Matrix models for Seiberg-Witten theory

Taro Kimura

Mathematical Physics Laboratory, RIKEN

November 2012

- References
  - T. Kimura, JHEP **1109** (2011) 015 [arXiv:1105.6091]
  - T. Kimura, Prog.Theor.Phys. **127** (2011) 271 [arXiv:1109.0004]

Today's goal

Solving $\mathcal{N} = 2$ gauge theory by matrix model

Today's goal

Solving $\mathcal{N} = 2$ gauge theory by matrix model

Why matrix model?

Partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S[\phi]}$$

Infinite dimensional integral

Partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S[\phi]}$$

Infinite dimensional integral

$$\Downarrow$$

Matrix model

$$Z = \int \mathcal{D}X \, e^{-\frac{1}{\hbar} \mathrm{Tr} \, V(X)}$$

Finite dimensional integral

Partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S[\phi]}$$

Infinite dimensional integral

$$\Downarrow$$

Matrix model

$$Z = \int \mathcal{D}X \, e^{-\frac{1}{\hbar} \mathrm{Tr} \, V(X)}$$

Finite dimensional integral

- Effective theory: QCD, cond-mat...

Partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar}S[\phi]}$$

Infinite dimensional integral

$$\Downarrow$$

Matrix model

$$Z = \int \mathcal{D}X \, e^{-\frac{1}{\hbar}\mathrm{Tr}\, V(X)}$$

Finite dimensional integral

- Effective theory: QCD, cond-mat...
- **Exact integration**: localization technique via SUSY

Exact results in SUSY

- 5 dim: S^5, S^5_{sq}, $\mathbb{R}^4 \times S^1$...
- 4 dim: $\mathbb{R}^4$, S^4, S^4_{sq}, $S^3 \times S^1$...
- 3 dim: S^3, S^3_{sq}, $\Sigma_2 \times S^1$...
- 2 dim: $\mathbb{R}^2$, S^2...

Partition function exactly evaluated

Exact results in SUSY

- 5 dim: S^5, $S^5_{\rm sq}$, $\mathbb{R}^4 \times S^1$...
- 4 dim: $\mathbb{R}^4$, S^4, $S^4_{\rm sq}$, $S^3 \times S^1$...
- 3 dim: S^3, $S^3_{\rm sq}$, $\Sigma_2 \times S^1$...
- 2 dim: $\mathbb{R}^2$, S^2...

Partition function exactly evaluated

Check non-perturbative aspects of QFT:
AdS/CFT, mirror, ...

Exact results in SUSY

- 5 dim: S^5, $S^5_{\rm sq}$, $\mathbb{R}^4 \times S^1$...
- 4 dim: $\mathbb{R}^4$, S^4, $S^4_{\rm sq}$, $S^3 \times S^1$...
- 3 dim: S^3, $S^3_{\rm sq}$, $\Sigma_2 \times S^1$...
- 2 dim: $\mathbb{R}^2$, S^2...

Partition function exactly evaluated

Check non-perturbative aspects of QFT:
AdS/CFT, mirror, ... **Seiberg-Witten**

- Progress on $\mathcal{N} = 2$ supersymmetric gauge theories

- Progress on $\mathcal{N} = 2$ supersymmetric gauge theories

'94 Seiberg-Witten

"*Solution*" of $\mathcal{N} = 2$ theory

'02 Nekrasov

Instanton counting and gauge theory partition function

'09 Alday-Gaiotto-Tachikawa

Meaning of SW curve, relation between 2d/4d theories

- Progress on $\mathcal{N} = 2$ supersymmetric gauge theories

'94 Seiberg-Witten

"*Solution*" of $\mathcal{N} = 2$ theory

'02 Nekrasov

Instanton counting and gauge theory partition function

'09 Alday-Gaiotto-Tachikawa

Meaning of SW curve, relation between 2d/4d theories

"Solution" is not yet solved?

- Progress on $\mathcal{N} = 2$ supersymmetric gauge theories

'94 Seiberg-Witten

"*Solution*" of $\mathcal{N} = 2$ theory

'02 Nekrasov

Instanton counting and gauge theory partition function

'09 Alday-Gaiotto-Tachikawa

Meaning of SW curve, relation between 2d/4d theories

"Solution" is not yet solved?

Today's goal

Seiberg-Witten solution from matrix model

Contents

Contents

$\mathcal{N} = 2$ gauge theory

- Generic form of $\mathcal{N} = 2$ theory

$$\mathcal{L} = \frac{1}{4\pi} \mathrm{Im} \left[\int d^2\theta d^2\bar{\theta} \frac{\partial \mathcal{F}(A)}{\partial A} \bar{A} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha \right]$$

$\mathcal{F}(A)$: prepotential

Low energy effective action

- Asymptotic freedom
  - UV (high energy) : weak coupling
  - IR (low energy) : strong coupling

$$\mathcal{F}(a) = \frac{1}{2}\tau_0 a^2 \quad \longrightarrow \quad \frac{1}{2}\tau_0 a^2 + \frac{i}{2\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right) + a^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k}$$

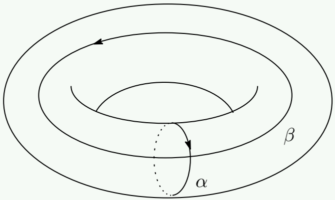
- Coupling constant

$$\begin{aligned} \tau &= \frac{\partial^2 \mathcal{F}}{\partial a^2} \\ &= \tau_0 + \frac{i}{\pi} \log\left(\frac{a^2}{\Lambda^2}\right) + \cdots \end{aligned}$$

Seiberg-Witten shows hot to get the prepotential $\mathcal{F}(a)$

Seiberg-Witten solution

$$a = \oint_\alpha dS, \quad a_D \equiv \frac{\partial \mathcal{F}}{\partial a} = \oint_\beta dS, \qquad dS = \frac{1}{2\pi i} z \frac{dw}{w}$$



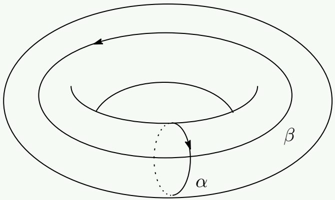
- Coupling constant

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{da_D/du}{da/du}$$

complex structure

Seiberg-Witten solution

$$a = \oint_\alpha dS, \quad a_D \equiv \frac{\partial \mathcal{F}}{\partial a} = \oint_\beta dS, \qquad dS = \frac{1}{2\pi i} z \frac{dw}{w}$$



- Coupling constant

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{da_D/du}{da/du}$$

complex structure

Seiberg-Witten curve for $\mathrm{SU}(2)$ theory

$$z^2 + u = \Lambda^n \left(w + \frac{1}{w} \right)$$

- $\mathrm{SU}(n)$ theory : $z^2 + u \to z^n + \cdots$ (monic polynomial)

Summary

- Solution to $\mathcal{N} = 2$ supersymmetric gauge theory
  $\longrightarrow$ characterized by Riemann surface: Seiberg-Witten curve

Summary

- Solution to $\mathcal{N} = 2$ supersymmetric gauge theory
  $\longrightarrow$ characterized by Riemann surface: Seiberg-Witten curve

How to get Seiberg-Witten curve from matrix model?

Contents

- How to obtain the matrix model for $\mathcal{N} = 2$ theory?

- How to obtain the matrix model for $\mathcal{N} = 2$ theory?

Path integral for $\mathcal{N} = 2$ theory exactly evaluated

- How to obtain the matrix model for $\mathcal{N} = 2$ theory?

Path integral for $\mathcal{N} = 2$ theory exactly evaluated

Partition function can be written in a **combinatorial** way

Nekrasov partition function for $\mathrm{SU}(n)$ gauge theory

$$Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda) = \sum_{\vec{\lambda}} \Lambda^{2n|\vec{\lambda}|} Z_{\vec{\lambda}}(\vec{a};\epsilon_1,\epsilon_2)$$

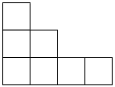
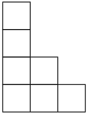
$$Z_{\vec{\lambda}}(\vec{a};\epsilon_1,\epsilon_2) = \prod_{l,m;i,j} \frac{1}{a_l - a_m - \epsilon_1(\check{\lambda}_j^{(m)} - i + 1) + \epsilon_2(\lambda_i^{(l)} - j)}$$

[Nekrasov '02] [Nekrasov-Okounkov '03]

- Summation over n-tuple partition

$$\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}), \qquad \lambda_1^{(l)} \geq \lambda_2^{(l)} \geq \cdots \geq \lambda_N^{(l)}$$

  - $\check{\lambda}_i^{(l)}$: transposed partition

$\lambda = (4,2,1)$:  $\longleftrightarrow$ $\check{\lambda} = (3,2,1,1)$: 

- How to get Seiberg-Witten solution from $Z(\vec{a}; \epsilon_1, \epsilon_2, \Lambda)$?

- How to get Seiberg-Witten solution from $Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda)$?

Seiberg-Witten solution (prepotential)

$$\mathcal{F}(\vec{a};\Lambda) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda)$$

[Nekrasov '02]
[Nekrasov-Okounkov] [Nakajima-Yoshioka] [Bravermann-Etingof]

- How to get Seiberg-Witten solution from $Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda)$?

Seiberg-Witten solution (prepotential)

$$\mathcal{F}(\vec{a};\Lambda) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z(\vec{a};\epsilon_1,\epsilon_2,\Lambda)$$

[Nekrasov '02]
[Nekrasov-Okounkov] [Nakajima-Yoshioka] [Bravermann-Etingof]

Today's goal

Seiberg-Witten curve from matrix model

Matrix integral representation

$$
\begin{aligned}
Z_{\text{matrix}} &= \int \mathcal{D}X \, e^{-\frac{1}{\hbar}\operatorname{Tr} V(X)} \\
&= \int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \, e^{-\frac{1}{\hbar}\sum_{i=1}^{N} V(x_i)}
\end{aligned}
$$

- Vandermonde determinant

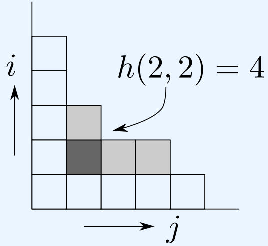
$$
\Delta(x)^2 = \prod_{i<j}^{N} (x_i - x_j)^2
$$

Abelian theory

$\mathrm{U}(1)$ partition function

$$Z_{\mathrm{U}(1)} = \sum_{\lambda} \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2}$$

Hook length



$$h(i,j) = \lambda_i - j + \check{\lambda}_j - i + 1$$

- cf. Plancherel measure (symmetric group: $\mathfrak{S}_\infty \supset \mathfrak{S}_N$)

- Weight function

$$\prod_{(i,j)\in\lambda}\frac{1}{h(i,j)^2} = \prod_{i<j}^{N}(\lambda_i-\lambda_j-i+j)^2\prod_{i=1}^{N}\frac{1}{\Gamma(\lambda_i+N-i+1)^2}$$

$$= \prod_{i<j}^{N}(\xi_i-\xi_j)^2\prod_{i=1}^{N}\frac{1}{\Gamma(\xi_i)^2} \quad (\xi_i=\lambda_i+N-i+1)$$

$$\equiv \Delta^2(\xi)\exp\left(-\frac{1}{\hbar}\sum_{i=1}^{N}V(\xi_i)\right),$$

$$V(\xi) = 2\hbar\log\Gamma(\xi)$$

- Weight function

$$\prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2} = \prod_{i<j}^{N} (\lambda_i - \lambda_j - i + j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(\lambda_i + N - i + 1)^2}$$

$$= \prod_{i<j}^{N} (\xi_i - \xi_j)^2 \prod_{i=1}^{N} \frac{1}{\Gamma(\xi_i)^2} \quad (\xi_i = \lambda_i + N - i + 1)$$

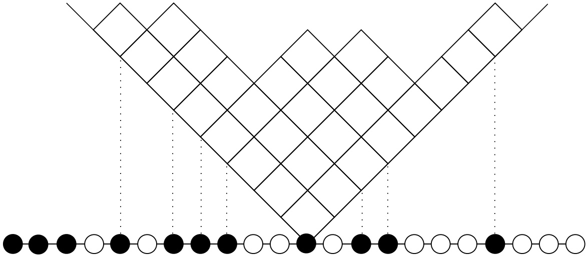
$$\equiv \Delta^2(\xi) \exp\left(-\frac{1}{\hbar} \sum_{i=1}^{N} V(\xi_i) \right),$$

$$V(\xi) = 2\hbar \log \Gamma(\xi)$$

- Scaling limit: $x_i = \hbar \xi_i$, $\hbar \longrightarrow 0$

$$V(\xi) = 2\Big(x \log x - x\Big) + \mathcal{O}(\hbar)$$

1-dim particle description



- Fermionization: $\xi_i = \lambda_i + N - i + 1$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \quad \longrightarrow \quad \xi_1 > \xi_2 > \cdots > \xi_N$$

Young diag $\Longleftrightarrow$ 1-dim particle

Matrix integral representation of partition function

$$Z_{\mathrm{U}(1)} \approx \int \prod_{i=1}^{N} dx_i \, \Delta(x)^2 \, e^{-\frac{1}{\hbar}\sum V_{\mathrm{U}(1)}(x_i)}$$

Matrix potential

$$V_{\mathrm{U}(1)}(x) = 2\left(x \log \frac{x}{\Lambda} - x\right)$$

- cf. $\mathbb{C}\mathbf{P}^1$ matrix model [Eguchi-Yang]

$$Z(\{g_k\}) = \sum_{\lambda} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)^2} \prod_{k=1}^{\infty} e^{-g_k C_k(\lambda)}$$

Non-Abelian theory

- $\mathrm{SU}(n)$ theory with $-\epsilon_1 = \epsilon_2 = \hbar$

$$Z_{\vec{\lambda}}(\vec{a};\hbar) = \prod_{l,m;i,j} \frac{1}{a_l - a_m + \hbar(\lambda_i^{(l)} - j + \check{\lambda}_j^{(m)} - i + 1)}$$

- Fermionic variable

$$\xi_i^{(l)} = \lambda_i^{(l)} + N - i + 1 + \frac{a_l}{\hbar}$$

$$\Downarrow$$

$$(\zeta_1, \zeta_2, \cdots, \zeta_{nN}) = (\xi_1^{(n)}, \cdots, \xi_N^{(n)}, \xi_1^{(n-1)}, \cdots\cdots, \xi_N^{(2)}, \xi_1^{(1)}, \cdots, \xi_N^{(1)})$$

where

$$\zeta_1 > \zeta_2 > \cdots > \zeta_{nN}$$

Matrix potential for $\mathrm{SU}(n)$ gauge theory

$$V_{\mathrm{SU}(n)}(x) = 2 \sum_{l=1}^{n} \left[(x - a_l) \log \left(\frac{x - a_l}{\Lambda} \right) - (x - a_l) \right]$$

[Klemm-Sułkowski]

Matrix model involving "external field"

$$V(x) = \sum_{l=1}^{n} U(x - a_l)$$

$$\sum_{i=1}^{N} V(x_i) = \sum_{l=1}^{n} \sum_{i=1}^{N} U(x_i - a_l) = \mathrm{Tr}\ U(X - A)$$

[Brézin-Hikami] [Zinn-Justin]

- How to get Seiberg-Witten curve?

- How to get Seiberg-Witten curve?

Asymptotic behavior at large N limit

Large N analysis

- Saddle point approximation : $N \to \infty$, $\hbar \to 0$

$$Z \approx \int \prod_{i=1}^{N} dx_i \, e^{-\frac{1}{\hbar^2}\mathcal{F}_{\text{eff}}(\{x_i\})} \quad \longrightarrow \quad e^{-\frac{1}{\hbar^2}\mathcal{F}_{\text{eff}}(\{x_i^{(0)}\})}$$

Large N analysis

- Saddle point approximation : $N \to \infty$, $\hbar \to 0$

$$Z \approx \int \prod_{i=1}^{N} dx_i \, e^{-\frac{1}{\hbar^2}\mathcal{F}_{\text{eff}}(\{x_i\})} \quad \longrightarrow \quad e^{-\frac{1}{\hbar^2}\mathcal{F}_{\text{eff}}(\{x_i^{(0)}\})}$$

- Effective potential

$$\begin{aligned} \frac{1}{\hbar^2}\mathcal{F}_{\text{eff}}(\{x_i\}) &= \frac{1}{\hbar}\sum_{i=1}^{N} V(x_i) - 2\sum_{i<j}^{N} \log(x_i - x_j) \\ &\to \int dx \, \rho(x) V(x) - 2\int_{x<y} dx dy \, \rho(x)\rho(y) \log(x-y) \end{aligned}$$

Large N analysis

- Extremal condition (equation of motion)

$$\frac{1}{2}V'(x) = \mathrm{P}\int dy\, \frac{\rho(y)}{x-y}$$

Large N analysis

- Extremal condition (equation of motion)

$$\frac{1}{2}V'(x) = \mathrm{P}\int dy\, \frac{\rho(y)}{x-y}$$

$$\Downarrow$$

$$V'(x) = \omega(x+i\epsilon) + \omega(x-i\epsilon)$$

- Resolvent

$$\omega(x) = \int dy\, \frac{\rho(y)}{x-y}$$

- Analytic (entire) function

$$\Lambda\left(e^{y(x)/2} + e^{-y(x)/2}\right) = x + \mathrm{const.}$$

where

$$y(x) = V'(x) - 2\omega(x) = \omega_{\mathrm{sing}}(x)$$

- Analytic (entire) function

$$\Lambda \left(e^{y(x)/2} + e^{-y(x)/2} \right) = x + \text{const.}$$

where

$$y(x) = V'(x) - 2\omega(x) = \omega_{\text{sing}}(x)$$

Spectral curve

$$x + \text{const.} = \Lambda \left(e^{y(x)/2} + e^{-y(x)/2} \right) \equiv \Lambda \left(w + \frac{1}{w} \right)$$

- $\mathrm{SU}(n)$ theory : $x + \text{const.} \longrightarrow x^n + \cdots$ (monic polynomial)

- Analytic (entire) function

$$\Lambda\left(e^{y(x)/2}+e^{-y(x)/2}\right)=x+\mathrm{const.}$$

where

$$y(x)=V'(x)-2\omega(x)=\omega_{\mathrm{sing}}(x)$$

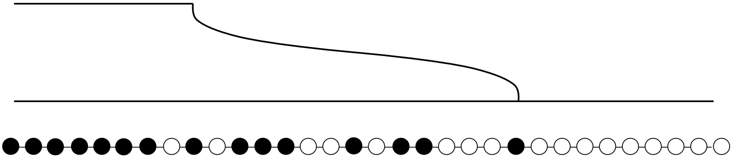
Spectral curve

$$x+\mathrm{const.}=\Lambda\left(e^{y(x)/2}+e^{-y(x)/2}\right)\equiv\Lambda\left(w+\frac{1}{w}\right)$$

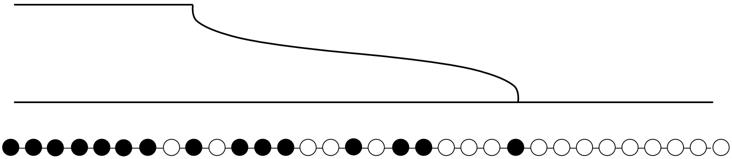
- $\mathrm{SU}(n)$ theory : $x+\mathrm{const.}\longrightarrow x^n+\cdots$ (monic polynomial)

Seiberg-Witten curve

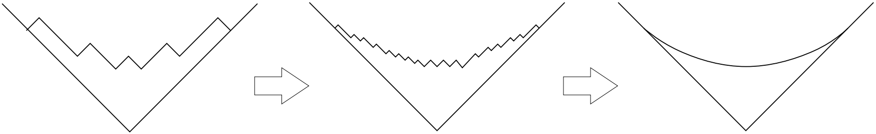
- Eigenvalue density



- Eigenvalue density



- Profile function



Summary

- Matrix model from the gauge theory partition function
- Spectral curve in the large N limit consistent with SW curve

Contents

β-deformation

- Changing matrix symmetry: symmetric, hermitian, self-dual

$$X = U^{-1} \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} U, \qquad U \in \begin{cases} \mathrm{O}(N) \\ \mathrm{U}(N) \\ \mathrm{Sp}(N) \end{cases}$$

β-deformation

- Changing matrix symmetry: symmetric, hermitian, self-dual

$$X = U^{-1} \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} U, \qquad U \in \begin{cases} \mathrm{O}(N) \\ \mathrm{U}(N) \\ \mathrm{Sp}(N) \end{cases}$$

Matrix integral

$$\begin{aligned} Z_{\mathrm{matrix}} &= \int \mathcal{D}X \, e^{-\frac{1}{\hbar}\mathrm{Tr}\, V(X)} \\ &= \int \prod_{i=1}^{N} dx_i \, \Delta(x)^{\mathbf{2}\beta} \, e^{-\frac{1}{\hbar}\sum_{i=1}^{N} V(x_i)} \end{aligned}$$

symmetric: $\beta = \frac{1}{2}$, hermitian: $\beta = 1$, self-dual: $\beta = 2$

- continuation to arbitrary $\beta \in \mathbb{R}_{>0}$

β-ensemble matrix model

- cf. Beta-function, Selberg integral..

$$B(a,b) = \int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$S(\alpha,\beta,\gamma) = \int_0^1 \prod_{i=1}^N dx_i \prod_{i<j}^N (x_i - x_j)^{2\gamma} \prod_{i=1}^N x_i^{\alpha-1}(1-x_i)^{\beta-1}$$

...naturally coming from gauge theory partition function

Example 1

β-deformation

$$Z_\lambda^{(\beta)} = \prod_{(i,j)\in\lambda} \frac{1}{h^\beta(i,j) h_\beta(i,j)}$$

- Inhomogeneous hook-length

$$h_\beta(i,j) = \lambda_i - j + \beta(\check{\lambda}_j - i) + 1, \quad h^\beta(i,j) = \lambda_i - j + \beta(\check{\lambda}_j - i) + \beta$$

- Gauge theory interpretation: Ω-background

$$\beta = -\frac{\epsilon_1}{\epsilon_2} \longrightarrow 1 \quad \text{when} \quad \epsilon_1 = -\epsilon_2 = \hbar$$

- Weight function

$$
\begin{aligned}
&\frac{1}{h_\beta(i,j)h^\beta(i,j)} \\
= &\prod_{i<j}^{N} \frac{\Gamma(\lambda_i-\lambda_j+\beta(j-i)+\beta)\Gamma(\lambda_i-\lambda_j+\beta(j-i)+1)}{\Gamma(\lambda_i-\lambda_j+\beta(j-i))\Gamma(\lambda_i-\lambda_j+\beta(j-i)+1-\beta)} \\
&\times \prod_{i=1}^{N} \frac{\Gamma(\beta)}{\Gamma(\lambda_i+\beta(N-i)+\beta)\Gamma(\lambda_i+\beta(N-i)+1)}
\end{aligned}
$$

Matrix measure

$$
\Delta^2(x) \longrightarrow \prod_{i<j}^{N}(x_i-x_j)^{2\beta}
$$

β-ensemble matrix model

Example 2

q-deformation

$$Z_\lambda^{(q,t)} = \prod_{(i,j)\in\lambda} \frac{(1-q)(1-q^{-1})}{(1-q^{\lambda_i-j+1}t^{\check{\lambda}_j-i})(1-q^{-\lambda_i+j}t^{-\check{\lambda}_j+i-1})}$$

- q-integer

$$[n]_q = \frac{1-q^n}{1-q} \longrightarrow n \quad (q \to 1)$$

- Reduction: $t = q^\beta$, $q = e^{-\hbar R} \longrightarrow 1 \quad \Longrightarrow \quad \beta$-deformation
  - cf. $5d$ partition function on $\mathbb{R}^4 \times S^1$

Matrix measure

$$\Delta^2(x) \longrightarrow \prod_{i<j}^{N} \left(\frac{2}{R} \sinh \frac{R}{2} (x_i - x_j) \right)^{2\beta}$$

Matrix potential

$$V^{(q,t)}(x) = -\frac{1}{R} \left[\mathrm{Li}_2 \left(e^{Rx} \right) - \mathrm{Li}_2 \left(e^{-Rx} \right) \right]$$

- Partition function

$$Z^{(q,t)} \approx \int \prod_{i=1}^{N} dx_i \prod_{i<j}^{N} \left(\frac{2}{R} \sinh \frac{R}{2} (x_i - x_j) \right)^{2\beta} e^{-\frac{1}{\hbar} \sum V(x_i)}$$

Example 3

Orbifold partition function

$$Z_{\text{orbifold}} = \sum_{\lambda} \prod_{\Gamma\text{-inv} \subset \lambda} \frac{(\Lambda/\hbar)^2}{h(i,j)^2}$$

Γ-invariant sector: $\quad h(i,j) \equiv 0 \pmod{r}$

- Gauge theory interpretation

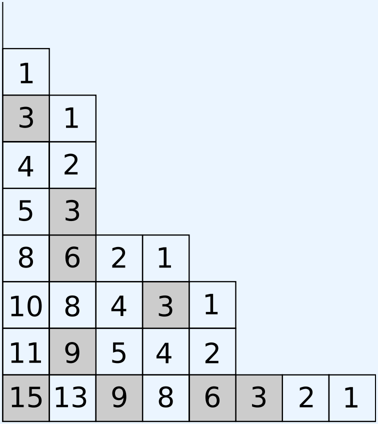
  $\longrightarrow$ instanton counging on orbifold $\mathbb{C}^2/\Gamma$ with $\Gamma = \mathbb{Z}_r$

- Resolving its singularity $\longrightarrow$ ALE space

  [Eguchi-Hanson] [Gibbons-Hawking]

  Γ: a finite subgroup of $\mathrm{SU}(2) \longrightarrow$ ADE classification

Γ-invariant sector for $\mathrm{U}(1)$ theory



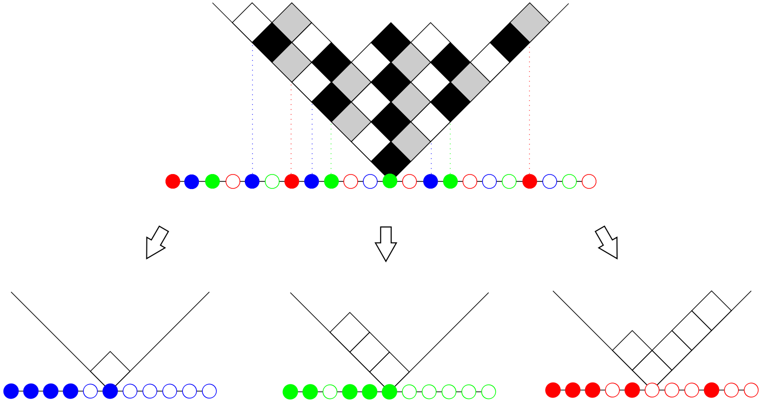
Numbers in boxes stand for their hook lengths $\lambda_i - j + \check{\lambda}_j - i + 1$.
Shaded boxes are invariant under the action of $\Gamma = \mathbb{Z}_3$.

- How to study asymptotics of the orbifold partition function?

- How to study asymptotics of the orbifold partition function?

q-deformation and its root of unity limit

1-dim particle description



n-tuple partition $\longrightarrow$ nr-tuple partition ($\longrightarrow$ r-tuple partition)

Matrix model for $\mathbb{Z}_r$-orbifold partition function

$$Z_{\mathrm{matrix}} = \int \prod_{v=1}^{r} \prod_{i=1}^{N^{(v)}} dx_i^{(v)} \, \Delta^2(x) \, \exp\left(-\frac{1}{\hbar} \sum_{v=1}^{r} \sum_{i=1}^{N^{(v)}} V_{\mathbb{Z}_r}(x_i^{(v)}) \right)$$

Matrix measure for $\mathbb{Z}_r$ theory

$$\begin{aligned} \Delta^2(x) &= \prod_{v=1}^{r} \prod_{i<j}^{N^{(v)}} \left(x_i^{(v)} - x_j^{(v)} \right)^{2(\beta-1)/r+2} \\ &\quad \times \prod_{v<w}^{r} \prod_{i,j} \left(x_i^{(v)} - x_j^{(w)} \right)^{2(\beta-1)/r} \end{aligned}$$

Matrix potential for $\mathbb{Z}_r$ theory

$$V_{\mathbb{Z}_r}(x) = \frac{2}{r} \sum_{l=1}^{n} \left[(x - a_l) \log \left(\frac{x - a_l}{\Lambda} \right) - (x - a_l) \right]$$

Multi-matrix model from orbifold partition function

- SW curve for orbifold $\mathbb{C}^2/\mathbb{Z}_r$

$$V_{\mathbb{Z}_r}(x) = \frac{2}{r}\sum_{l=1}^{n}\left[(x-a_l)\log\left(\frac{x-a_l}{\Lambda}\right)-(x-a_l)\right]$$

- SW curve for orbifold $\mathbb{C}^2/\mathbb{Z}_r$

$$V_{\mathbb{Z}_r}(x) = \frac{2}{r} \sum_{l=1}^{n} \left[(x - a_l) \log \left(\frac{x - a_l}{\Lambda} \right) - (x - a_l) \right]$$

$$\Downarrow$$

$$P_n(z) = \Lambda^n \left(e^{ry/2} + e^{-ry/2} \right) \equiv \Lambda^n \left(w^r + \frac{1}{w^r} \right)$$

Seiberg-Witten curve with r-th root branch

[Kimura '11]

Contents

- Solution to $\mathcal{N} = 2$ supersymmetric gauge theory

  $\longrightarrow$ characterized by Riemann surface: SW curve

- Large N analysis of matrix model from combinatorics

  $\longrightarrow$ SW curve given by spectral curve of matrix model

- Some extentions

  $\longrightarrow$ β and q-deformations and orbifolding

Discussions

- Vortex on orbifold $\mathbb{C}/\mathbb{Z}_r$
  [Kimura-Nitta '11] [Fujimori-Kimura-Nitta-Ohashi '12]
- Condensed-matter physics: Quantum Hall effect [Kimura '12]
- Combinatorics, statistical mechanics, other matrix models...