Wall-Crossing & Quiver Invariants

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with
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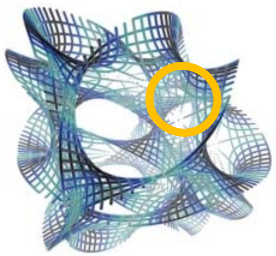
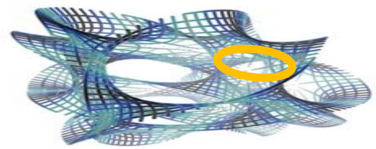
wall-crossing of BPS states with 4 or less supersymmetries

marginal stability wall

$$\gamma_1 \qquad \gamma_1 + \gamma_2$$
$$\gamma_2 \qquad \gamma_2 + \gamma_3$$
$$\gamma_3 \qquad \gamma_1 + \gamma_3$$
$$\vdots \qquad \vdots$$
$$\gamma_1 + \gamma_2 + \gamma_3$$
$$\vdots$$
$$\sum k_i \gamma_i$$
$$\vdots$$

$$\gamma_1$$
$$\gamma_2$$
$$\gamma_3$$
$$\vdots$$





Kontsevich-Soibelman, 2008

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha,\beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta} \qquad K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2} \right)$$

marginal stability wall

+ side

$$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_{\gamma'} K_{\gamma'}^{\Omega^-(\gamma')}$$

– side

with the 2nd helicity trace, for BPS states in D=4 N=2 SUSY

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2$$

$$\rightarrow (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

or, more generally, the protected spin character

$$SO(4) = SU(2)_{\rm rotation} \times SU(2)_{\rm R-symmetry}$$

$$J \qquad\qquad\qquad I$$

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

2nd helicity trace

protected spin character

Gaiotto, Moore, Neitzke 2010

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]

x [angular momentum l multiplet]

prototype : D=4 N=2 SU(2) → U(1)

$$W = (0, \pm 2) \qquad \Omega(W) = -2$$

$$D_n = (\mp 1, \pm 2n) \qquad \Omega(D_n) = 1$$

$$D_0 = (\pm 1, 0)$$

$$D_1 = (\mp 1, \pm 2)$$

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta}$$

$$\cancel{K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2}\right)}$$

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

marginal stability wall

+ side $\qquad \displaystyle\prod_\gamma e^{\bar{\Omega}^+(\gamma) V_\gamma} = \prod_{\gamma'} e^{\bar{\Omega}^-(\gamma') V_{\gamma'}} \qquad$ - side true, in general ?

how to see from physical BPS state building/counting ?

& why rational invariants ?

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

input data ?

$$\Omega^+(\gamma) = \Omega^-(\gamma)$$

wall-crossing

BPS quivers

quiver invariants

wall-crossing examples

calibrated geometry : special Lagrange submanifolds in CY3

type II string theories on CY3 : wrapped D-branes

D=4 N=2 supergravity : BPS black holes

D=4 N=2 Seiberg-Witten theory : BPS particles

⋮

D=4 N=4 …… ¼ BPS ……

⋮

D=2 N=2 Landau-Ginzburg : BPS kinks

1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center classical dyon solitons

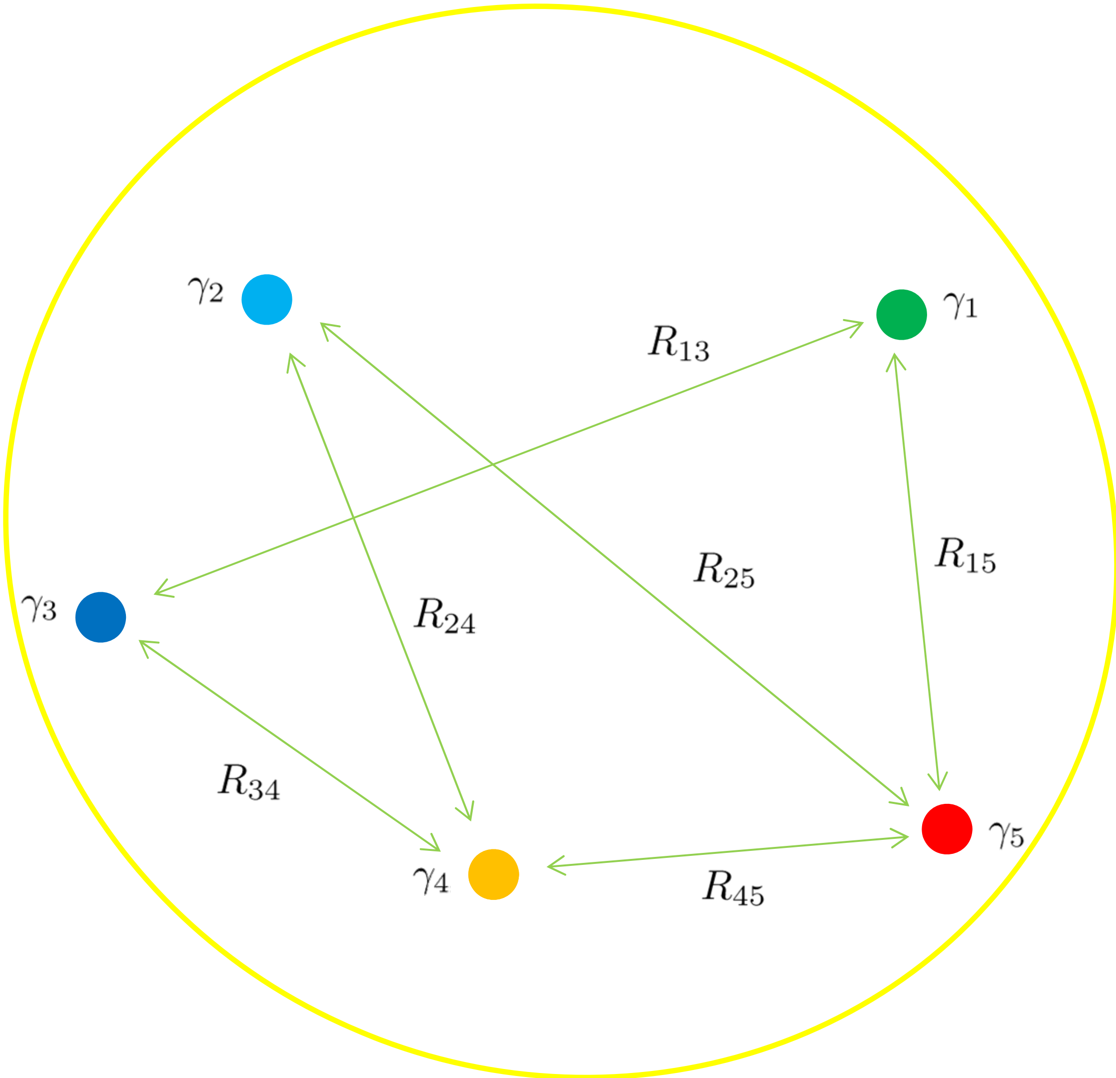
1999 Bak + Lee + Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.

N=2 SU(n) BPS states via multi-center monopole dynamics

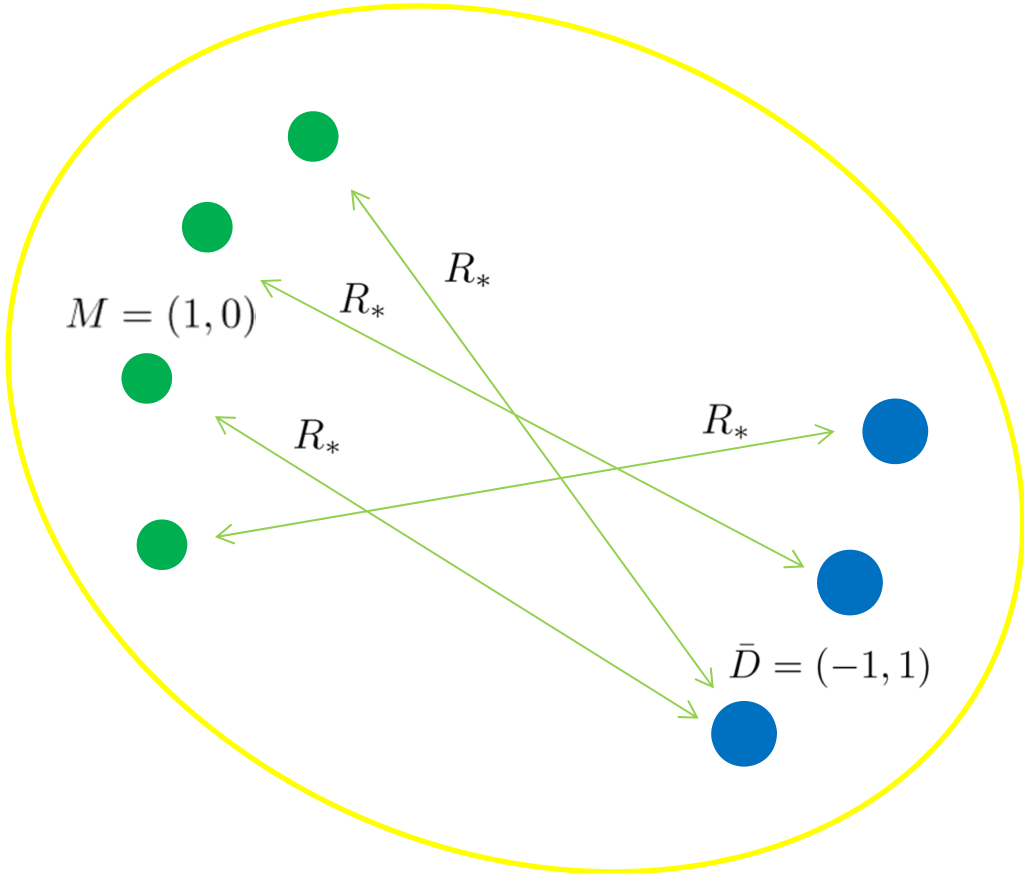
generic BPS "particles" are loose bound states of charge centers



$$R^3 = \{\vec{X}\}$$

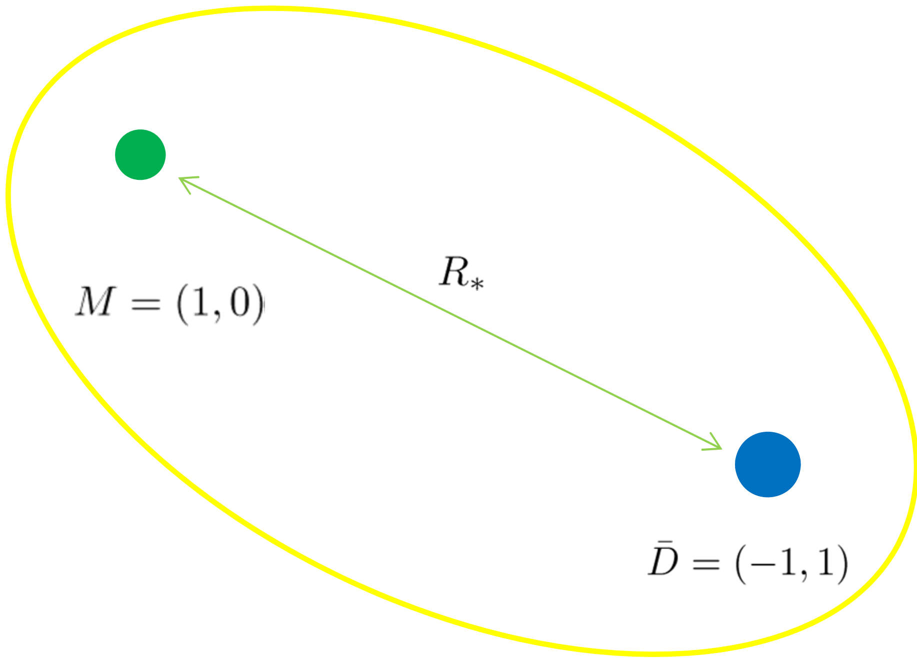
in particular, for SU(2) Seiberg-Witten

$$D_n = (n+1)M + n\bar{D}$$



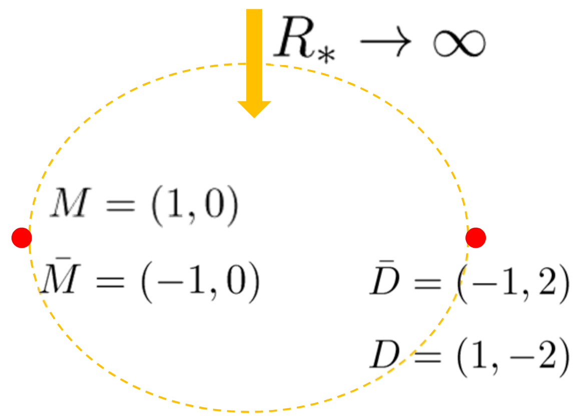
in particular, for SU(2) Seiberg-Witten

$$W^+ = M + \bar{D}$$



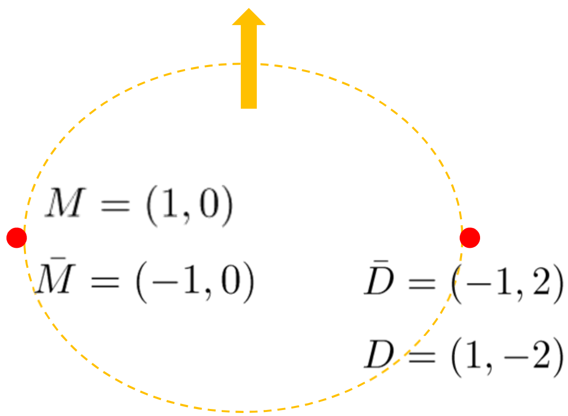
wall-crossing ← dissociation of supersymmetric bound states

$$W^+ = M + \bar{D}$$
$$W^- = \bar{M} + D$$
$$D_n = (n+1)M + n\bar{D}$$
$$\bar{D}_n = (n-1)M + n\bar{D}$$



$R_* \to 0$

wall-crossing → emergence of supersymmetric bound states



$$W^+ = M + \bar{D}$$

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1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.

N=2 SU(n) BPS states via multi-center monopole dynamics

2001 Denef

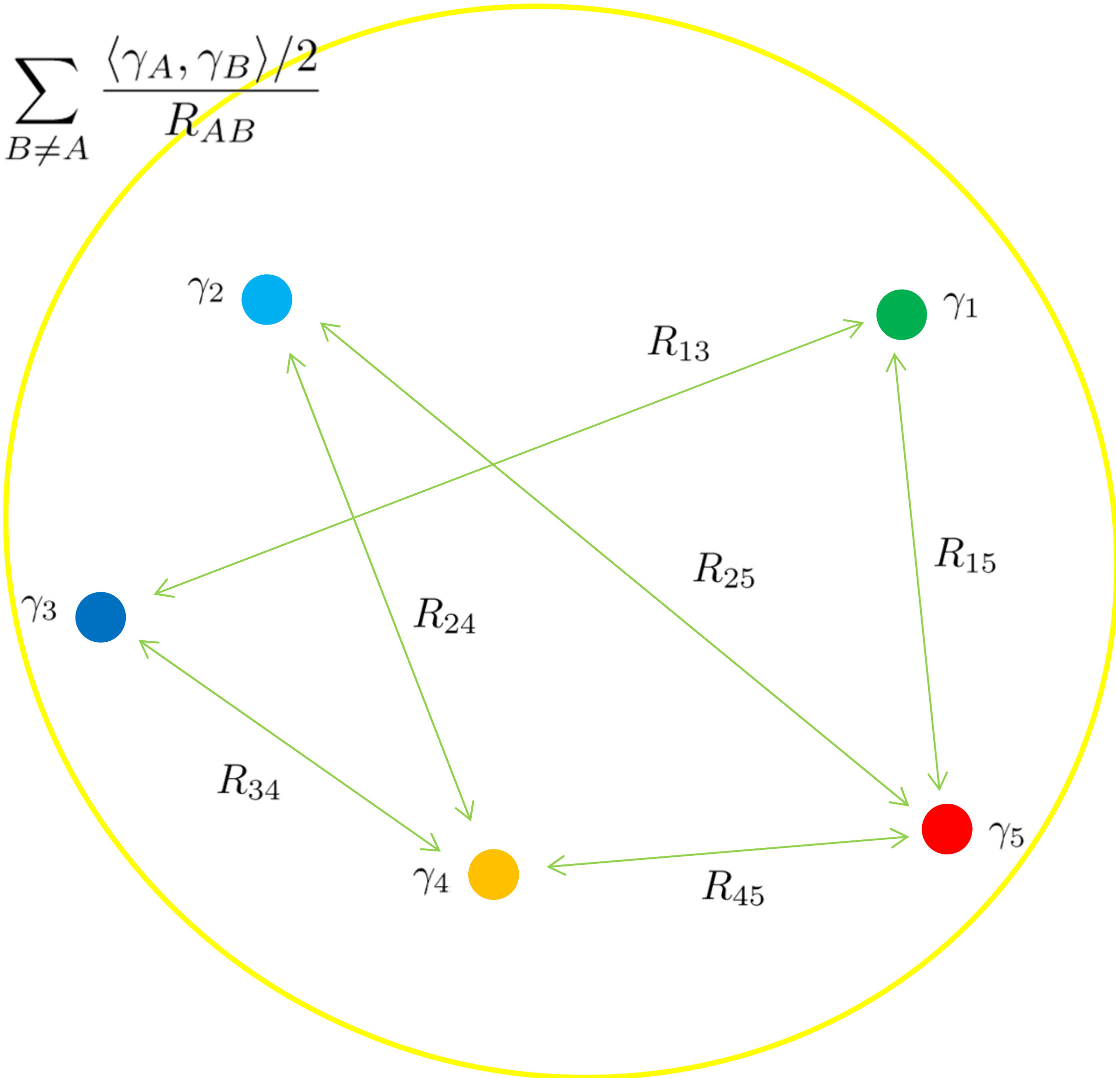
N=2 supergravity via classical multi-center black holes attractor solutions

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}} \qquad\qquad \zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

generic BPS "particles" are loose bound states of charge centers

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

$$R^3 = \{\vec{X}\}$$



wall-crossing problem

= how to count & classify such many-body bound states

index theorem & low energy dynamics of SW BPS particles

also, effectively, index theorem for the Coulomb phase of BPS quivers

BPS states as semiclassical & asymptotic solutions

with electric charges n^i and magnetic charges m^i

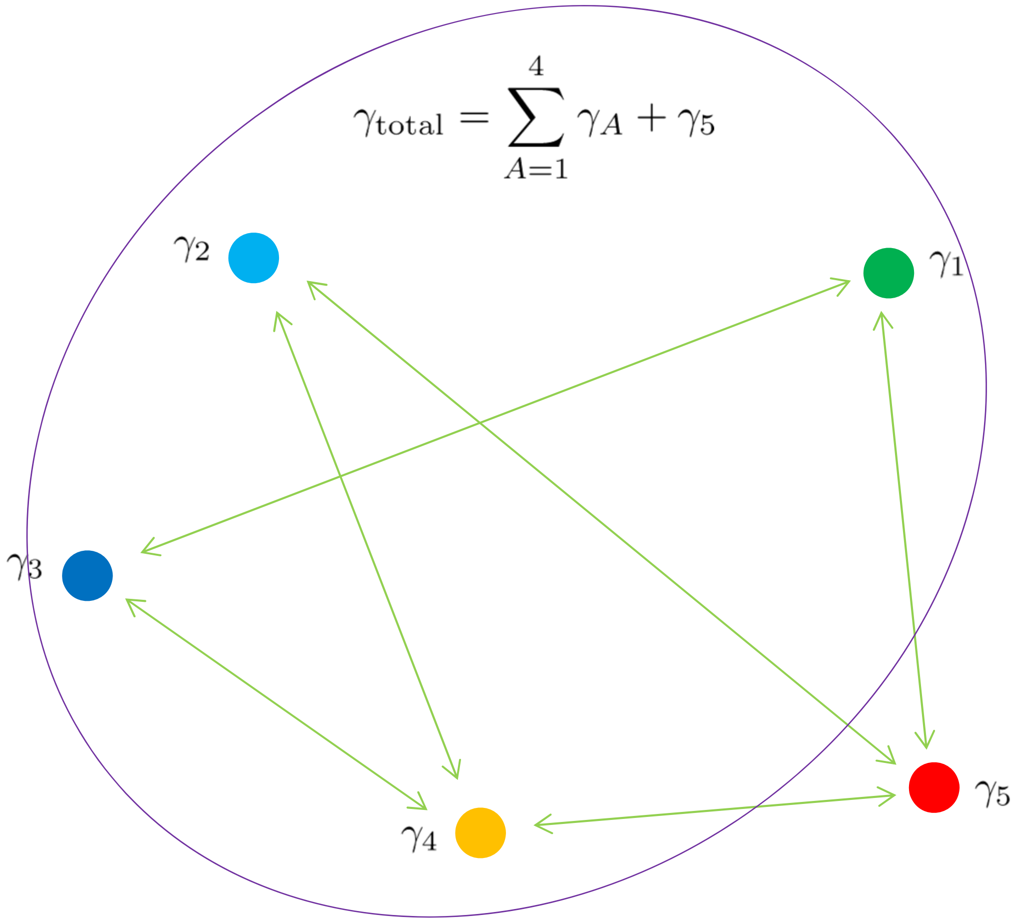
$$\mathcal{E} = |Z| \qquad Z \equiv \langle m^i \phi_D^i + n^i \phi^i \rangle = |Z|\zeta$$

$$\boxed{\begin{aligned} F_a^i &= i\zeta^{-1}\partial_a \phi^i \\ (F_D)_a^i &= i\zeta^{-1}\partial_a \phi_D^i \end{aligned}}$$

$$F_a^i \equiv B_a^i + iE_a^i \qquad \mathrm{Re}\int_{S^2} F^i = 4\pi m^i$$

$$(F_D)_a^i \equiv \tau^{ij} F_j^a \qquad \mathrm{Re}\int_{S^2} F_D^i = -4\pi n^i$$

as a preliminary step, treat one dyon dynamical at a time



$$R^3 = \{\vec{X}\}$$

a probe charge to a system of background "core" dyons

$$\gamma_h = (p, 2q)$$

$$\boxed{\begin{gathered}\sum_{A \neq h} \gamma_A = \sum_{A \neq h} (m_A, 2n_A) \\ F_a^i = i\zeta^{-1} \partial_a \phi^i \\ (F_D)_a^i = i\zeta^{-1} \partial_a \phi_D^i\end{gathered}}$$

$$\mathcal{Z}_{\gamma_h} \equiv q_i \phi^i + p^i \phi_D^i$$

$$\vec{\partial}^2 \, \mathrm{Im} \left[\zeta^{-1} \mathcal{Z}_{\gamma_h} \right] = \sum_{\mathrm{A}} \delta(\vec{x} - \vec{x}_A) \langle \gamma_h, \gamma_A \rangle / 8\pi$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A\neq h}\frac{\langle\gamma_h,\gamma_A\rangle/2}{|\vec{x}-\vec{x}_A|}$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - \left(|\mathcal{Z}_h| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h]\right) - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A\neq h}\frac{\langle\gamma_h,\gamma_A\rangle/2}{|\vec{x}-\vec{x}_A|}$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - (|\mathcal{Z}_h| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h]) - \dot{\vec{x}}\cdot\vec{W}$$

$$\zeta^{-1}\mathcal{Z}_h = |\mathcal{Z}_h|e^{i\alpha}, \quad |\alpha| \ll 1$$

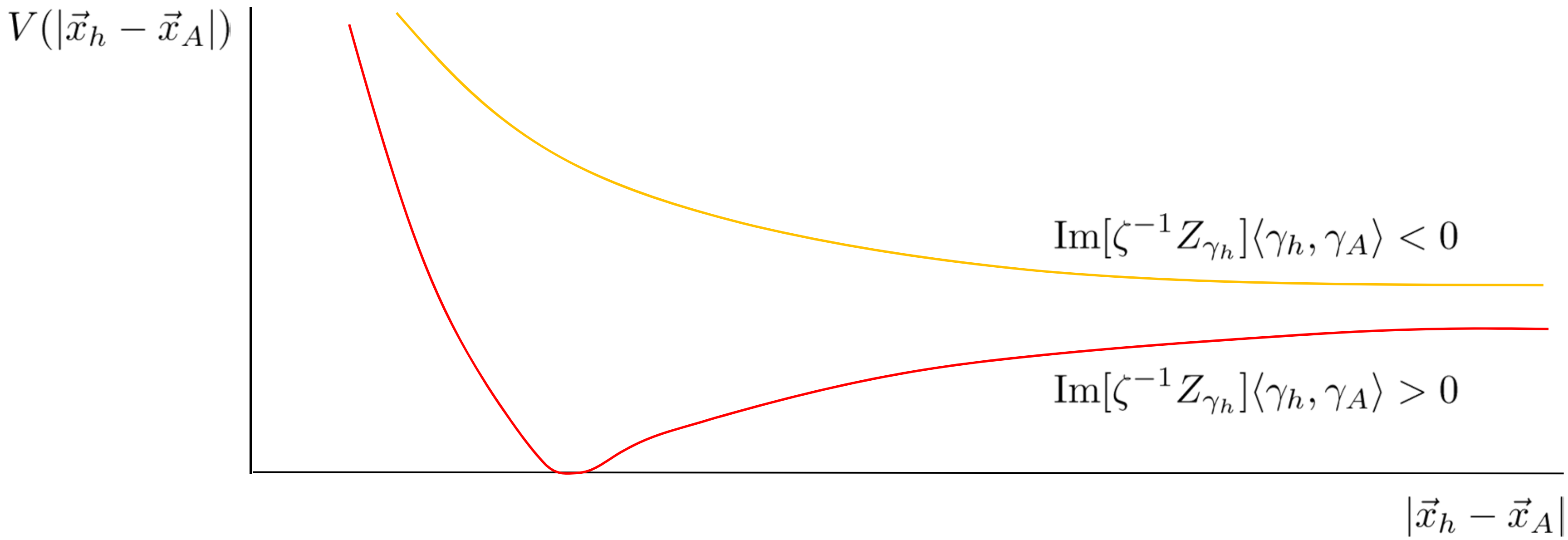
$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_h])^2}{2|\mathcal{Z}_h|} - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A\neq h}\frac{\langle\gamma_h,\gamma_A\rangle/2}{|\vec{x}-\vec{x}_A|}$$

wall-crossing is almost a classical phenomenon !

$$V = \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_h])^2}{2|\mathcal{Z}_h|} \quad \sim \left(\mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \frac{\langle\gamma_h,\gamma_A\rangle/2}{|\vec{x}-\vec{x}_A|}\right)^2$$



N=4 susy with 3n bosons & 4n fermions ?

4n N=1 supermultiplets

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa} \qquad \Lambda^A = i\lambda^A + i\theta b^A \qquad A = 1, 2, \ldots, n$$

position of A-th dyon

or n N=4 supermultiplet

$$\hat{\Phi}^{Aa} = -\frac{i}{4}(\epsilon\sigma^a)^{\alpha\beta}\Phi^A_{\alpha\beta} \ ; \qquad \Phi^A_{\alpha\beta} = (D_\alpha\bar{D}_\beta + \bar{D}_\beta D_\alpha)V^A$$

N=4 susy with 3n bosons & 4n fermions ?

Kim+Park+P.Y.+Wang 2011

$$\int dt\ \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

Smilga; Ivanov;
Papadopoulos; ……
circa1988-1991

is manifestly N=4 supersymmetric

$$\int dt\ \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

is N=4 supersymmetric iff

$$\vec{\partial}_A \cdot \vec{\partial}_B \mathcal{K}_C = 0$$

$$\vec{\partial}_A \times \vec{\partial}_B \mathcal{K}_C = 0$$

$$\vec{\partial}_A \mathcal{K}_B = \frac{1}{2}\left(\vec{\partial}_A \times \vec{W}_B + \vec{\partial}_B \times \vec{W}_A\right) = \vec{\partial}_A \mathcal{K}_B$$

ab initio, real space N=4 susy dynamics for n dyons

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

Denef 2002 :
the Coulomb phase
interaction of BPS quivers

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|}$$

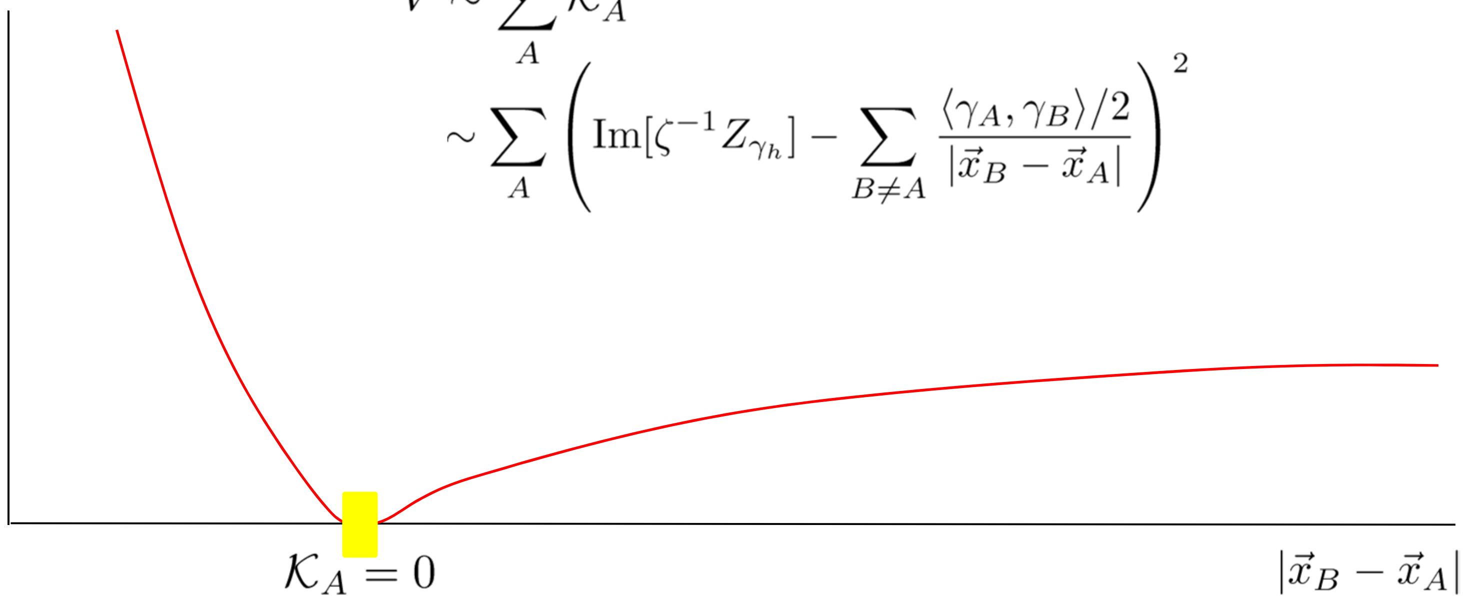
$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

3n → 3 + 2(n-1) ?

BPS → susy → zero energy → wavefunction supported on $\mathcal{K}_A = 0$ submanifold ?

$$V \sim \sum_A \mathcal{K}_A^2$$
$$\sim \sum_A \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_h}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_B - \vec{x}_A|} \right)^2$$



3n → 3 + 2(n-1)

only after sacrificing all but one supersymmetries !!!

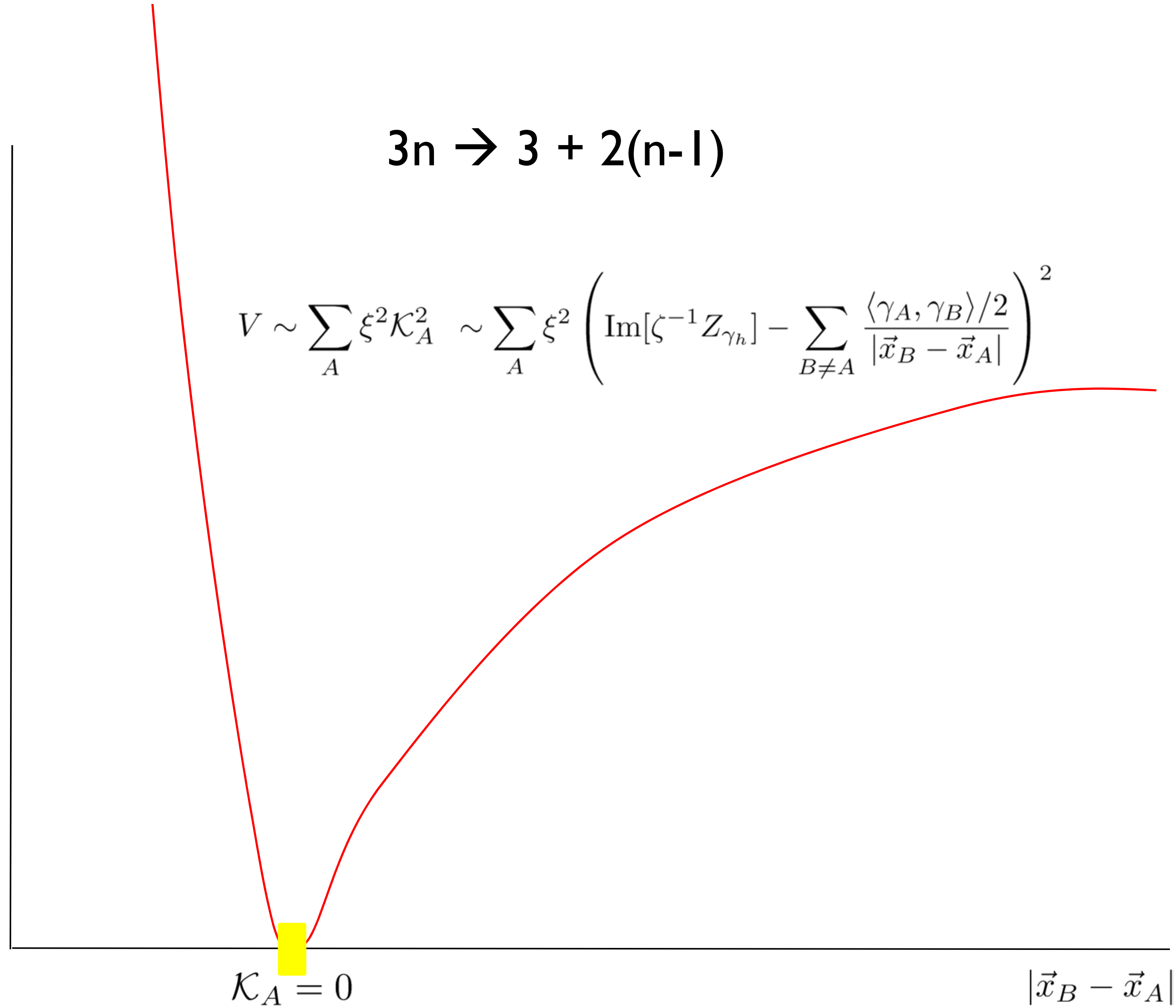
$$\mathcal{L}_{deformed}^{for\ index\ only}$$

$$= \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa}) \quad + \int d\theta \ \boxed{(i\xi \cdot \mathcal{K}(\Phi)_A \Lambda^A} - iW(\Phi)_{Aa} D\Phi^{Aa})$$

$$\vec{\partial}_A \ (\xi \cdot \mathcal{K}_B) \neq \frac{1}{2}\left(\vec{\partial}_A \times \vec{W}_B + \vec{\partial}_B \times \vec{W}_A\right) = \vec{\partial}_A \mathcal{K}_B$$

3n → 3 + 2(n-1)

$$V \sim \sum_A \xi^2 \mathcal{K}_A^2 \;\sim \sum_A \xi^2 \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_h}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_B - \vec{x}_A|} \right)^2$$



counting problem reduces to a N=1 Dirac index of a nonlinear sigma model on the manifold $\mathcal{K}_A = 0$

3n bosons + 4n fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}^{for\ index\ only}_{deformed}\Big|_{\xi\to\infty} \quad \to \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \simeq \frac{1}{2} g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu - \dot{x}^\mu \cdot \mathcal{A}_\mu + \frac{i}{2} g_{\mu\nu}\psi^\mu \left(\dot{\psi}^\nu + \dot{z}^\alpha \Gamma^\nu_{\alpha\beta}\psi^\beta\right) + i\mathcal{F}_{\mu\nu}\psi^\mu\psi^\mu$$

$$\mathcal{F} \equiv \sum_A dW_A\Big|_{\mathcal{K}_A=0} = d\mathcal{A}$$

basic state counting index

Kim+Park+P.Y.+Wang 2011

Manschot, Pioline, Sen 2010

$$I_n(\{\gamma_A\}) = \mathrm{tr}\,\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\,\left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \boxed{\mathbf{A}(\mathcal{M})}$$

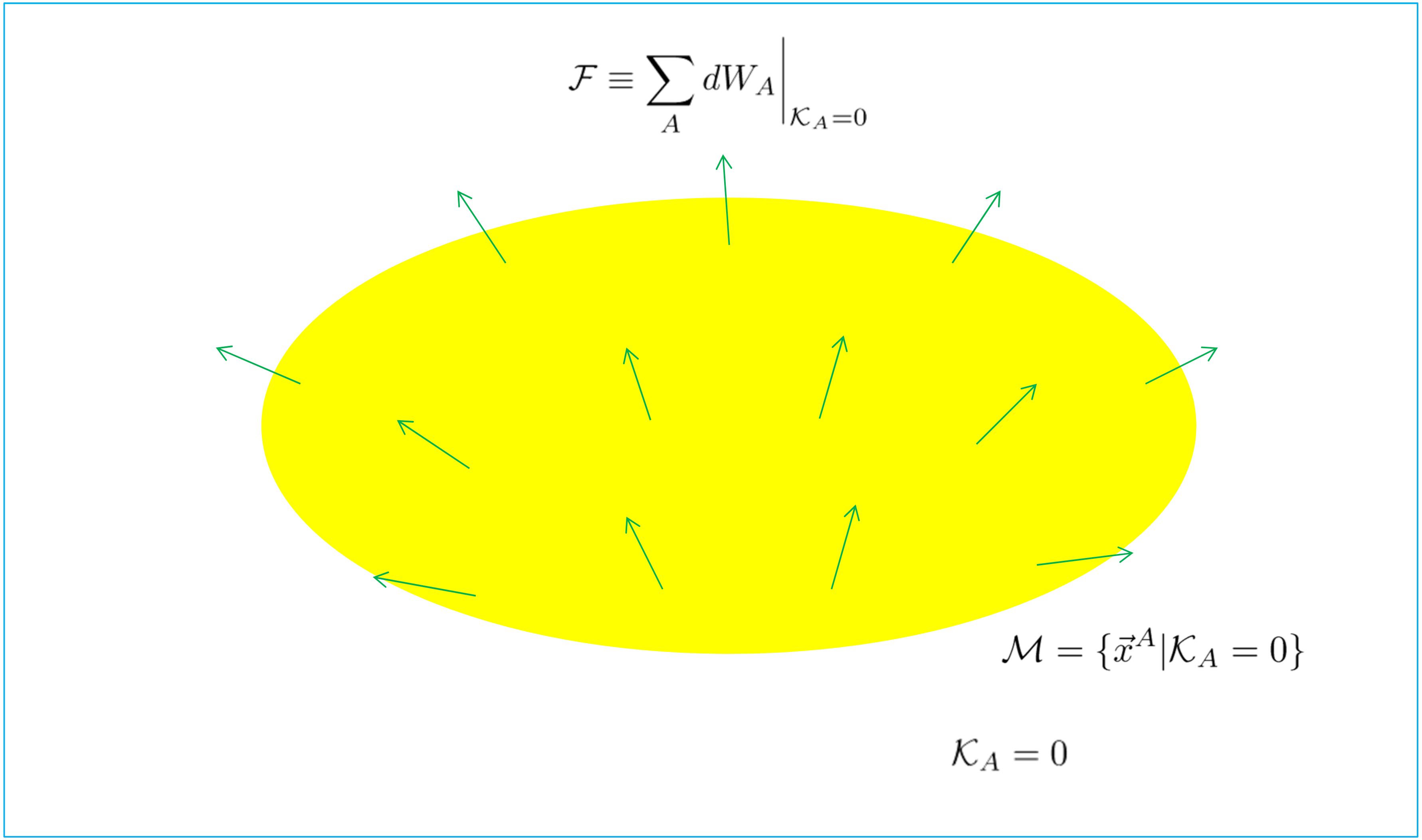
trivial for a complete intersection
in flat ambient space

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F})$$

$$= \frac{1}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n} \mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

3n → 3 + 2(n-1)


$\mathcal{F} \equiv \sum_A dW_A \Big|_{\mathcal{K}_A=0}$
$\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\}$
$\mathcal{K}_A = 0$

the 2nd helicity trace & the protected spin character

$$SO(4) = SU(2)_{\text{rotation}} \times SU(2)_{\text{R-symmetry}}$$

$$J \qquad\qquad\qquad I$$

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

2nd helicity trace — protected spin character

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

symmetry of $R^{3n} = \{\vec{x}^A\}$ quantum mechanics

$$x^{a=1,2,3} \quad \leftarrow \quad SU(2)_J$$

$$\psi^{\mu=1,2,3,4} \quad \leftarrow \quad SO(4) = SU(2)_J \times SU(2)_I$$

symmetry of $\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\}$

$$x^{a=1,2,3} \quad \leftarrow \quad SU(2)_J$$

$$\psi^{\mu=1,2,3,4} \quad \leftarrow \quad SO(4) = SU(2)_J \times SU(2)_I$$

$$\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\} \qquad \downarrow \qquad \mathcal{J} = J + I$$

$$\theta, \phi \quad \leftarrow \quad SO(3)_{\mathcal{J}}$$

$$\psi^{\theta,\phi} \quad \leftarrow \quad SO(3)_{\mathcal{J}}$$

2^{nd} helicity trace (protected spin character) → (equivariant) index on $\mathcal{M}$

Kim+Park+P.Y.+Wang 2011

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I_3)}\right]$$

$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I_3)}(-1)^{2I_3}y^{2(J_3+I_3)}\right]$$

reduction to $\mathcal{M}$

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

equivariant index for distinct particles

Kim+Park+P.Y.+Wang 2011

Manschot, Pioline, Sen 2010

$$\Omega(y) \quad = \quad (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3}e^{-\beta\mathcal{Q}^2})$$

$$= \quad \prod_A \Omega(\gamma_A)\times\frac{(-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}}{(2\pi)^{n-1}(n-1)!}\int_{\mathcal{M}_n=\{\vec{x}|\mathcal{K}_A=0\}/R^3}\mathcal{F}^{n-1}$$

$$\mathcal{F}\equiv\sum_{A>B}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

Bose/Fermi statistics,

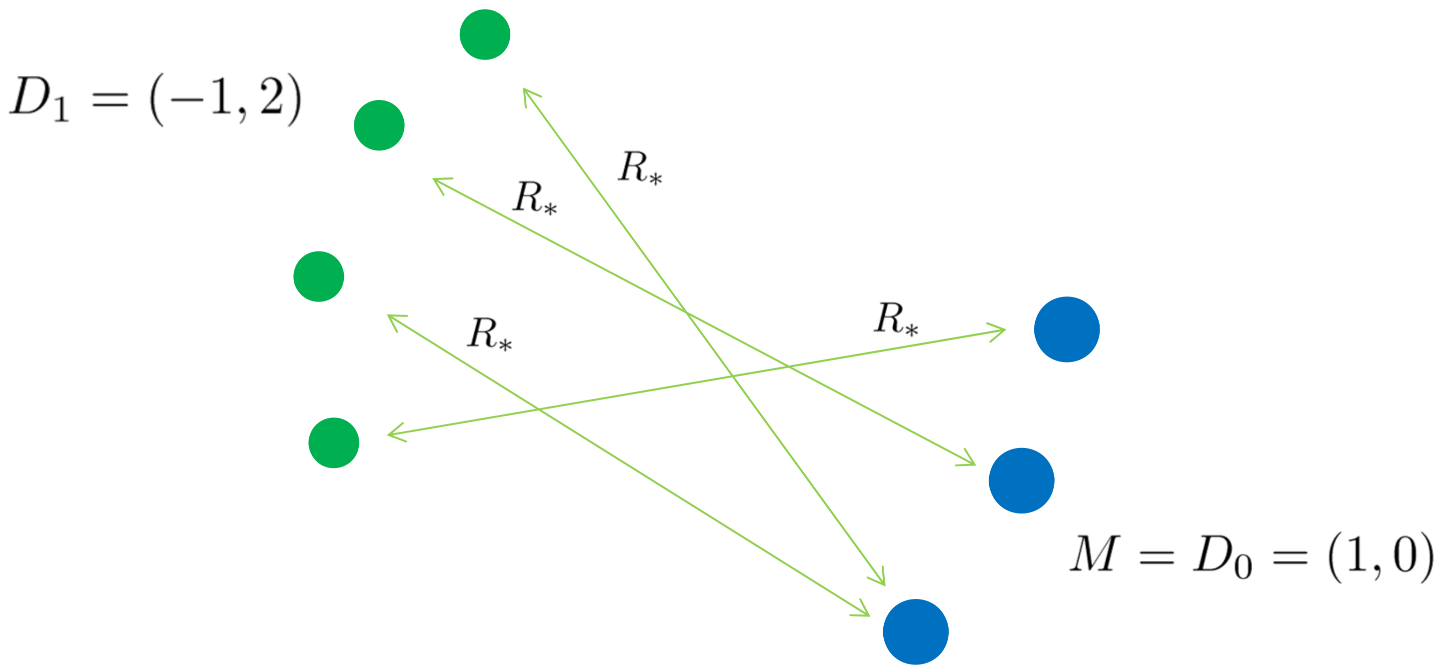
rational invariants,

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

and the wall-crossing formula

Bose/Fermi statistics from identical constituent particles is essential, for example, to solve SU(2)→ U(1) problem

$$D_n = (n-1)D_0 + nD_1$$



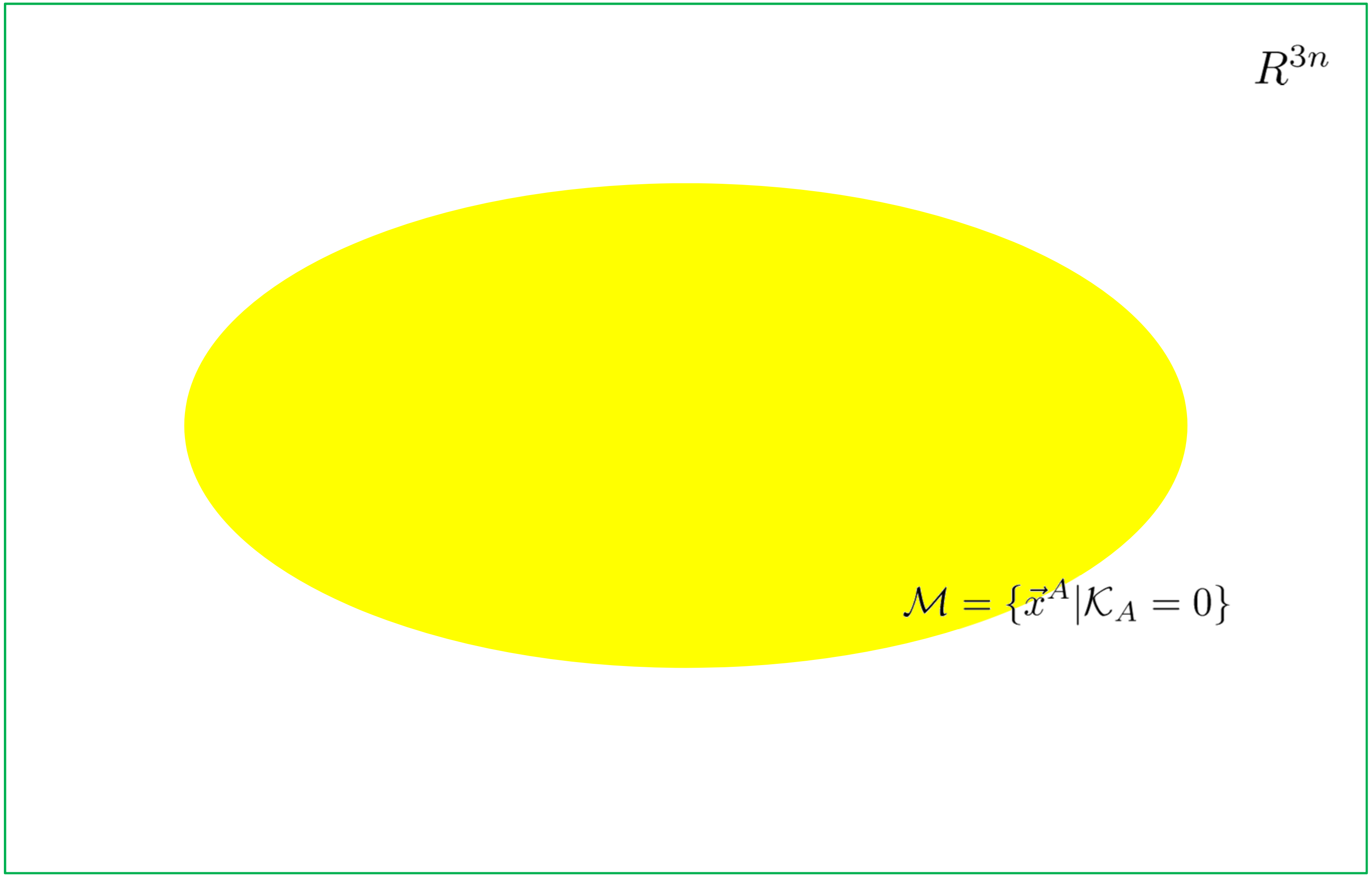
incorporating statistics

$$\mathrm{tr}\left[(-1)^F e^{-\beta Q^2}\mathcal{P}\right]$$

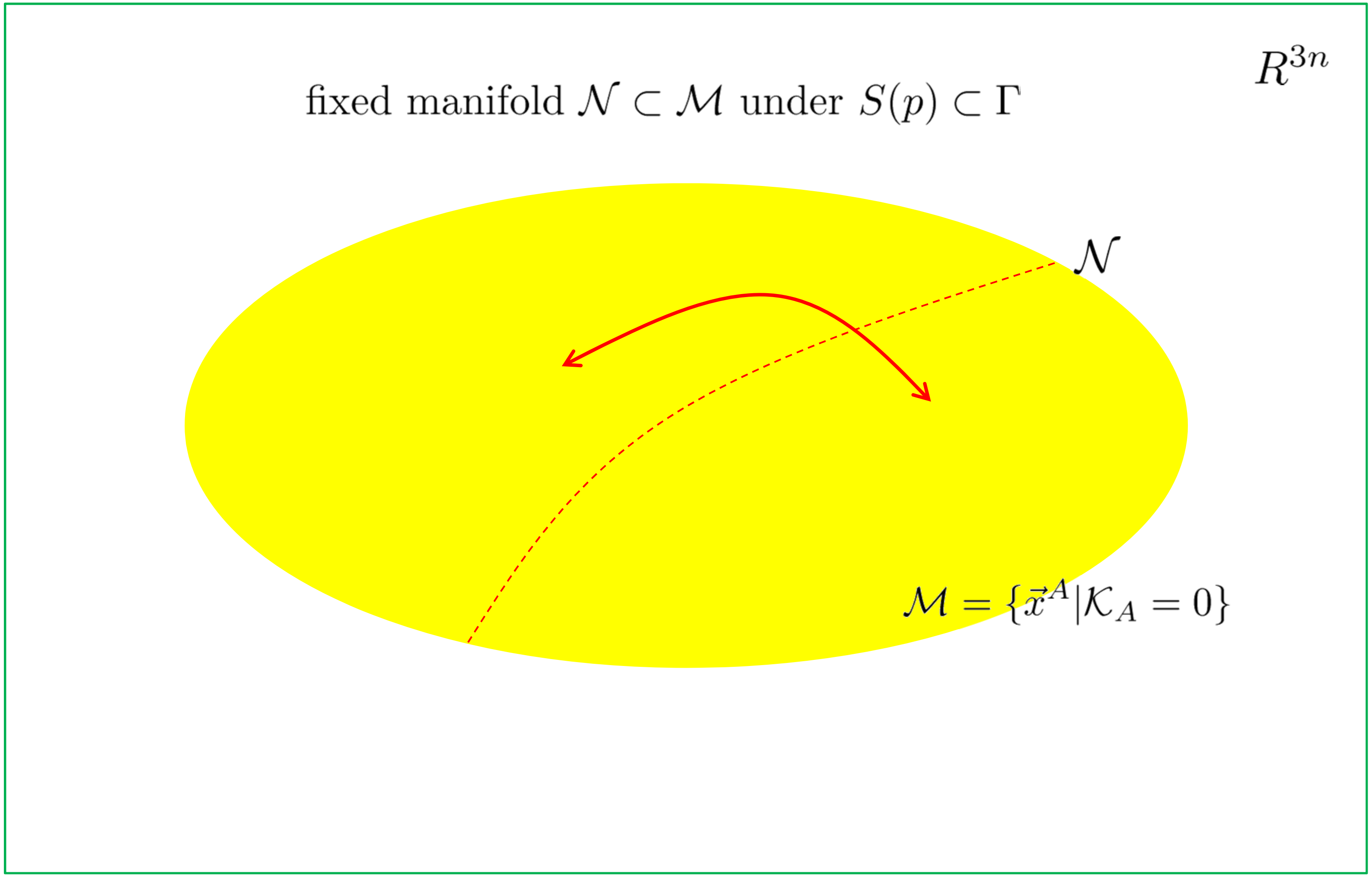
$$\mathcal{P} = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}(\pm 1)^{|\sigma|}\sigma$$

free bulk + sum over fixed submanifolds

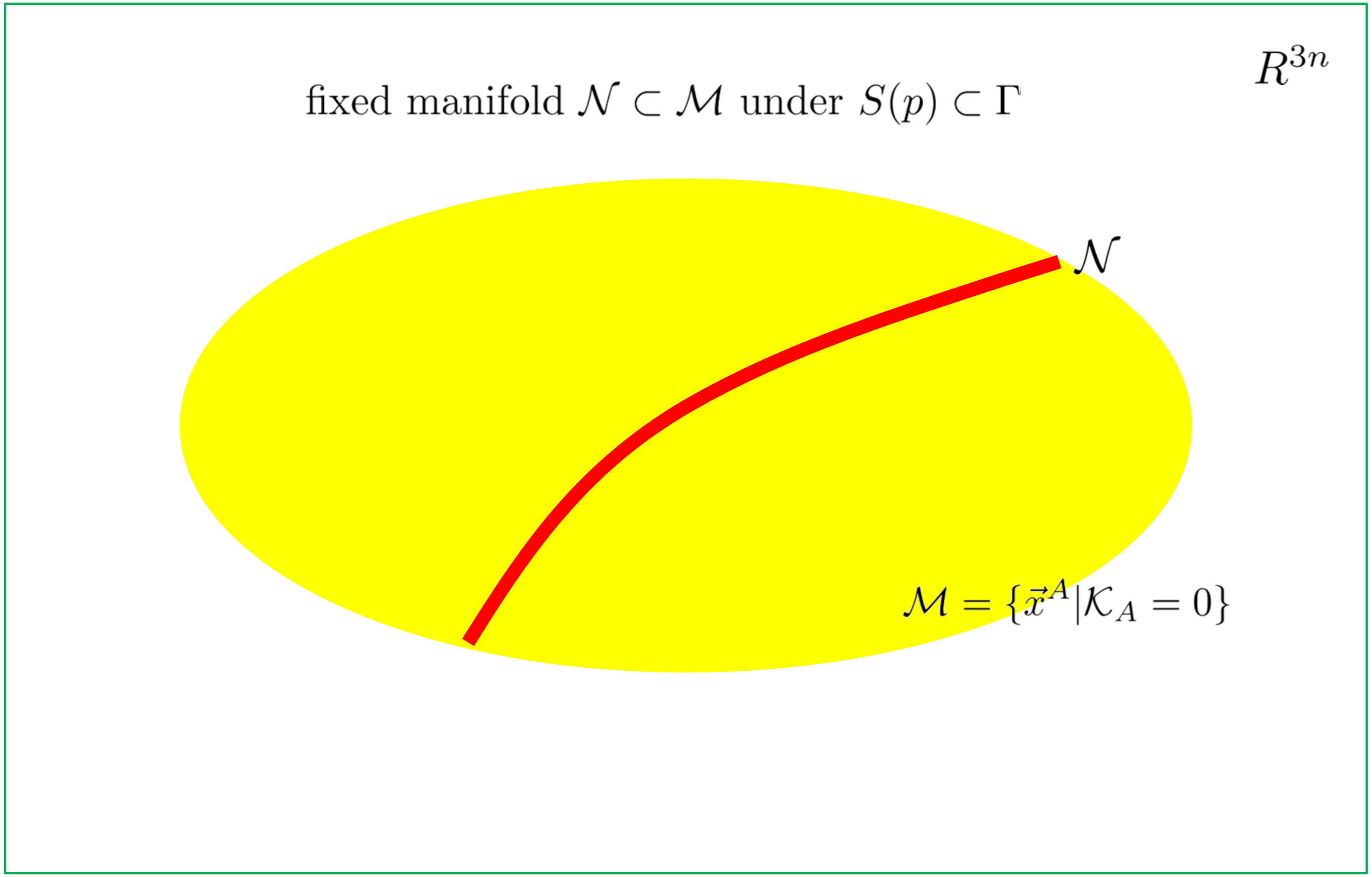
incorporating statistics



incorporating statistics



incorporating statistics



incorporating statistics

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

$$\Gamma' = \Gamma / S(p) \qquad\qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}}\,\mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

incorporating statistics for a pair

$$\mathcal{P}_2^{(\pm)} \ : \ x \to -x, \ \psi \to -\psi$$

$$\Delta_{\mathcal{N}}^{(\pm)}\Big|_{p=2} \leftarrow \lim_{\beta\to 0} \mathrm{tr}_{R^d;n_f} \left[(-1)^{F^\perp} e^{\beta\partial^2/2}\mathcal{P}_2^{(\pm)}\right]$$

P.Y. 1997

$$= \lim_{\beta\to 0} \int_{R^d} d^d x \ \langle -x|e^{\beta\partial^2/2}|x\rangle \times (\pm 2^{n_{fermion}/2-1})$$

$$= \lim_{\beta\to 0} \frac{\pm 2^{n_{fermion}/2-1}}{(2\pi\beta)^{d/2}} \int_{R^d} d^d x \ e^{-(x+x)^2/2\beta}$$

$$= \lim_{\beta\to 0} \frac{\pm 2^{n_{fermion}/2-1}}{2^d}$$

$$\boxed{\to \quad \frac{\pm 1}{2^2} \qquad \begin{array}{ccccc} n_f & = & 2 & 4 & 8 & 16 \\ d & = & 2 & 3 & 5 & 9 \end{array}}$$

incorporating statistics for p identical particles

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

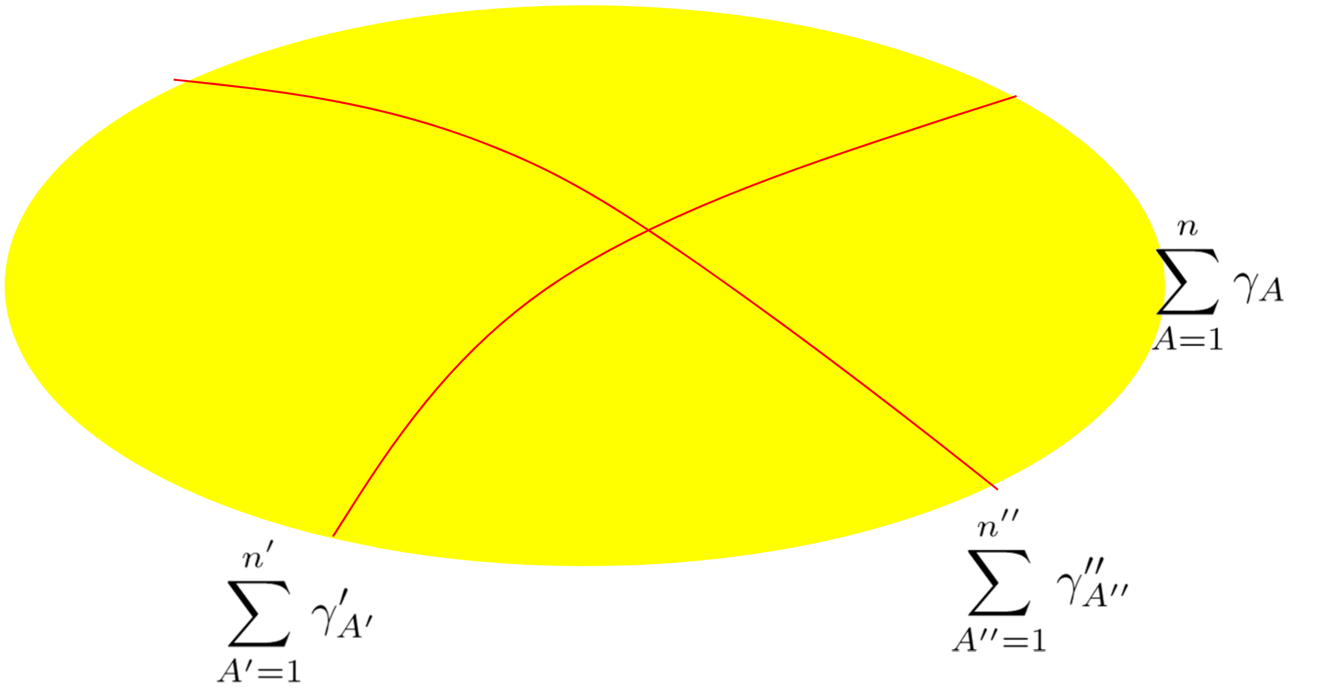
$$\Delta_{\mathcal{N}}^{(\pm)} = \mathrm{tr}_{\mathcal{N}^{\perp}} \left[(-1)^{F^{\perp}} e^{-\beta H^{\perp}} \mathcal{P}_{S(p)}^{(\pm)} \right] \quad = \quad \frac{\pm 1}{p^2}$$

P.Y. 1997 / Green+Gutperle 1997

incorporating statistics

$$\sum_{A=1}^{n} \gamma_A = \cdots = \sum_{A'=1}^{n'} \gamma'_{A'} = \cdots = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

R^{3n}



wall-crossing from real space (Coulomb phase) dynamics

Manschot, Pioline, Sen 2010/2011

Kim+Park+P.Y.+Wang 2011

$$\Omega^-\left(\sum\gamma_A\right)-\Omega^+\left(\sum\gamma_A\right) \quad = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1}\frac{\prod_A\bar{\Omega}^-(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}}ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1}\frac{\prod_{A'}\bar{\Omega}(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'}ch(\mathcal{F}')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1}\frac{\prod_{A''}\bar{\Omega}(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''}ch(\mathcal{F}'')$$

$$\vdots$$

$$\sum_{A=1}^{n}\gamma_A=\sum_{A'=1}^{n'}\gamma'_{A'}=\sum_{A''=1}^{n''}\gamma''_{A''}=\cdots$$

$$\bar{\Omega}(\gamma)=\sum_{p|\gamma}\Omega(\gamma/p)/p^2$$

wall-crossing formula from low energy dynamics of BPS particles

(~ wall-crossing formula for the Coulomb phase of BPS quivers)

with partition sums and rational invariants incorporating Bose/ Fermi statistics

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

with all charges γ_A on a single plane of charge lattice

this has been shown to be equivalent to the Kontsevich-Soibelman

(Ashoke Sen, December 2011)

BPS quivers

wall-crossing examples

calibrated geometry : special Lagrange submanifolds in CY3

type II string theories on CY3 : wrapped D-branes

D=4 N=2 supergravity : BPS black holes

D=4 N=2 Seiberg-Witten theory : BPS particles

⋮

D=4 N=4 …… ¼ BPS ……

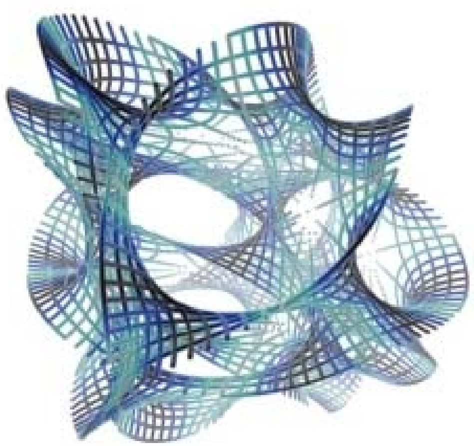
⋮

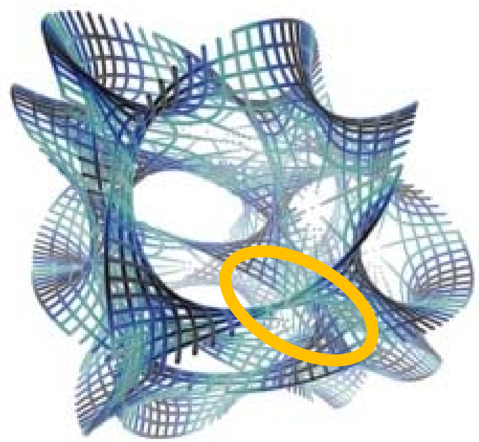
D=2 N=2 Landau-Ginzburg : BPS kinks

calibrated 3-cycles in Calabi-Yau 3-fold

$$J^{(1,1)} \qquad\qquad J^{(1,1)}\Big| = 0$$

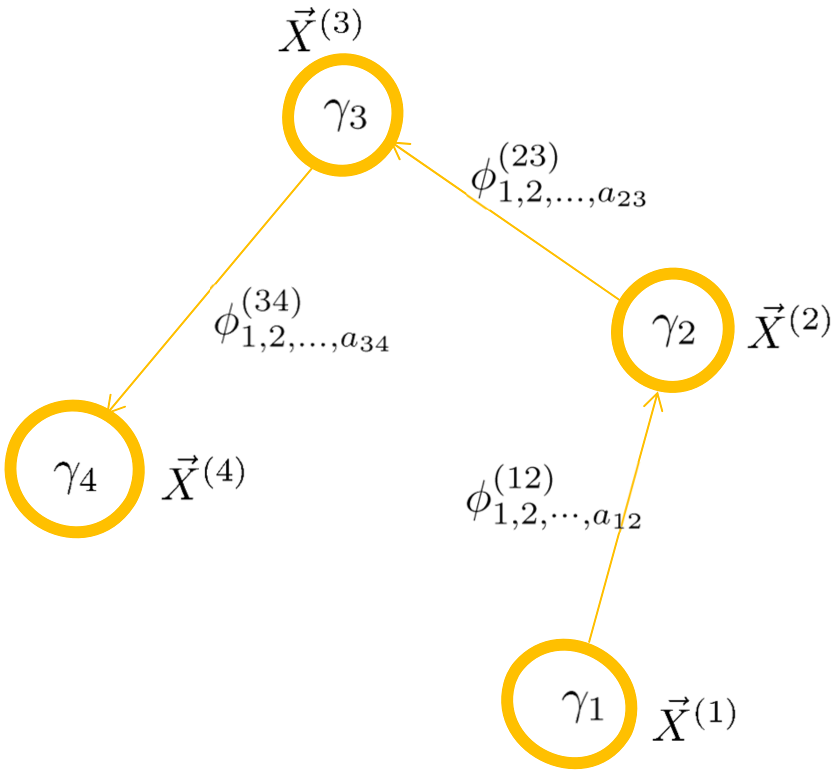
$$\Omega^{(3,0)} \qquad\qquad \zeta^{-1}\Omega^{(3,0)}\Big| = \text{volume density of}$$





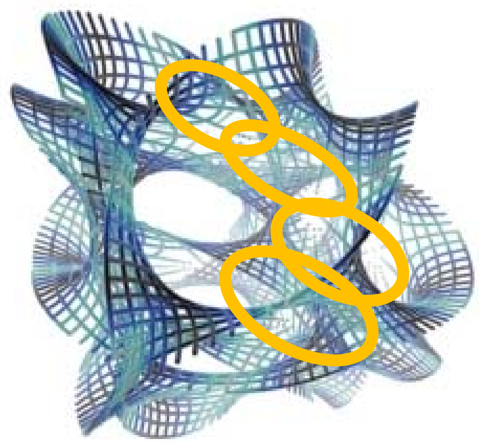
D3 wrapped on 3-cycles in CY3 → BPS quiver quantum mechanics

Denef 2002



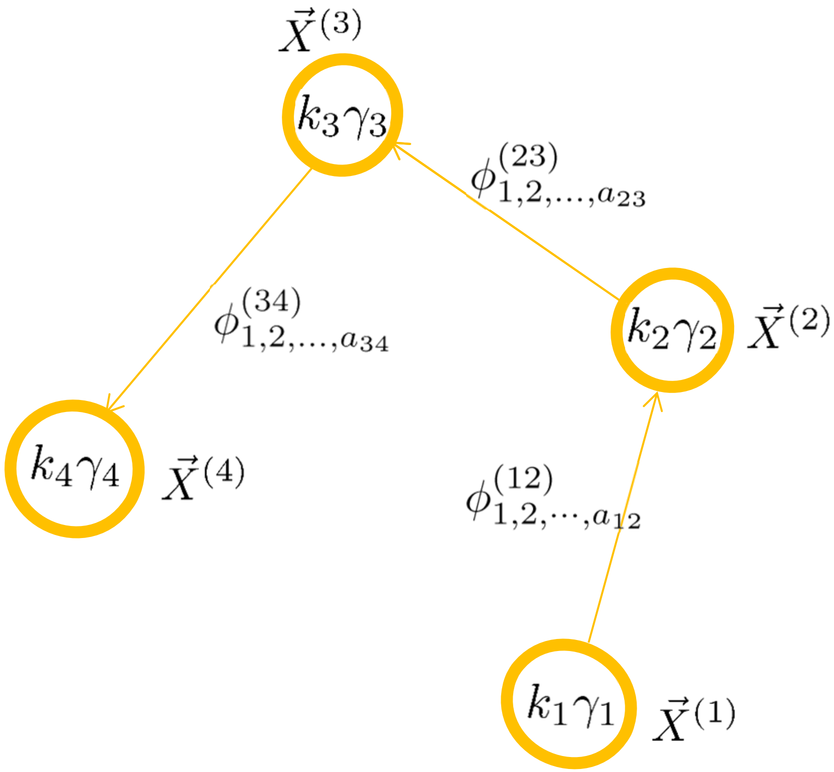
$$\begin{array}{ccccccc} \vec{X}^{(1)} & & \vec{X}^{(2)} & & \vec{X}^{(3)} & & \vec{X}^{(4)} \\ U(1) & \times & U(1) & \times & U(1) & \times & U(1) \\ & \phi^{(12)}_{1,2,\cdots,a_{12}} & & \phi^{(23)}_{1,2,\ldots,a_{23}} & & \phi^{(34)}_{1,2,\ldots,a_{34}} & \end{array}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



D3 wrapped on 3-cycles in CY3 → BPS quiver quantum mechanics

Denef 2002

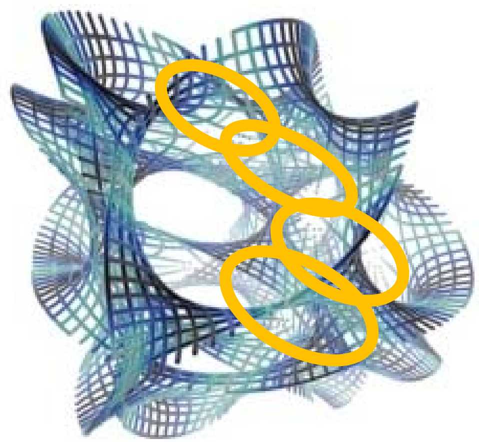


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



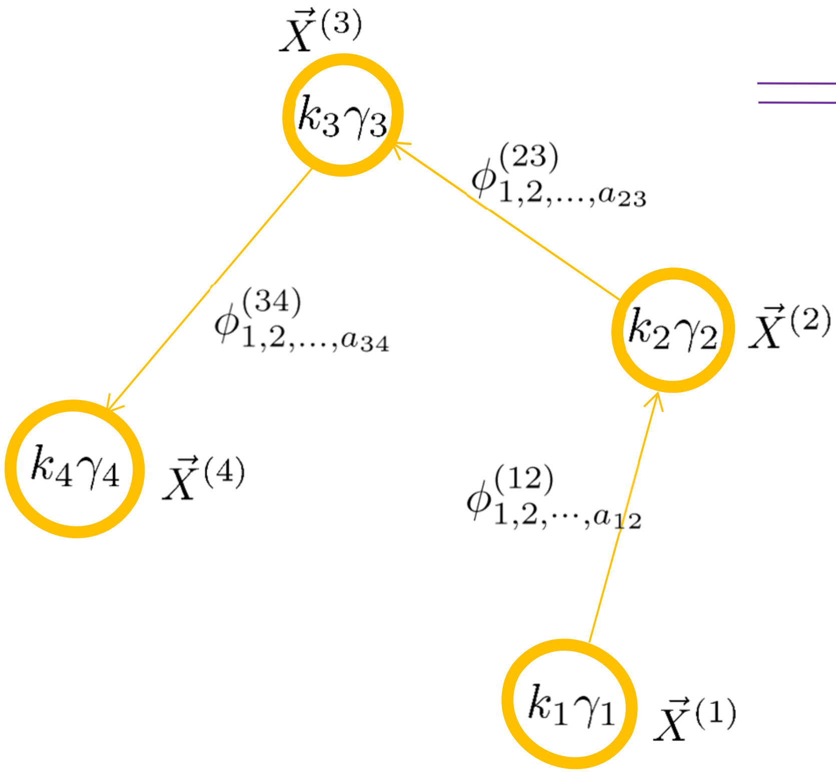
Coulomb phase

Denef 2002

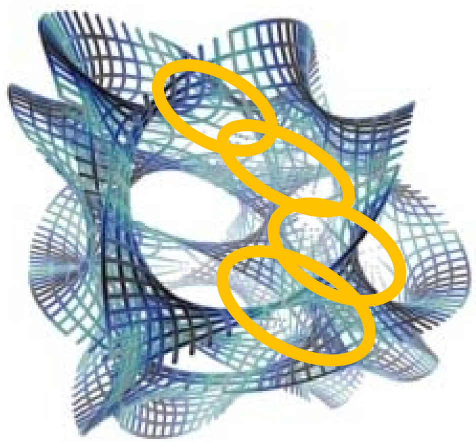
$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

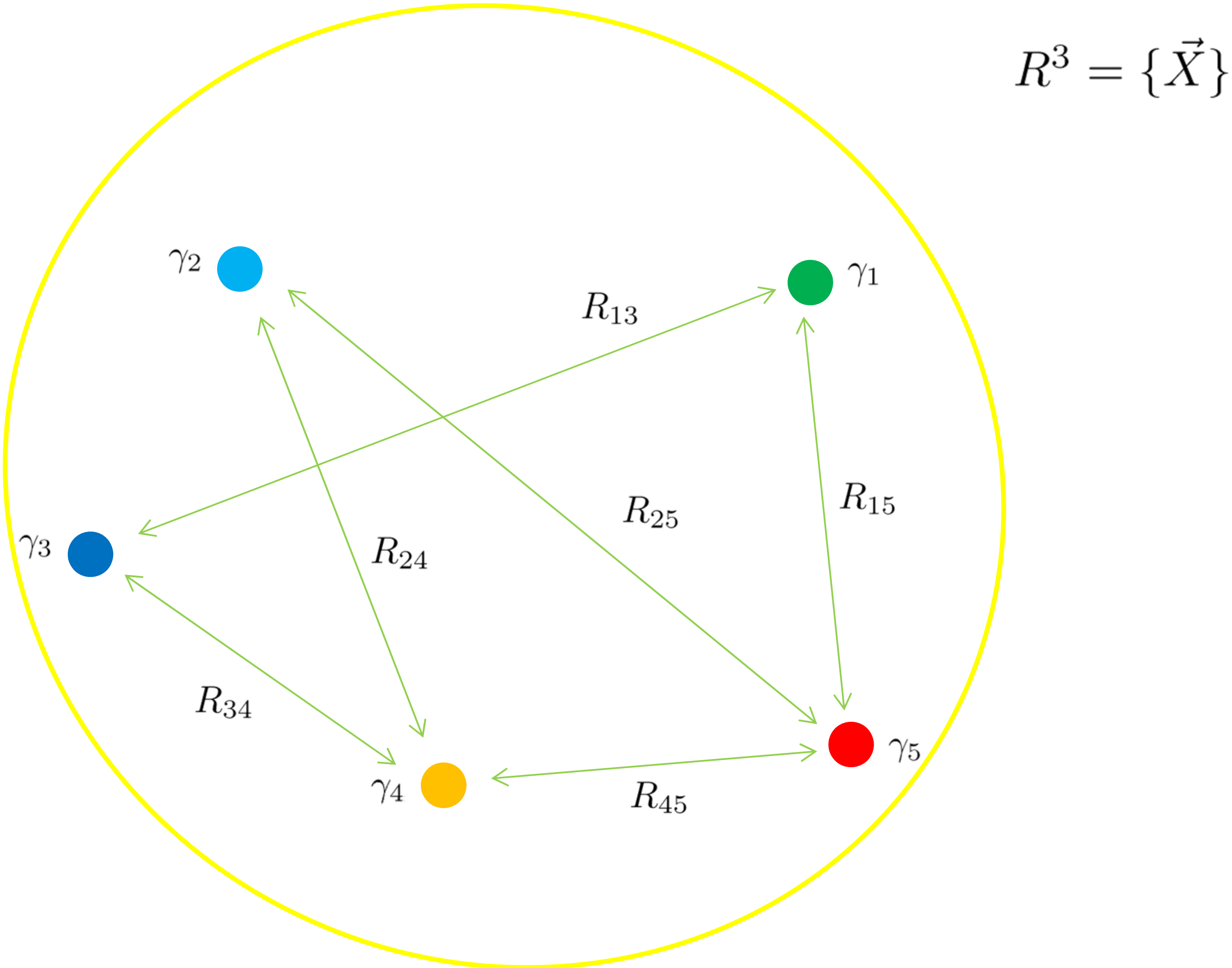
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$



$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



which was, in effect, addressed in the first half of this talk



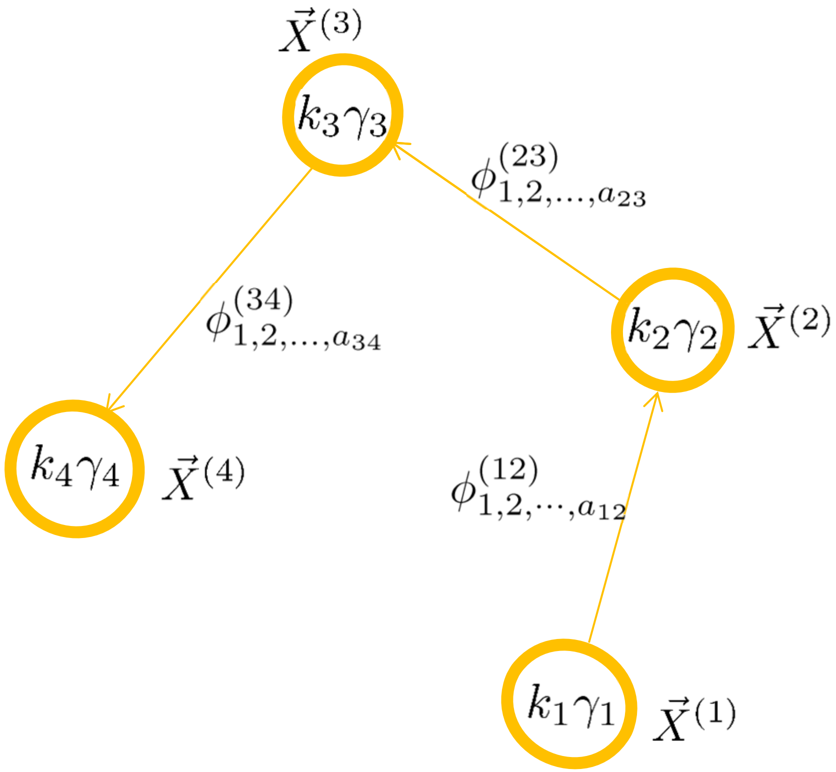
Higgs phase

Denef 2002

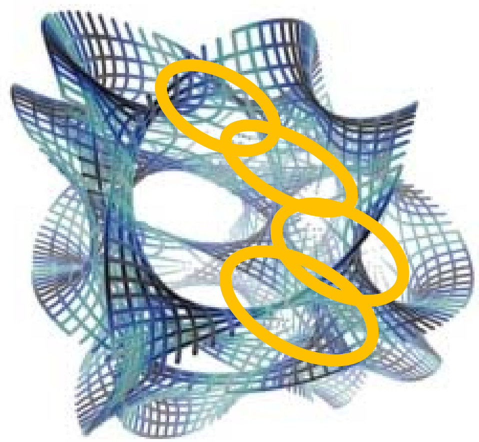
$$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

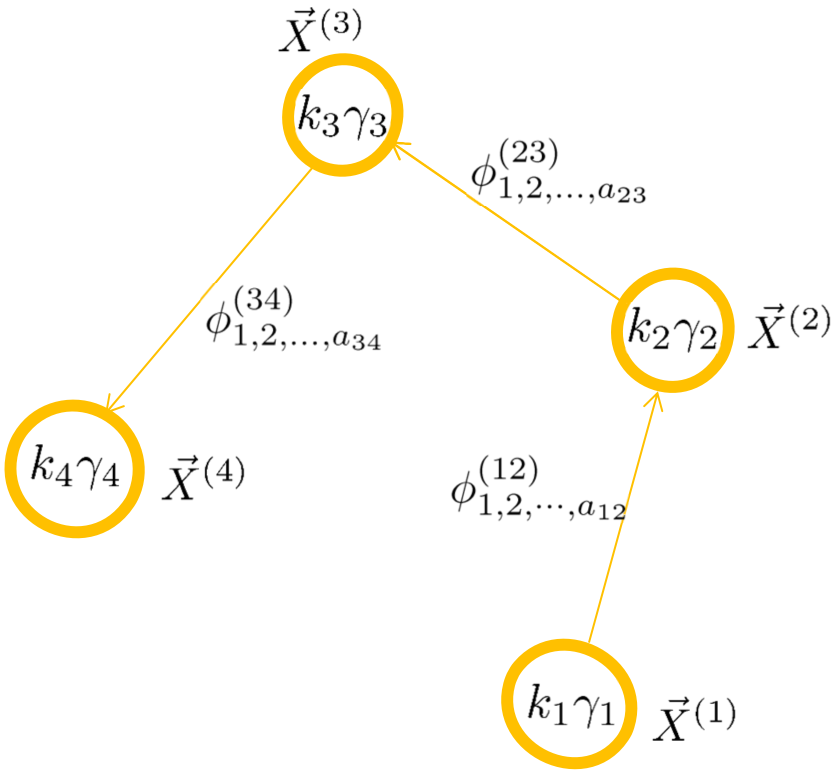


$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Higgs phase : $\mathcal{M}_{\rm H} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$

Denef 2002



$$\Omega_{\rm Higgs}\left(\sum_i k_i\gamma_i; \xi^{(i)}\right)$$

$$\sim \chi\left(\mathcal{M}_{\rm H}\right)$$

$$= \sum_l (-1)^l \dim\left[H^l\left(\mathcal{M}_{\rm H}\right)\right]$$

Higgs phase : $\mathcal{M}_{\mathrm{H}} = \{\phi^{(12)}, \ldots | D^{(i)} = \xi^{(i)} 1_{k_i \times k_i}\} / \prod_i U(k_i)$

Denef 2002

marginal stability wall

$$\xi^{(1)} > \xi^{(2)} > \xi^{(3)} > \xi^{(4)}$$

$$\chi\left(\mathcal{M}_{\mathrm{H}}\right) = 0$$

$$\chi\left(\mathcal{M}_{\mathrm{H}}\right) = a_{12} \times a_{23} \times a_{34}$$

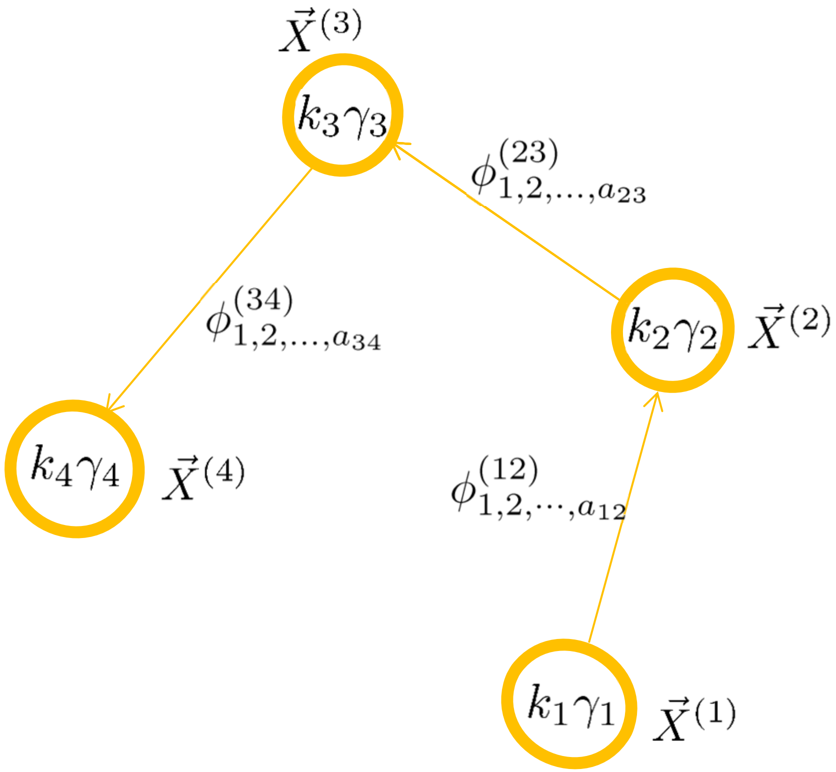
$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

$$\mathcal{M}_{\mathrm{H}}\left(\sum_i \gamma_i; \xi^{(i)}\right) = 0$$

$$\mathcal{M}_{\mathrm{H}}\left(\sum_i \gamma_i; \xi^{(i)}\right)$$

$$= CP^{a_{12}-1} \times CP^{a_{32}-1} \times CP^{a_{34}-1}$$

equivariant index



$$\Omega_{\mathrm{Higgs}}[y]\left(\sum_i k_i\gamma_i;\xi^{(i)}\right)$$

$$= \mathrm{tr}(-1)^{p+q-d}y^{2p-d}$$

$$= \sum_{p,q}(-1)^{p+q-d}y^{2p-d}h^{(p,q)}$$

$$h^{(p,q)} = \mathrm{dim}H^{(p,q)}(M)$$

$$\Omega_{\mathrm{Higgs}} = \Omega_{\mathrm{Coulomb}}$$

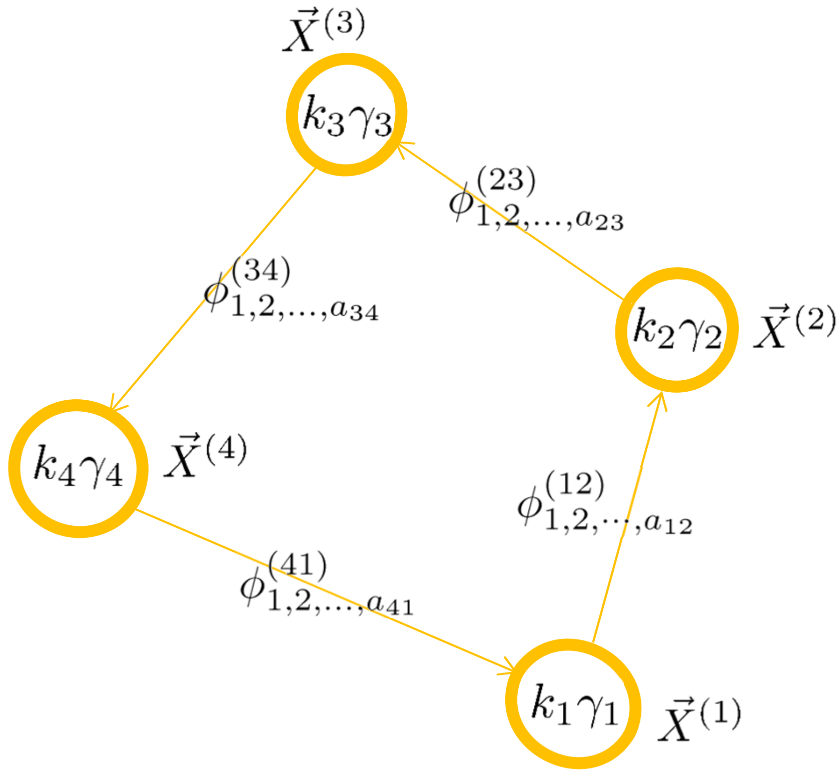
F. Denef 2002 + A. Sen 2011

which apparently fails for some quivers with loops

Denef + Moore 2007

$$|\Omega_{\text{Higgs}}| \ \geq \ |\Omega_{\text{Coulomb}}|$$

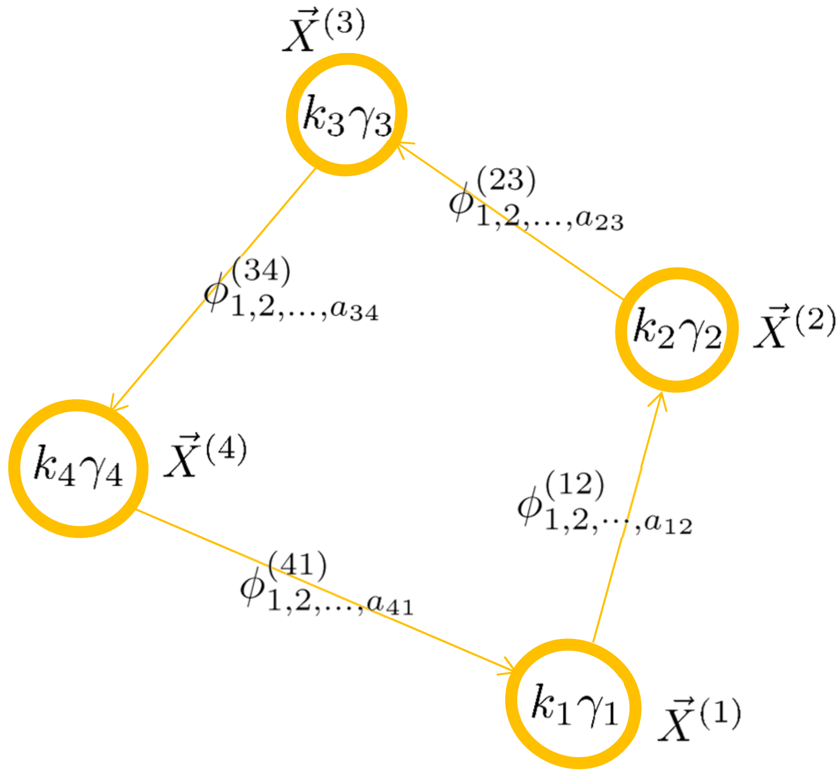
$$(\ |\Omega_{\text{Higgs}}| \ >> \ |\Omega_{\text{Coulomb}}|\ )$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

what physical & mathematical properties characterize these extra BPS states in the Higgs phase ?

$$|\Omega_{\text{Higgs}}| - |\Omega_{\text{Coulomb}}| = ???$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

also, all known wall-crossing formulae need input data

$$\Omega^+(\gamma) = \Omega^-(\gamma)$$

how to count & figure out these wall-crossing safe states ?

example 1 : elementary objects such as certain 2r+f hypermultiplet dyons in Seiberg-Witten theory of rank r and f flavors

$$\Omega^{+} = 1$$

$$\Omega^{-} = 1$$

example 1 : elementary objects such as certain 2r+f hypermultiplet dyons
in Seiberg-Witten theory of rank r and f flavors

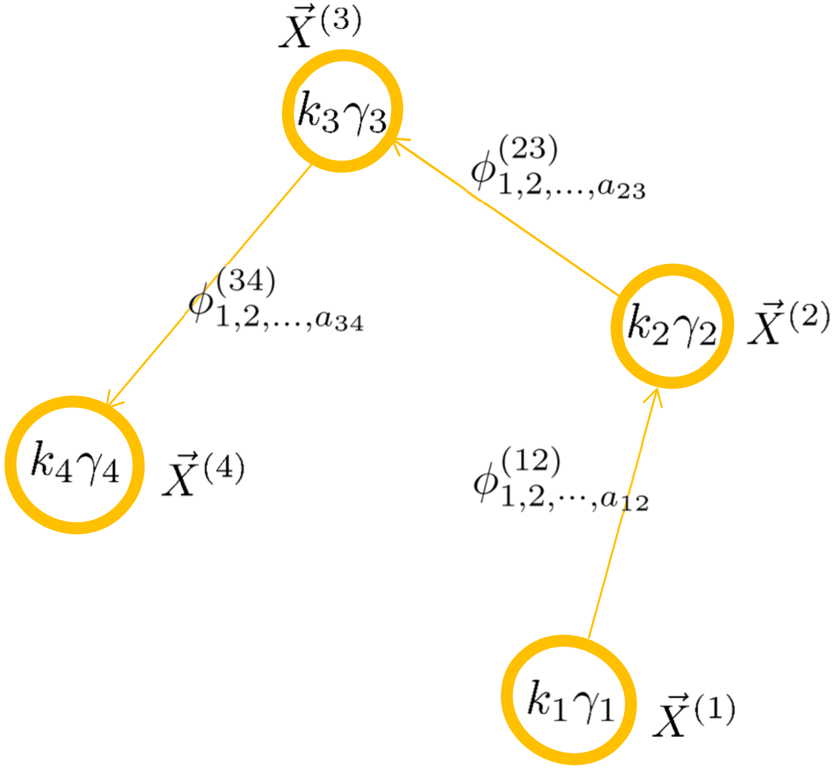
example 2 : single-center black holes

$$\Omega^{+} = \boxed{\Omega_{\mathrm{Invariant}}} + \Omega^{+}_{\mathrm{Coulomb}}$$

$$\Omega^{-} = \boxed{\Omega_{\mathrm{Invariant}}} + \Omega^{-}_{\mathrm{Coulomb}}$$

quiver invariants

Higgs phase of BPS quiver quantum mechanics

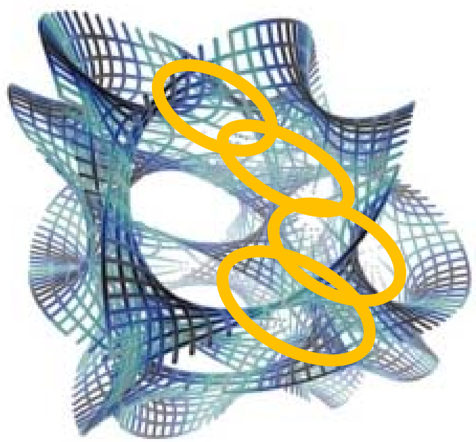


$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$

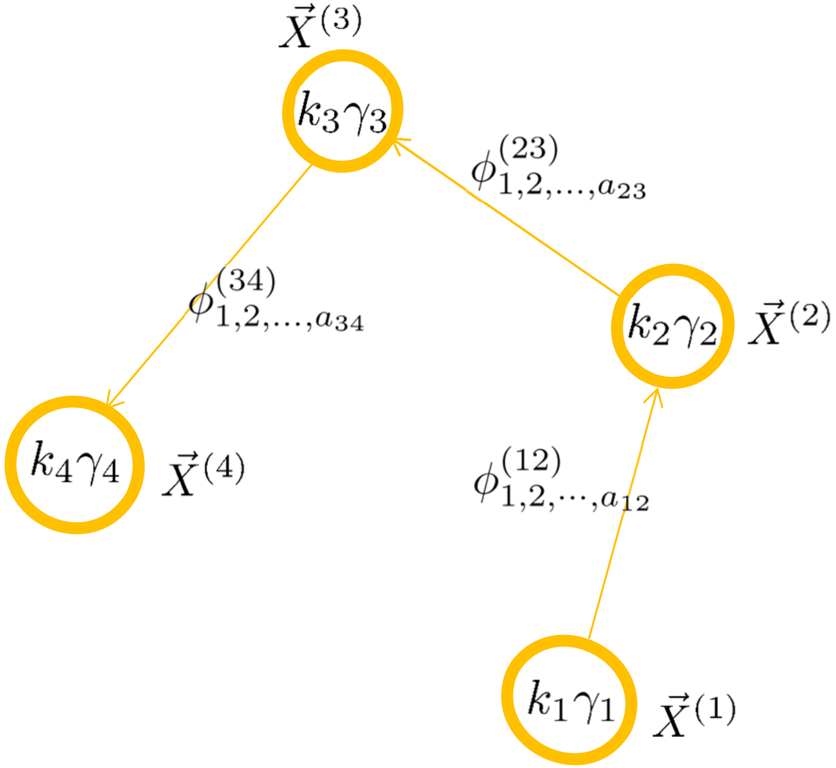
$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Higgs phase of BPS quiver quantum mechanics

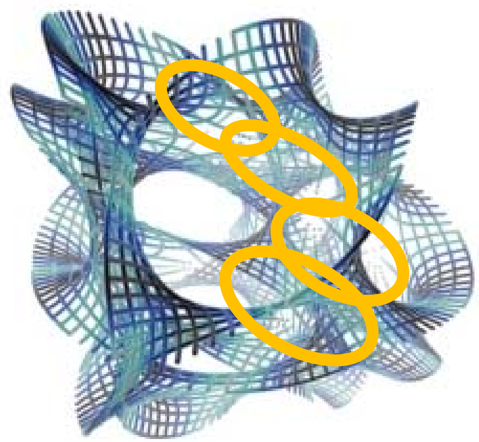


$$\xi^{(1)} \qquad \xi^{(2)} \qquad \xi^{(3)} \qquad \xi^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

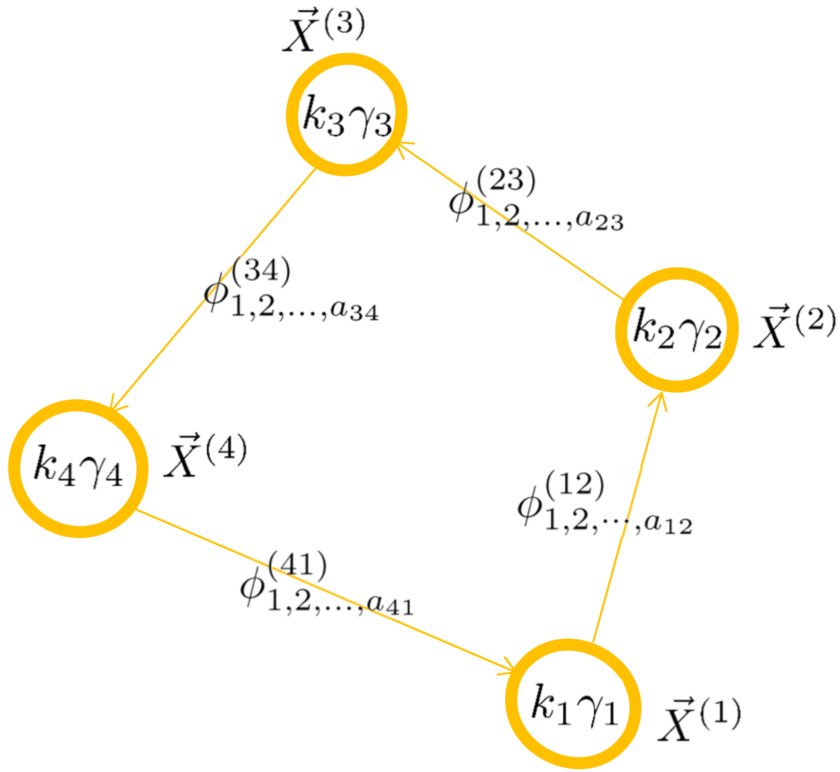
$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



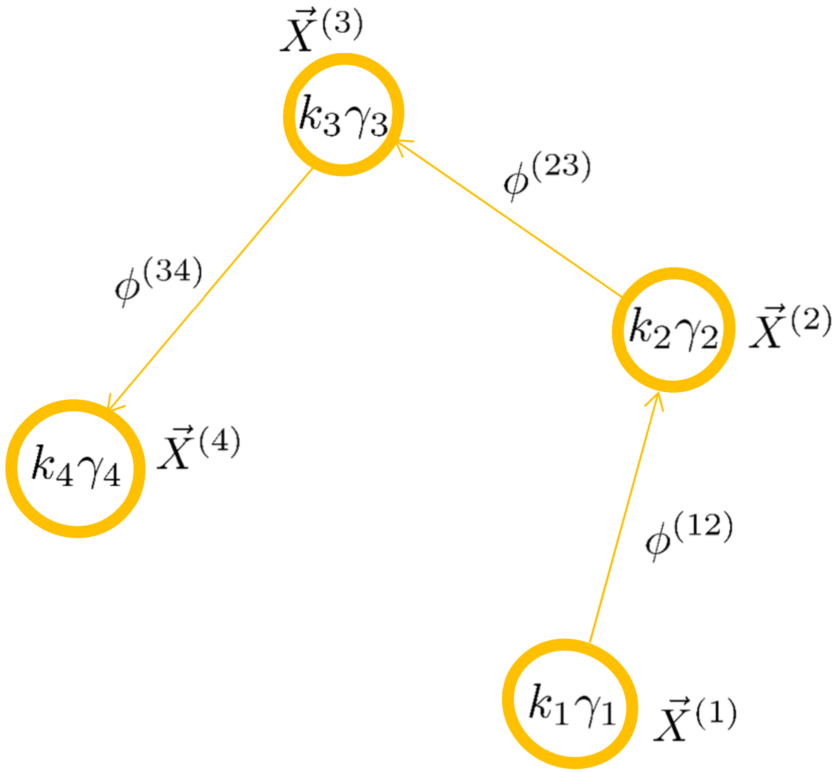
$$|\Omega_{\text{Higgs}}| \;\geq\; |\Omega_{\text{Coulomb}}|$$

$$\begin{tikzcd}
\{\phi^{(12)}, \ldots\} \arrow[d, "D \;\sim\; / \prod_i U(k_i)_C"] \\
X_{\text{H}} \arrow[d, "F \;\sim\; \partial_\phi W = 0"] \\
\mathcal{M}_{\text{H}}
\end{tikzcd}$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

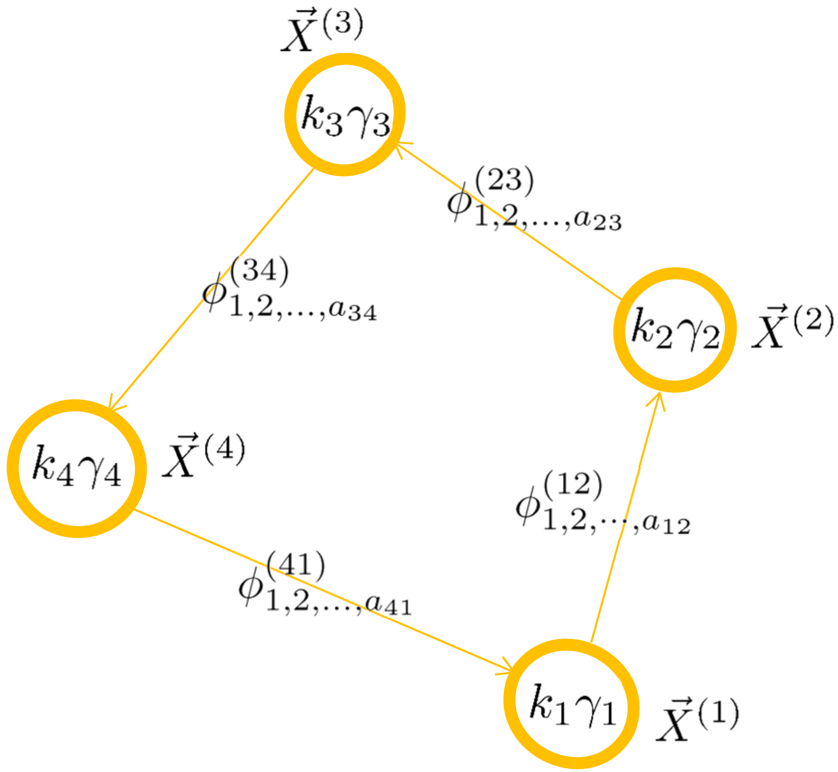
$$\Omega_{\mathrm{Higgs}} = \Omega_{\mathrm{Coulomb}}$$



$$\begin{tikzcd}
\{\phi^{(12)},\ldots\} \arrow[d, "D \quad \sim \ / \prod_i U(k_i)_C"] \\
\mathcal{M}_{\mathrm{H}}
\end{tikzcd}$$

$$|\Omega_{\mathrm{Higgs}}| \;\geq\; |\Omega_{\mathrm{Coulomb}}|$$

$$\begin{tikzcd}
\{\phi^{(12)}, \ldots\} \arrow[d, "D \;\sim\; /\prod_i U(k_i)_C"] \\
X_{\mathrm{H}} \\
\mathcal{M}_{\mathrm{H}} \arrow[u, "\text{embedding map}"', "i \;\sim\; \partial_\phi W = 0"]
\end{tikzcd}$$



$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

two conjectures

$$\begin{tikzcd} \{\phi^{(12)},\ldots\} \arrow[d, "D"] \\ X_{\mathrm{H}} \\ \mathcal{M}_{\mathrm{H}} \arrow[u, "i", "\text{embedding map}"'] \end{tikzcd}$$

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

S.L. Lee + Z.L. Wang + P.Y., 2012

Bena + Berkooz + de Boer +
El-Showk + d. Bleeken, 2012

two conjectures

S.L. Lee + Z.L. Wang + P.Y., 2012

$$
\begin{tikzcd}
\{\phi^{(12)},\ldots\} \arrow[d, "D"] \\
X_{\mathrm{H}} \\
\mathcal{M}_{\mathrm{H}} \arrow[u, "i"', "\text{embedding map}"]
\end{tikzcd}
$$

$$H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$$

$$= i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\mathrm{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \qquad \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\Omega_{\mathrm{Coulomb}} \qquad\qquad \Omega_{\mathrm{Invariant}}$$

complete proof / explicit counting formula exist
for all cyclic Abelian quivers

S.L. Lee + Z.L. Wang + P.Y., 2012
Manschot + Pioline + Sen, 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}}$$

$$= \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

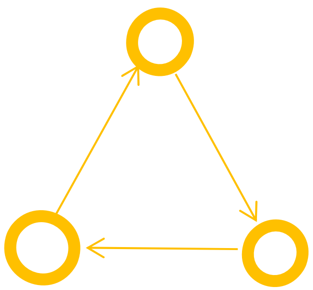
Lefschetz hyperplane theorem !!!

physics of wall-crossing and wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$\dim H^{(p,q)}(\mathcal{M}_H)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$



$a_{12} = a_{23} = a_{31} = 8$

physics of wall-crossing and wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

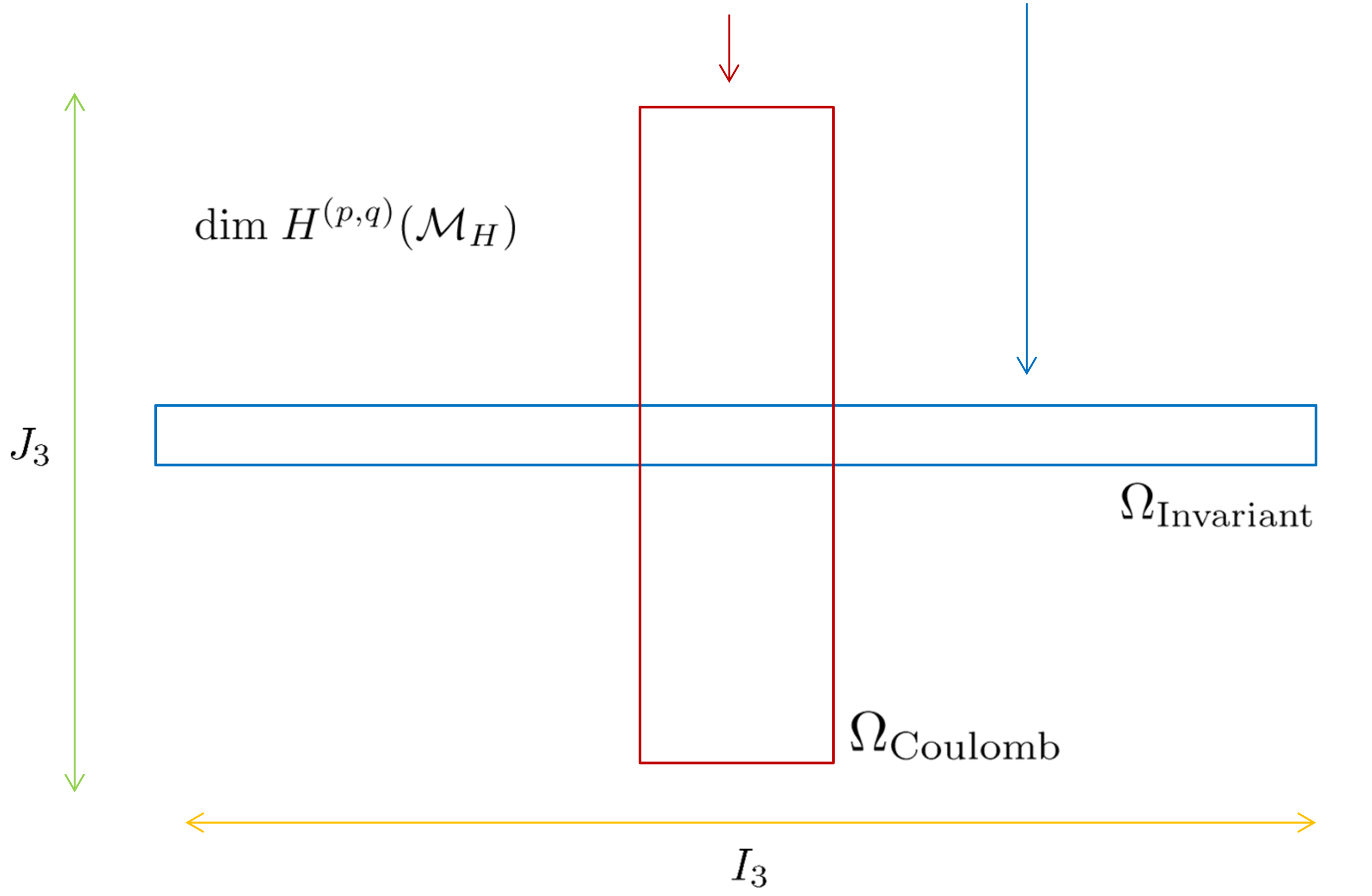
$$\dim H^{(p,q)}(\mathcal{M}_H)$$

$$J_3 \qquad \begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

$$I_3$$

physics of wall-crossing and wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$



physics of wall-crossing and wall-crossing-safe states

Seung-Joo Lee + Zhao-Long.Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

many body bound states wall-crossing	single-center black holes? wall-crossing-safe
vertical in the Hodge diamond	horizontal middle in the Hodge diamond

physics of wall-crossing and wall-crossing-safe states

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\rm Intrinsic}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d} y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\rm Coulomb} \qquad\qquad \Omega_{\rm Invariant} = \Omega_{\rm Higgs} - \Omega_{\rm Coulomb}$$

many body bound states
wall-crossing

single-center black holes?
wall-crossing-safe

angular momentum

R-charge

physics of wall-crossing and wall-crossing-safe states

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}}$$

$$= \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

many body bound states
wall-crossing

single-center black holes?
wall-crossing-safe

angular momentum
(R-charge = 0 for SW ?)

R-charge
angular momentum = 0

examples : wall-crossing-safe states of cyclic Abelian quivers

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\begin{aligned}
\Omega(y)\Big|^{\{a_{i,i+1}\}=(15,16,17)}_{\mathrm{Intrinsic}} &= \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p.q)}_{\mathrm{Intrinsic}}\\
&= 1665y^{-12}\\
&+ 724674y^{-10}\\
&+ 60686563y^{-8}\\
&+ 1523273844y^{-6}\\
&+ 13886938949y^{-4}\\
&+ 50685934038y^{-2}\\
&+ 77668453887\\
&+ 50685934038y^{2}\\
&+ 13886938949y^{4}\\
&+ 1523273844y^{6}\\
&+ 60686563y^{8}\\
&+ 724674y^{10}\\
&+ 1665y^{12}
\end{aligned}$$

examples : wall-crossing-safe states of cyclic Abelian quivers

Seung-Joo Lee + Zhao-Long Wang + P.Y., 2012

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\Omega(y)\Big|_{\mathrm{Intrinsic}}^{\{a_{i,i+1}\}=(8,9,10,11,12)} = \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p,q)}_{\mathrm{Intrinsic}}$$

$$\begin{aligned}
= \;& 32294250/y^{22} + 58872952926/y^{20} + 23086762587054/y^{18} \\
&+3146301650299568/y^{16} + 186529800766285403/y^{14} \\
&+5480846262397291070/y^{12} + 86780383421802203555/y^{10} \\
&+783408269154731872224/y^{8} + 4192271239441338802849/y^{6} \\
&+13657486692285216220742/y^{4} + 27560691162972524163666/y^{2} \\
&+34791235315880411958041 + 27560691162972524163666y^{2} \\
&+13657486692285216220742y^{4} + 4192271239441338802849y^{6} \\
&+783408269154731872224y^{8} + 86780383421802203555y^{10} \\
&+5480846262397291070y^{12} + 186529800766285403y^{14} \\
&+3146301650299568y^{16} + 23086762587054y^{18} \\
&+58872952926y^{20} + 32294250y^{22}
\end{aligned}$$

summary

wall-crossing formulae from direct index computation

 for SW BPS dyons with ab initio low energy dynamics

 for the Coulomb phase of quiver descriptions

subtleties with index theorems in the Coulomb phase

equivalence to Kontsevich-Soibelman (when $\Omega_{\mathrm{Invariant}} = 0$),
and rational invariants from statistics orbifolding

quiver invariants, or wall-crossing safe BPS states in the Higgs phase

non-Abelian quivers / stringy realizations ?