Toward the understanding of QCD phase structure at finite temperature and density

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WHOT-QCD collaboration
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QCD phase structure at high temperature and density

**Lattice QCD Simulations**

- Phase transition lines $T$
- Equation of state
- Direct simulation: Impossible at $\mu \neq 0$. 

![Diagram showing phase transitions and possible states in QCD](image-url)
Histogram method

- Problem of Complex Determinant at $\mu \neq 0$
  - Boltzmann weight: complex at $\mu \neq 0$
  - Monte-Carlo method is not applicable.

$$\langle O \rangle_{(m,T,\mu)} = \frac{1}{Z} \int DU \, O \left( \text{det} M(m,\mu) \right)^{N_f} e^{-S_g}$$

- Distribution function in Density of state method (Histogram method)
  $X$: order parameters, total quark number, average plaquette etc.

$$W(X;m,T,\mu) \equiv \int DU \, \delta(X-\hat{X})(\text{det}M(m,\mu))^{N_f} e^{-S_g}$$

- Expectation values

$$\langle O[X] \rangle_{(m,T,\mu)} = \frac{1}{Z} \int dX \, O[X] W(X,m,T,\mu)$$

$$Z(m,T,\mu) = \int dX \, W(X,m,T,\mu)$$
$(\beta, m, \mu)$-dependence of the Distribution function

- Distributions of plaquette $P$ (1x1 Wilson loop for the standard action)

$$W(P', \beta, m, \mu) \equiv \int DU \delta(\hat{P} - P') (\det M(m, \mu))^N_f e^{6N_{\text{site}} \beta \hat{P}}$$

$$S_g = -6N_{\text{site}} \beta \hat{P}$$

$$R(P, \beta, \beta_0 m, m_0, \mu) \equiv W(P, \beta, m, \mu) / W(P, \beta_0, m_0, 0) \quad \text{(Reweight factor)}$$

$$R(P') = e^{6N_{\text{site}} (\beta - \beta_0) P'} \left\langle \delta(\hat{P} - P') \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right\rangle_{(\beta_0, \mu=0)} \equiv e^{6N_{\text{site}} (\beta - \beta_0) P'} \left\langle \left( \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^N_f \right\rangle_{P'}$$

Effective potential:

$$V_{\text{eff}}(P, \beta, m, \mu) = -\ln[W(P, \beta m, \mu)] = V_{\text{eff}}(P, \beta_0, m_0, 0) - \ln R(P, \beta, \beta_0 m, m_0, \mu)$$

$$\ln R(P) = 6N_{\text{site}} (\beta - \beta_0) P + \ln \left\langle \left( \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^N_f \right\rangle_P$$

independent of $\beta_0$
Sign problem

\[
\langle \left( \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^{N_f} \rangle_{X \text{ fixed}} = \langle e^{i\theta} \left| \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right|^{N_f} \rangle_{X \text{ fixed}}
\]

\(\theta\): complex phase of \((\det M)^{N_f}\)

- Sign problem: If \(e^{i\theta}\) changes its sign frequently,

\[
W(X) \sim \langle e^{i\theta} \left| \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right| \rangle_{X \text{ fixed}} \ll \text{(statistical error)}
\]
Overlap problem

\[
\langle OR \rangle = \frac{1}{Z} \int ORW(X) \, dX = \frac{1}{Z} \int \exp(-V_{\text{eff}}(X) + \ln(OR)) \, dX
\]

\[V_{\text{eff}}(X) = -\ln W(X)\]

- \( W \) is computed from the histogram.
- Distribution function around \( X \) where 
  \( V_{\text{eff}}(X) - \ln(OR) \) is minimized: important.
- \( V_{\text{eff}} \) must be computed in a wide range.

\( V_{\text{eff}}(X) \)
\[\downarrow \]
\( \text{reliable range} \)

\( -\ln(OR) \)
\[\downarrow \]

If \( X \)-dependence is large.

\( V_{\text{eff}}(X) - \ln(OR) \)
\[\downarrow \]
\( \text{reliable range} \)

Out of the reliable range
Distribution function in quenched simulations

Effective potential in a wide range of $P$: required.

Plaquette histogram at $K=1/m_q=0$.

![Histogram Graph](image1)

Derivative of $V_{\text{eff}}$ at $\beta=5.69$

![Derivative Graph](image2)

$N_{\text{site}} = 24^3 \times 4$, 5 $\beta$ points, quenched.

$dV_{\text{eff}}/dP$ is adjusted to $\beta=5.69$, using

These data are combined by taking the average.

$$
\frac{dV_{\text{eff}}}{dP}(\beta_2) = \frac{dV_{\text{eff}}}{dP}(\beta_1) - 6N_{\text{site}}(\beta_2 - \beta_1)
$$
Distribution functions for $\Omega_R$ and $P$

Expectation value of Polyakov loop and its susceptibility by the reweightuing method at $\mu=0$.

- If $W(P,\Omega)$ is a Gaussian distribution,
  - The peak position of $W(P,\Omega)$ $(<P>, <\Omega>)$
  - The width of $W(P,\Omega)$ susceptibilities $\chi_P, \chi_\Omega$
- If $W(P,\Omega)$ have two peaks, first order transition
μ-dependence of the effective potential

\[ Z(T, \mu) = \int dX \, W(X, T, \mu), \quad V_{\text{eff}}(X) = -\ln W(X) \]

\( X \): order parameters, total quark number, average plaquette, quark determinant etc.

Crossover

\[ V_{\text{eff}}(X, T, \mu) \]

Correlation length: short

\( V(X) \): Quadratic function

Critical point

Correlation length: long

Curvature: Zero

1st order phase transition

Two phases coexist

Double well potential

hadron

QGP

CSC?
Quark mass dependence of the critical point

- Where is the physical point?
- Extrapolation to finite density
  - investigating the quark mass dependence near $\mu=0$
- Critical point at finite density?

![Diagram showing the dependence of quark masses on quark mass and chempotentital](image-url)
Distribution function in the heavy quark region

- We study the properties of $W(X)$ in the heavy quark region.
- Performing quenched simulations + Reweighting.
- We find the critical surface.
- Standard Wilson quark action + plaquette gauge action, $S_g = -6N_{\text{site}}\beta P$
- Lattice size: $24^3 \times 4$
- 5 simulation points; $\beta=5.68-5.70$.

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

Hopping parameter expansion

$$N_f \ln \left( \frac{\det M(K, \mu)}{\det M(0,0)} \right) = N_f \left( 288N_{\text{site}}K^4 P + 12 \cdot 2^{N_t}N_s^3K^{N_t}(\cosh(\mu/T)\Omega_R + i\sinh(\mu/T)\Omega_I) + \cdots \right)$$

$P$: plaquette, $\Omega=\Omega_R+i\Omega_I$: Polyakov loop

$$\det M(0,0) = 1$$
Effective potential near the quenched limit ($\mu=0$)

WHOT-QCD, Phys.Rev.D84, 054502(2011)

**at phase transition point**

Quenched Simulation
($m_q=\infty, K=0$

$K \sim 1/m_q$ for large $m_q$

**Critical point for $N_f=2$:**
$K_{cp}=0.0658(3)(8)$

$$\frac{T_c}{m_\pi} \approx 0.02$$

$$V_{\text{eff}}(P, \beta, m, \mu) = -\ln[W(P, \beta m, \mu)] = V_{\text{eff}}(P, \beta_0, m_0, 0) - \ln R(P, \beta, \beta_0 m, m_0, \mu)$$

$$\ln R(P) = 6 N_{\text{site}} (\beta - \beta_0) P + \ln \left\langle \left( \frac{\det M(m, \mu)}{\det M(m_0, 0)} \right)^{N_f} \right\rangle_P$$

- detM: Hopping parameter expansion,
- First order transition at $K = 0$ changes to crossover at $K > 0$. 

$V_{\text{eff}}(P, \beta, m, \mu) = -\ln[W(P, \beta m, \mu)] = V_{\text{eff}}(P, \beta_0, m_0, 0) - \ln R(P, \beta, \beta_0 m, m_0, \mu)$
Order of the phase transition
Polyakov loop distribution

Effective potential of $|\Omega|$ on the pseudo-critical line at $\mu=0$

- The pseudo-critical line is determined by $\chi_\Omega$ peak.

- Double-well at small $K$ – First order transition
- Single-well at large $K$ – Crossover

Critical point: $\kappa^4 \approx 2.0 \times 10^{-5}$
Polyakov loop distribution in the complex plane
($\mu=0$)

\[ \kappa^4 = 0.0 \]

\[ \kappa^4 = 5.0 \times 10^{-6} \]

\[ \kappa^4 = 1.0 \times 10^{-5} \]

\[ \kappa^4 = 1.5 \times 10^{-5} \]

\[ \kappa^4 = 2.0 \times 10^{-5} \]

\[ \kappa^4 = 2.5 \times 10^{-5} \]

Critical point

- on $\beta_{pc}$ measured by the Polyakov loop susceptibility.
Distribution function of $\Omega_R$ at finite density

$$W(\Omega_R, \beta, K, \mu) = \int DU \delta(\Omega_R - \hat{\Omega}_R)(\det M(k))^{N_f} e^{-6N_{\text{site}}\hat{P}}$$

- Hopping parameter expansion

$$\frac{W(\beta, K, \mu)}{W(\beta_0, 0, 0)} = \langle \exp\left[6(\beta - \beta_0) + 288N_fK^4\right]N_{\text{site}}\hat{P} - 12 \times 2^{N_i} N_f N_s^3 K^{N_i} \cosh(\mu/T)\hat{\Omega}_R + i\theta \rangle_{\Omega_R; \beta_0, K=\mu=0}$$

- Adopting $\beta_0 \equiv \beta + 48N_fK^4$, 

$$\left(\theta = 12 \cdot 2^{N_i} N_s^3 N_f K^{N_i} \sinh(\mu/T)\hat{\Omega}_I\right)$$

- Effective potential:

$$V_{\text{eff}}(\Omega_R; \beta, K, \mu) = -\ln W(\Omega_R; \beta, K, \mu)$$

$$V_{\text{eff}}(\beta, K, \mu) = V_{\text{eff}}(\beta_0, 0, 0) - 12 \times 2^{N_i} N_f N_s^3 K^{N_i} \cosh(\mu/T)\Omega_R - \ln\langle e^{i\theta} \rangle_{\Omega_R; \beta_0, K=\mu=0}$$

$$\equiv V_0(\beta, K, \mu) - \ln\langle e^{i\theta} \rangle_{\Omega_R; \beta_0, K=\mu=0}$$

Phase-quenched part \hspace{1cm} Phase average

- $V_0$ is $V_{\text{eff}}(\mu=0)$ when we replace $K^{N_i} \Rightarrow K^{N_i} \cosh(\mu/T)$

(at $\mu=0$, $V_{\text{eff}}(\beta, K, 0) = V_{\text{eff}}(\beta_0, 0, 0) - 12 \times 2^{N_i} N_f N_s^3 K^{N_i} \Omega_R$)
Avoiding the sign problem

\( \theta: \text{complex phase} \quad \theta \equiv \text{Im ln det } \mathcal{M} \approx 12 \cdot 2^N s^3 N_f K^N \sinh(\mu/T) \Omega \)

- Sign problem: If \( e^{i\theta} \) changes its sign,
  \[
  \langle e^{i\theta} \rangle_{P,\Omega_R \text{ fixed}} \ll (\text{statistical error})
  \]

- Cumulant expansion
  \[
  \langle e^{i\theta} \rangle_{P,\Omega_R} = \exp \left[ i\langle \theta \rangle_C - \frac{1}{2} \langle \theta^2 \rangle_C - \frac{i}{3!} \langle \theta^3 \rangle_C + \frac{1}{4!} \langle \theta^4 \rangle_C + \cdots \right]
  \]
  \[
  \text{cumulants} \quad \langle \theta \rangle_C = \langle \theta \rangle_{P,\Omega_R}, \quad \langle \theta^2 \rangle_C = \langle \theta^2 \rangle_{P,\Omega_R} - \langle \theta \rangle^2_{P,\Omega_R}, \quad \langle \theta^3 \rangle_C = \langle \theta^3 \rangle_{P,\Omega_R} - 3\langle \theta^2 \rangle_{P,\Omega_R} \langle \theta \rangle_{P,\Omega_R} + 2\langle \theta \rangle^3_{P,\Omega_R}, \quad \langle \theta^4 \rangle_C = \cdots
  \]
  
  - Odd terms vanish from a symmetry under \( \mu \leftrightarrow -\mu \) (\( \theta \leftrightarrow -\theta \))

Source of the complex phase

If the cumulant expansion converges, No sign problem.
Convergence in the large volume ($V$) limit

The cumulant expansion is good in the following situations.

- If the phase is given by $\theta = \sum x \theta_x$
  
  - No correlation between $\theta_x$.

$$\langle e^{i\theta} \rangle_{P,\Omega_R} = \langle e^{i\sum x \theta_x} \rangle_{P,\Omega_R} \approx \prod_x \langle e^{i\theta_x} \rangle_{P,\Omega_R} = \exp \left[ \sum_x \sum_n \frac{i^n}{n!} \langle \theta_x^n \rangle_C \right]$$

$$\langle e^{i\theta} \rangle_{P,\Omega_R} = \exp \left[ \sum_n \frac{i^n}{n!} \langle \theta^n \rangle_C \right] \Rightarrow \langle \theta^n \rangle_C \approx \sum_x \langle \theta_x^n \rangle_C \sim O(V)$$

  - Ratios of cumulants do not change in the large $V$ limit.
  - Convergence property is independent of $V$,
    although the phase fluctuation becomes larger as $V$ increases.
  - The application range of $\mu$ can be measured on a small lattice.

- When the distribution function of $\theta$ is perfectly Gaussian, the average of the phase is give by the second order, $\langle e^{i\theta} \rangle_{P,\Omega_R} = \exp \left[ -\frac{1}{2} \langle \theta^2 \rangle_C \right]$
Cumulant expansion

\[ \beta^* = 5.69 \]

\[ K^4(\mu) \sinh(\mu/T) = 0.00002 \]

\[ \ln\left( e^{i\theta} \right)_{\Omega_R} = -\frac{1}{2} \langle \theta^2 \rangle_C + \frac{1}{4!} \langle \theta^4 \rangle_C - \frac{1}{6!} \langle \theta^6 \rangle_C + \ldots \]

\[ K^4(\mu) \sinh(\mu/T) = 0.00005 \]

\[ K^4(\mu) \sinh(\mu/T) = 0.0001 \]

- The effect from higher order terms is small near the critical point of phase-quenched part.

\[ K_{cp}^{N_t}(0) = K_{cp}^{N_t}(\mu) \cosh(\mu/T) > K_{cp}^{N_t}(\mu) \sinh(\mu/T) \approx 0.00002 \]
Effect from the complex phase factor

- Polyakov loop effective potential for each $K^N_t \cosh(\mu/T)$ at the pseudo-critical ($\beta, K$).
  - Solid lines: complex phase omitted, i.e., $\sinh(\mu/T) = 0$
  - Dashed lines: complex phase is estimated with $\sinh(\mu/T) = \cosh(\mu/T)$

The effect from the complex phase factor is very small except near $\Omega_R=0$. 

\[
\theta \approx 12 \cdot 2^{N_t} N_s^3 N_f K^N_t \sinh(\mu/T) \Omega_I
\]

\[
V_{\text{eff}}(\Omega_R) = V_0(\Omega_R) - \ln \langle e^{i\theta} \rangle_{\Omega_R: \text{fixed}}
\]

\[
\approx V_0(\Omega_R) + \frac{1}{2} \langle \theta^2 \rangle_{\Omega_R: \text{fixed}}
\]
Critical line in 2+1-flavor finite density QCD

- The effect from the complex phase is very small for the determination of $K_{c_p}$.

\[ N_f = 2 \text{ at } \mu = 0: \quad K_{c_p} = 0.0658(3)(8) \]

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

\[ N_f = 2+1 \]

\[
\ln \left( \frac{(\text{det } M(K_{ud}))^2 \text{det } M(K_s)}{\text{det } M(0)^3} \right) = 288 N_{\text{site}} (2K_{ud}^4 + K_s^4) P + 12 \times 2^{N_f} N_s^3 \left( 2K_{ud}^{N_f} \cosh \left( \frac{\mu_{ud}}{T} \right) + K_s^{N_f} \cosh \left( \frac{\mu_s}{T} \right) \right) \Omega_R + \cdots
\]

The critical line is described by

\[
2K_{ud}^{N_f} \cosh \left( \frac{\mu_{ud}}{T} \right) + K_s^{N_f} \cosh \left( \frac{\mu_s}{T} \right) = 2K_{c_p(N_f=2)}^{N_f}
\]

Critical line for $\mu_u = \mu_d = \mu_s = \mu$

Critical line for $\mu_u = \mu_d = \mu$, $\mu_s = 0$
Distribution function in the light quark region
WHOT-QCD Collaboration, in preparation,
(Nakagawa et al., arXiv:1111.2116)

• Perform phase quenched simulations
• Add the effect of the complex phase by the reweighting.
• Calculate the probability distribution function.

• Goal
  – The critical point
  – The equation of state

    Pressure, Energy density, Quark number density, Quark number susceptibility, Speed of sound, etc.
Probability distribution function by phase quenched simulation

• We perform phase quenched simulations with the weight:

$$W(P', F', \beta, m, \mu) = \int DU \delta(\hat{P} - P')\delta(\hat{F} - F') (\text{det} M(m, \mu))^N_f e^{-S_g}$$

$$= \int DU \delta(\hat{P} - P')\delta(\hat{F} - F') e^{i\theta} \text{det} M(m, \mu)^N_f e^{-S_g}$$

$$= \langle e^{i\theta} \rangle_{P', F'} \times W_0(P', F', \beta, m, \mu)$$

**expectation value with fixed $P,F$**

**histogram**

$P$: plaquette

$$F(\mu) = \frac{N_f}{N_{\text{site}}} \ln \left| \frac{\text{det} M(\mu)}{\text{det} M(0)} \right|$$

$$\theta \equiv N_f \text{ Im } \ln \text{ det } M$$

Distribution function of the phase quenched.

$$W_0(P', F') = \int DU \delta(\hat{P} - P')\delta(\hat{F} - F') \text{det} M|_{N_f} e^{6N_{\text{site}}\beta P}$$
$\mu$-dependence of the effective potential

Curvature of the effective potential

Crossover

$$-\ln[W(P,\beta)]$$

Critical point

$$-\ln[W_0(P,\beta)] - \ln[\langle e^{i\theta} \rangle]$$

1\textsuperscript{st} order phase transition

$$-\ln[W_0(P,\beta)] - \ln[\langle e^{i\theta} \rangle]$$

hadron

QGP

CSC

$\mu$

$T$

phase effect

Curvature: Zero

phase effect

Curvature: Negative
Curvature of the effective potential

- If the distribution is Gaussian,

\[
W_0(P, F) \approx \sqrt{\frac{6N_{\text{site}}}{2\pi\chi_P}} \exp\left[-\frac{6N_{\text{site}}}{2\chi_P} (P - \langle P \rangle)^2\right] \times \sqrt{\frac{N_{\text{site}}}{2\pi\chi_F}} \exp\left[-\frac{N_{\text{site}}}{2\chi_F} (F - \langle F \rangle)^2\right]
\]

\[
\chi_P = 6N_{\text{site}} \langle (P - \langle P \rangle)^2 \rangle \\
\chi_F = N_{\text{site}} \langle (F - \langle F \rangle)^2 \rangle
\]

\[
\frac{\partial^2 (-\ln W_0)}{\partial P^2}(\langle P \rangle, \langle F \rangle) = \frac{6N_{\text{site}}}{\chi_P}
\]

\[
\frac{\partial^2 (-\ln W_0)}{\partial F^2}(\langle P \rangle, \langle F \rangle) \approx \frac{N_{\text{site}}}{\chi_F}
\]

at the peak of the distribution
Complex phase distribution

- We should not define the complex phase in the range from $-\pi$ to $\pi$.
- When the distribution of $q$ is perfectly Gaussian, the average of the complex phase is given by the second order (variance),

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[ -\frac{1}{2} \left\langle \theta^2 \right\rangle_C \right]$$

- Gaussian distribution $\rightarrow$ The cumulant expansion is good.
- We define the phase

$$\theta(\mu) = N_f \Im \left( \ln \frac{\det M(\mu)}{\det M(0)} \right) = N_f \int_0^{\mu/T} \Im \left[ \frac{\partial \ln \det M}{\partial (\mu/T)} \right] \frac{d\mu}{T}$$

- The range of $\theta$ is from $-\infty$ to $\infty$. 
Distribution of the complex phase

- Well approximated by a Gaussian function.
- Convergence of the cumulant expansion: good.

\[ \langle e^{i\theta} \rangle_{P,F} \approx \exp \left[ -\frac{1}{2} \langle \theta^2 \rangle \right] \]

\[ \frac{1}{2} \langle \theta^2 \rangle_{\langle P \rangle, \langle F \rangle} \approx \frac{1}{2} \langle \theta^2 \rangle_{\beta_0, \mu_0} \]

at the peak of \( W_0 \) in each simulation.
Simulations

$8^3 \times 4$ lattice \quad $m_\pi / m_\rho \approx 0.8$

- Simulation point in the $(\beta, \mu_0/T)$

- Peak of $W_0(P,F)$ for each $\mu$

2-flavor QCD Iwasaki gauge + clover Wilson quark action

Random noise method is used.
The curvature for $F$ decreases as $\mu$ increases.
Effect from the complex phase

- Rapidly changes around the pseudo-critical point.
Critical point at finite $\mu$

- zero curvature: expected at a large $\mu$. 

Diagram:
- $T$ axis
- $\mu$ axis
- Small $\langle \theta^2 \rangle_c$
- Large $\chi_F^{-1}$
- Complex phase gives negative curvature
- Confinement phase
- Small $\chi_F^{-1}$
- Large $\langle \theta^2 \rangle_c$
- Deconfinement phase
- Pion condensation phase in phase quenched
Curvature of the effective potential

- Without the complex phase effect

\[
\frac{\partial^2 \left(- \ln W_0 \right)}{\partial F^2} \left( \langle P \rangle, \langle F \rangle \right) \approx \frac{N_{\text{site}}}{\chi_F}
\]

\[
\chi_F = N_{\text{site}} \left\langle \left( F - \langle F \rangle \right)^2 \right\rangle
\]
Phase average

- 2\textsuperscript{nd} order cumulant

\[
\ln \left< e^{i\theta} \right>_{P,F} \approx -\frac{1}{2} \left< \theta^2 \right>_{P,F} \\
\frac{1}{2} \frac{1}{\left< P \right> \left< F \right>} \approx \frac{1}{2} \left< \theta^2 \right>_{\beta_0, \mu_0}
\]
Curvature of the effective potential

- The effect of the phase included.

zero curvature
Critical point
Peak position of $W(P,F)$

- The slopes are zero at the peak of $W(P,F)$.

$$\frac{\partial \ln W}{\partial P}(P,F,\beta,\mu) = \frac{\partial \ln W_0}{\partial P}(P,F,\beta,\mu) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial P}$$

$$= \frac{\partial \ln W_0}{\partial P}(P,F,\beta_0,\mu_0) + 6N_{site}(\beta - \beta_0) + \frac{\partial \ln R}{\partial P}(P,F,\mu,\mu_0) + \frac{\partial \ln \langle e^{i\theta} \rangle_{P,F}}{\partial P}$$

$$-0$$

- If these terms are canceled,

$$W(P,F,\beta,\mu) \approx W_0(P,F,\beta_0,\mu_0) \times \text{(const.)}$$

- $W(\beta, \mu)$ can be computed by simulations around $(\beta_0, \mu_0)$. 
QCD phase diagram

\[ W_0(P, F, \beta, m, \mu) \times \left\langle e^{i\theta} \right\rangle_{P,F} = W(P, F, \beta, m, \mu) \]

phase-quenched QCD

finite-density QCD

\[ \langle e^{i\theta} \rangle = 0 \]

pion condensed phase

color superconductor phase?
Summary

• We studied the quark mass and chemical potential dependence of the nature of QCD phase transition.

• The shape of the probability distribution function changes as a function of the quark mass and chemical potential.

• To avoid the sign problem, the method based on the cumulant expansion of $\theta$ is useful.

• Our results by phase quenched simulations suggest the existence of the critical point at high density.

• To find the critical point at finite density, further studies in light quark region are important applying this method.
Backup
Complex phase

• Gaussian distribution → The cumulant expansion is good.
• We define the phase

\[ \theta(\mu) = N_f \text{Im} \ln \frac{\det M(\mu)}{\det M(0)} = N_f \int_0^{\mu/T} \text{Im} \left[ \frac{\partial \ln \det M}{\partial (\mu/T)} \right] \mu d\left( \frac{\mu}{T} \right) \]

– The range of \( \theta \) is from \(-\infty\) to \(\infty\).
• At the same time, we calculate \( F \) as a function of \( \mu \),

\[ F(\mu) = N_f \ln \left| \frac{\det M(\mu)}{\det M(0)} \right| = N_f \int_0^{\mu/T} \text{Re} \left[ \frac{\partial \ln \det M}{\partial (\mu/T)} \right] \mu d\left( \frac{\mu}{T} \right) \]

• The reweighting factor is also computed,

\[ C(\mu) = N_f \ln \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right| = N_f \int_{\mu_0/T}^{\mu/T} \text{Re} \left[ \frac{\partial \ln \det M}{\partial (\mu/T)} \right] \mu d\left( \frac{\mu}{T} \right) \]
Distribution function for $P$ and $\Omega_R$

$$W(P, \Omega_R, \beta, \kappa) = \int DU \delta(P - \hat{P}) \delta(\Omega_R - \hat{\Omega}_R) (\det M(\kappa))^N_f e^{-6N_{\text{site}}\hat{P}}$$

$$\frac{W(\beta, K, \mu)}{W(\beta_0, 0, 0)} = \frac{\left< \delta(P-\hat{P})\delta(\Omega_R-\hat{\Omega}_R)e^{6N_{\text{site}}(\beta-\beta_0)P}\left(\frac{\det M(K, \mu)}{\det M(0,0)}\right)^{N_f}\right>_{(\beta_0, K=\mu=0)}}{\left< \delta(P-\hat{P})\delta(\Omega_R-\hat{\Omega}_R)\right>_{(\beta_0, K=\mu=0)}} \equiv \left< e^{6N_{\text{site}}(\beta-\beta_0)\hat{P}}\left(\frac{\det M(K, \mu)}{\det M(0,0)}\right)^{N_f}\right>_{P, \Omega_R}$$

- Effective potential  
  $$V_{\text{eff}}(P, \Omega_R; \beta, \kappa) = -\ln W(P, \Omega_R; \beta, \kappa)$$

- Hopping parameter expansion
  $$V_{\text{eff}}(\beta, \kappa) - V_{\text{eff}}(\beta_0, 0) = -\left(6(\beta - \beta_0) + 288N_fK^4\right)N_{\text{site}}P - 12 \times 2^{N_i} N_f N_s^3 K^{N_i} \cosh(\mu/T)\Omega_R - \ln \left< e^{i\theta} \right>_{P, \Omega_R}$$

  \[\equiv V_0(\beta, \kappa) - \ln \left< e^{i\theta} \right>_{P, \Omega_R} \quad \left(\theta = 12 \cdot 2^{N_i} N_s^3 N_f K^{N_i} \sinh(\mu/T)\hat{\Omega}_I \right)\]

  Phase-quenched part

- 2 parameters in $V_0$:  
  \[\beta + 48N_fK^4 \equiv \beta^*, \quad K^{N_i} \cosh(\mu/T)\]
  
  – $V_0$ is the same as $V_{\text{eff}}(\mu=0)$ when $K^{N_i} \Rightarrow K^{N_i} \cosh(\mu/T)$

- 1 parameter in $\theta$:  
  \[K^{N_i} \sinh(\mu/T) = K^{N_i} \cosh(\mu/T) \tanh(\mu/T) < K^{N_i} \cosh(\mu/T)\]
Order of phase transitions and Distribution function

\[ W(P, \Omega_R, \beta, \kappa) = \int DU \delta(P - \hat{P}) \delta(\Omega - \hat{\Omega}) (\det M(\kappa))^{N_f} e^{-6N_{\text{site}}P} \]

\[ V_{\text{eff}}(P, \Omega_R; \beta, \kappa) = -\ln W(P, \Omega_R; \beta, \kappa) \]

- Peak position of \( W \): 
  \[ \frac{dV_{\text{eff}}}{dP} = \frac{dV_{\text{eff}}}{d\Omega_R} = 0 \]

- Lines of zero derivatives for first order

- Crossover
- 1 intersection
- First order transition
- 3 intersections
Derivatives of $V_{\text{eff}}$ in terms of $P$ and $\Omega_R$

Phase-quenched part: when $\ln\langle e^{i\theta} \rangle$ is neglected,

$$V_{\text{eff}}(\beta, \kappa) - V_{\text{eff}}(\beta_0, 0) = -(6(\beta - \beta_0) + 288N_fK^4)N_{\text{site}} P - 12 \times 2^{N_t} N_fN_s^3K^{N_t} \cosh(\mu/T)\Omega_R$$

$$\frac{dV_{\text{eff}}(\beta, K)}{dP} \frac{dV_{\text{eff}}(\beta_0, 0)}{dP} = -(6(\beta - \beta_0) + 288N_fK^4)N_{\text{site}}$$

measured at $\kappa = 0$

$$\frac{dV_{\text{eff}}}{dP}$$

$\Omega_R$ $P$

- Contour lines of $\frac{dV_{\text{eff}}}{dP}$ and $\frac{dV_{\text{eff}}}{d\Omega_R}$ at $(\beta, \kappa) = (\beta_0, 0)$ correspond to the lines of the zero derivatives at $(\beta, \kappa)$. 
lines of \(\frac{dV_{\text{eff}}}{dP} = 0\) and \(\frac{dV_{\text{eff}}}{d\Omega_R} = 0\) in the \((P,\Omega)\) plane

\[
\frac{dV_{\text{eff}}(\beta_0,0)}{dP} = 6N_{\text{site}}(\beta - \beta_0 + 48N_fK^4_s) = 6N_{\text{site}}(\beta^* - \beta_0)
\]

\[
\frac{dV_{\text{eff}}(\beta_0,0)}{d\Omega_R} = 12 \times 2^N_t N_f N_s^3 K^N_t \cosh\left(\frac{\mu}{T}\right)
\]

- Small K: lines of \(\frac{dV_{\text{eff}}}{d\Omega_R} = 0\) : S-shape \(\rightarrow\) first order
- Large K: lines of \(\frac{dV_{\text{eff}}}{d\Omega_R} = 0\) : straight line \(\rightarrow\) crossover