Extra Dimension Approach to CMB Power-Spectrum Physics *

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0. Introduction

Cosmic Microwave Background Radiation の観測データ

暗黒物質、暗黒エネルギー（〜宇宙項）

重力理論、場の理論の欠陥：　発散の問題（紫外、赤外）

dS4　上　での場の量子論場の理論的定式化がまだまだ不完全
'03 J. Maldacena, Non-Gaussian ...
'06 S. Weinberg , in-in formalism
Schwinger-Keldysh formalism in '07 A.M. Polyakov

A.M. Polyakov, '09
Dark energy, like the black body radiation 150 years ago, hides secrets of fundamental physics

$$
\begin{aligned}
\mathsf{dS}_4 \ &: \ ds^2 = -dt^2 + \mathrm{e}^{2H_0 t}(dx^2 + dy^2 + dz^2) \\
\mathsf{AdS}_4 \ &: \ ds^2 = dz^2 + \mathrm{e}^{-2H_0|z|}(-dt^2 + dx^2 + dy^2) \\
\mathsf{AdS}_5 \ &: \ ds^2 = dw^2 + \mathrm{e}^{-2\omega|w|}(-dt^2 + dx^2 + dy^2 + dz^2) \\
\omega \ &: \ \text{warp factor} \quad w \ : \ \text{extra coordinate}
\end{aligned}
\tag{1}
$$

1. Background Field Formalism

$$S[\Phi; g_{\mu\nu}] = \int d^4x\sqrt{g}\left(\frac{-1}{16\pi G_N}(R-2\lambda) - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{m^2}{2}\Phi^2 - V(\Phi)\right),$$

$$V(\Phi) = \frac{\sigma}{4!}\Phi^4 \quad , \quad \sigma > 0 \quad . \tag{2}$$

Background Expansion: $\Phi = \Phi_{cl} + \varphi \quad , \quad$ NOT expand $g_{\mu\nu}$ (3)

$$\mathrm{e}^{i\Gamma[\Phi_{cl};g_{\mu\nu}]} = \int \mathcal{D}\varphi \exp i\left\{S[\Phi_{cl}+\varphi; g_{\mu\nu}] - \frac{\delta S[\Phi_{cl}; g_{\mu\nu}]}{\delta\Phi_{cl}}\varphi\right\} \quad . \tag{4}$$

Φ_{cl} is perturbatively solved, at the tree level, as

$$\Phi_{cl}(x) = \Phi_0(x) + \int D(x-x') \ \sqrt{g}\frac{\delta V(\Phi_{cl})}{\delta \Phi_{cl}}\bigg|_{x'} d^4x' \quad ,$$

$$\sqrt{g}(\nabla^2 - m^2)\Phi_0 = 0 \quad , \quad \sqrt{g}(\nabla^2 - m^2)D(x-x') = \delta^4(x-x') \quad . \tag{5}$$

2. dS$_4$ Geometry, Conformal Time and Z$_2$ Symmetry

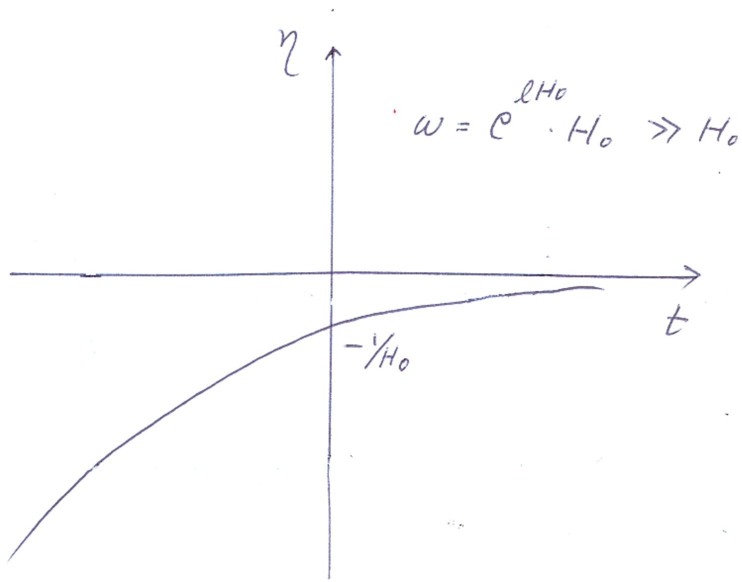
background field $g_{\mu\nu}$: dS$_4$

time variable: $t \quad \rightarrow \quad \eta \; (conformal\ time)$

$$d\eta = \mathrm{e}^{-H_0 t} dt \quad , \quad \eta = -\frac{1}{H_0}\mathrm{e}^{-H_0 t} \quad , \quad -\infty < t < \infty \quad , \quad -\infty < \eta < 0 \quad , \tag{6}$$

See Fig.1

Figure 1: Conformal time (η) versus ordinary time (t). Eq.(6): $\eta = -\mathrm{e}^{-H_0 t}/H_0$.



The metric transforms to the conformally-flat type.

$$ds^2 = -dt^2 + \mathrm{e}^{2H_0 t}(dx^2 + dy^2 + dz^2)$$

$$= \frac{1}{(H_0\eta)^2}(-d\eta^2 + dx^2 + dy^2 + dz^2) = \tilde{g}_{\mu\nu}(\chi) d\chi^\mu d\chi^\nu \quad ,$$

$$(\chi^\mu) = (\chi^0, \chi^1, \chi^2, \chi^3) = (\eta, x, y, z) \quad , \tag{7}$$

The perturbative solution Φ_{cl}, (5), is given by

$$\Phi_{cl}(\chi) = \Phi_0(\chi) + \int \tilde{D}(\chi, \chi') \; \frac{1}{(H_0\eta')^4} \frac{\delta V(\Phi_{cl})}{\delta \Phi_{cl}}\bigg|_{\chi'} d^4\chi' \; ,$$

$$\sqrt{-\tilde{g}}(\tilde{\nabla}^2 - m^2)\Phi_0 = -\left\{ \partial_\eta \frac{1}{(H_0\eta)^2} \partial_\eta + \frac{m^2}{(H_0\eta)^4} - \frac{1}{(H_0\eta)^2} \vec{\nabla}^2 \right\} \Phi_0 = 0. \tag{8}$$

$\tilde{D}(\chi, \chi')$ is the *propagator* on the dS$_4$ geometry $\tilde{g}_{\mu\nu}(\chi)$.

$$\sqrt{-\tilde{g}}(\tilde{\nabla}^2 - m^2)\tilde{D}(\chi, \chi') =$$

$$-\left\{\partial_\eta \frac{1}{(H_0\eta)^2}\partial_\eta + \frac{m^2}{(H_0\eta)^4} - \frac{1}{(H_0\eta)^2}\vec{\nabla}^2\right\}\tilde{D}(\chi,\chi') = \delta^4(\chi-\chi') \quad , \tag{9}$$

To regularize IR behavior, we introduce

$$\mathrm{Z}_2 \text{ Symmetry} \ : \quad t \leftrightarrow -t \quad , \quad \text{Periodicity} \ : \quad t \to t+2l \quad , \tag{10}$$

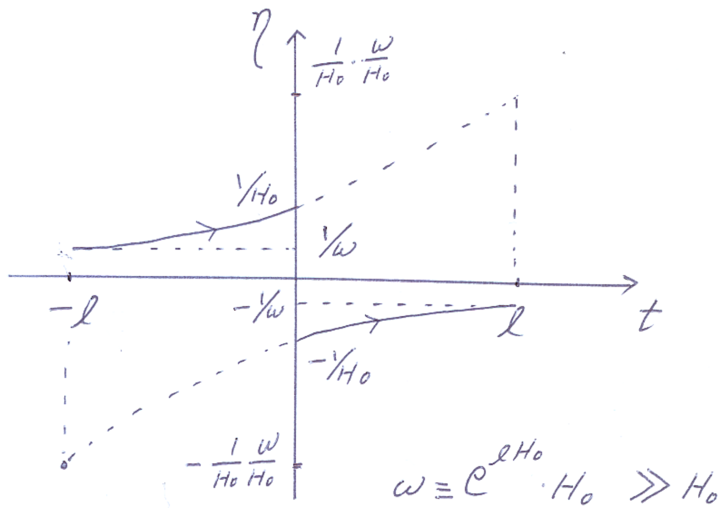
l : the period parameter (IR parameter).

$$\eta = \left\{ \begin{array}{llll} -\frac{1}{H_0}\mathrm{e}^{-H_0 t} \ , & d\eta = -H_0\eta dt \ , & 0<t<l \ , & \frac{-1}{H_0}<\eta<\frac{-1}{\omega} \\ +\frac{1}{H_0}\mathrm{e}^{H_0 t} \ , & d\eta = H_0\eta dt \ , & -l<t<0 \ , & \frac{1}{\omega}<\eta<\frac{1}{H_0} \end{array} \right.$$

$$\omega \equiv \mathrm{e}^{lH_0}H_0 \ \gg \ H_0 \quad , \tag{11}$$

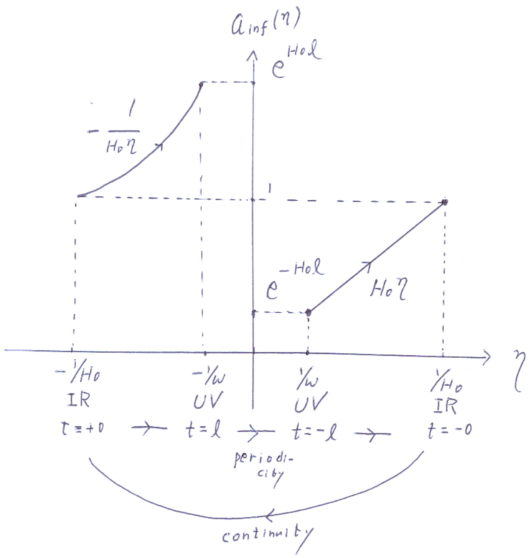
$\eta \leftrightarrow -\eta \iff$ the Z_2 symmetry in (10). See Fig.2.

Figure 2: Conformal time (η) versus ordinary time (t). Time-reversal correspondence is valid. Eq.(11)



See Fig.3

Figure 3: Scale factor in dS_4 geometry.



Let us switch to the spacially-Fourier-transformed expression:

$$\Phi_0(\eta, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \mathrm{e}^{i\vec{p}\cdot\vec{x}} \phi_{\vec{p}}(\eta) \quad , \quad \tilde{D}(\chi, \chi') = \int \frac{d^3\vec{p}}{(2\pi)^3} \mathrm{e}^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \tilde{D}_{\vec{p}}(\eta, \eta') \quad , (12)$$

$\tilde{D}_{\vec{p}}(\eta, \eta')$: 'Momentum/Position propagator'
$\phi_{\vec{p}}(\eta)$ satisfies the following Bessel eigenvalue equation.

$$\left\{ {\partial_\eta}^2 - \frac{2}{\eta}\partial_\eta + \frac{m^2}{(H_0\eta)^2} + M^2 \right\} \phi_M(\eta) = \{ s(\eta)^{-1}\hat{L}_\eta + M^2 \} \phi_M(\eta) = 0 \quad ,$$
$$M^2 \equiv \vec{p}^2 \quad , \quad s(\eta) \equiv \frac{1}{(H_0\eta)^2} \quad , \quad \hat{L}_\eta \equiv \partial_\eta s(\eta) \partial_\eta + \frac{m^2}{(H_0\eta)^4} \quad . \quad (13)$$

$\tilde{D}_{\vec{p}}(\eta, \eta')$ satisfies

$$\{ \hat{L}_\eta + \vec{p}^2 s(\eta) \} \tilde{D}^{\mp}_{\vec{p}}(\eta, \eta') = \begin{cases} \epsilon(\eta)\epsilon(\eta')\hat{\delta}(|\eta| - |\eta'|) & \text{for P}=- \\ \hat{\delta}(|\eta| - |\eta'|) & \text{for P}=+ \end{cases} \quad (14)$$

The Bessel equation (13) gives us the free field wave function as

$$\phi_M(\eta) = \eta^{3/2} Z_\nu(M\eta) \quad , \quad \nu = \sqrt{\left(\frac{3}{2}\right)^2 - \frac{m^2}{H_0^2}} \quad , \tag{15}$$

$$\nu = 0, 1/2, \sqrt{5}/2, 3/2 \iff m = (3/2)H_0, \sqrt{2}H_0, H_0, 0$$

3. Boundary Condition, Bunch-Davies Vacuum, Casimir Energy

Boundary Condition for $P = -$ Free Wave Function

$$\begin{aligned} \Phi_0(\eta \to \pm 1/\omega \ , \vec{x}) = 0 \quad & \text{Dirichlet} \\ \Phi_0(\eta \to \pm 1/H_0 \ , \vec{x}) = 0 \quad & \text{Dirichlet} \end{aligned} \tag{16}$$

Boundary Condition for $P = +$ Free Wave Function

$$\begin{aligned} \partial_\eta \Phi_0|_{\eta \to \pm 1/\omega} = 0 \quad & \text{Neumann} \\ \partial_\eta \Phi_0|_{\eta \to \pm 1/H_0} = 0 \quad & \text{Neumann} \end{aligned} \tag{17}$$

Casimir energy: σ-independent part of the 1-loop effective action in (4)

$$\exp\{-H_0^{-3}E^{dS4}_{Cas}\} = \int \mathcal{D}\varphi \exp i \int d^4x\sqrt{g}\left(-\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - \frac{m^2}{2}\varphi^2\right)$$

$$= \det{}^{-1/2}\{\sqrt{-g}(\nabla^2 - m^2)\} = \exp\left[\int \frac{d^3\vec{p}}{(2\pi)^3}2\int_{-1/H_0}^{-1/\omega} d\eta\{-\frac{1}{2}\ln(-s(\eta)^{-1}\hat{L}_\eta - \vec{p}^2)\}\right] \quad (18$$

From the formula: $\int_0^\infty(\mathrm{e}^{-t} - \mathrm{e}^{-tM})/t\ dt = \ln M,\ \det\ M > 0,$

$$-H_0^{-3}E^{dS4}_{Cas} = \int \frac{d^3\vec{p}}{(2\pi)^3}2\int_{-1/H_0}^{-1/\omega} d\eta\{\frac{1}{2}\int_0^\infty \frac{1}{\tau}\mathrm{e}^{\tau(s(\eta)^{-1}\hat{L}_\eta+\vec{p}^2)}\}$$
$$= \int_0^\infty \frac{d\tau}{\tau}\frac{1}{2}\mathrm{Tr}H_{\vec{p}}(\eta,\eta';\tau) \quad , \qquad (19)$$

where $H_{\vec{p}}(\eta,\eta';\tau)$ is the Heat-Kernel:

$$\left\{\frac{\partial}{\partial\tau} - (s^{-1}\hat{L}_\eta + \vec{p}^2)\right\} H_{\vec{p}}(\eta,\eta';\tau) = 0 \quad ,$$
$$H_{\vec{p}}(\eta,\eta';\tau) = (\eta|\mathrm{e}^{(s^{-1}\hat{L}_\eta+\vec{p}^2)\tau}|\eta') \quad . \tag{20}$$

Bunch-Davies Vacuum: the complete and orthonormal eigen functions $\phi_n(\eta)$ of the operator $s^{-1}\hat{L}_\eta$.

$$\phi_n(\eta) \equiv (n|\eta) = (\eta|n) \quad , \quad \{s(\eta)^{-1}\hat{L}_\eta + {M_n}^2\}\phi_n(\eta) = 0 \quad ,$$
$$\left(\int_{-1/H_0}^{-1/\omega} + \int_{1/\omega}^{1/H_0}\right)\frac{d\eta}{(H_0\eta)^2}(n|\eta)(\eta|k) = 2\int_{-1/H_0}^{-1/\omega}\frac{d\eta}{(H_0\eta)^2}(n|\eta)(\eta|k) = (n|k) = \delta_{n,k} \quad ,$$

$$(\eta|\eta') = \begin{cases} (H_0\eta)^2\epsilon(\eta)\epsilon(\eta')\hat{\delta}(|\eta|-|\eta'|) & \text{for } P=- \\ (H_0\eta)^2\delta(|\eta|-|\eta'|) & \text{for } P=+ \end{cases}$$

$$\left(\int_{-1/H_0}^{-1/\omega}+\int_{1/\omega}^{1/H_0}\right)\frac{d\eta}{(H_0\eta)^2}|\eta)(\eta| = 2\int_{-1/H_0}^{-1/\omega}\frac{d\eta}{(H_0\eta)^2}|\eta)(\eta| = 1 \quad ,$$

$$\sum |n)(n| = 1 \quad ,(2$$

4. Wick Rotation

$$-H_0^{-3}E_{Cas}^{dS4} = \int \frac{d^3\vec{p}}{(2\pi)^3} 2\int_{-1/H_0}^{-1/\omega} d\eta \{\frac{1}{2}\int_0^\infty \frac{1}{\tau}\mathrm{e}^{\tau(s(\eta)^{-1}\hat{L}_\eta + \vec{p}^2)}\}$$

$$= \int_0^\infty \frac{d\tau}{\tau}\frac{1}{2}\mathrm{Tr}H_{\vec{p}}(\eta, \eta'; \tau) \quad , \tag{22}$$

Diverges very badly ! To regularize it, we do

Wick rotation for space-components of momentum

$$p_x \ , \ p_y \ , \ p_z \ \longrightarrow ip_x \ , \ ip_y \ , \ ip_z \tag{23}$$

The regularized expression is Casimir energy for AdS_4.
For example, $ds^2 = dz^2 + \mathrm{e}^{-2H_0|w|}(-dt^2 + dx^2 + dy^2)$
From the isotropy requirement for the space world, we do further regularization by replacing the 1+3 dim dS model by the 1+4 AdS model:

$$ds^2 = dw^2 + \mathrm{e}^{-2H_0|w|}(-dt^2 + dx^2 + dy^2 + dz^2) \quad , \quad w \ : \quad \text{Extra Coordinate} \tag{24}$$

AdS_5 Casimir energy is already obtained (S.I.2008). Continue to Conclusion Section.

5. Metric Fluctuation and Averaging Over the 4D space-time using the Generalized Path-Integral

Metric field $g_{\mu\nu}(x)$: the background one (dS$_4$). It is defined by the variational equation of the effective action $\Gamma[\Phi_{cl}; g_{\mu\nu}]$ (4).

$$S = \Gamma[\Phi_{cl}(x); g_{\mu\nu}(x)] \equiv \int d^4x \mathcal{L}[x^\mu] \quad . \tag{25}$$

The action for a quantum mechanical system: dynamical variables (x^i: $i = 1, 2, 3$) and time ($x^0 = t$). the small fluctuation of x^i, keeping $x^0 = t$ fixed, in the dS$_4$ geometry $g^{inf}_{\mu\nu}(x)$.

$$x^i \to x^i + \sqrt{\epsilon} f^i(\vec{x}, t) = x^{i'} \quad , \quad t = t' \ (x^0 = x^{0'}) \quad , \tag{26}$$

where $\vec{x} = (x^i)$. ϵ: a small parameter. This fluctuation can be absorbed into the metric fluctuation as the requirement of the invariance of the line element (general coordinate invariance).

$$g^{inf}_{\mu\nu}(x) dx^{\mu\prime} dx^{\nu\prime} = g_{\mu\nu}{}'(x) dx^\mu dx^\nu \quad , \quad g_{\mu\nu}{}'(x) = g^{inf}_{\mu\nu}(x) + \epsilon h_{\mu\nu}(x) \quad , \tag{27}$$

$$\begin{aligned} h_{00} = \mathrm{e}^{2H_0 t}\partial_0 f^i \cdot \partial_0 f^i \quad , \quad h_{0i} = h_{i0} = \mathrm{e}^{2H_0 t}\partial_i f^j \cdot \partial_0 f^j \quad , \\ h_{ij} = \mathrm{e}^{2H_0 t}\partial_i f^k \cdot \partial_j f^k \quad , \\ \text{constraint :} \quad \{\frac{1}{2}(\partial_i f^j + \partial_j f^i) dx^j + 2\partial_0 f^i dt\} dx^i = 0 \quad , \end{aligned} \tag{28}$$

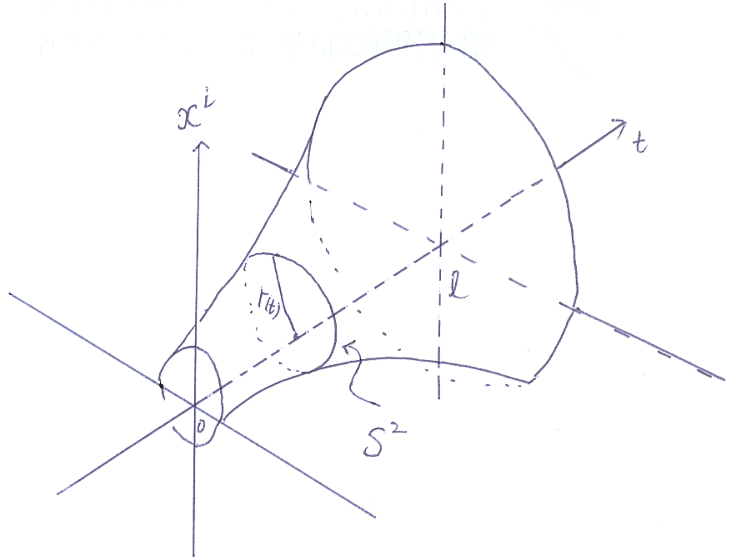
We see the coordinates-fluctuation produces the metric- fluctuation (around the homogeneous and isotropic (dS$_4$) metric) , as far as the above constraint is preserved. The constraint comes from the difference in the perturbation order between the metric fluctuation and the coordinate fluctuation.

Cause of the fluctuation: the underlying unknown micro dynamics (just like Brownian motion of nano-particles in liquid and solid). We treat it as the statistical phenomena. The coordinates are fluctuating in a statistical ensemble. By specifying the statistical distribution, we compute the statistical average. (NOTE: not the quantum effect but the statistical one.) In order to specify the statistical ensemble in the geometrically-meaningful way, we introduce 3 dimensional hypersurface in dS_4 space-time based on the isotropy requirement.

$$x^2 + y^2 + z^2 = r(t)^2 \quad , \tag{29}$$

$r(t)$: the radius of S^2 in the 3D plane at t. See Fig.4.

Figure 4: Hyper-surface in dS_4 space-time. Eq.(29).



Take the hyper-surface $\{r(t):\ 0 \le t \le l\}$ as a path. On the path (29), the induced metric g_{ij} is given by

$$ds^2 = g^{infl}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \mathrm{e}^{2H_0 t} dx^i dx^i$$
$$= (-\frac{1}{r^2 \dot{r}^2} x^i x^j + \delta^{ij} \mathrm{e}^{2H_0 t}) dx^i dx^j \equiv g_{ij} dx^i dx^j \quad , \tag{30}$$

The constraint in (28) reduces to

$$\{\frac{1}{2}(\partial_i f^j + \partial_j f^i) v^i + 2\partial_0 f^i\} v^i = 0 \quad , \quad v^i \equiv \frac{dx^i}{dt} \quad ,$$
$$\text{即ち} \quad \vec{v} \cdot D_t \vec{f} = 0 \quad , \quad D_t = \vec{v} \cdot \vec{\nabla} \; + \; \partial_0 \quad . \tag{31}$$

cf. fluid dynamics eq.

As the geometrical quantity, we can take the area A of the hypersurface.

$$A[x^i, \dot{x}^i] = \int \sqrt{\det g_{ij}}\; d^3\vec{x} = \frac{2\sqrt{2}}{3}\int_0^l \mathrm{e}^{-3H_0 t}\sqrt{\dot{r}^2 - \mathrm{e}^{-2H_0 t}}\; dt \quad . \tag{32}$$

More generally,

$$H[x^i, \dot{x}^i] = \int \sqrt{\det g_{ij}}(\lambda + \frac{1}{\kappa}R(g_{ij}) + O(\partial_i{}^4))d^3\vec{x} \quad . \tag{33}$$

For the general operator $\mathcal{O}[\Phi_{cl}(x); g_{\mu\nu}^{inf}(x)]$, the statistically averaged quantity is defined by the generalized path integral:

$$< \mathcal{O} >= \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(0)=\rho, r(l)=\rho} \mathcal{D}x^i(t) \times$$

$$\mathcal{O}[\Phi_{cl}(x); g_{\mu\nu}^{inf}(x)] \exp(-\frac{1}{2\alpha'} A[x^i, \dot{x}^i]) \quad . \tag{34}$$

μ, Λ : IR and UV cutoffs. α': the surface tension parameter.

'CMB spectrum':

$$< \Phi_{cl}(\vec{x}(t)) \Phi_{cl}(\vec{x}(t')) > \tag{35}$$

This unacceptable situation demands AdS$_5$ extra-dimension model again.

$$\begin{aligned} &< \Phi_{cl}(x^\mu(w)) \Phi_{cl}(x^\mu(w')) > = < \Phi_{cl}(t(w), \vec{x}(w)) \Phi_{cl}(t(w'), \vec{x}(w')) > \quad , \\ &\mu \;=\; 0 \;,\; 1 \;,\; 2 \;,\; 3 \qquad\qquad t(w) \;=\; t(w') \end{aligned} \tag{36}$$

6. Casimir Energy, RG-flow of the Cosmological Constant and Conclusion

$$E^W_{Cas}/\Lambda T^{-1} = -\alpha\omega^4 \left(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)\right) = -\alpha\omega'^4 \quad ,$$
$$\omega' = \omega\sqrt[4]{1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)} \quad . \qquad (37)$$

we find the renormalization group function for the warp factor ω as

$$|c| \ll 1 \quad , \quad |c'| \ll 1 \quad , \quad \omega' = \omega(1 - c\ln(\Lambda/\omega) - c'\ln(\Lambda/T)) \quad ,$$
$$\beta(\beta\text{-function}) \equiv \frac{\partial}{\partial(\ln\Lambda)}\ln\frac{\omega'}{\omega} = -c - c' \quad . \qquad (38)$$

We should notice that, in the flat geometry case, the IR parameter (extra-space size) l is renormalized . In the present warped case, however, the corresponding parameter T is not renormalized, but the warp parameter ω is renormalized. Depending on the sign of $c+c'$, the 5D bulk curvature ω flows as follows. When $c+c'>0$, the bulk curvature ω decreases (increases) as the the measurement energy scale Λ increases (decreases). When $c+c'<0$, the flow goes in the opposite way.

$$\frac{1}{G_N}\lambda_{obs} \sim \frac{1}{G_N {R_{cos}}^2} \sim m_\nu^4 \sim (10^{-3}eV)^4 \quad , \tag{39}$$

where R_{cos} is the cosmological size (Hubble length), m_ν is the neutrino mass.

$$\frac{1}{G_N}\lambda_{th} \sim \frac{1}{{G_N}^2} = {M_{pl}}^4 \sim (10^{28}eV)^4 \quad . \tag{40}$$

The famous huge discrepancy factor: $\lambda_{th}/\lambda_{obs} \sim 10^{124}$. If we apply the present approach, we have the warp factor ω, and the result (37) strongly suggests the following choice:

$$\text{INPUT 1} \quad \Lambda = M_{pl}$$

$$\text{INPUT 2(Newton's law exp.)}\ \omega \sim \frac{1}{\sqrt[4]{G_N {R_{cos}}^2}} = \sqrt{\frac{M_{pl}}{R_{cos}}} \sim m_\nu \sim 10^{-3}\text{eV}$$

$$\text{FACT} \quad S \sim \int d^4x\sqrt{-g}\frac{1}{G_N}\lambda_{obs} \sim R_{COS}^4\omega^4$$

$$\text{Result(37)requires} \quad \mathrm{e}^{-S} \leftrightarrow \mathrm{e}^{-E_{Cas}/T^4} = \exp\{-T^{-4}\Lambda T^{-1}\omega^4\}$$

$$\Longrightarrow \ T^5 = \frac{M_{pl}}{R_{cos}^4} \quad \text{OUTPUT} \qquad (41)$$

We do not yet succeed in obtaining the right sign, but succeed in obtaining the finiteness and its gross absolute value of the cosmological constant. Now we understand that the smallness of the cosmological constant comes from the renormalization flow for the non asymptotic-free case ($c + c' < 0$ in (38)).