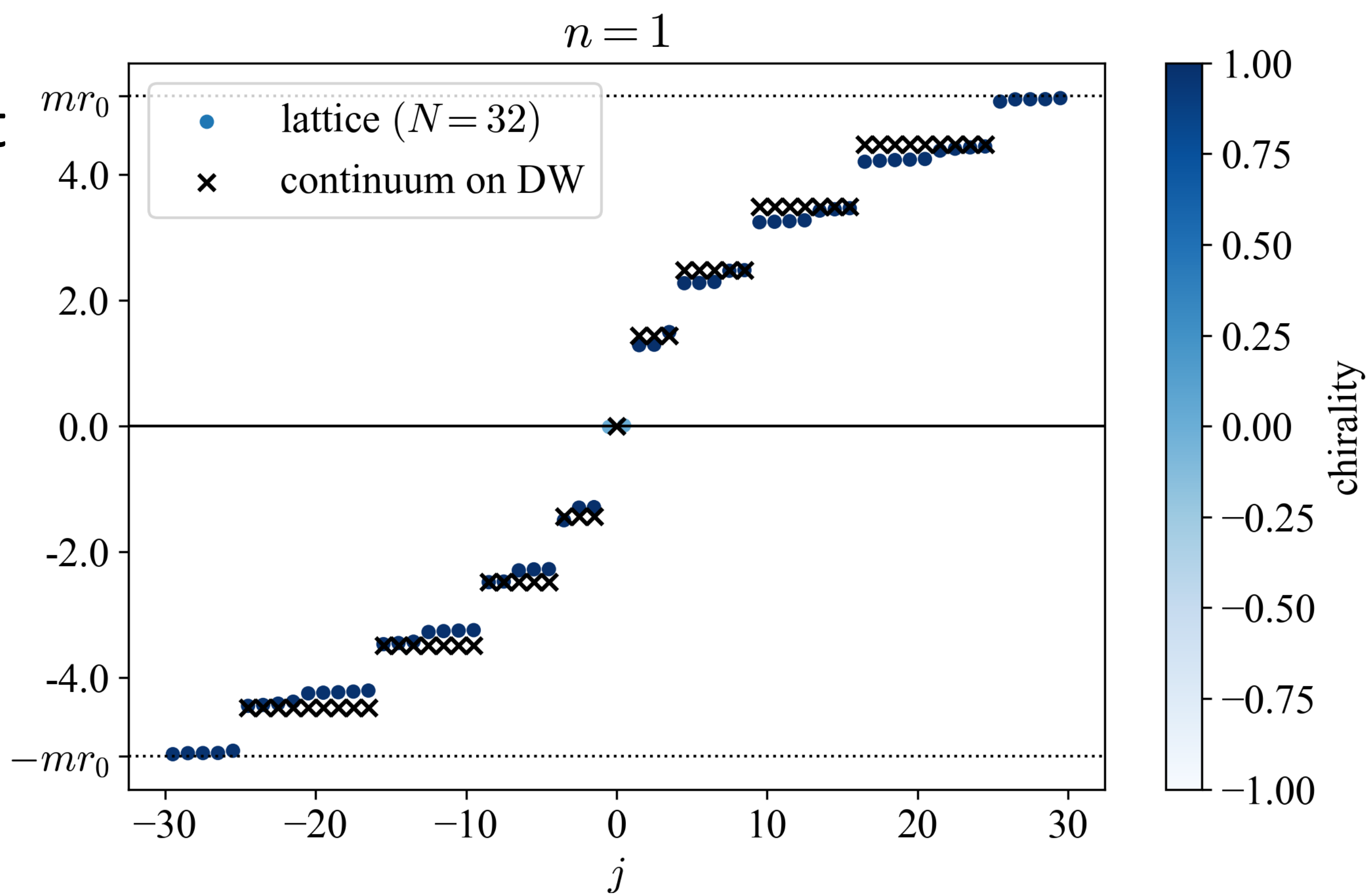
Crosses: analytic result on DW

Gradation: chirality.

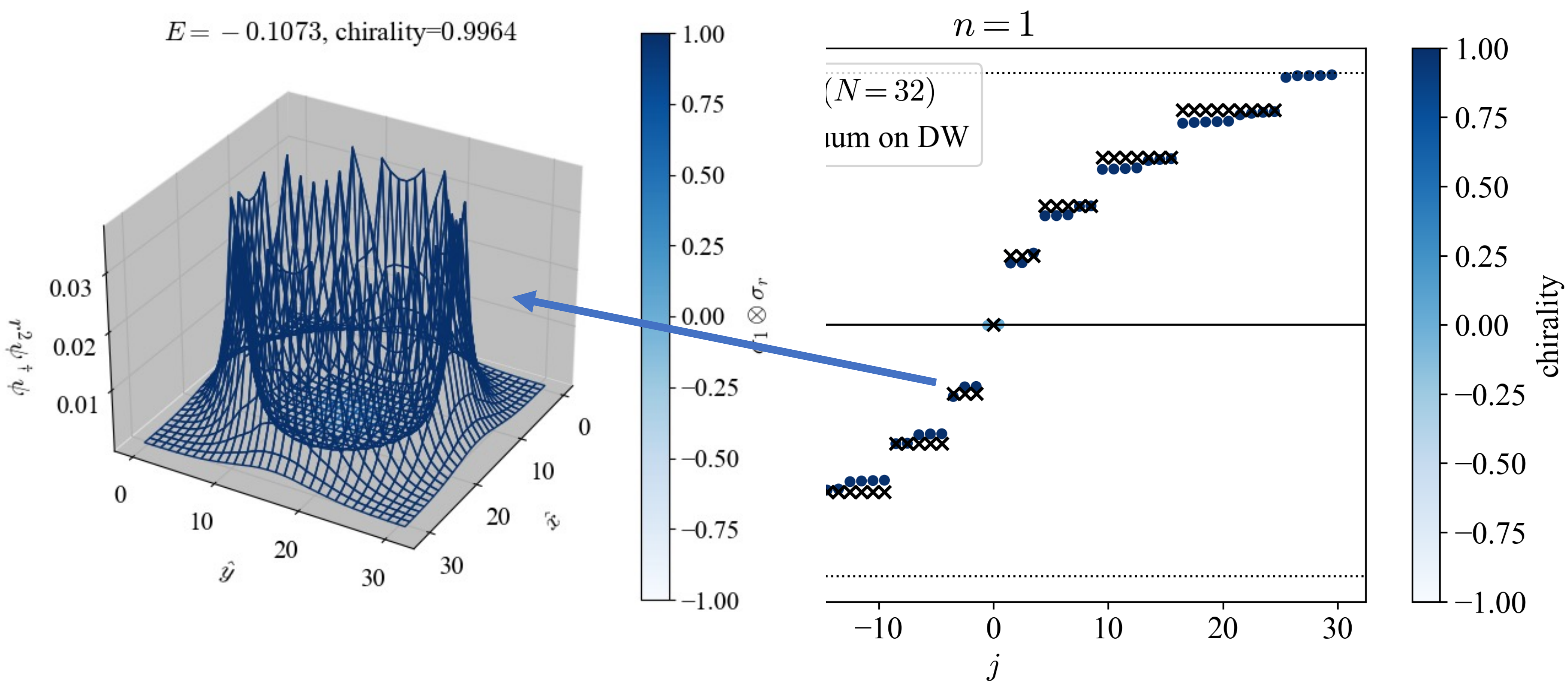
$$\langle \sigma_1 \otimes \sigma_r \rangle_k$$

Monopole charge n=1.

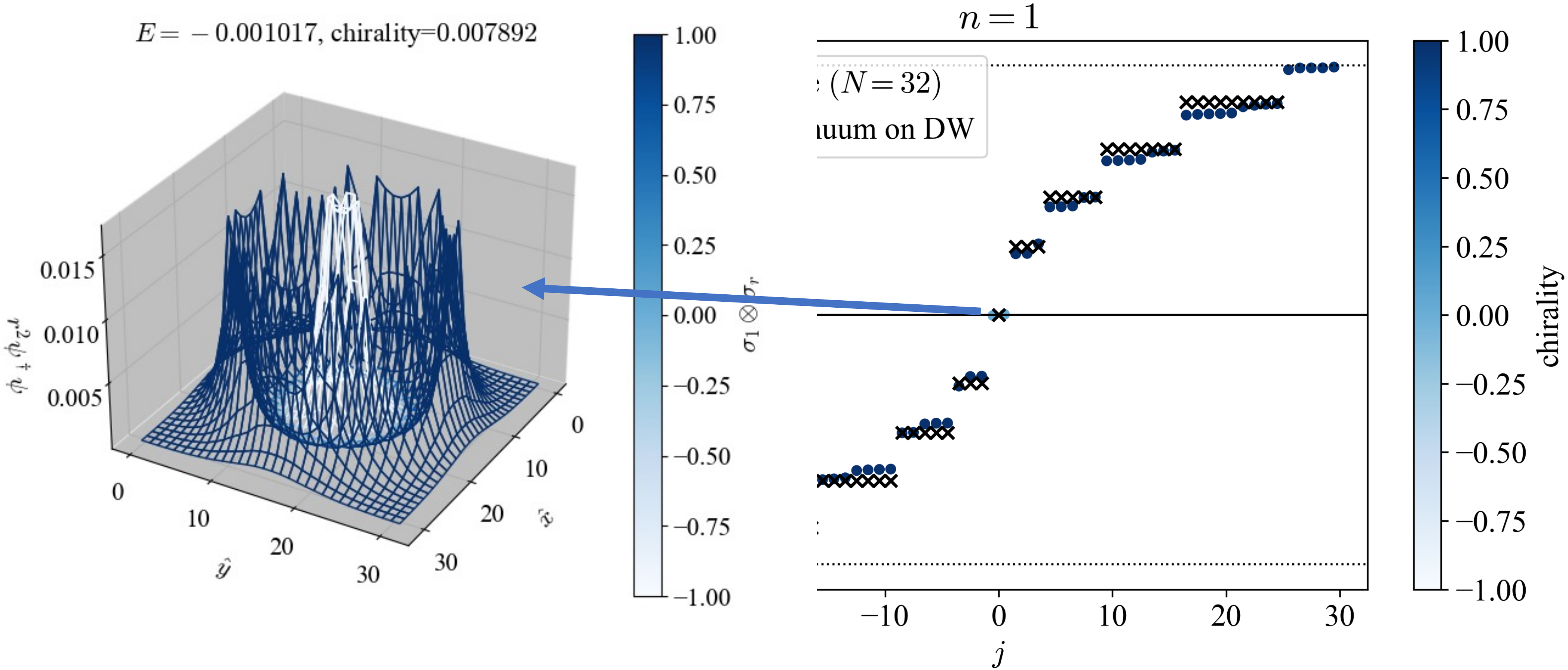
We have two near Zero modes.



Near-zero eigenvalues/functions



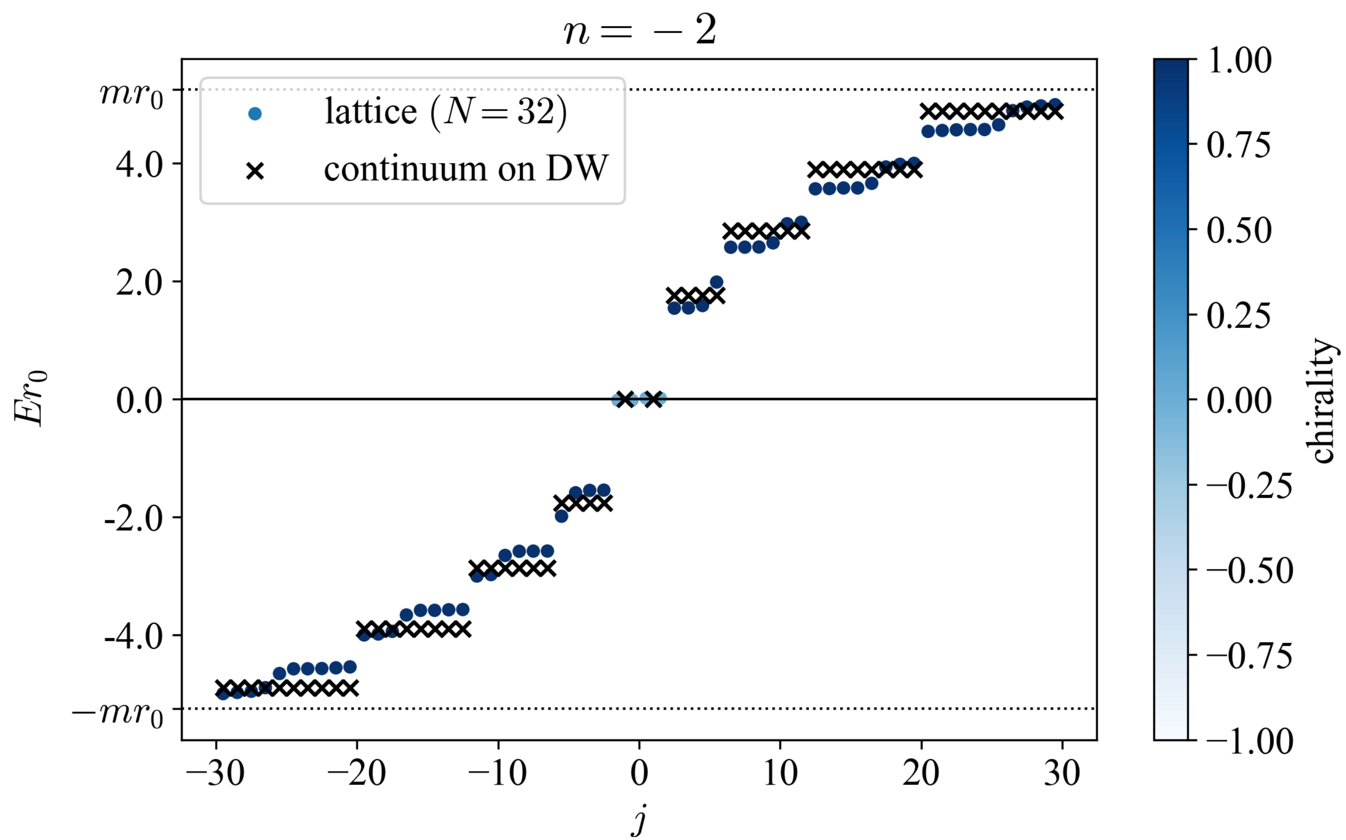
Near-zero eigenvalues/functions



Near-zero eigenvalues/functions

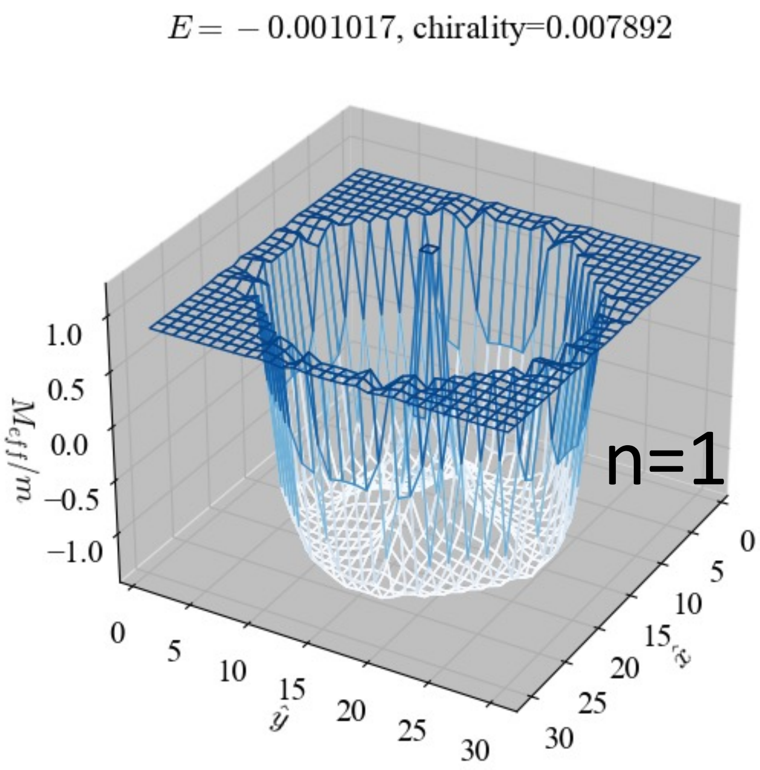
n=-2

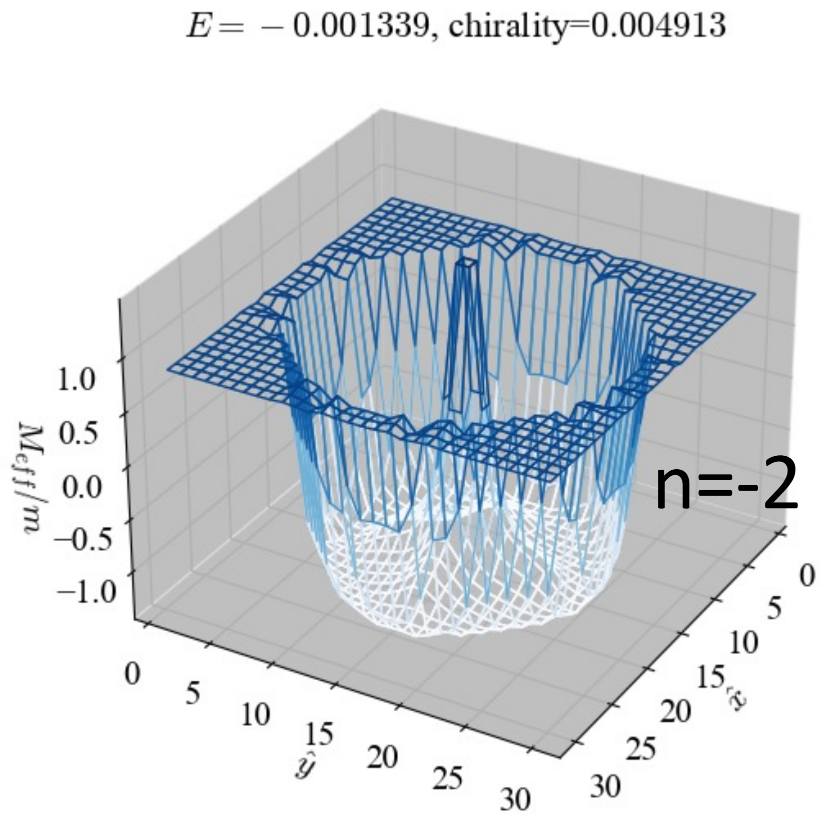
We have 4 near-zero modes.



Effective mass at z=(L+1)/2 slice

$$\frac{\phi_0(\boldsymbol{x})^\dagger \left[-\sum_{i=1,2,3} \frac{1}{2}\nabla_i^f \nabla_i^b + m\right] \phi_0(\boldsymbol{x})}{\phi_0(\boldsymbol{x})^\dagger \phi_0(\boldsymbol{x})}$$



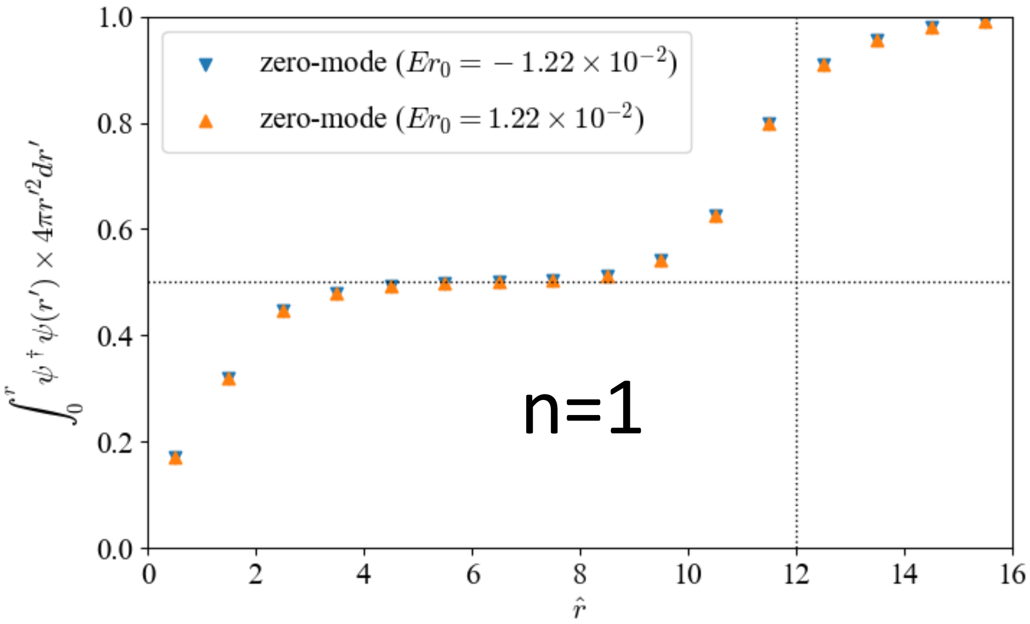


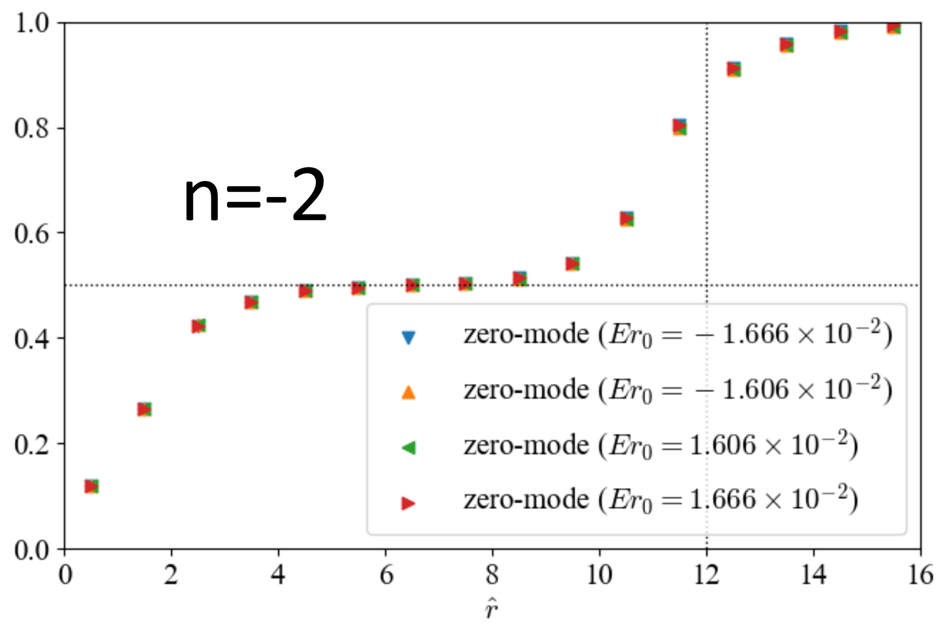
Electric charge around the monopole

Cumlative distribution

$$C_k(r) = \int_{|\boldsymbol{x}|<r} d^3x \phi_k(\boldsymbol{x})^\dagger \phi_k(\boldsymbol{x})$$

Saturates to 1/2
until r ～ 0.8 r0





Contents

Re-interpretation of the Maxwell theory

Original description by Witten

$$\partial_\mu F^{\mu\nu} = -\frac{\theta}{8\pi^2}\partial_\mu \tilde{F}^{\mu\nu} \qquad q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2}\int d^3x \nabla \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}$$

Our reinterpretation = θ term has a defect.

$$\partial_\mu F^{\mu\nu} = -\frac{1}{8\pi^2}\partial_\mu \left[\theta(r)\tilde{F}^{\mu\nu}\right] \qquad q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{1}{4\pi^2}\int d^3x \nabla\theta(r) \cdot \boldsymbol{B}$$

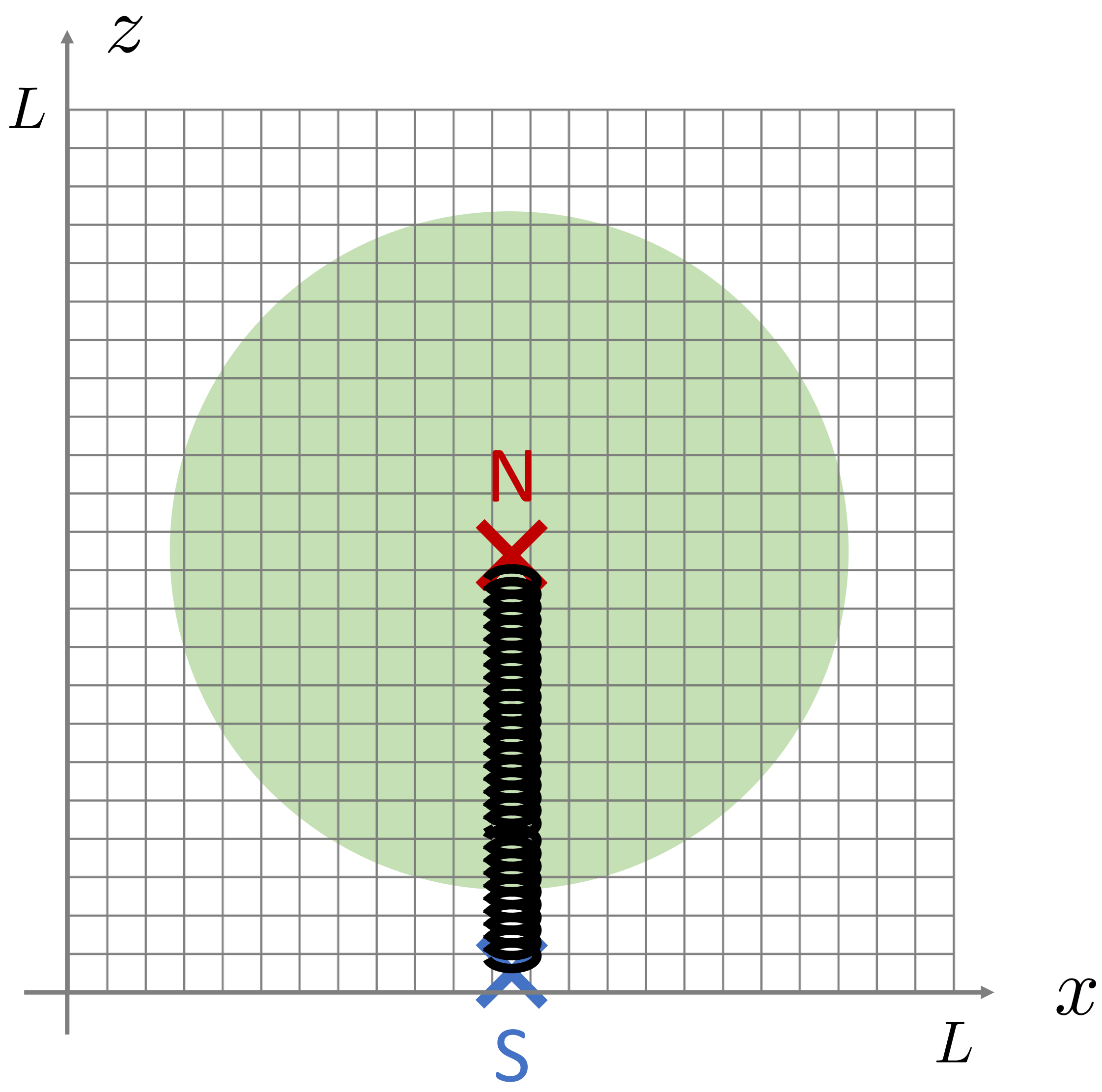
$$= -\frac{1}{4\pi^2}\int d^3x \pi\delta(r - r_1)\boldsymbol{e}_r \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}$$

The result is the same!

But our case does not require a true monopole with $\nabla \cdot \boldsymbol{B} \neq 0$

Witten effect without true monopoles?

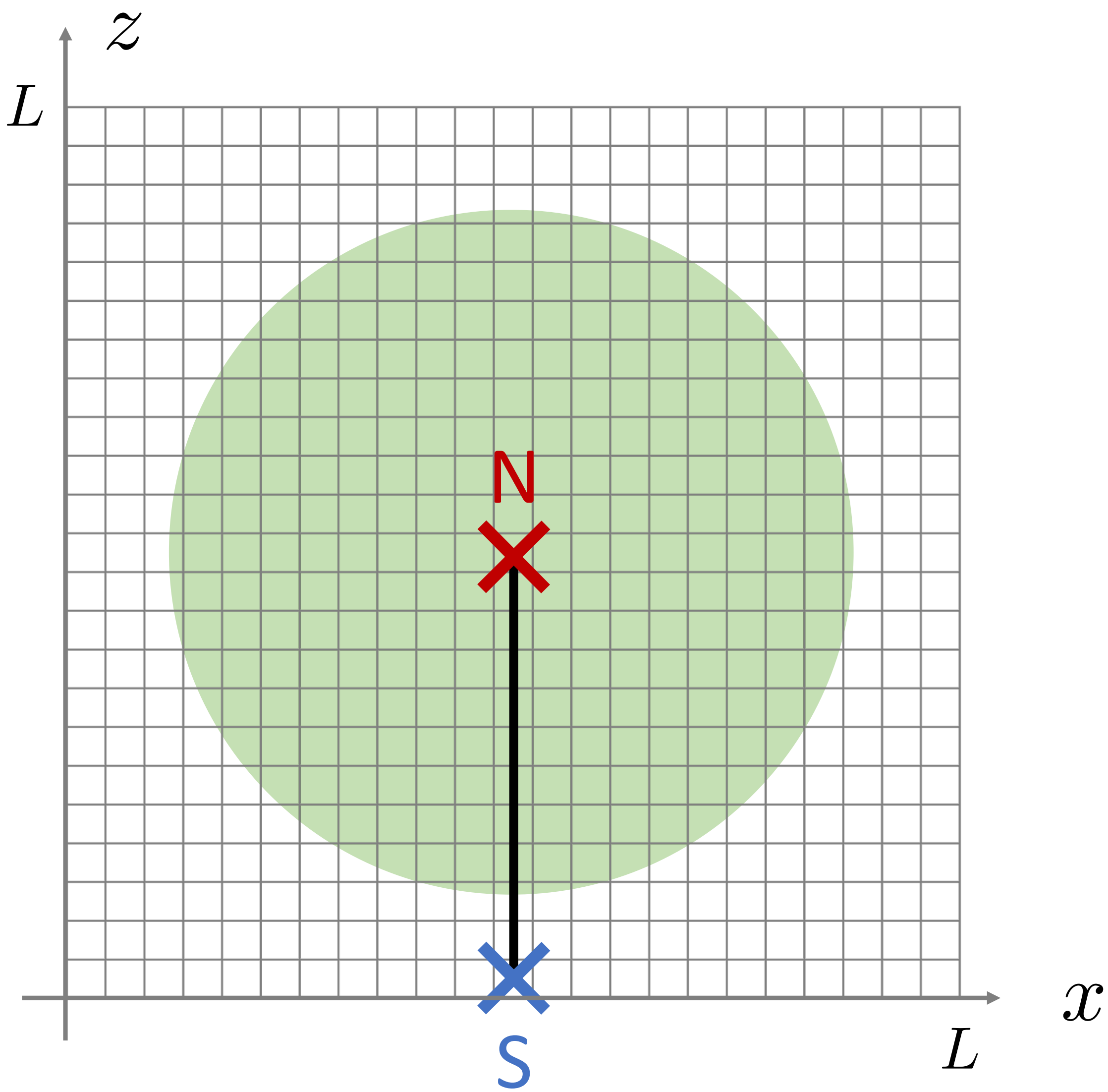
Can a thin magnetic flux generated by a solenoid capture electric charge?



It is difficult to make a solenoid of atomic scale

But we can simulate it on a lattice!

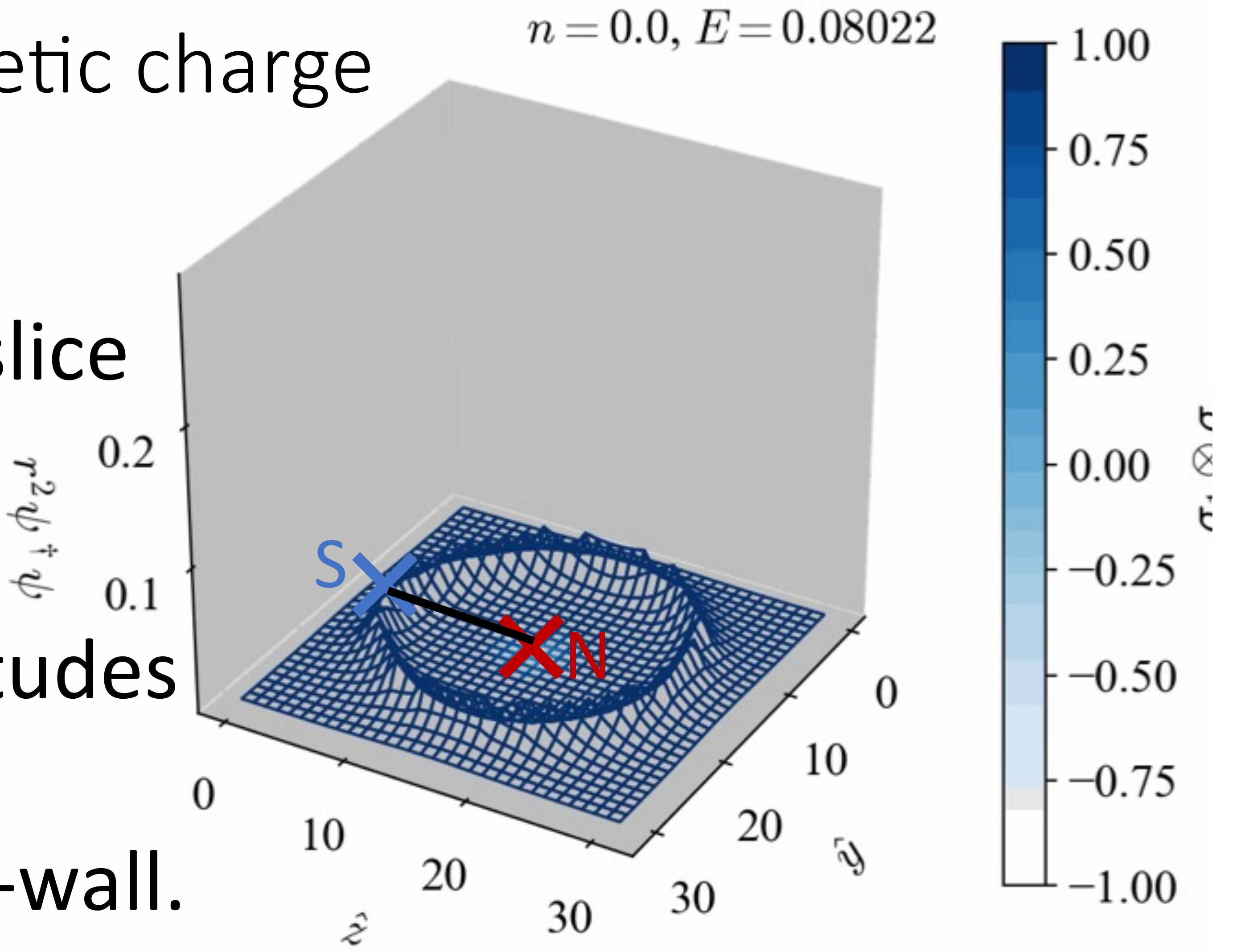
Let us see what happens if we change n from 0 to 1 continuously.



Continuous magnetic charge

Wave function
at the x =(L+1)/2 slice
(weighted by r^2)

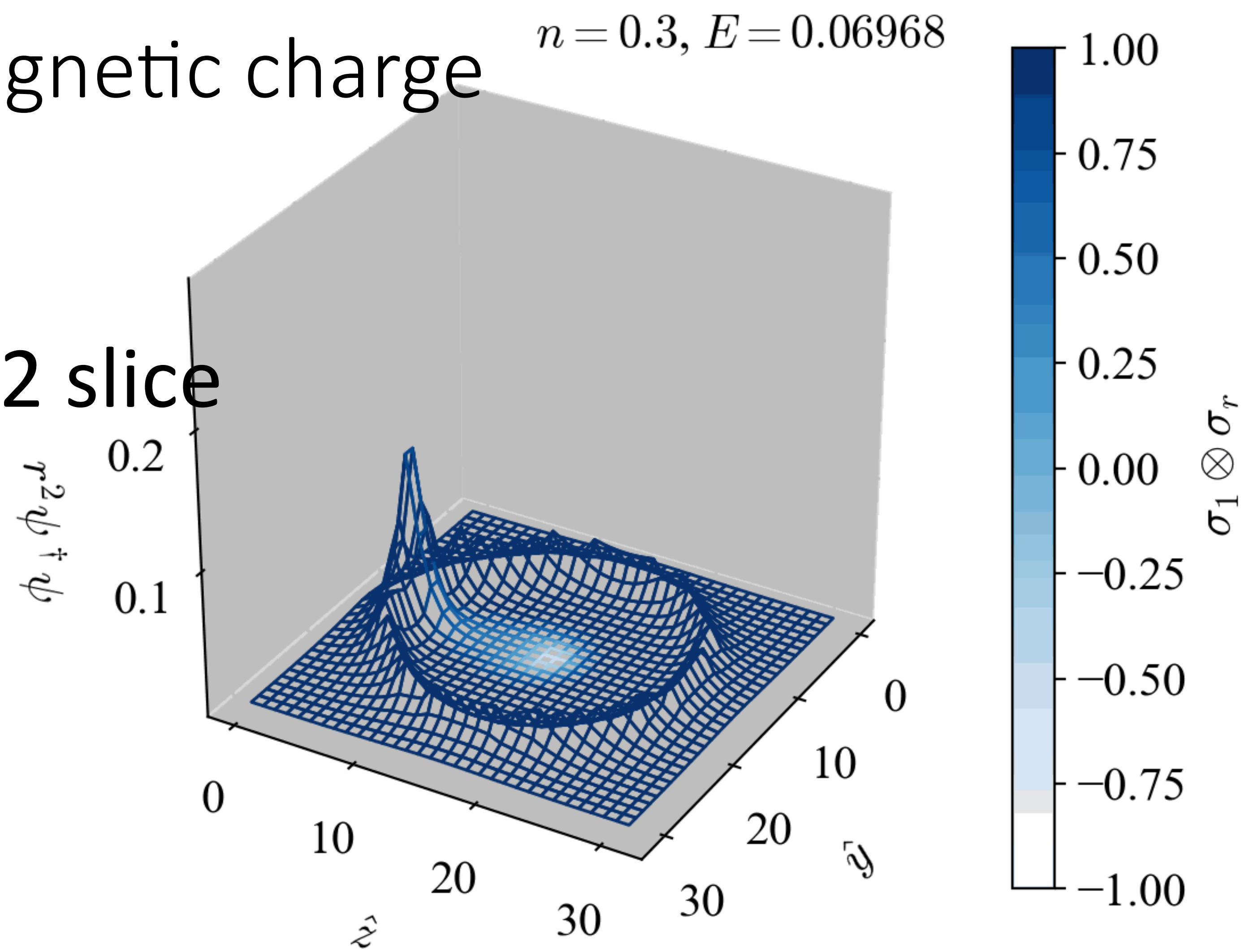
At n=0, the amplitudes are all located at the outer domain-wall.



Continuous magnetic charge

Wave function
at the x =(L+1)/2 slice

n=0.3

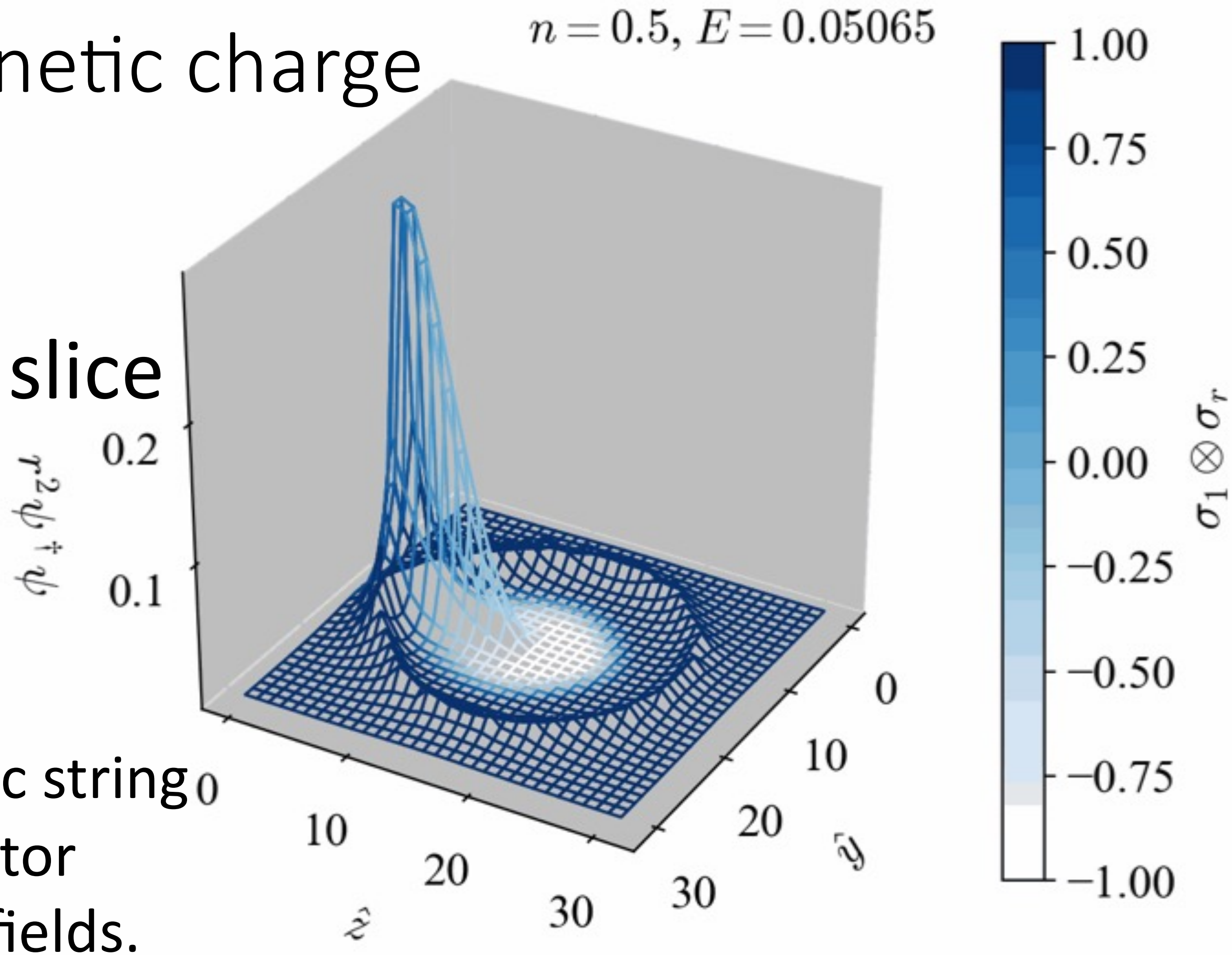


Continuous magnetic charge

Wave function
at the x =(L+1)/2 slice

n=0.5

Neiborhood of the Dirac string
becomes normal insulator
to attract the electron fields.

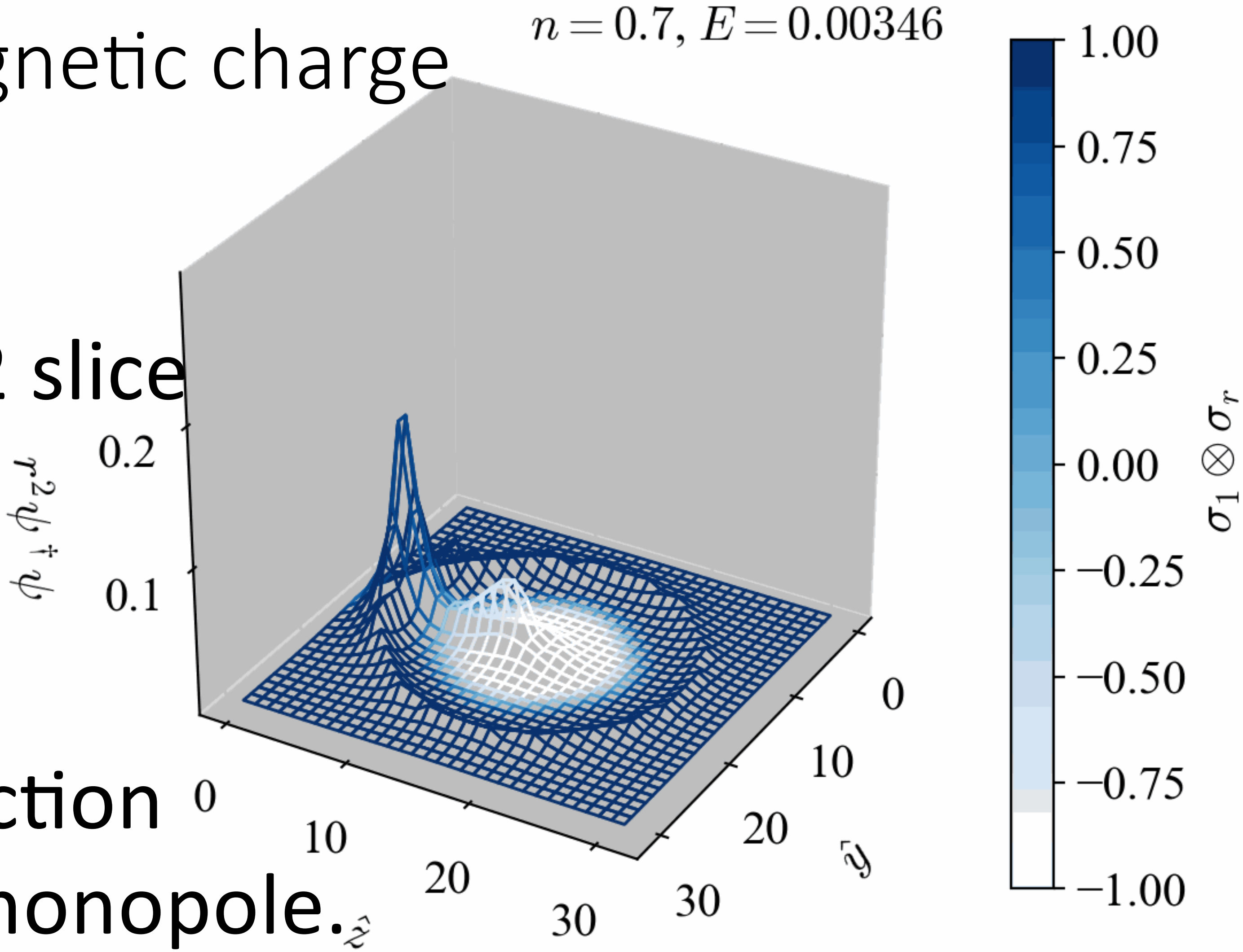


Continuous magnetic charge

Wave function
at the x =(L+1)/2 slice

n=0.7

Half of wavefunction
reaches to the monopole.

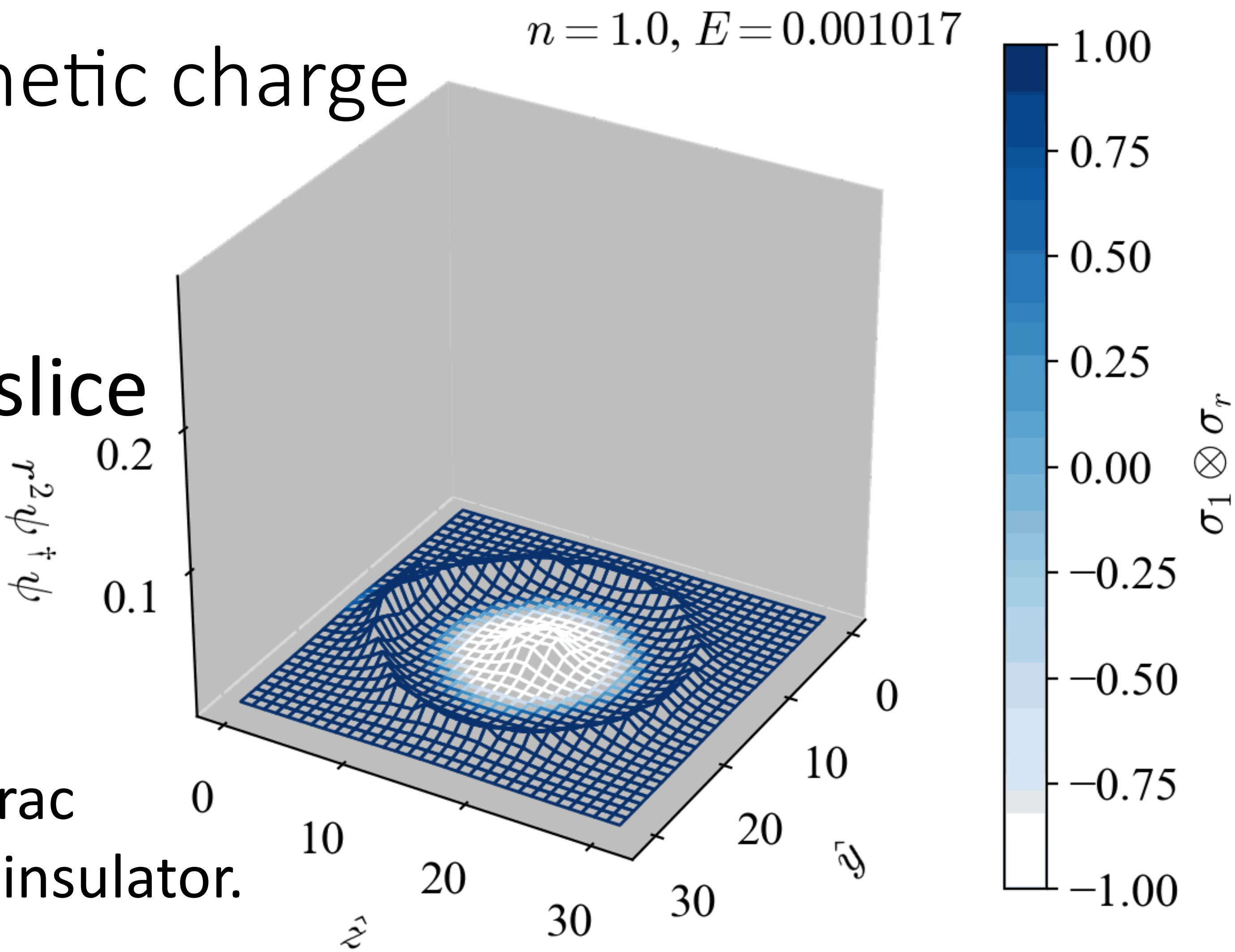


Continuous magnetic charge

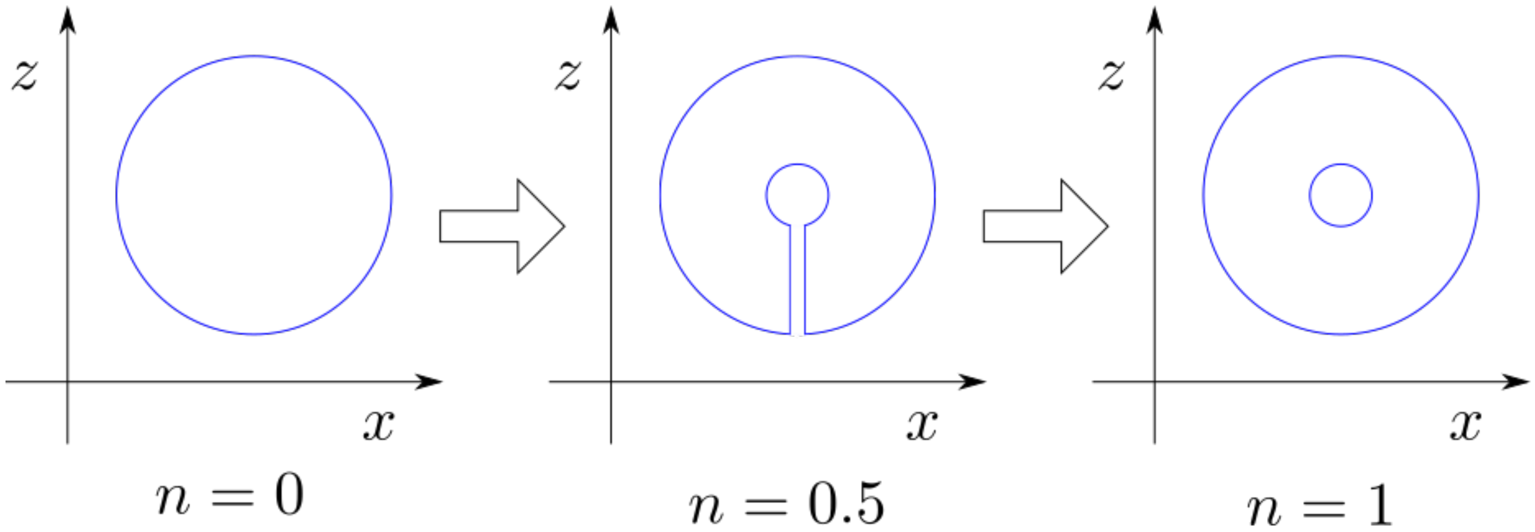
Wave function at the x =(L+1)/2 slice

n=1.0

Neighborhood of the Dirac string is back to normal insulator.



Domain-wall topology change



Inside domain-wall = topological insulator,
Outside = normal insulator.
Dirac sting becomes normal insulator > separated into two spheres
This topology change “pumps” the electric charge to monopole.

Contents

✓ 1. Introduction
   We try to microscopically understand the Witten effect (with domain-wall fermions).

✓ 2. Bound state solution of naïve Dirac eq. (review of Yamagishi 1983)
   A zero mode localized at the monopole was found but the relation to the Witten effect is not obvious.

✓ 3. Regularized Dirac eq. and fundamental reason for electron bound state
   Dense magnetic field forms a finite-sized domain-wall whose chiral edge modes are identified as the origin of the electric charge.

✓ 4. Atiyah-Singer index theorem responsible for 1/ 2 electric charge
   Zero mode must be paired with another at the surface, and the 50% vs. 50% tunneling mixing at half-filling is responsible for the half integral charge. 。

✓ 5. Reinterpretation of Effective Maxwell theory and continuous magnetic charge
   Witten effect may be observed without true monopoles. A thin solenoid pumps 1/2 electic charge from outer domain-wall to inner domain-wall.

6. Summary and discussion

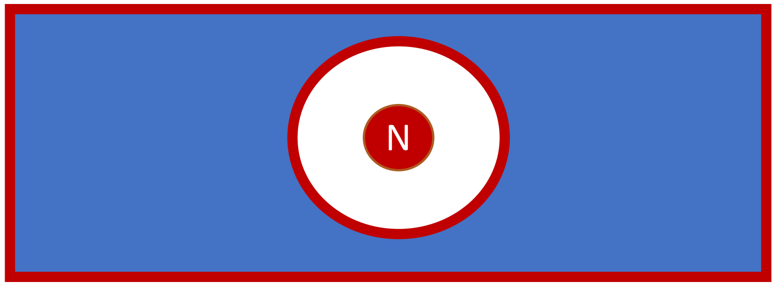
Summary

We have revealed the microscopic mechanism of the Witten effect: how a magnetic monopole acquires a half-integral electric charge.

In our analysis, the (second derivative) Wilson term is added to distinguish normal and topological insulators and to eliminate short-distance singularity.

Summary

1. Magnetic field (via the Wilson term) gives a positive shift to form a domain-wall around the monopole (only when m<0).
2. Origin of electric charge can be identified as the edge localized modes on the S^2 domain-wall and naturally explains the chirality.
3. The edge-localized zero modes are protected by the Atiyah-Singer index theorem on the small but finite-sized S^2 and its cobordism nature explains why the charge is half-integer(while the total charge is still integer, having the other half at the surface.)

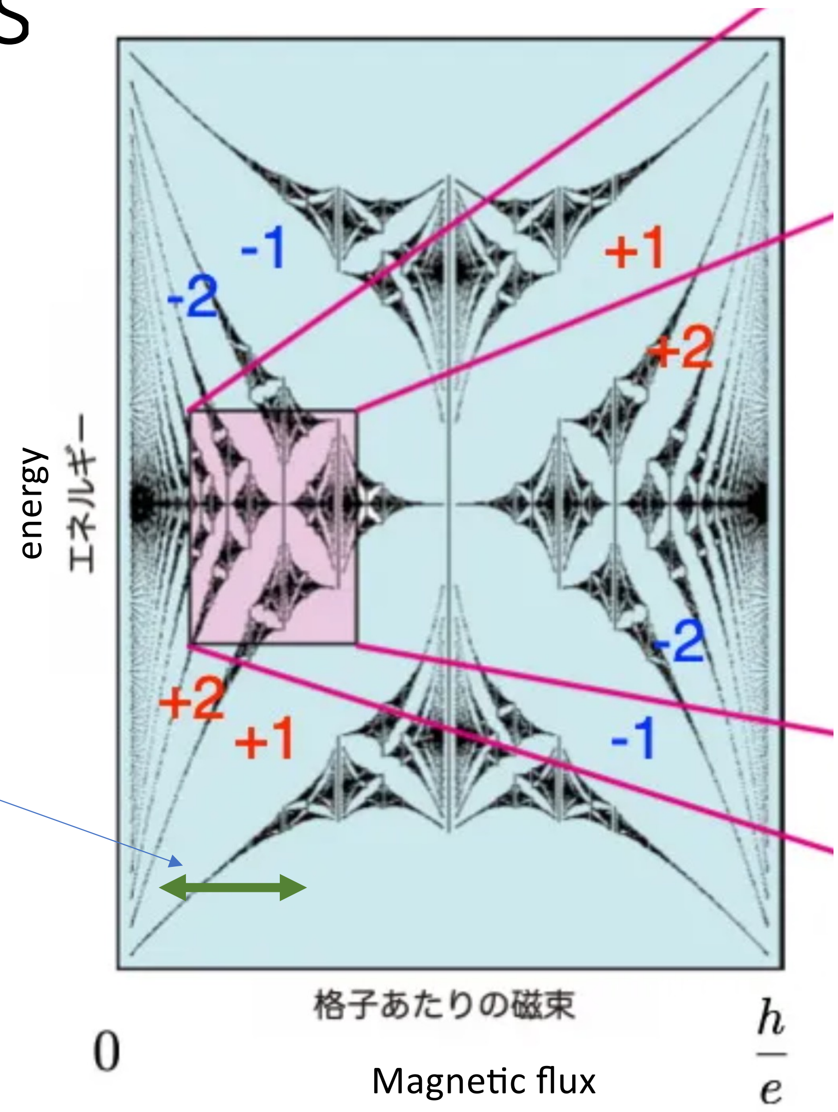


Comparison with previous works 1

Changing topological phase by
Magnetic field is observed
in quantum Hall systems.

Our analysis corresponds to here.

From webpage of Hatugai lab, U. Tsukuba



Comparison with previous works 2

With the Wilson term,

Zhao and Shen 2012, Tyner & Goswami 2022

obtained essentially the same solution with ours,

$$\psi^{\mathrm{mono}}_{j,j_3} = \frac{Be^{-\frac{M_{\mathrm{PV}}r}{2}}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta,\phi)$$

However, they did not notice

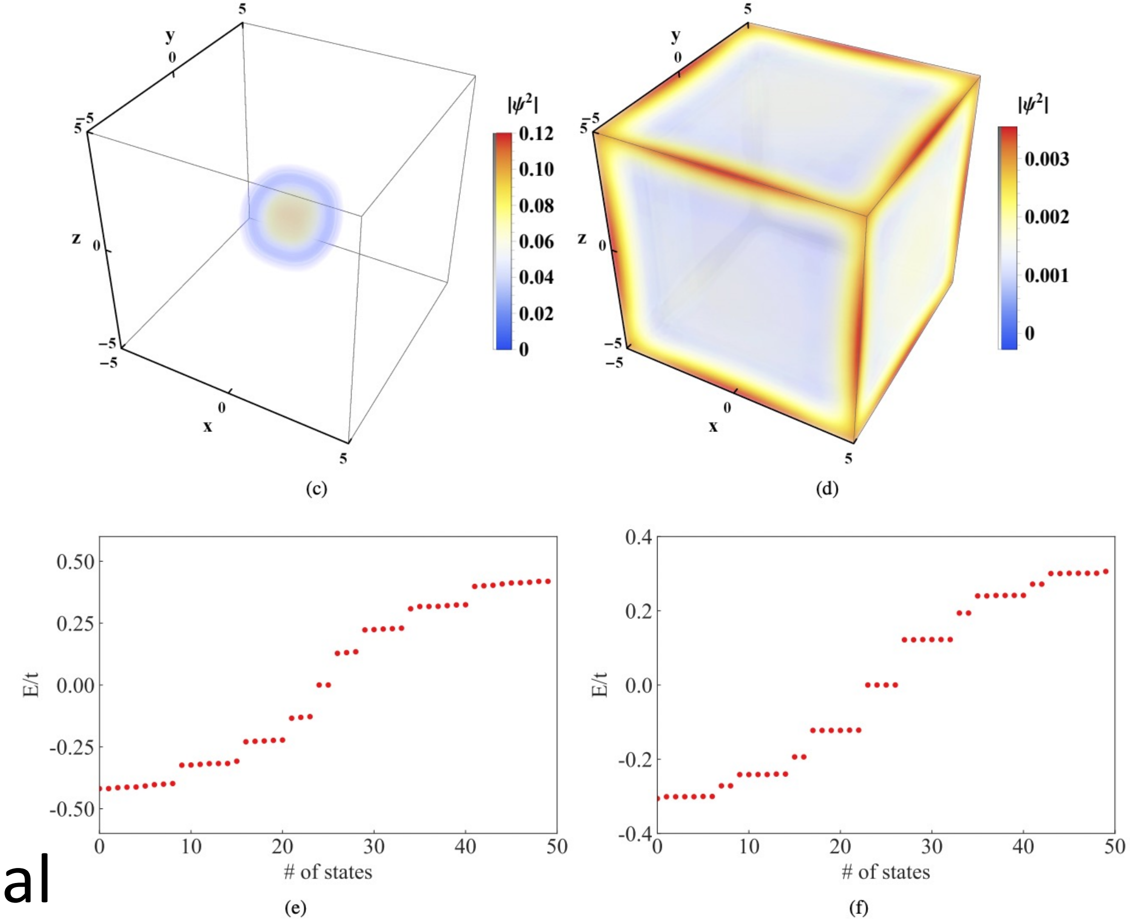
1) The peak is located at finite radius

2) The solution satisfies the Dirac equation on S^2.

Comparison with previous works 3

Numerical analysis by
Rosenberg and Franz 2010,
Zhao and Shen 2012,
Tyner & Goswami 2022
found zero-mode pairings
with the surface states .

But did not notice the topological
origin (=Atiyah-Singer index on S2).



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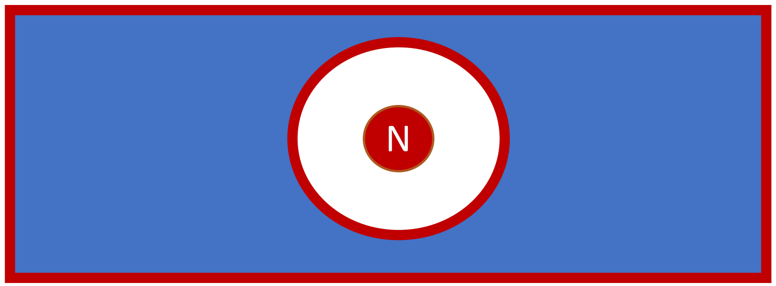
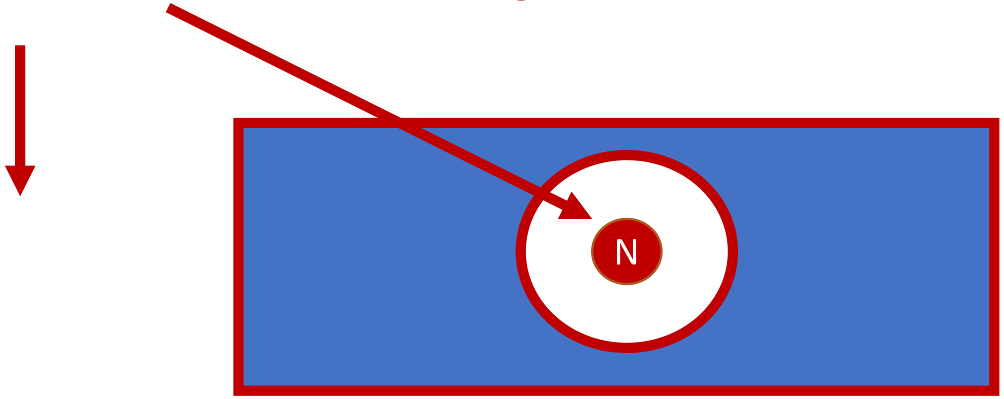


Take-home message :

Outside is important.



Topological insulator ~~= manifold with boundary~~.

= a domain of electron system.

Backup slides

Subtlety of the charge

The effective throy indicates $\partial_\mu F^{\mu\nu} = -\frac{\theta}{8\pi^2}\partial_\mu \tilde{F}^{\mu\nu}$

$$q_e = \int d^3x \nabla \cdot \boldsymbol{E} = -\frac{\theta}{4\pi^2}\int d^3x \nabla \cdot \boldsymbol{B} = -\frac{\theta q_m}{2\pi}$$

At θ=π mod 2πZ

q_m=1 monopole has -1/2 mod Z electric charge.

Breaking of particle-hole symmetry would prefer -1/2 (half of one electron).