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Three-Point Functions in N=4 SYM from Integrability

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based on works with Yoichi Kazama
arXiv:1110.3949 [hep-th]
arXiv:1205.6060 [hep-th]

What?:

Three point functions

(structure constant) of N=4 SYM

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

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If AdS/CFT is true, there should be a corresponding quantity in string theory on AdS.

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If AdS/CFT is true, there should be a corresponding quantity in string theory on AdS.

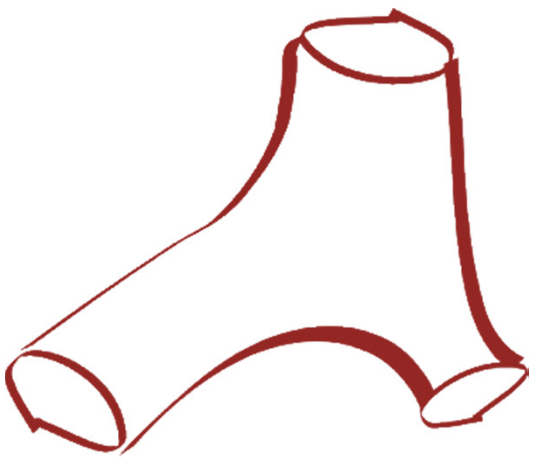
Let us calculate it at strong coupling

$$\lambda \to \infty$$

to check/understand AdS/CFT.

How?

- Classical string in AdS $(\lambda \to \infty)$
- Worldsheet correlation functions $\langle V_I V_J V_K \rangle$ in the classical limit
- Integrability



"3-legged" string

Why? (1/3)

Since N=4 SYM is conformal,

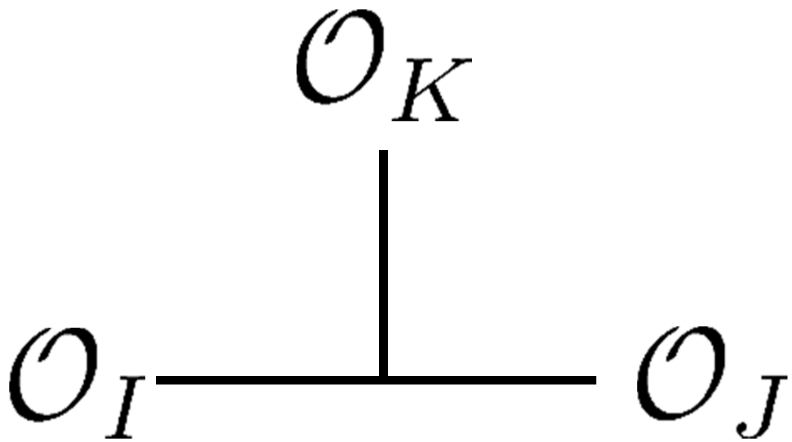
Δ_I : scaling dimension
C_{IJK}: structure constant

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle = \frac{\delta_{IJ}}{|x-y|^{2\Delta_I}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

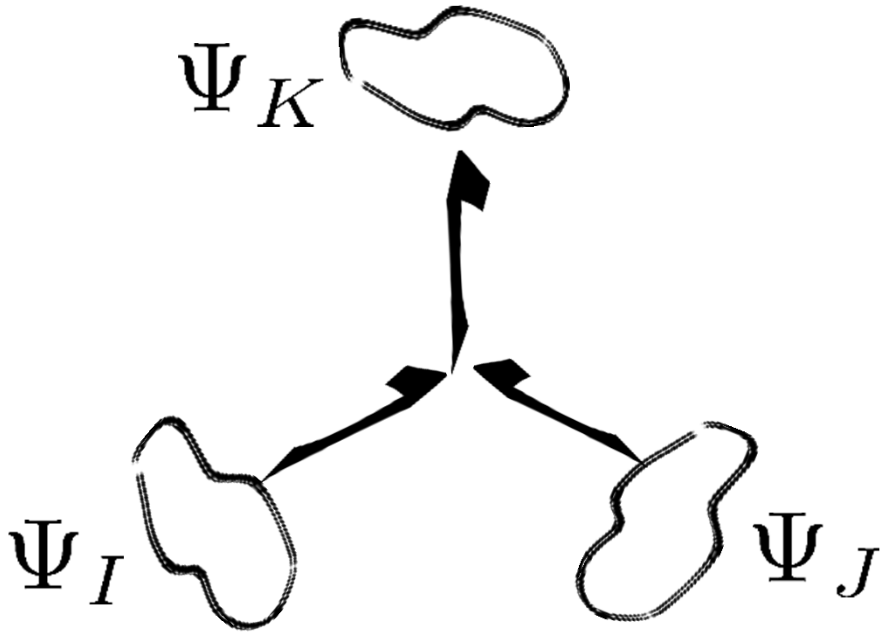
Δ_I and C_{IJK} together determine the theory through the OPE.

$$C_{IJK} \equiv$$



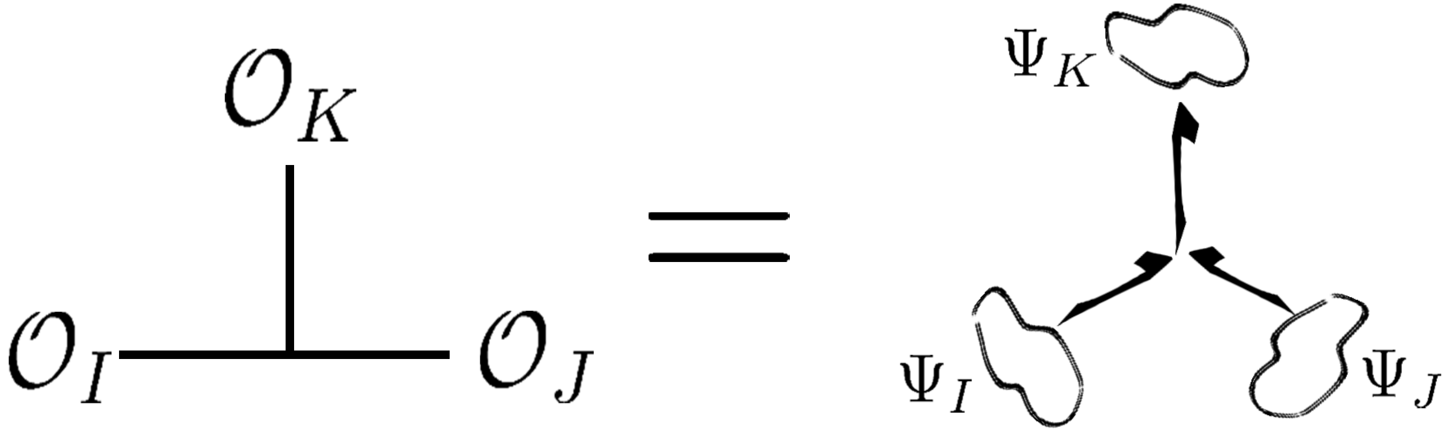
Why? (2/3)

Related to the interaction vertex of three strings on AdS
by AdS/CFT.



Why? (3/3)

Hopefully, important for understanding
the mechanism of AdS/CFT.



I will discuss this point more in detail later.

Outline

Introduction: AdS/CFT and correlation functions

Two point functions (review of the known results, ~2009)

Gauge theory 1-loop → Classical string → Beyond 1-loop /classical limit

Three point functions (2011~)

Gauge theory 0-loop → $\boxed{\text{Classical string}}$ → ...?

Our Works

Summary and Prospect

Introduction:
AdS/CFT and correlation functions

AdS$_5$/CFT$_4$ correspondence:

$\mathcal{N} = 4\ \mathrm{SU}(\mathrm{N}_c)$
super Yang-Mills

4d gauge theory

$=$

superstring on
$AdS_5 \times S^5$

10d string theory
(quantum gravity)

$\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

$\mathcal{N}=4$ SU(N_c) super Yang-Mills

superstring on $AdS_5 \times S^5$

$$\lambda = g_{\mathrm{YM}}^2 N_c \longleftrightarrow S_{\mathrm{string}} = \sqrt{\lambda}\int d^2z\,\partial X_\mu \bar{\partial} X^\mu + \mathrm{fermion}$$

't Hooft coupling constant

string tension

$$N_c \longleftrightarrow g_s = \frac{1}{N_c}$$

color

string loop effect

Today, we focus only on

$N_c \to \infty$: Large N. No string loop.

$\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence

- $\mathcal{N}=4$ SU(N_c) super Yang-Mills — superstring on $AdS_5 \times S^5$

$$\lambda = g_{\mathrm{YM}}^2 N_c \longleftrightarrow S_{\mathrm{string}} = \sqrt{\lambda}\int d^2z\,\partial X_\mu \bar{\partial} X^\mu + \mathrm{fermion}$$

't Hooft coupling constant — string tension

$$N_c \longleftrightarrow g_s = \frac{1}{N_c}$$

color — string loop effect

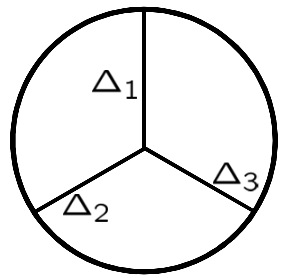
Today, we focus only on

$N_c \to \infty$: Large N. No string loop.

- For ½ BPS operators, **GKP-Witten relation** provides a mapping between two theories.

Correlation functions of ½ BPS operators $\longleftrightarrow$ Boundary to bulk correlators of SUGRA

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$$



- The relation to supergravity modes is widely used in the applications of holography (AdS/cond-mat, AdS/QCD).

- However, the original AdS/CFT correspondence predicts much stronger correspondence.

$$\mathcal{O}_{\mathrm{non-BPS}} \quad \longleftrightarrow$$

Non-BPS operators **Stringy states**

- Such quantities are not protected by supersymmetry. Difficult to obtain exact results.

  Use integrability.

Two point functions

Two point functions from Integrability

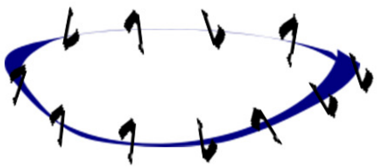
Integrability has been proven to be useful for the calculation of two point functions.

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle \sim \frac{\delta_{IJ}}{|x-y|^{2\Delta_I}}$$

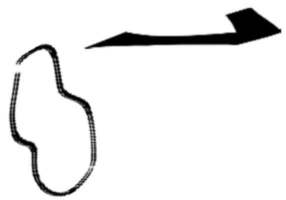
$\delta\Delta_{\text{one-loop}} \simeq H_{\text{spin-chain}}$

$\Delta =$ Energy of classical string solution

$\lambda \ll 1$



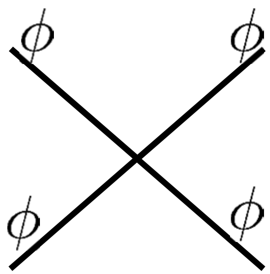
$\lambda \gg 1$

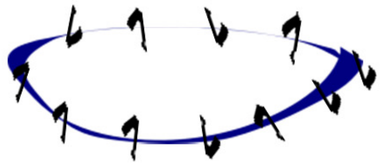


Prediction at any coupling from Thermodynamic Bethe Ansatz

$\lambda \ll 1$

spin-chain from 1-loop gauge theory





$\mathcal{N} = 4$ SYM

Lagrangian

$$\mathcal{L} = \frac{1}{2g_{\rm YM}^2}\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi_i)^2 + [\phi_i\,,\phi_j]^2 + \bar{\psi}^a i\gamma^\mu\partial_\mu\psi^a + \bar{\psi}^a\Gamma^i_{ab}[\phi_i\,.\psi^b]\big)$$

$$i,j = 1,\cdots,6 \quad a,b = 1,\cdots,4$$

All in the adjoint representation of SU(N).

Supersymmetry

Q^a : Four sets of supersymmetries

R-symmetry $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$

ϕ_i : SO(6) vector

ψ^a : SO(6) Weyl spinor

Γ^i_{ab} : SO(6) gamma matrix

helicity	fields
1	A_μ
$\frac{1}{2}$	$\psi^1\,,\psi^2\,,\psi^3\,,\psi^4$
0	$\phi_1\,,\phi_2\,,\phi_3\,,\phi_4\,,\phi_5\,,\phi_6$
$-\frac{1}{2}$	$\bar{\psi}^1\,,\bar{\psi}^2\,,\bar{\psi}^3\,,\bar{\psi}^4$
-1	A_μ

Renormalization and mixing of operators

Consider a composite operator: $\text{e.g. } \mathcal{O} = \mathrm{tr}\,(\phi_1 \phi_2 \cdots)$

Renormalize to obtain finite 2-point functions.

$\mathcal{O}_a^{\mathrm{ren}} = Z_a{}^b \mathcal{O}_b$ Mixing effect.

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$$\mathcal{O}^{\mathrm{ren}}_a = Z_a{}^b \mathcal{O}_b$$

Mixing effect.

anomalous dimensions: $$\frac{d \ln Z}{d \ln \Lambda} = \underline{\gamma(\lambda)}$$

Λ: UV cut-off

a function of λ

$$\langle \mathcal{O}^{\mathrm{ren}}(x_1) \mathcal{O}^{\mathrm{ren}}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2(\Delta_0 + \gamma)}}$$

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In summary, we need to

i) Calculate Z_{ab} pertubatively.

ii) Diagonalize Z_{ab} and calculate its eigenvalues.

Calculation of ${Z_a}^b$

Consider operators made up of scalars: $\mathrm{tr}\left(\phi_{i_1}\phi_{i_2}\phi_{i_3}\cdots\right)$

SO(6) “spins” (in the vector rep.) aligned in the trace

Calculation of $Z_a{}^b$

Consider operators made up of scalars: $\mathrm{tr}\left(\phi_{i_1}\phi_{i_2}\phi_{i_3}\cdots\right)$

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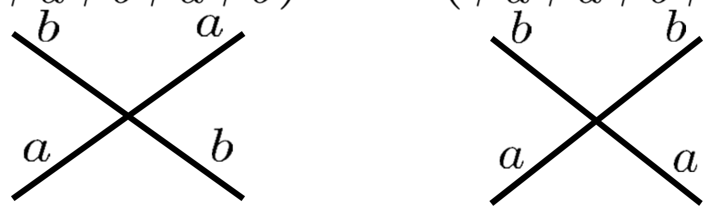
4-scalar interaction:

$$\mathrm{tr}([\phi_a,\phi_b]^2)/2 = \mathrm{tr}(\phi_a\phi_b\phi_a\phi_b) - \mathrm{tr}(\phi_a\phi_a\phi_b\phi_b)$$



$+1$ -1

Different factors depending on the order of spins

Calculation of $Z_a{}^b$

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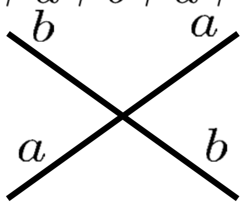

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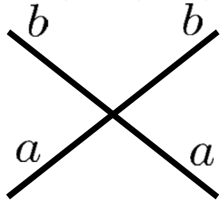
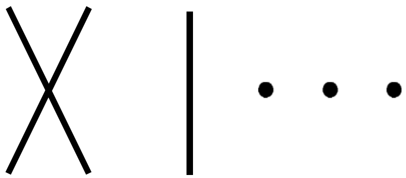


-1

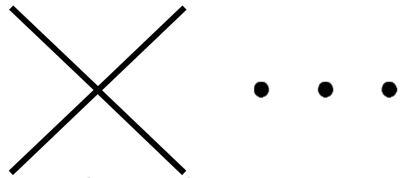
Different factors depending on the order of spins

In **large N**, only adjacent two fields can interact at one loop.



$\mathrm{tr}(\phi_{i_1}\phi_{i_2}\phi_{i_3}\cdots)$

Leading contribution

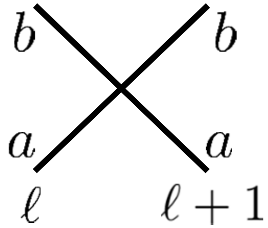
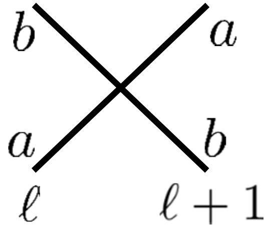


$\mathrm{tr}(\phi_{i_1}\phi_{i_2}\phi_{i_3}\cdots)$

Suppressed by 1/N

Nearest-neighbor interaction

Including all the other interactions,

$$\gamma = \frac{\lambda}{16\pi^2} \sum_{\ell}$$  $$-2$$ 

Hamiltonian of SO(6) spin-chain

Solvable by Bethe-Ansatz.

If operators are made up **only of the following two fields**,

$X := \phi_1 + i\phi_2$ spin “up”

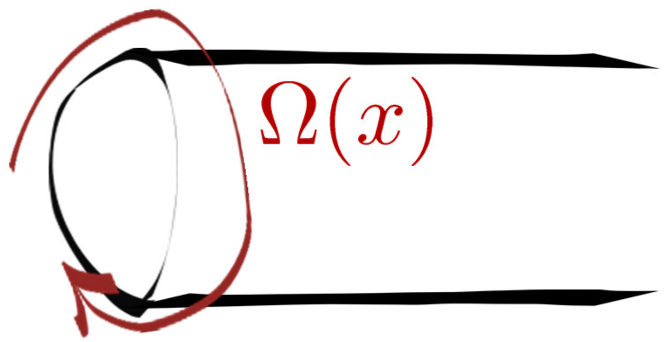
$Z := \phi_3 + i\phi_4$ spin “down”

$$\gamma \propto \sum_i S_x^{(i)} S_x^{(i+1)} + S_y^{(i)} S_y^{(i+1)} + S_z^{(i)} S_z^{(i+1)}$$

Heisenberg spin-chain

$$\lambda \gg 1$$

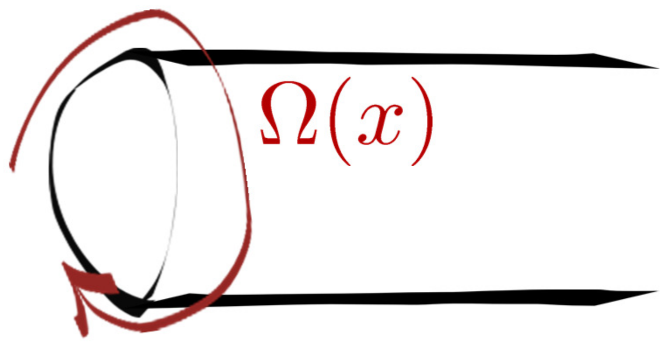
Two point function from classical string



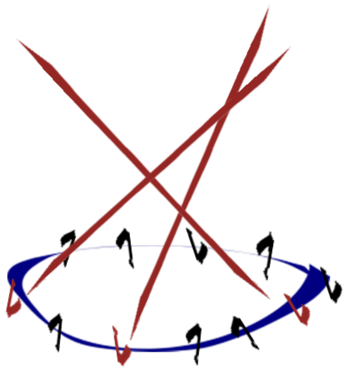
I will discuss later in detail.

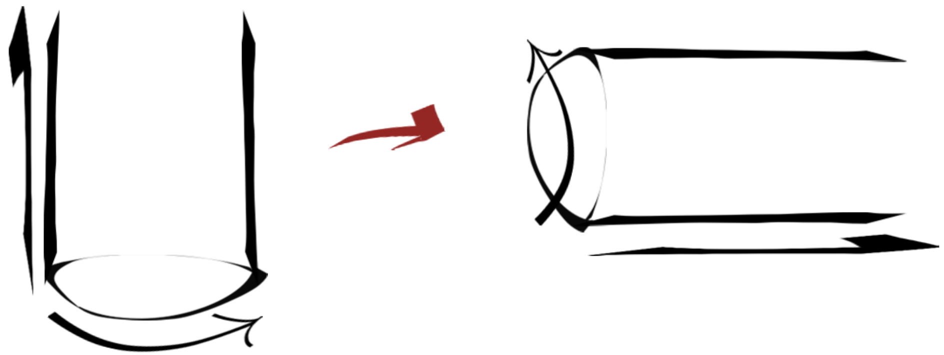
$$\lambda \gg 1$$

Two point function from classical string

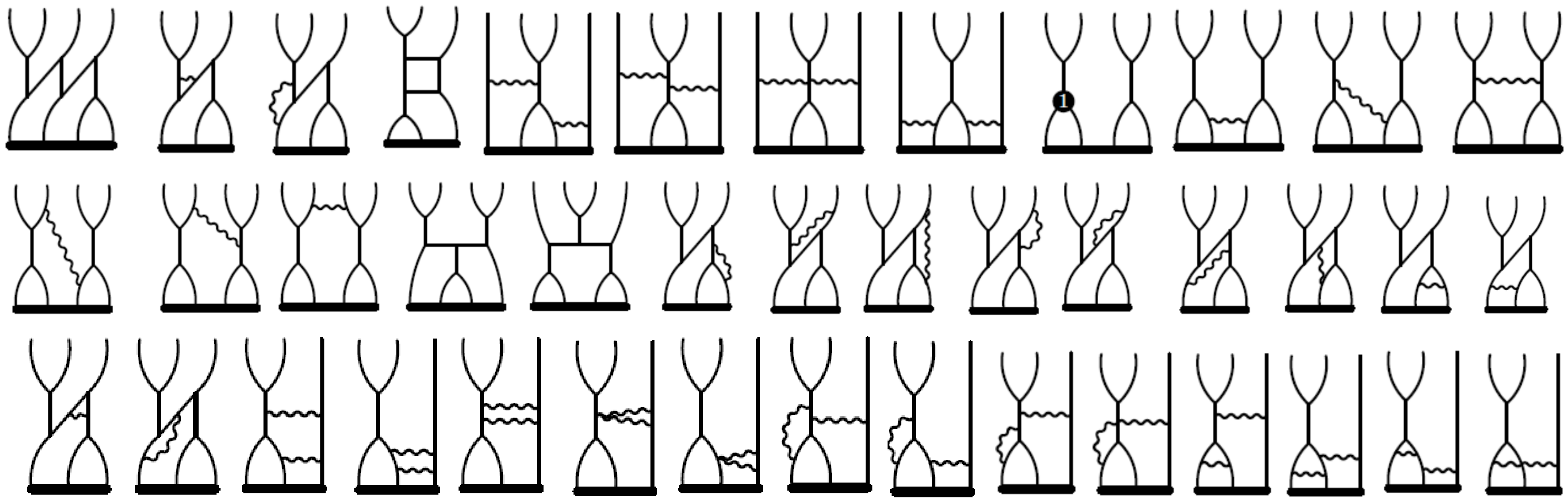


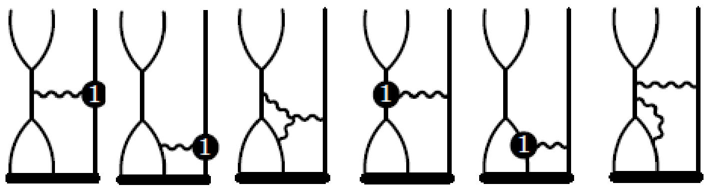
Beyond 1-loop





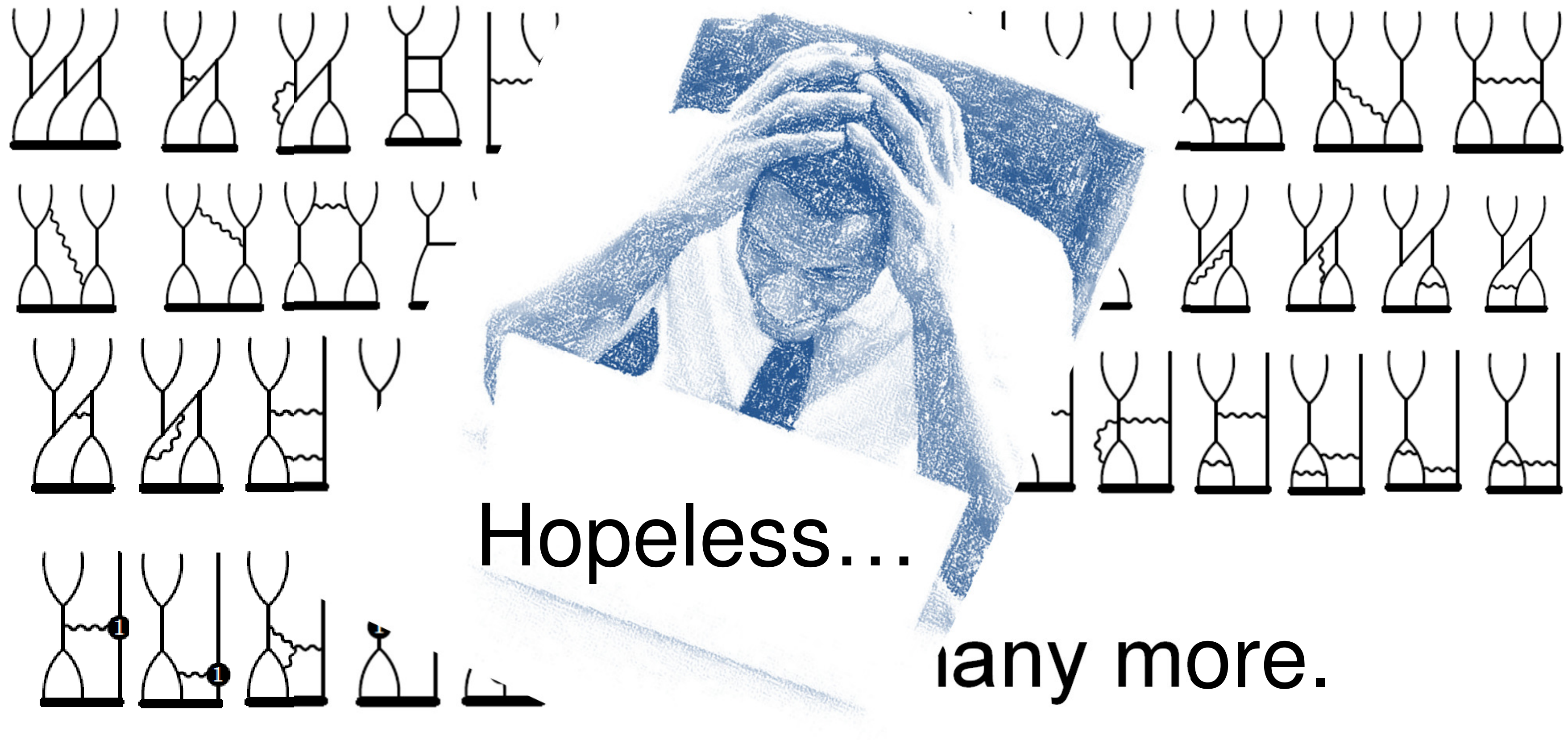
Higher-loop calculation





and many more.

Higher-loop calculatio



Hopeless…

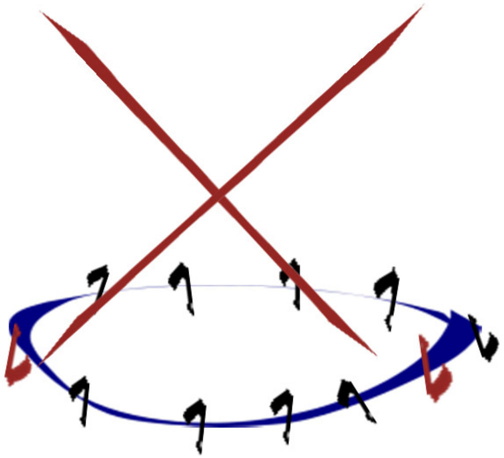
any more.

if you are not Russian…

Fortunately, there is an easier (but technical) way.

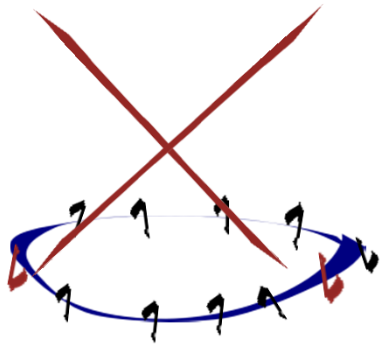
Consider a scattering problem on the spin-chain.

Instead of trying to construct the spin-chain Hamiltonian.



All-loop calculation(1)

Consider a scattering problem on the spin-chain.



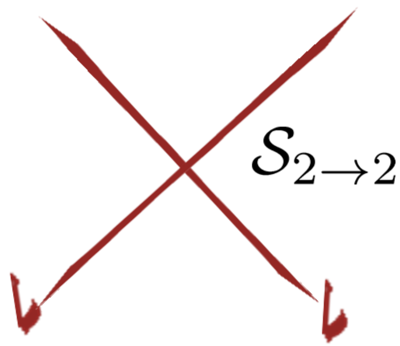
Results from

Symmetry
SO(6), SO(4,2), SUSY

\+

1-loop gauge theory
Classical string

Determine



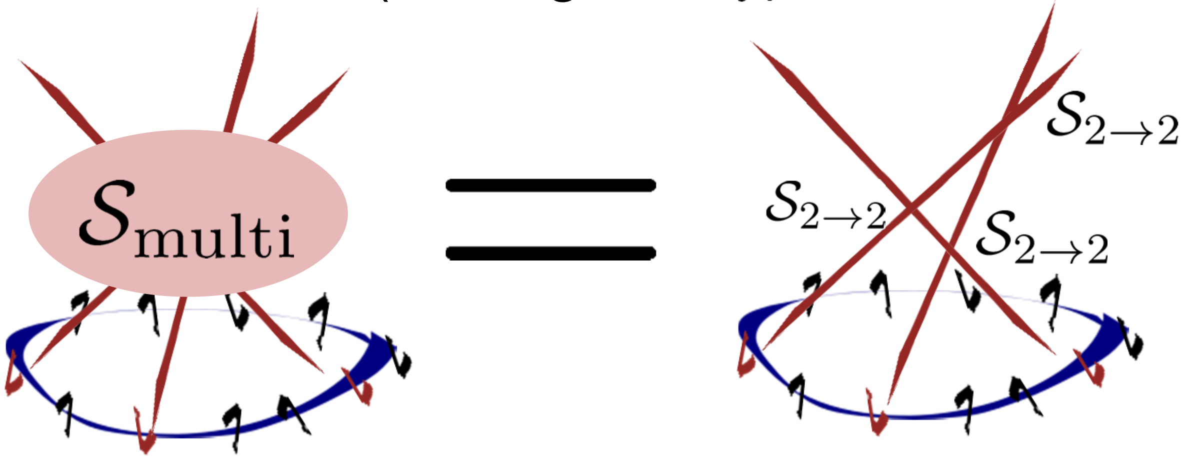
2→2 S-matrix

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2 p/2}$$

Dispersion relation

All-loop calculation(2)

Assume factorization (≈ integrability). Verified up to 3-loop.



Multi-particle S-matrix   Product of 2→2 S-matrices

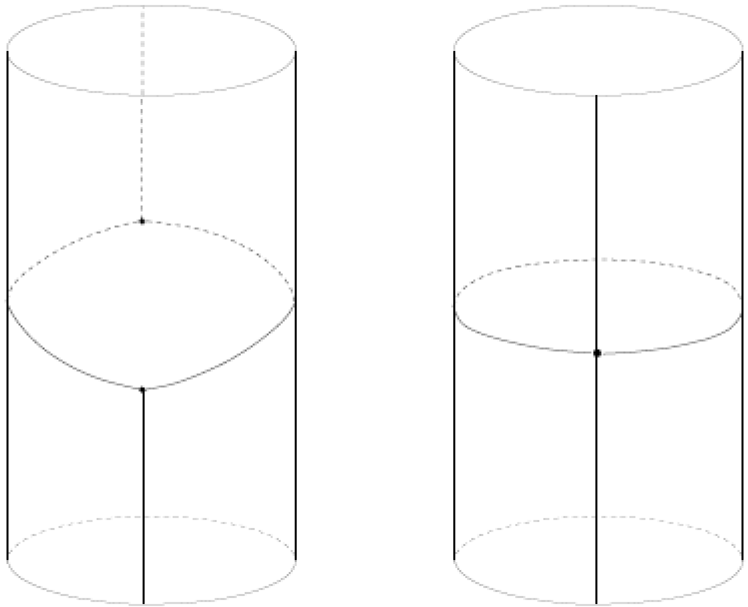
Δ of infinitely long operators are determined.

$$\mathrm{tr}\,(\cdots XXXZZX\mathcal{D}ZXX\mathcal{D}ZZZ\cdots)$$

$$\rightarrow \Delta$$

All-loop calculation(3)

To calculate the finite size effect, we can use Luscher formula.



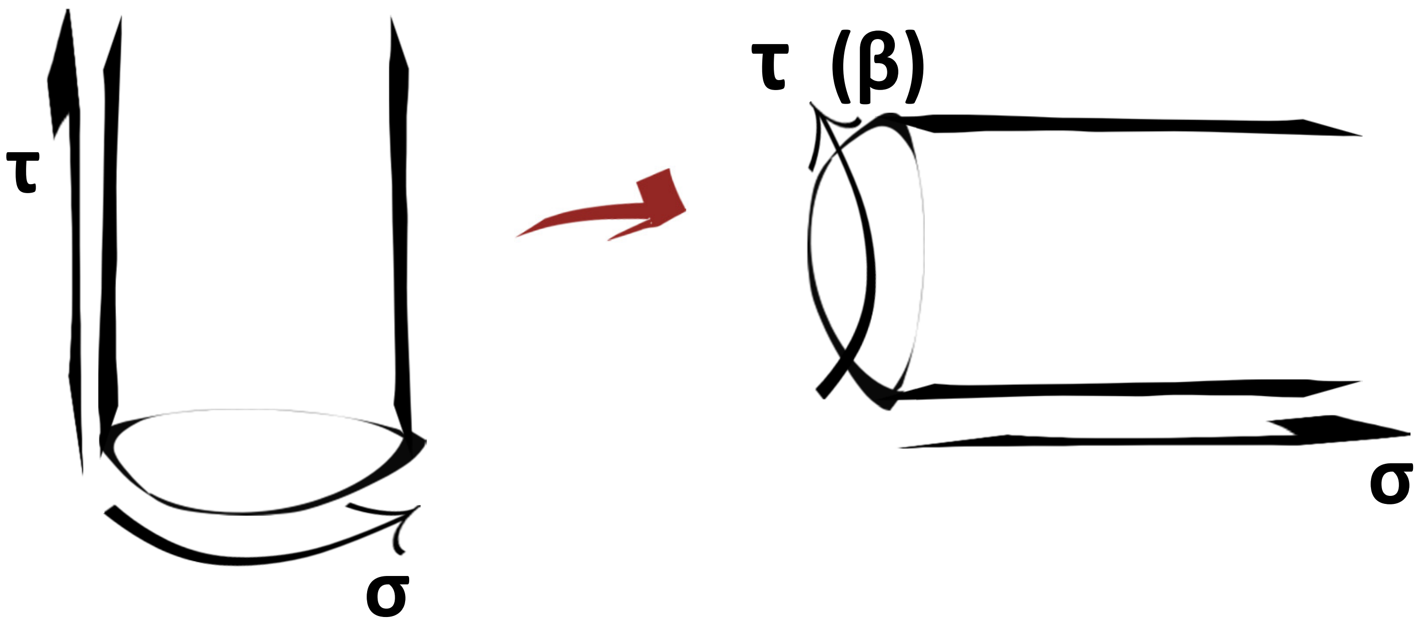
Correction by virtual particles.

Finite-size effect $\xleftrightarrow{\text{Luscher formula}}$ S-matrix

Also used in lattice gauge theory

All-loop calculation(4)

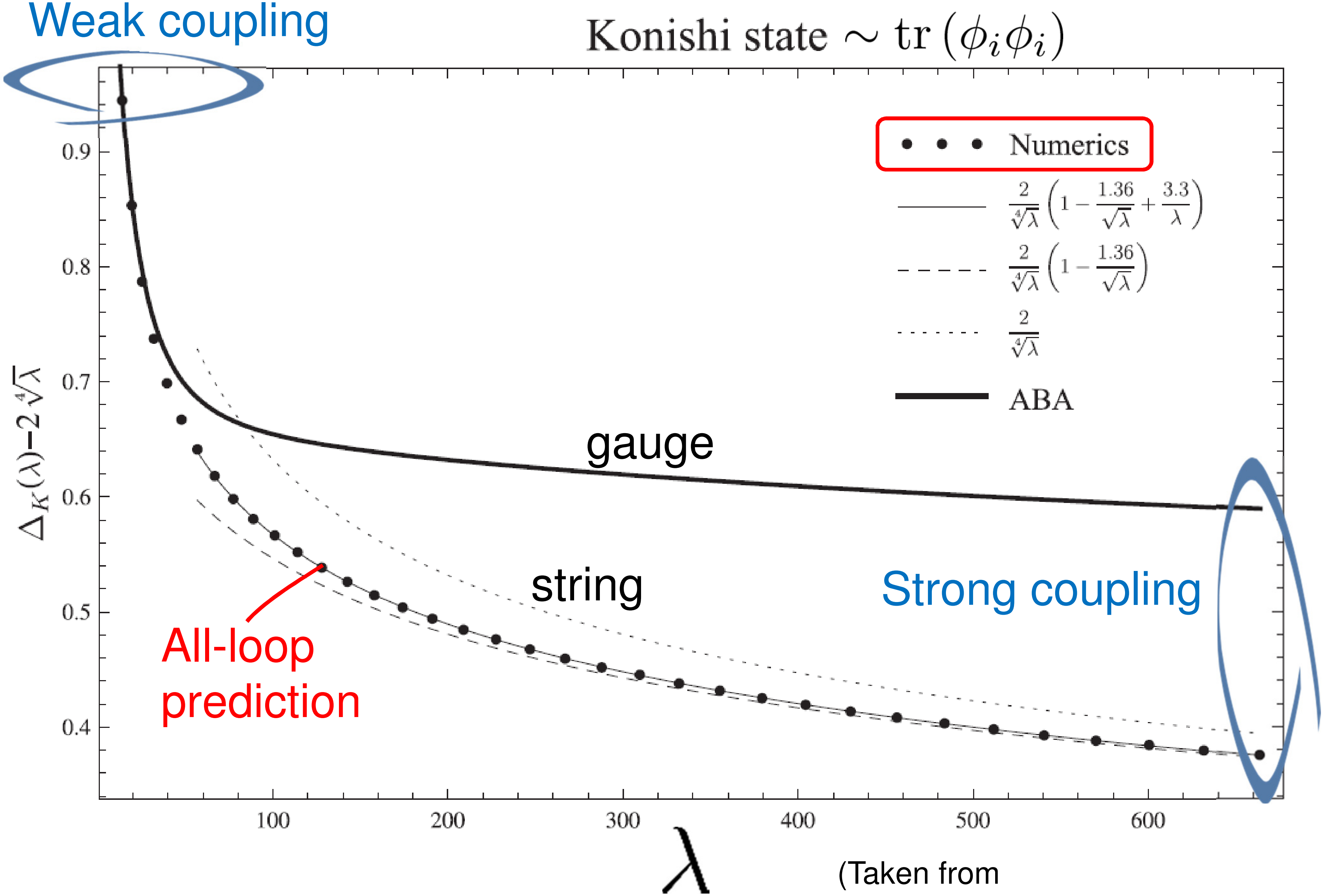
More powerful way in this case: Thermodynamic Bethe Ansatz
Exchange “space” and “time” of the spin chain.



Finite size effect → Finite temperature effect

Calculable by Thermodynamic Bethe Ansatz.
(coupled nonlinear integral equations)

Weak coupling



(Taken from
[Gromov-Kazakov-Vieira '08])

Remarkable and impressive results.

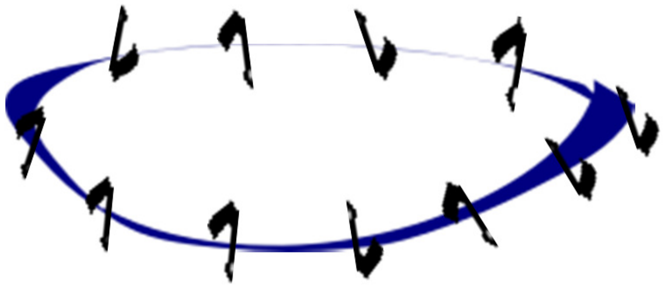
But…

Remarkable and impressive results.

But…

“What did we learn about the fundamental mechanism of AdS/CFT?”

For instance…



Long-range spin-chain
(Discrete)

=

string
(Continuous)

Without taking the
continuum limit

Why ?

Perhaps not much.

Because

Perhaps not much.

Because

i) (The assumption of) integrability is too powerful for the spectrum problem.

Need to consider quantities for which integrability is less manifest.

Perhaps not much.

Because

i) (The assumption of) integrability is too powerful for the spectrum problem.

Need to consider quantities for which integrability is less manifest.

ii) We have studied the duality only through one particular observable, Δ.

Need to compare both sides "more directly".

Perhaps not much.

Because

i) (The assumption of) integrability is too powerful for the spectrum problem.

→ Need to consider quantities for which integrability is less manifest.

ii) We have studied the duality only through one particular observable, Δ.

Need to compare both sides "more directly".

→ Compare wave functions.

Wave functions

Wave functions for the spin chain (the gauge theory)

$$\Psi_{\text{spin}} = \text{tr}\,(XXZ) + \cdots \quad \mathcal{O}^{\text{ren}}$$

Exact form of the renormalized operator

Wave functions for the string

$$\Psi_{\text{string}} = \Big| \quad \Big\rangle$$

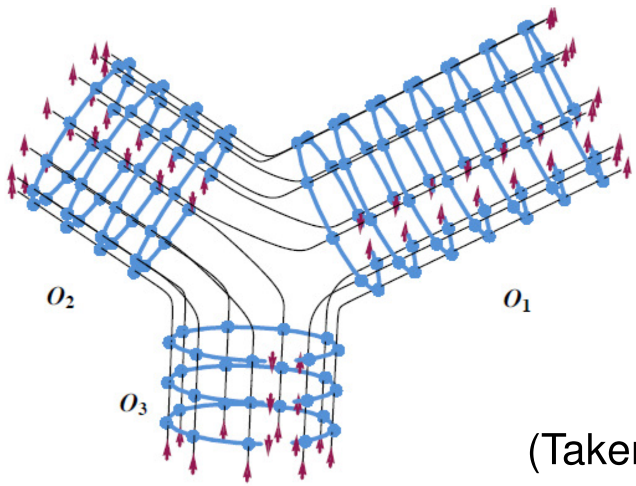
Encodes the shape and the motion of the string

For the spectrum problem: We didn't really need wave functions.
For three point functions: Wave functions are important.

To study 3-pnt functions is a good starting point

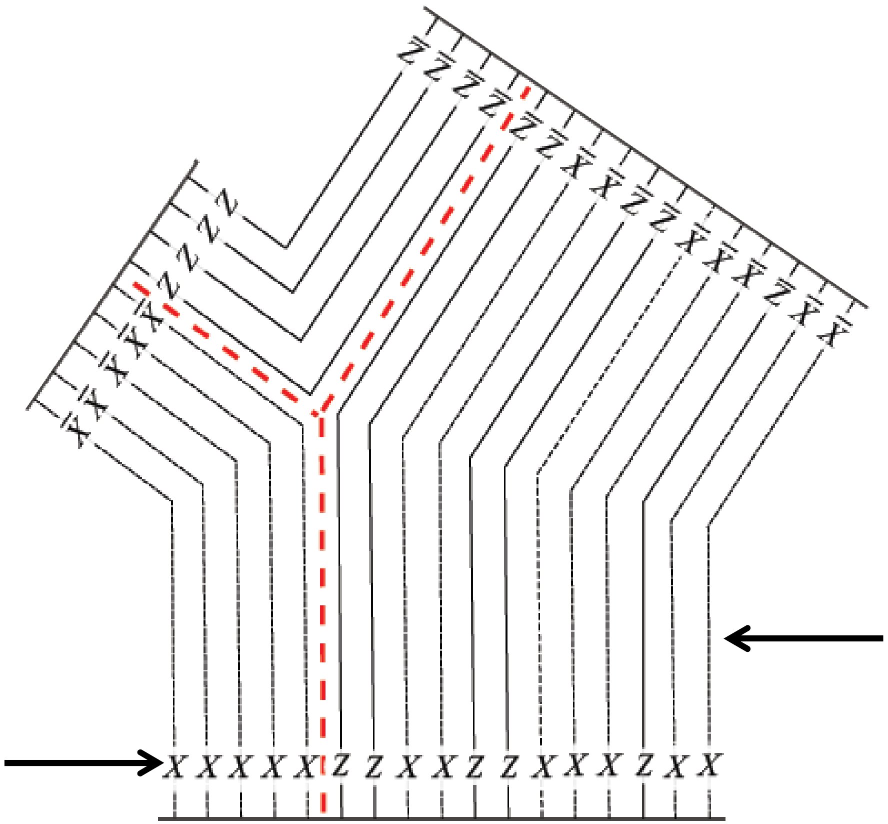
Three point functions

Lowest order calculation of the gauge theory



(Taken from [Gromov, Vieira '12])

Lowest order calculation of three point functions



Tree-level propagator

Operator at 1-loop $\mathcal{O}_{1-\mathrm{loop}}$

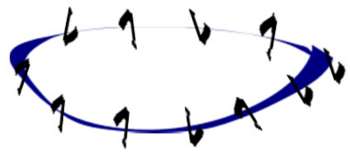
(Taken from [Kostov '12])

At 0-loop a huge number of operators are degenerate.
We need to use operators at 1-loop.
(degenerate perturbation theory)

In terms of spin-chain...

- Construct the wave function of the spin-chain.

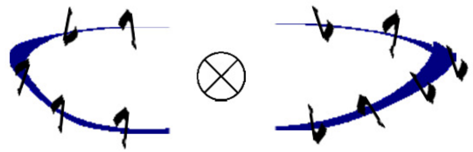
$$|\Psi_I\rangle$$



- Divide the spin chain into two parts.

$$|\Psi_I\rangle \to \sum_a |\Psi^{(l)}_{I,a}\rangle \otimes |\Psi^{(r)}_{I,a}\rangle$$

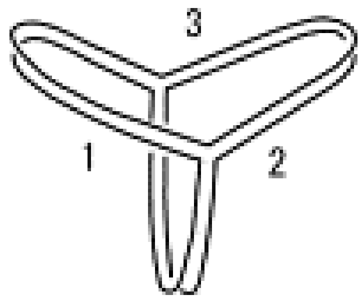
entangled state



- Flip the right-part (Ket to Bra).

$$\to \sum_a |\Psi^{(l)}_{I,a}\rangle\langle\Psi^{(r)}_{I,a}| =: \widehat{\Psi}_I$$

- Calculate the overlap by taking a trace.



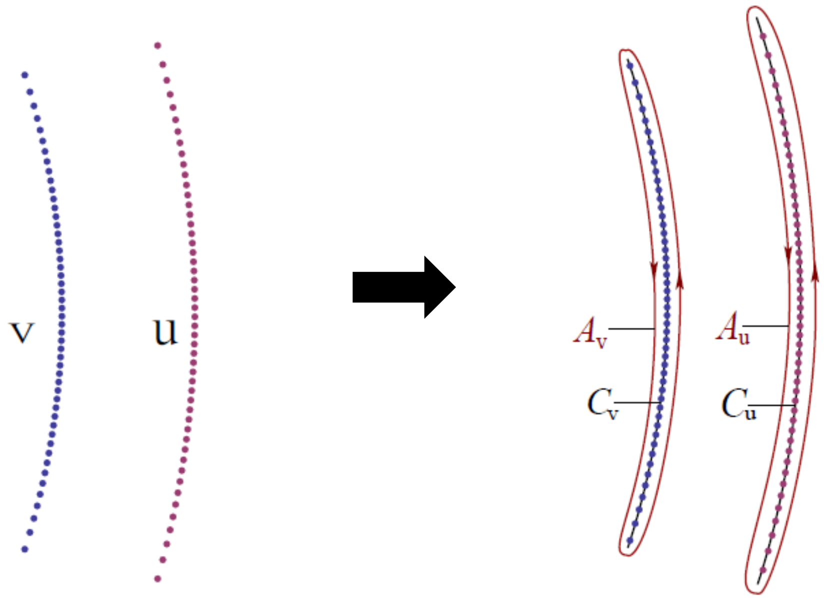
$$C_{IJK} \sim \mathrm{Tr}\left(\widehat{\Psi}_I\widehat{\Psi}_J\widehat{\Psi}_K\right)$$

Wave functions are important

Result for long operators

For 3 long operators…

$$\ln C_{123} \sim \oint \frac{du}{2\pi}\mathrm{Li}_2(e^{i(p_1(u)+p_2(u))}) - \oint \frac{du}{2\pi}\mathrm{Li}_2(e^{-ip_3(u)})$$
$$-\frac{1}{2}\oint \frac{du}{2\pi}\mathrm{Li}_2(e^{2ip_1(u)}) - \frac{1}{2}\oint \frac{du}{2\pi}\mathrm{Li}_2(e^{2ip_2(u)}) - \frac{1}{2}\oint \frac{du}{2\pi}\mathrm{Li}_2(e^{2ip_3(u)})$$



Contour integrals on a certain Riemann surface.

Three-point function from integrability

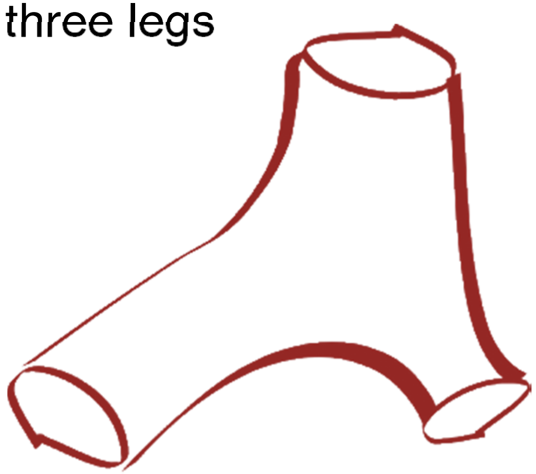
$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

$\lambda \ll 1$ $\longleftrightarrow$ $\lambda \gg 1$

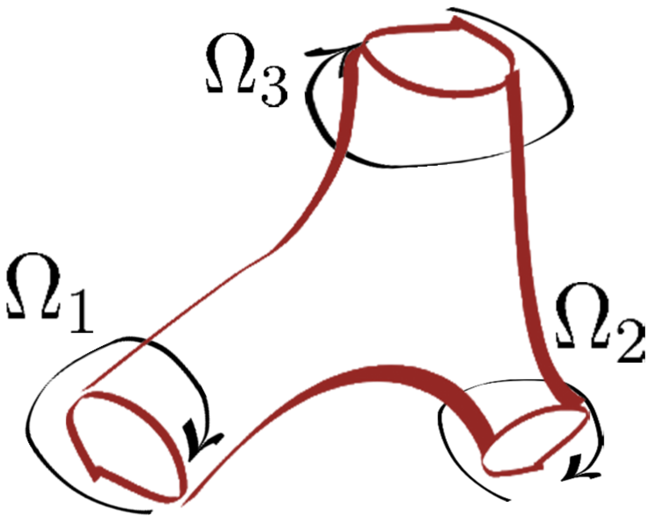
The overlap of three spin-chain wavefunctions $\Psi^{\text{spin-chain}}_{I,J,K}$

originally by [Okuyama-Tseng], [Roiban-Volovich], [Alday et al.]
recently by [Escobedo-Gromov-Sever-Vieira], [Kostov]
(one-loop)
[Gromov-Vieira], [Serban]
(two-loop)

Classical string solution with three legs



[Janik-Wereszczynski], [Kazama-SK]



Three point function from three-legged string

$AdS_5 \times S^5$

Definition

$$AdS_5 : \; X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = -1$$

symmetry: SO(4,2)

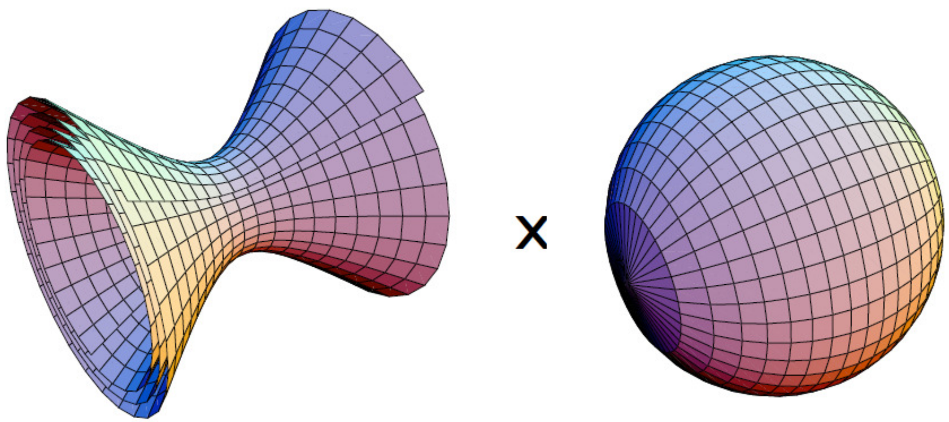
$$S^5 : \; Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = 1$$

symmetry: SO(6)

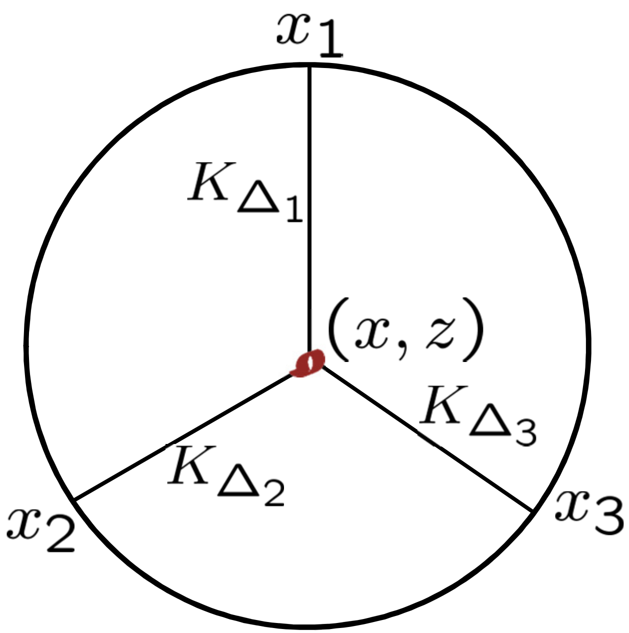
Poincare coordinate (x_0, x_1, x_2, x_3, z)

$$X_{-1} + X_4 = \frac{1}{z}, \quad X_i = \frac{x_i}{z} \quad (i = 0, 1, 2, 3)$$

$$z = 0 : \; \text{boundary of AdS}$$



GKP-Witten for SUGRA



$$K_{\Delta_I}(x_I; x, z) \sim \left(\frac{z}{(x - x_I)^2 + z^2} \right)^{\Delta_I}$$

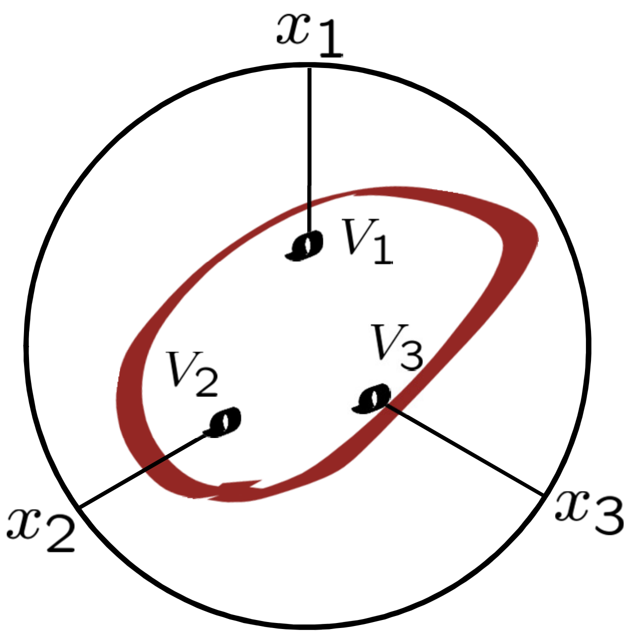
$$\times \, (\underline{\text{Factor}})$$

spin, R-charge, etc.

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle_{\text{gauge theory}}$$

$$= \int \frac{dz d^4 x}{z^5} K_{\Delta_1}(x_1; x, z) K_{\Delta_2}(x_2; x, z) K_{\Delta_3}(x_3; x, z)$$

GKP-Witten for strings



$$K_{\Delta_I}(x_I; x, z) \sim \left(\frac{z}{(x - x_I)^2 + z^2} \right)^{\Delta_I} \times (\text{Factor})$$

$$V_I[x_I; X(\sigma)] \sim \left(\frac{z(\sigma)}{(x(\sigma) - x_I)^2 + z(\sigma)^2} \right)^{\Delta_I} \times (\underline{\text{Factor}})$$

spin, R-charge, string oscillation.

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle_{\text{gauge theory}}$$
$$= \frac{1}{\text{Möbius}} \int \prod_i d^2 z_i \, \langle V_1\left[X^\mu(z_1)\right] V_2\left[X^\mu(z_2)\right] V_3\left[X^\mu(z_3)\right] \rangle_{\text{worldsheet}}$$

Strong coupling limit

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle_{\text{gauge theory}}$$

$$= \frac{1}{\text{Möbius}} \int \prod_i d^2 z_i \,\langle V_1\left[X^\mu(z_1)\right] V_2\left[X^\mu(z_2)\right] V_3\left[X^\mu(z_3)\right]\rangle_{\text{worldsheet}}$$

$$\langle V_1(z_1) V_2(z_2) \cdots \rangle = \int \mathcal{D}X\, V_1(z_1) V_2(z_2) \cdots e^{-S_{\text{string}}}$$

$$S_{\text{string}} = \sqrt{\lambda} \int d^2 z \partial X^\mu \bar{\partial} X_\mu$$

Strong coupling limit

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle_{\text{gauge theory}}$$

$$= \frac{1}{\text{Möbius}} \int \prod_i d^2 z_i \, \langle V_1\left[X^\mu(z_1)\right] V_2\left[X^\mu(z_2)\right] V_3\left[X^\mu(z_3)\right] \rangle_{\text{worldsheet}}$$

$$\langle V_1(z_1) V_2(z_2) \cdots \rangle = \int \mathcal{D}X \, V_1(z_1) V_2(z_2) \cdots e^{-S_{\text{string}}}$$

$$S_{\text{string}} = \sqrt{\lambda} \int d^2 z \partial X^\mu \bar{\partial} X_\mu$$

$\lambda \to \infty$

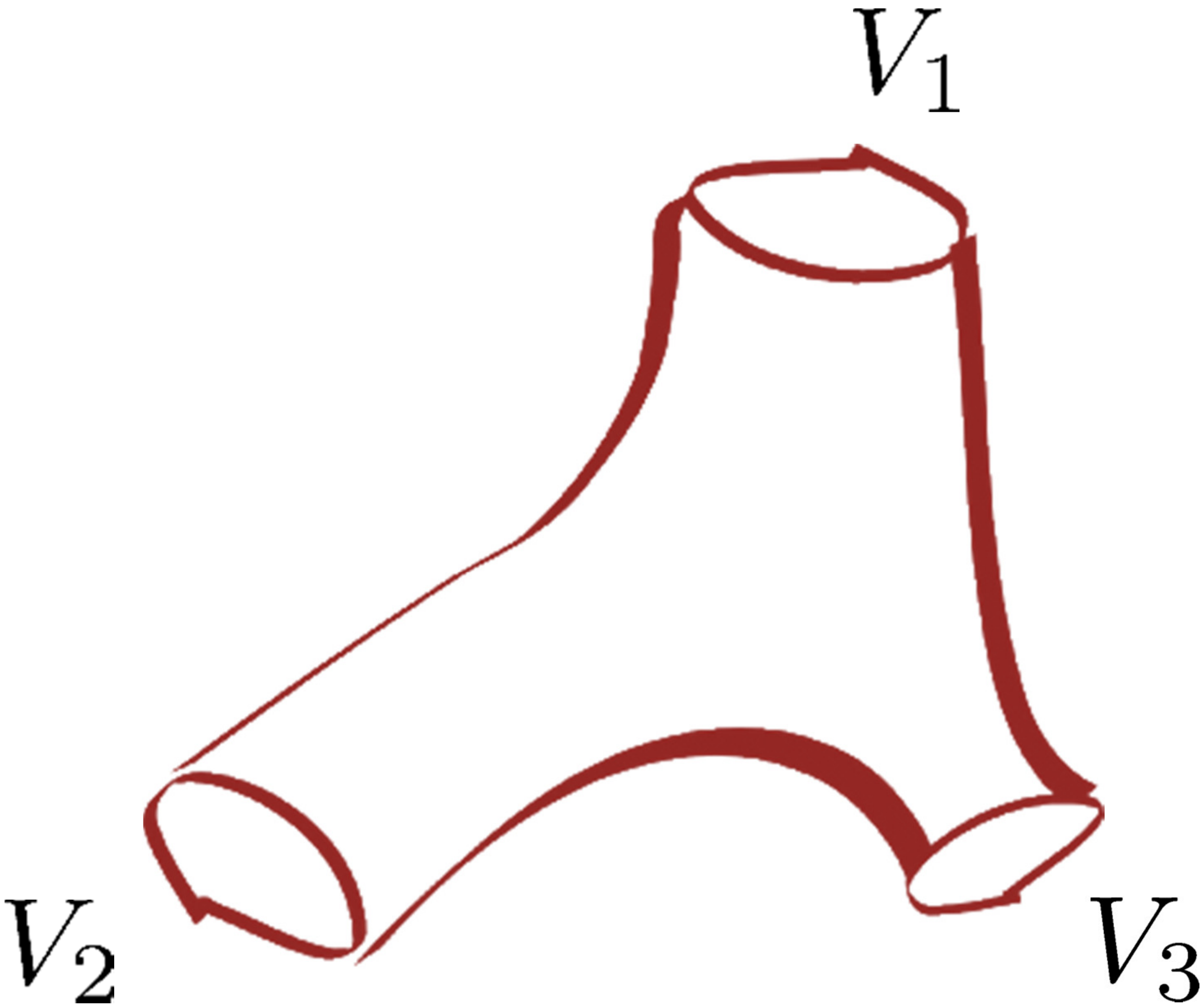
Dominated by a saddle point

$$\partial \bar{\partial} X_\mu + \cdots = -\frac{1}{\sqrt{\lambda}} \sum_i \frac{\delta}{\delta X^\mu} \ln V_i(z_i)$$

$$V_1[X_*(z_1)] V_2[X_*(z_2)] V_2[X_*(z_3)] e^{-S[X_*]}$$

$$X_* : \ \text{saddle point trajectory}$$

X_*



Saddle configuration

Two difficulties

$$V_1[X_*(z_1)]V_2[X_*(z_2)]V_2[X_*(z_3)]e^{-S[X_*]}$$

1. It is difficult to construct X_*

2. We do not know the exact form of V_I

Two difficulties

$$V_1[X_*(z_1)]V_2[X_*(z_2)]V_2[X_*(z_3)]e^{-S[X_*]}$$

1. It is difficult to construct X_*

Instead of trying to construct X_*, directly calculate $S[X_*]$ by integrability.

2. We do not know the exact form of V_I

Construct wave functions instead of V_I

Two difficulties

$$V_1[X_*(z_1)]V_2[X_*(z_2)]V_2[X_*(z_3)]e^{-S[X_*]}$$

1. It is difficult to construct X_*

Instead of trying to construct X_*, directly calculate $S[X_*]$ by integrability.

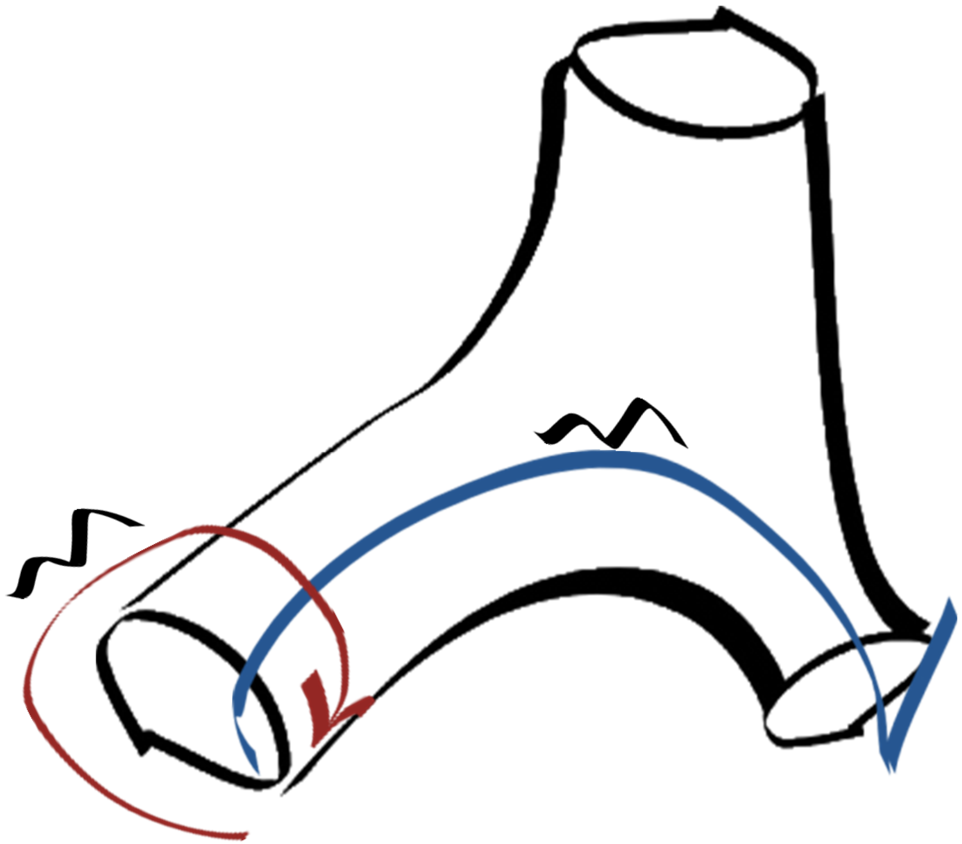
2. We do not know the exact form of V_I

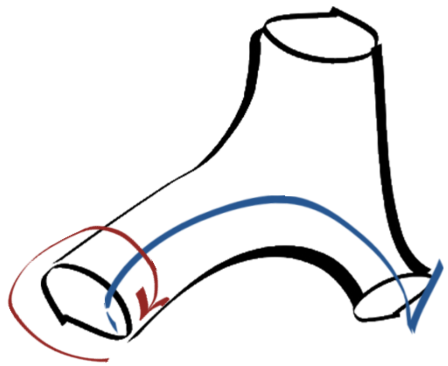
Construct wave functions instead of V_I

What should we do to know the property of something unknown?

(Experimental) Physicist's Approach

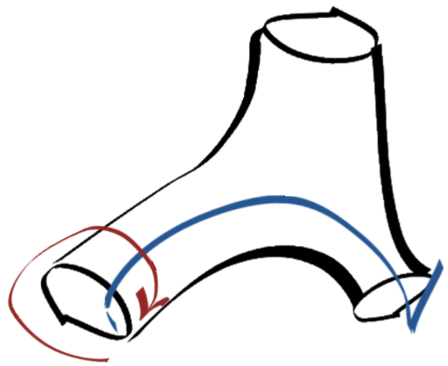
In this case, we can perform two kinds of such experiments.

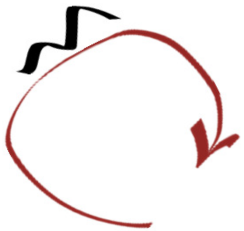




 :Local. Characterizes string states, V[X] .

 :Global. Encodes the information on the action, S[X*].





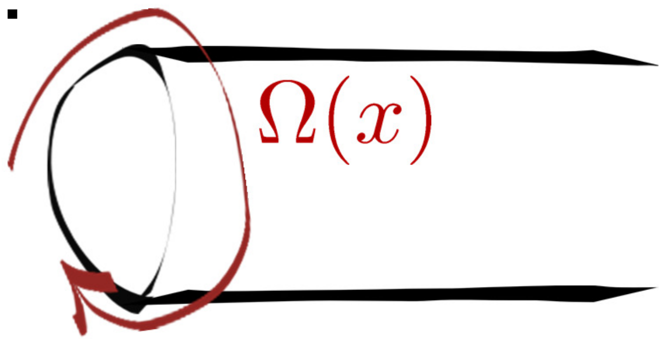
:Local. Characterizes string states, V[X] .

Input

:Global. Encodes the information on the action, S[X].

Output

 -experiment:



String e.o.m as “gauge fields”

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : \; X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

Eq. of motion: $\partial\bar{\partial}X^\mu + (\partial X^\nu \bar{\partial} X_\nu) X^\mu = 0$

$z = \tau + i\sigma$

Nonlinear. Difficult.

String e.o.m as “gauge fields”

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : \; X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

$$\texttt{Factor}[\partial\bar{\partial}X^{\mu} + (\partial X^{\nu}\bar{\partial}X_{\nu})X^{\mu} = 0]$$

Shift + Enter

String e.o.m as “gauge fields”

Consider,  $\subset \mathrm{AdS}_3$

$$\mathrm{AdS}_3 : X_{-1}^2 + X_0^2 - X_1^2 - X_4^2 = -1$$

$$\left[\partial + \frac{J_z}{1-x}, \bar{\partial} + \frac{J_{\bar{z}}}{1+x}\right] = 0$$

J : 2×2 matrix $\qquad x$: arbitrary parameter

$$J_z = g^{-1}\partial_z g, \quad g = \begin{pmatrix} X_{-1}+X_4 & X_0+X_1 \\ -X_0+X_1 & X_{-1}-X_4 \end{pmatrix}$$

“Gauge field”

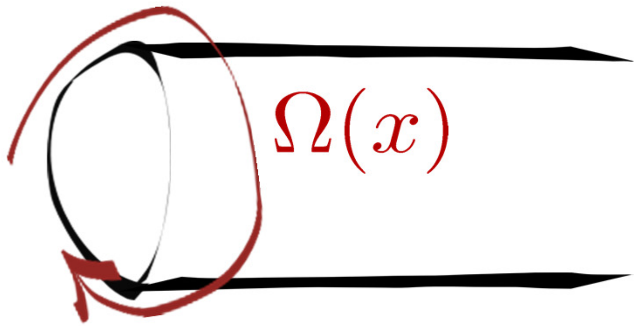
E.O.M $\rightarrow$ Field strength = 0

Auxiliary linear problem:

$$\left(\partial + \frac{J_z}{1-x}\right)\Psi(z;x) = 0 \qquad \left(\bar{\partial} + \frac{J_{\bar{z}}}{1+x}\right)\Psi(z;x) = 0$$

 -experiment:

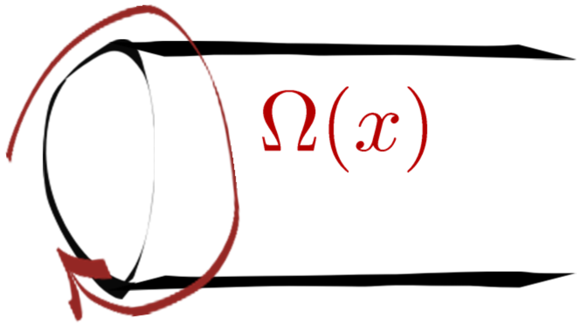
$$\Psi(e^{2\pi i}z;x) = \Omega(x)\Psi(z;x)$$



$$\Omega(x) := \mathcal{P}\exp\left(\int_{\sigma=0}^{\sigma=2\pi} \frac{J_z}{1-x}dz + \frac{J_{\bar{z}}}{1+x}d\bar{z}\right)$$

Monodromy matrix

Quasi-momentum:



$$\Psi(e^{2\pi i} z; x) = \Omega(x) \Psi(z; x)$$

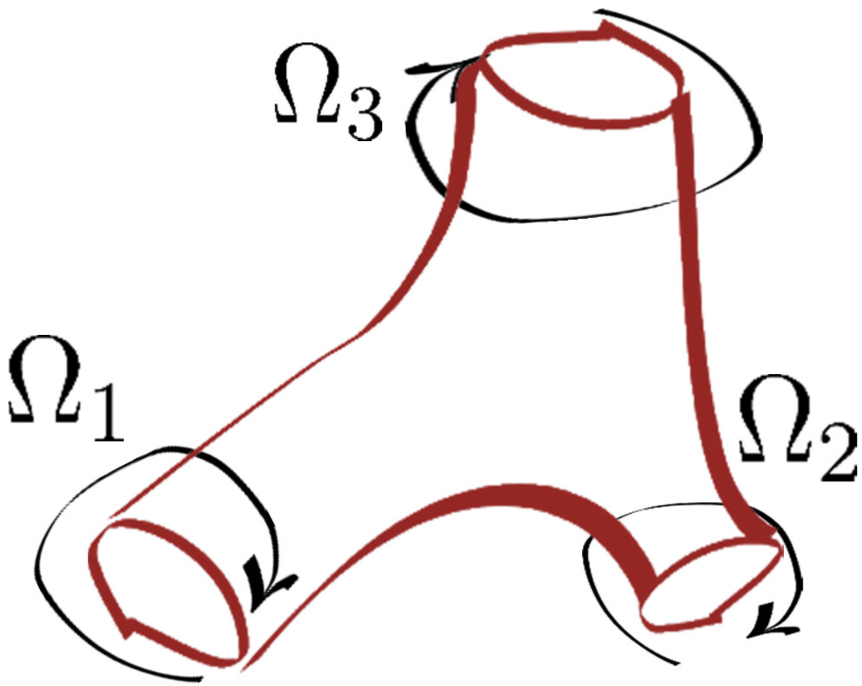
$$\Omega(x) \sim \begin{pmatrix} e^{ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}$$

$p(x)$: quasi-momentum

All the information on the string state (charge, oscillation mode) is encoded in p(x)

$$p(x) = \Delta + x Q_2 + \cdots$$

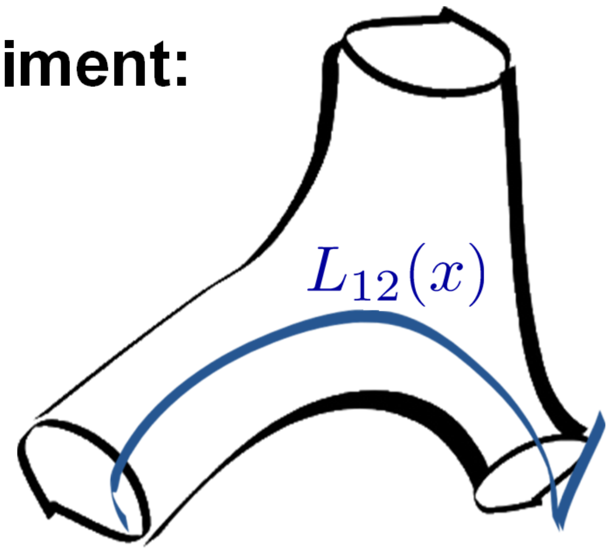
For three-point functions, we can perform such an experiment for each of the legs.



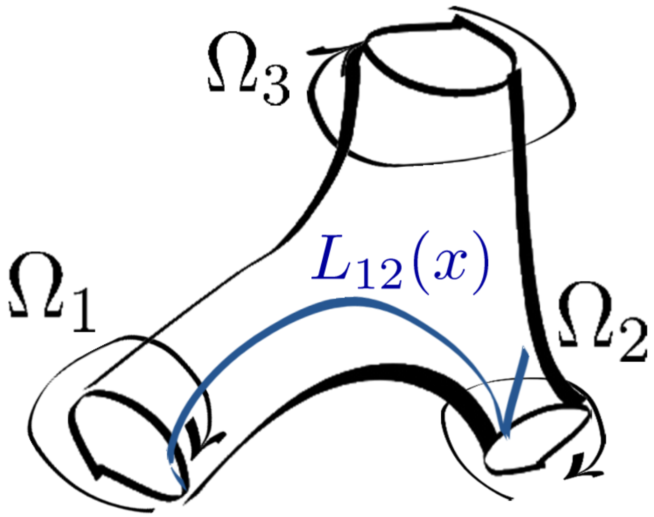
$$p_1(x), p_2(x), p_3(x)$$

Input for three point functions.

 -experiment:



Definition of L_{IJ}



$$\Omega_i \sim \begin{pmatrix} e^{ip_i(x)} & 0 \\ 0 & e^{-ip_i(x)} \end{pmatrix}$$

$$L_{12}(x) := \langle 1_+ \, , 2_+ \rangle$$

cf. $\check{L}_{12}(x) := \langle 2_- \, , 1_- \rangle$

$i_\pm$: eigenvectors of Ω_i

$$\Omega_i(x) i_\pm = e^{\pm i p_i} i_\pm$$

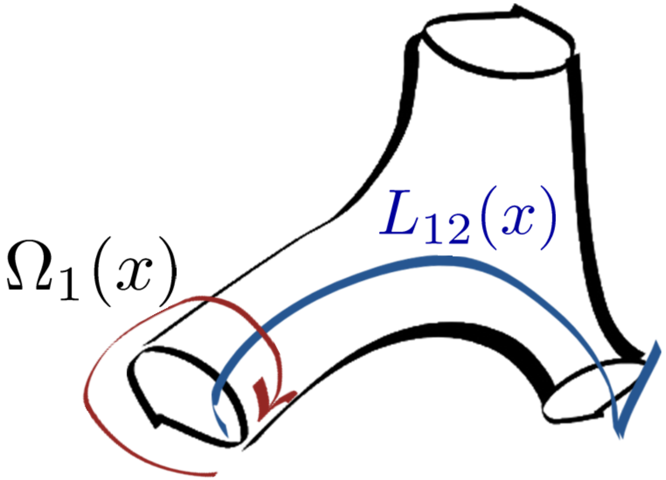
$$\langle \eta \, , \lambda \rangle := \epsilon_{\alpha\beta} \eta^\alpha \lambda^\beta$$

Skew symmetric product

 is what we need to know

It turns out that

[Detail]



$$S[X^*] \propto \text{Area} \sim Q_1^{(1)} \tilde{Q}_{12}^{(2)} + \cdots$$

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi \tilde{Q}_{12}^{(2)} + \cdots\right]$$

$$\operatorname{tr} \Omega_1(x) = \exp\left[\frac{Q_1^{(1)}}{\xi} + \cdots\right]$$

$\tilde{Q}_{IJ}$: "charges" along an open path

$$\xi = \sqrt{\frac{1-x}{1+x}}$$

Input

$$p_1(x), p_2(x), p_3(x)$$

Output

$$L_{IJ}(x) := \langle I_+ \, , J_+ \rangle$$

$$\rightarrow S[X^*]$$

- Take the basis with which Ω_1 is diagonal. $\Omega_1 = \begin{pmatrix} e^{ip_1} & 0 \\ 0 & e^{-ip_1} \end{pmatrix}$
- Ω_2 is constrained by the following condition.

$$\Omega_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \boxed{\begin{array}{l} a+d = e^{ip_2} + e^{-ip_2} \\ ad - bc = 1 \end{array}}$$

- From the triviality of the monodromy at infinity,

$$\Omega_1\Omega_2\Omega_3 = 1, \qquad \Omega_3 = \Omega_2^{-1}\Omega_1^{-1} = \begin{pmatrix} e^{-ip_1}d & -e^{ip_1}b \\ -e^{-ip_1}c & e^{ip_1}a \end{pmatrix}$$

$$\boxed{e^{-ip_1}d + e^{ip_1}a = e^{ip_3} + e^{-ip_3}}$$

- a and d can be determined from above two equations.

$$a = -i\frac{\cos p_3 - e^{-ip_1}\cos p_2}{\sin p_1} \qquad d = -i\frac{e^{ip_1}\cos p_2 - \cos p_3}{\sin p_1}$$

- Only the product, bc, can be determined. Individual value depends on the normalization of the basis.

$$U\Omega_i U^{-1} \qquad U = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

- Certain combinations of Wilson lines are free from such ambiguity.

$$L_{12}(x)\check{L}_{12}(x) = \frac{\sin\frac{p_1+p_2+p_3}{2}\sin\frac{p_1+p_2-p_3}{2}}{\sin p_1(x)\sin p_2(x)}$$

$$L_{12}(x) := \langle 1_+ \,, 2_+ \rangle \qquad \check{L}_{12}(x) := \langle 2_- \,, 1_- \rangle$$

- To separate out the individual term,

Consider analytic properties w.r.t. x.

- Let me first explain why we can separate out the individual term if we know the analytic properties.

e.g. Large spin twist-2 operators

$$\begin{aligned} L_{12}(x) &: \text{ regular on the upper half plane of } \xi \\ \check{L}_{12}(x) &: \text{ regular on the lower half plane of } \xi \end{aligned}$$

$$\xi = \sqrt{\frac{1-x}{1+x}}$$

*Details of the analytic properties differ depending on string states we consider.

Wiener-Hopf decomposition

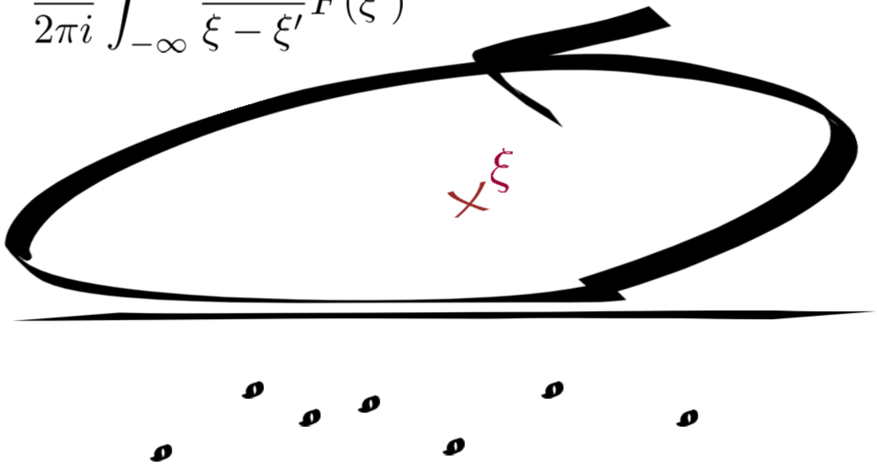
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\xi' \frac{1}{\xi'-\xi}\left(F(\xi')+G(\xi')\right) = \begin{cases} F(\xi)\,, & (\mathrm{Im}\,\xi > 0) \\ -G(\xi)\,, & (\mathrm{Im}\,\xi < 0) \end{cases}$$

$F:$ regular on $\mathrm{Im}\ \xi > 0$ $\qquad$ $G:$ regular on $\mathrm{Im}\ \xi < 0$

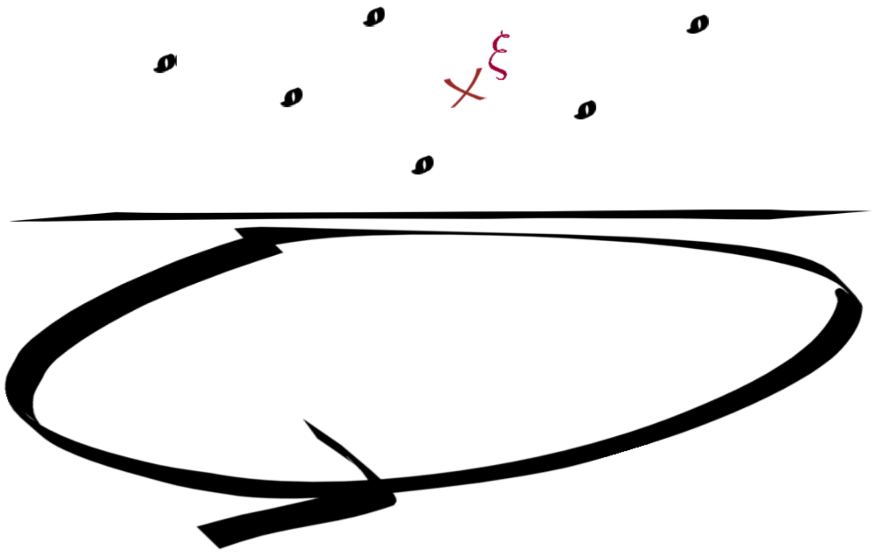
Apply

$$\log L_{12} + \log \check{L}_{12} = \log \frac{\sin\frac{p_1+p_2+p_3}{2}\,\sin\frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi-\xi'}F(\xi')$$



$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi-\xi'}G(\xi')$$



Now, let me explain how to determine the analytic properties of L_{IJ}.

Disclaimer:
The following discussion will be the most technical part in my talk.

Analytic properties from WKB-analysis

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi \tilde{Q}_{12}^{(2)} + \cdots\right]$$

- Expansion of L(x) can be regarded as WKB-expansion with respect to ξ.

Analytic properties from WKB-analysis

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi\tilde{Q}_{12}^{(2)} + \cdots\right]$$

- Expansion of L(x) can be regarded as WKB-expansion with respect to ξ.
- In general, it also has a series of nonperturbative terms.

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi\tilde{Q}_{12}^{(2)} + \cdots + \underline{\sum_n c_n \exp[-n\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \cdots]}\right]$$

- Owing to these "instanton corrections", L(x) exhibits a rich analytic structure (poles and zeros).

Analytic properties from WKB-analysis

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi\tilde{Q}_{12}^{(2)} + \cdots\right]$$

- Expansion of L(x) can be regarded as WKB-expansion with respect to ξ.
- In general, it also has a series of nonperturbative terms.

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi\tilde{Q}_{12}^{(2)} + \cdots + \underline{\sum_n c_n \exp[-n\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \cdots]}\right]$$

- Owing to these "instanton corrections", L(x) exhibits a rich analytic structure (poles and zeros).
- Therefore, to determine the analytic property of L(x), we need to know when it suffers from the instanton corrections and when it doesn't.

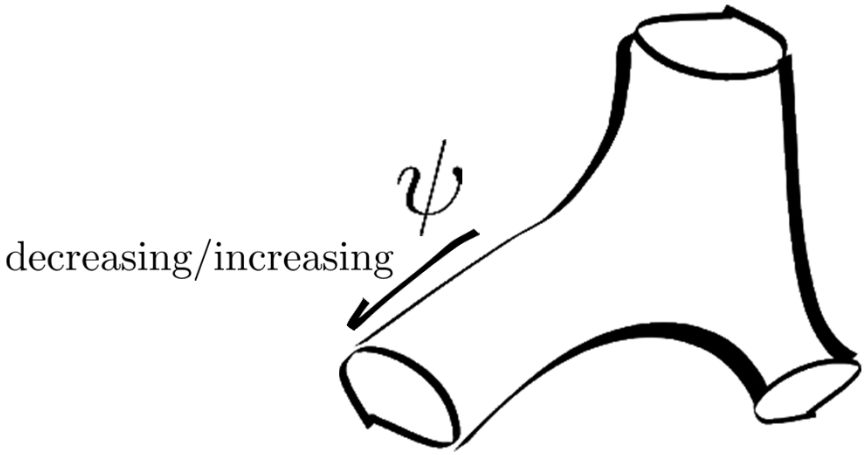
- Basically, such instanton corrections arise due to a mixing of solutions of the auxiliary linear problem.

$$\psi = A\, i_- + B\, i_+ \qquad\qquad i_+ \sim \exp\,[-S] \quad i_- \sim \exp\,[+S]$$

$$\psi \sim A \exp\,[S + \log[1 + \exp(-S)] + \cdots]$$

- A solution which exponentially decreases around a singularity is free from such instanton corrections since it cannot mix with an exponentially increasing solution.

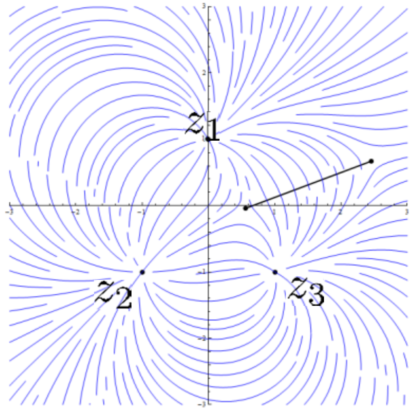
$$(\text{large}) + (\text{small}) \sim (\text{large})$$



To be more precise…

i) Draw “WKB-curves” defined by

$$\mathrm{Im}\, dS = 0\,, \quad \psi = \exp\left(\int dS\right)$$



On these lines, the phase of the solution is fixed and “exponentially increasing/decreasing” has a clear meaning.

ii) L(x) is regular (no poles/zeros)
if two solutions are exponentially decreasing ones and two poles are connected by WKB-curves.

$$L_{12}(x) := \langle 1_+ \,, 2_+ \rangle$$

decreasing around z_1 decreasing around z_2

$$z_1 \xleftrightarrow[\text{WKB-curves}]{} z_2$$

Result for three spinning strings (GKP string)

Determination of the analytic properties is generally complicated.
For certain simple operators (GKP strings), it becomes easier.

$$
\begin{aligned}
S_{\rm reg} \quad &= -\frac{\pi}{12} + \pi \left[-\kappa_1 K(\kappa_1) - \kappa_2 K(\kappa_2) - \kappa_3 K(\kappa_3) \right. \\
&+ \frac{\kappa_1+\kappa_2+\kappa_3}{2} K(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) \\
&+ \left|\frac{-\kappa_1+\kappa_2+\kappa_3}{2}\right| K(\left|\frac{-\kappa_1+\kappa_2+\kappa_3}{2}\right|) \\
&+ \left|\frac{\kappa_1-\kappa_2+\kappa_3}{2}\right| K(\left|\frac{\kappa_1-\kappa_2+\kappa_3}{2}\right|) \\
&+ \left. \left|\frac{\kappa_1+\kappa_2-\kappa_3}{2}\right| K(\left|\frac{\kappa_1+\kappa_2-\kappa_3}{2}\right|) \right] \ .
\end{aligned}
$$

$$
K(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\theta \, e^{-\theta} \log\left(1 - e^{-4\pi x \cosh\theta}\right)
$$

Final Result

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K}|x_{23}|^{\Delta_J+\Delta_K-\Delta_I}|x_{31}|^{\Delta_K+\Delta_I-\Delta_J}}$$

Expected spacetime dependence is reproduced

$$\begin{aligned}
\log C_{LSGKP} = &-\frac{17\sqrt{\lambda}}{12} - \sqrt{\lambda}\Big[\kappa_1 L(\kappa_1) + \kappa_2 L(\kappa_2) + \kappa_3 L(\kappa_3) \\
&- \frac{\kappa_1+\kappa_2+\kappa_3}{2} L(\frac{\kappa_1+\kappa_2+\kappa_3}{2}) \\
&- \frac{-\kappa_1+\kappa_2+\kappa_3}{2} L(\frac{-\kappa_1+\kappa_2+\kappa_3}{2}) \\
&- \frac{\kappa_1-\kappa_2+\kappa_3}{2} L(\frac{\kappa_1-\kappa_2+\kappa_3}{2}) \\
&- \frac{\kappa_1+\kappa_2-\kappa_3}{2} L(\frac{\kappa_1+\kappa_2-\kappa_3}{2})\Big] \\
&- \Big[\ell_1^- \log\sinh 2\pi\kappa_1 + \ell_2^- \log\sinh 2\pi\kappa_2 + \ell_3^- \log\sinh 2\pi\kappa_3 - (\ell_1^- + \ell_2^- + \ell_3^-)\log A \\
&- \frac{\ell_1^- + \ell_2^- - \ell_3^-}{2}\log\sinh\left(\pi(\kappa_1+\kappa_2-\kappa_3)\right) - \frac{\ell_1^- - \ell_2^- + \ell_3^-}{2}\log\sinh\left(\pi(\kappa_1-\kappa_2+\kappa_3)\right) \\
&- \frac{-\ell_1^- + \ell_2^- + \ell_3^-}{2}\log\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right)\Big] \qquad (2.35)
\end{aligned}$$

$$L(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh^2\theta}{\cosh\theta}\log\left(1-e^{-4\pi x\cosh\theta}\right)$$

$$\ell_i^- = \frac{\sqrt{\lambda}}{2\pi}\sinh\pi\kappa_i$$

$$\begin{aligned}
A = &\sqrt{\sinh\left(\pi(\kappa_1+\kappa_2+\kappa_3)\right)} \\
&- \sqrt{\sinh\left(\pi(-\kappa_1+\kappa_2+\kappa_3)\right)\sinh\left(\pi(\kappa_1-\kappa_2+\kappa_3)\right)\sinh\left(\pi(\kappa_1+\kappa_2-\kappa_3)\right)}
\end{aligned}$$

$$\Delta_i - S_i = \sqrt{\lambda}\kappa_i$$

$$\Delta_i + S_i = \frac{\sqrt{\lambda}}{\pi}\sinh\pi\kappa_i$$

Summary and Prospect

Summary

- Discussed the method to calculate 3-point functions using a classical string.
- Integrability was useful in the calculation.

Prospect

- Currently, the result is available only for specific operators. Generalization to other operators (BMN-like operators) is in progress.
- Comparison with calculations from gauge theory.
- Similar structure in the gauge theory calculation?

  Recent development on the gauge theory side: [Gromov-Vieira 12]
- Four point functions. Crossing symmetry, Bootstrap in higher dim CFT?

  [Caetano-Toledo 12]

  Recent attempt to solve 3d Ising model by boostrap: [El-Showk et al.]
- Generalization to other theories (ABJM etc.).

  Liouville correlation functions from integrability [SK, Honda in progress]
- Understand the structure of wave functions and AdS/CFT.

Thank you for listening

Action $S[X_*]$ to contour integrals

$$S[X_*] \sim \int d^2z \, \mathrm{tr}\,(J_z J_{\bar{z}}) \qquad J_z = g^{-1}\partial_z g,$$

Virasoro condition:

$$\mathrm{tr}\,(J_z J_z) = T(z) \qquad T(z) + T_{S^5}(z) = 0$$

$$T(z): \ \text{AdS-part of the stress energy tensor}$$

Diagonalize J_z

$$U^{-1} J_z U = \begin{pmatrix} \sqrt{T} & 0 \\ 0 & -\sqrt{T} \end{pmatrix} \qquad U^{-1} J_{\bar{z}} U = \begin{pmatrix} u & * \\ * & -u \end{pmatrix}$$

Introduce a closed one-form ω

$$\omega \equiv u d\bar{z} + v dz \qquad d\omega = 0$$

$$\int d^2z\,\mathrm{tr}\,(J_z J_{\bar z}) = 2\int d^2z\,\sqrt{T}u = i\int \sqrt{T}dz\wedge\omega$$

Consider a double cover of the worldsheet; $y^2 = T(z)$.

$$= \frac{i}{2}\int_D \sqrt{T}dz\wedge\omega = \frac{i}{2}\int_{\partial D}\Lambda(z)\omega$$

Stokes theorem

$$\Lambda(z) \equiv \int \sqrt{T}dz$$

To apply Stokes theorem, we need to choose ∂D so that Λ(z) is single-valued on D.

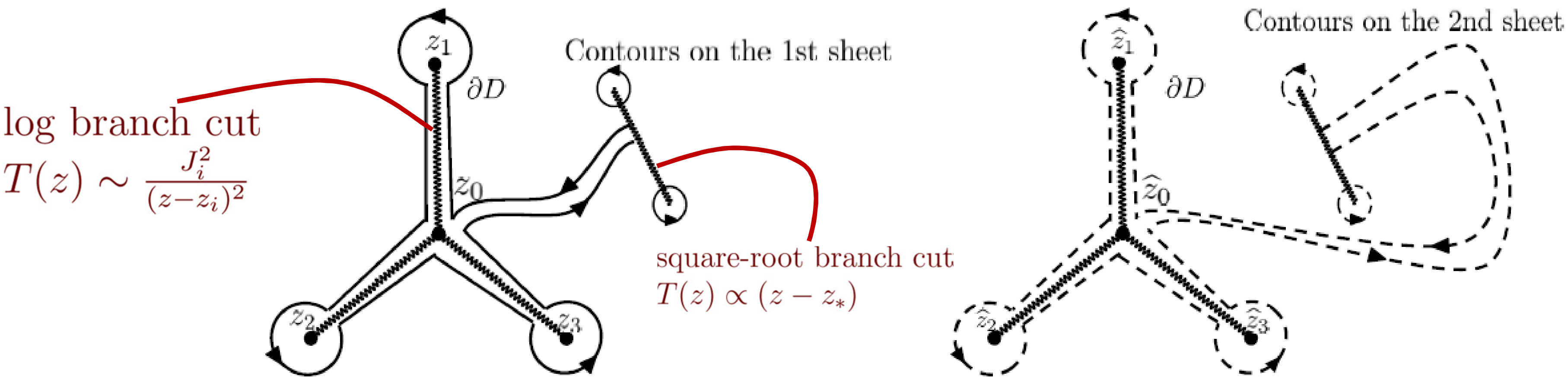
D and ∂D

They are determined so that Λ(z) is single-valued on D.

$$\Lambda(z) \equiv \int^{z} \sqrt{T(z')}dz'$$

$$T(z) = \left(\frac{J_1^2 z_{23}}{(z-z_1)} + \frac{J_2^2 z_{31}}{(z-z_2)} + \frac{J_3^2 z_{12}}{(z-z_3)} \right) \frac{1}{(z-z_1)(z-z_2)(z-z_3)}$$

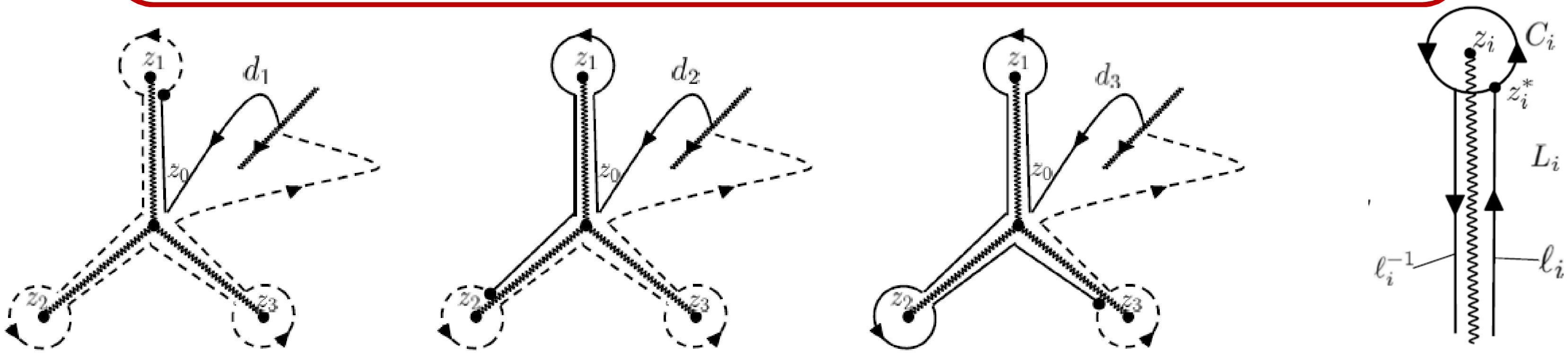
For three point functions,
AdS-part of the stress energy tensor is determined by S^5-charges, J_i.



Extended Riemann-bilinear identity

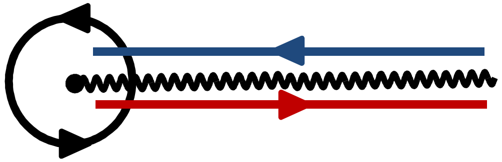
The contour integral can be further simplified to products of integrals.

$$\boxed{S \propto \frac{i}{2}\int_{\partial D} \Lambda\omega = \frac{i}{2}\sum_{j=1}^{3}\int_{C_j}\sqrt{T}dz\int_{d_j}\omega + \cdots}$$



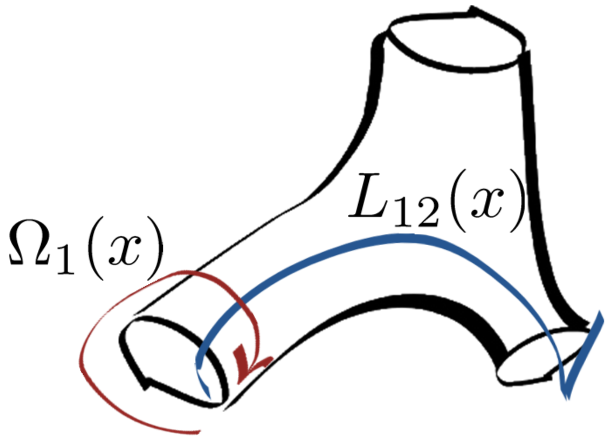
Idea of derivation

$$\int \Lambda(z)\omega = \int \delta\Lambda(z)\omega = \int \sqrt{p}dz \int \omega$$



Interestingly, the contour integrals we need appear in the expansion of L_{IJ}.

$$\boxed{S \propto \frac{i}{2}\int_{\partial D} \Lambda\omega = \frac{i}{2}\sum_{j=1}^{3}\int_{C_j}\sqrt{T}dz\int_{d_j}\omega + \cdots}$$



$$L_{12}(x) := \langle 1_+\,, 2_+\rangle$$

$$S \propto \mathrm{Area} \sim Q_1^{(1)}\tilde{Q}_{12}^{(2)} + \cdots$$

$$\Omega_1(x) = \exp\left[\frac{Q_1^{(1)}}{\xi} + \cdots\right] \qquad \xi = \sqrt{\frac{1-x}{1+x}}$$

$$L_{12}(x) = \exp\left[\frac{\tilde{Q}_{12}^{(1)}}{\xi} + \xi\tilde{Q}_{12}^{(2)} + \cdots\right]$$

$\tilde{Q}_{IJ}$: "charges" along an open path

[End]

Back up slides

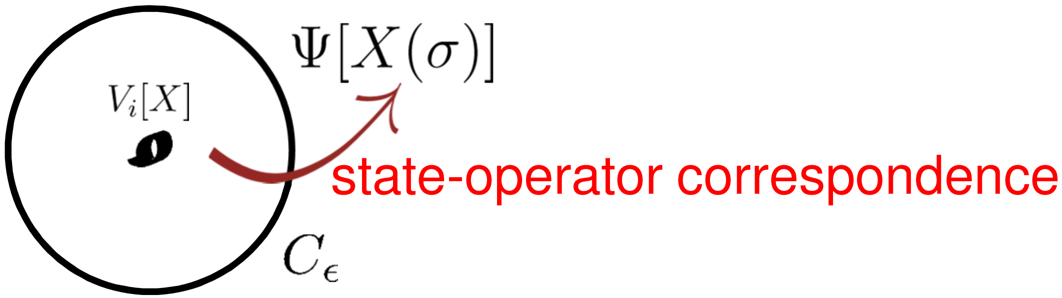
Calculation of the vertex operator part

$$V_{\mathrm{finite}}$$

Regularizing divergences

$$2\int d^2z\sqrt{p\bar{p}} \qquad\qquad V_I[X_*]$$

- To see the cancellation of divergences, we cut out a small circle of radius ε_i and condier wavefunctions instead of vertex operators.



- Wavefunctions in the classical limit are given by the solution of the **Hamilton-Jacobi eq.**

$$\Psi[X(\sigma)] \sim e^{iW[X(\sigma)]} \qquad W[X(\sigma)] : \text{characteristic function}$$

$$\boxed{cf. \quad -\frac{\hbar}{2m}\partial^2\psi + V\psi = E\psi \Rightarrow (\partial W)^2 = 2m(E-V) \qquad\qquad \psi \sim e^{\frac{i}{\hbar}W}}$$

- However, it is quite hard to directly solve the H-J eq.

Action-Angle variables

- If we canonical-transform to action-angle variables,

$$S_i \ : \text{constant} \qquad\qquad \theta_i \ : \ \text{linear time evolution}$$

$$\Psi[X] = \langle X|\Psi\rangle = \int d\theta_i \langle X|\theta_i\rangle\langle\theta_i|\Psi\rangle = \int d\theta_i \langle X|\theta_i\rangle \Psi'(\theta_i)$$

it is easy to construct wavefunctions.

$$\Psi'(\theta_i) = e^{i\sum_i S_i\theta_i}$$

- Thanks to integrability, we can construct action-angle variables using the "Sklyanin's magic recipe".

Brief sketch of magic recipe (1/2)

- Express the coordinate of AdS as follows.

$$g = \begin{pmatrix} X_{-1} + X_4 & X_1 + iX_2 \\ X_1 - iX_2 & X_{-1} - X_4 \end{pmatrix}$$

- The following connection is flat because of the e.o.m.

$$\left[\partial + \frac{1}{1-x} g^{-1}\partial g \,, \bar{\partial} + \frac{1}{1+x} g^{-1}\bar{\partial} g\right] = 0$$

* This connection is related to the previous one by a gauge transformation.

- Consider the normalized solution of the auxiliary linear problems.

$$\left(\partial + \frac{1}{1-x} g^{-1}\partial g\right)\psi(\sigma, \tau; x) = 0$$

$$\left(\bar{\partial} + \frac{1}{1+x} g^{-1}\bar{\partial} g\right)\psi(\sigma, \tau; x) = 0$$

$\vec{n} \cdot \vec{\psi} = 1$ $\qquad$ $\vec{n}$: arbitrary constant vector

Brief sketch of magic recipe (2/2)

- The angle variables can be constructed from the poles of the normalized solution.

$$\psi(0,\tau;x_i)=\infty \xrightarrow{\text{Abel map on the spectral curve}} \theta_i = F_i(x_j)$$

- The action variables can also be constructed.

$$S_i: \ \text{filling fraction} \qquad \{\theta_i, S_j\} = \delta_{ij}$$

- The remaining task is to determine $\vec{n}$.

  It is determined by requiring that the wavefunctions constructed by this recipe have the transformation property as the corresponding gauge theory operators.

Normalization and symmetry (1/2)

For instance, consider a vertex operator which corresponds to a gauge theory operator inserted at the origin.

$$\Psi'(\theta_i) \longleftrightarrow \mathcal{O}(0)$$

The gauge theory operator is

1. Invariant under the special conf. $\mathcal{O}(0) \to \mathcal{O}(0)$
2. Covariant under the translation. $\mathcal{O}(0) \to \mathcal{O}(x)$

Under these transformations, g and the solution transforms as

1. $\begin{pmatrix} 1 & \bar{\epsilon} \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -\epsilon & 1 \end{pmatrix} \psi$

2. $\begin{pmatrix} 1 & 0 \\ \bar{\epsilon} & 1 \end{pmatrix} g \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -\epsilon \\ 0 & 1 \end{pmatrix} \psi$

Normalization and symmetry (2/2)

Normalization condition which is invariant under the special conf. and covariant under the translation is

$$\vec{n} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

1. $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\epsilon & 1 \end{pmatrix} \psi = \begin{pmatrix} 1 & 0 \end{pmatrix} \psi$
2. $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\epsilon \\ 0 & 1 \end{pmatrix} \psi \neq \begin{pmatrix} 1 & 0 \end{pmatrix} \psi$

The wavefunction can be constructed uniquely by the above precedures.

$$\Psi'_I(\theta_i)$$

From the asymptotic behavior of the solution around the vertex operators, one can evaluate angle-variables.

$$\theta_i^*$$

Using the wavefunction constructed from the magic recipe, one can evaluate the contribution from vertex operators.

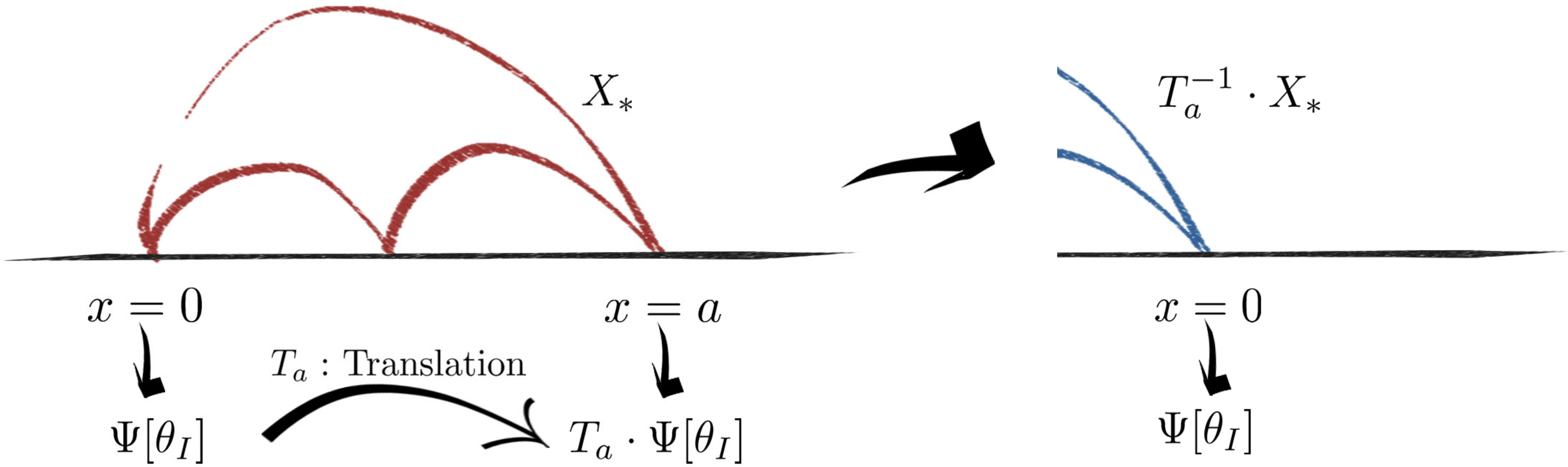
$$\Psi'_I(\theta_i^*)$$

The divergences cancel nicely, and the expected spacetime dependence can be reproduced.

$$C_{LSGKP} \times \frac{1}{\left(z^{(1)} - z^{(2)}\right)^{h_1+h_2-h_3}\left(z^{(2)} - z^{(3)}\right)^{h_2+h_3-h_1}\left(z^{(3)} - z^{(1)}\right)^{h_3+h_1-h_2}}$$
$$\times \frac{1}{\left(\bar{z}^{(1)} - \bar{z}^{(2)}\right)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\left(\bar{z}^{(2)} - \bar{z}^{(3)}\right)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\left(\bar{z}^{(3)} - \bar{z}^{(1)}\right)^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

$$z^{(i)} \equiv x_1^{(i)} + ix_2^{(i)}, \qquad \bar{z}^{(i)} \equiv x_1^{(i)} - ix_2^{(i)}, \qquad h_i = \frac{\Delta_i + S_i}{2}, \qquad \bar{h}_i = \frac{\Delta_i - S_i}{2}$$

Evaluation of vertex operators



- θ_I transforms in quite a complicated way under translation.

  – Instead of evaluating a transformed wavefunction on the original trajectory, we evaluate the original wavefunction on an inversely-transformed trajectory.

$$T_a \cdot \Psi[\theta_I]\Big|_{\text{on } X_*} = \Psi[\theta_I]\Big|_{\text{on } T_a^{-1} X_*}$$