RECENT TOPICS ON LATTICE FERMIONS

Taro Kimura (RIKEN)

Unusual

RECENT TOPICS ON LATTICE FERMIONS

Taro Kimura (RIKEN)

Remarks on Lattice Fermions

Problems in Lattice Fermion

- Naive discretization of fermion action is not good:
  - Species doublers
  - Chirality on a lattice
  - Discretization error
  - etc

Lattice Fermions

Min-doubled

Naive

Wilson

Overlap (DW)

Staggered

St. Wilson

St. Overlap

Lattice Fermions

Min-doubled

Naive $\rightarrow$ Wilson Overlap (DW)

Staggered St. Wilson St. Overlap

Lattice Fermions

Min-doubled

Naive ➔ Wilson ➔ Overlap (DW)

Staggered St. Wilson St. Overlap

Lattice Fermions

Min-doubled

Naive → Wilson → Overlap (DW)

↓

Staggered   St. Wilson   St. Overlap

Lattice Fermions

Min-doubled

Naive $\rightarrow$ Wilson $\rightarrow$ Overlap (DW)

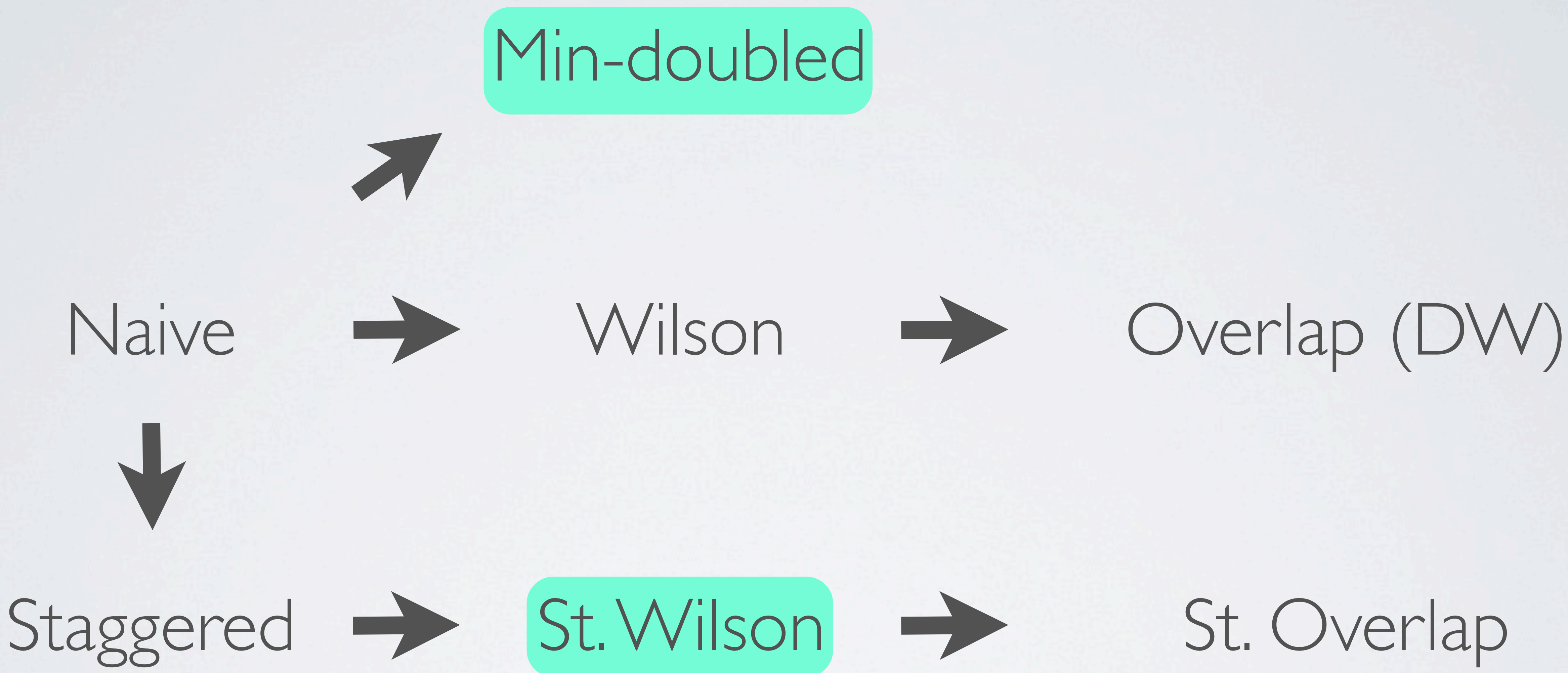
$\downarrow$

Staggered $\rightarrow$ St. Wilson $\rightarrow$ St. Overlap

Lattice Fermions

$$
\begin{tikzcd}
 & \text{Min-doubled} & \\
\text{Naive} \arrow[ur] \arrow[r] \arrow[d] & \text{Wilson} \arrow[r] & \text{Overlap (DW)} \\
\text{Staggered} \arrow[r] & \text{St. Wilson} \arrow[r] & \text{St. Overlap}
\end{tikzcd}
$$

Lattice Fermions



Contents

- Staggered Wilson fermion
  - Chiral limit and Aoki phase
- Minimally-doubled fermion
  - Applicability to finite density
- References
  - M. Creutz, T. Kimura, and T. Misumi, Phys. Rev. **D83** (2011) 094506
  - T. Misumi, T. Z. Nakano, T. Kimura, and A. Ohnishi, Phys. Rev. **D86** (2012) 034501
  - T. Misumi, T. Kimura, and A. Ohnishi, Phys. Rev. **D86** (2012) 094505

PHYSICAL REVIEW D **86,** 094505 (2012)

QCD phase diagram with two-flavor lattice fermion formulations

Tatsuhiro Misumi*

Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA

Taro Kimura†

Mathematical Physics Laboratory, RIKEN Nishina Center, Saitama 351-0198, Japan

Akira Ohnishi‡

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

(Received 11 June 2012; published 6 November 2012)

We propose a new framework for investigating two-flavor lattice QCD with finite temperature and density. We consider the Karsten-Wilczek fermion formulation, in which a species-dependent imaginary chemical potential term can reduce the number of species to two without losing chiral symmetry. This lattice discretization is useful for study on finite-(T, μ) QCD since its discrete symmetries are appropriate for the case. To show its applicability, we study strong-coupling lattice QCD with temperature and chemical potential. We derive the effective potential of the scalar meson field and obtain a critical line of the chiral phase transition, which is qualitatively consistent with the phenomenologically expected phase diagram. We also discuss that $O(1/a)$ renormalization of imaginary chemical potential can be controlled by adjusting a parameter of a dimension-3 counterterm.

PHYSICAL REVIEW D **86,** 094505 (2012)

QCD phase diagram with two-flavor lattice fermion formulations

Tatsuhiro Misumi[*]

Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA

Taro Kimura[†]

Mathematical Physics Laboratory, RIKEN Nishina Center, Saitama 351-0198, Japan

Akira Ohnishi[‡]

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

(Received 11 June 2012; published 6 November 2012)

We propose a new framework for investigating two-flavor lattice QCD with finite temperature and density. We consider the Karsten-Wilczek fermion formulation, in which a species-dependent imaginary chemical potential term can reduce the number of species to two without losing chiral symmetry. This lattice discretization is useful for study on finite-(T, μ) QCD since its discrete symmetries are appropriate for the case. To show its applicability, we study strong-coupling lattice QCD with temperature and chemical potential. We derive the effective potential of the scalar meson field and obtain a critical line of the chiral phase transition, which is qualitatively consistent with the phenomenologically expected phase diagram. We also discuss that $O(1/a)$ renormalization of imaginary chemical potential can be controlled by adjusting a parameter of a dimension-3 counterterm.

STAGGERED WILSON FERMION

Wilson Fermion

- Number of doublers : 16 → 1 [Wilson '74]
- Flavor sensitive "mass term"
  - Momentum dependent mass term

$$M_{\mathrm{W}}(p) = r \sum_{\mu=1}^{4} (1 - \cos p_\mu) \longrightarrow \bar{\psi} D_\mu^2 \psi$$

Staggered Wilson Fermion

- Number of doublers : 4 → 2 or 1 [Adams '10] [Hoelbling '10]
- Flavor sensitive "mass term"
  - Momentum dependent mass term

$$M_{\mathrm{stW}}(p) = \quad ? \qquad \longrightarrow \qquad \bar{\psi} D_\mu^2 \psi$$

Staggered Fermion

- Lattice action

$$S = \frac{1}{2}\sum_{n,\mu} \eta_\mu \bar{\chi}_n (\chi_{n+\hat{\mu}} - \chi_{n-\hat{\mu}}) + \sum_n m \bar{\chi}_n \chi_n$$

$$\eta_\mu = (-1)^{n_1+\cdots+n_{\mu-1}} \sim \gamma_\mu$$

- Chirality : $\epsilon_n = (-1)^{n_1+\cdots+n_4} \sim \gamma_5$

Staggered Wilson term

1. Adams type flavored-mass term [Adams '10] [Golterman-Schmit '84]

$$
\begin{aligned}
M_{\mathrm{A}}(p) &= \epsilon_x \eta_1 \eta_2 \eta_3 \eta_4 \sum_{\mathrm{sym}} \cos p_1 \cos p_2 \cos p_3 \cos p_4 \\
&\simeq \mathbb{1} \otimes \xi_5
\end{aligned}
$$

4-link hopping

$$
\xi_5 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

2-flavor staggered
(w/o rooting)

Staggered Wilson term

2. Hoelbling type flavored-mass term [Hoelbling '10]

$$
\begin{aligned}
M_{\mathrm{H}}(p) &= i\epsilon_{\mu\nu}\eta_\mu\eta_\nu \sum_{\mathrm{sym}} \cos p_\mu \cos p_\nu \\
&\simeq \mathbb{1} \otimes \xi_\mu\xi_\nu
\end{aligned}
$$

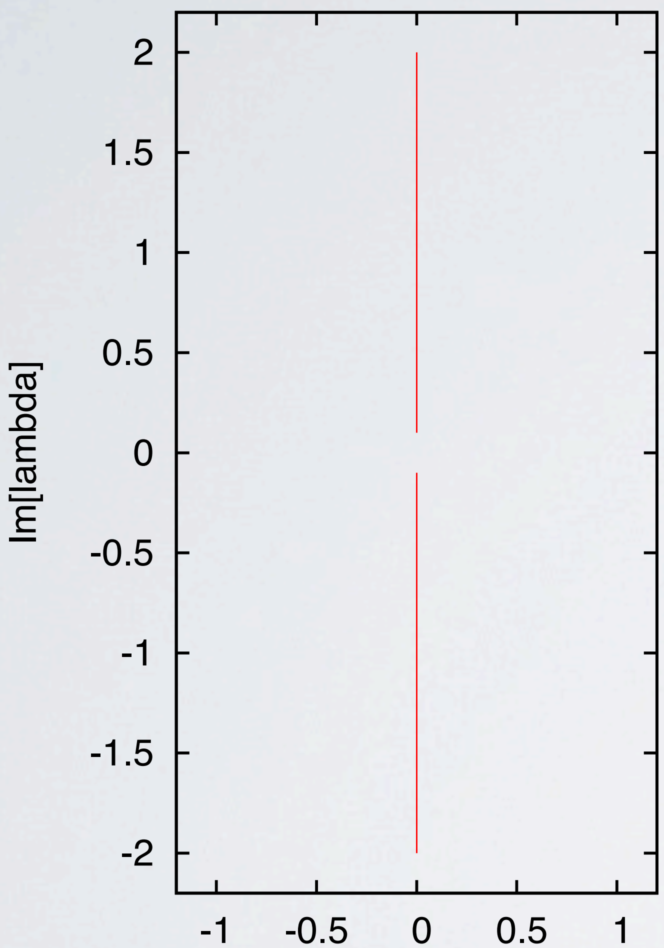
2-link hopping

$$
\xi_\mu\xi_\nu = \begin{pmatrix} +2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
$$

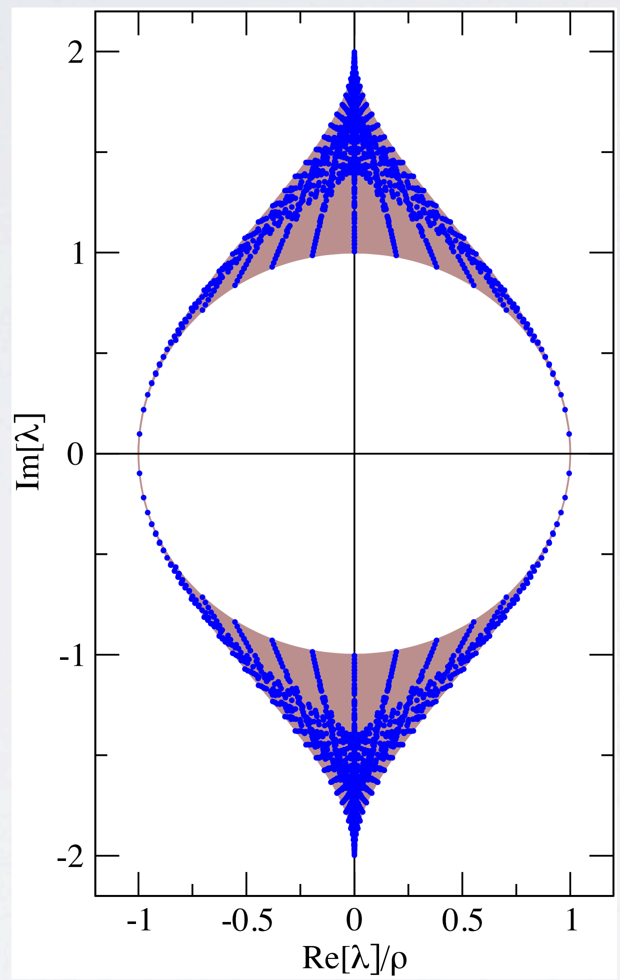
1-flavor staggered
(w/o rooting)

Dirac Spectrum

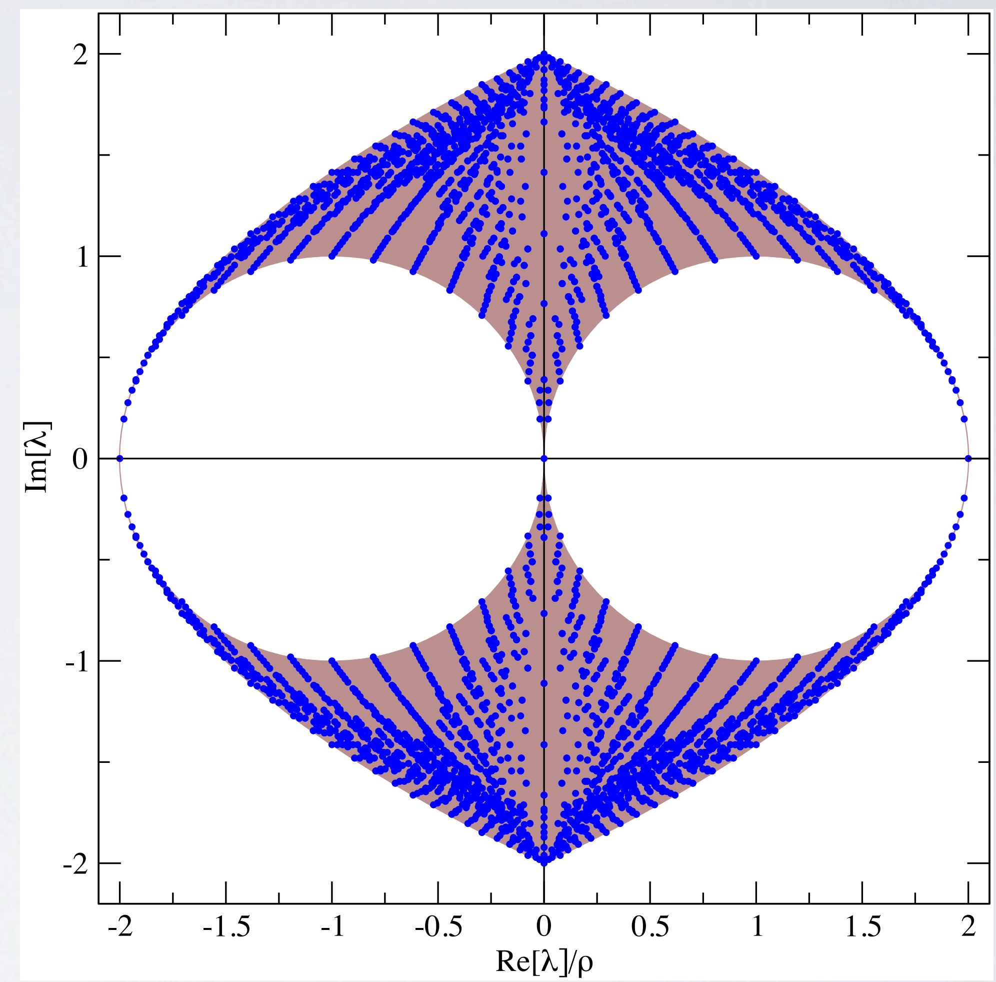
[de Forcrand-Kurkela-Panero '12]



Staggered



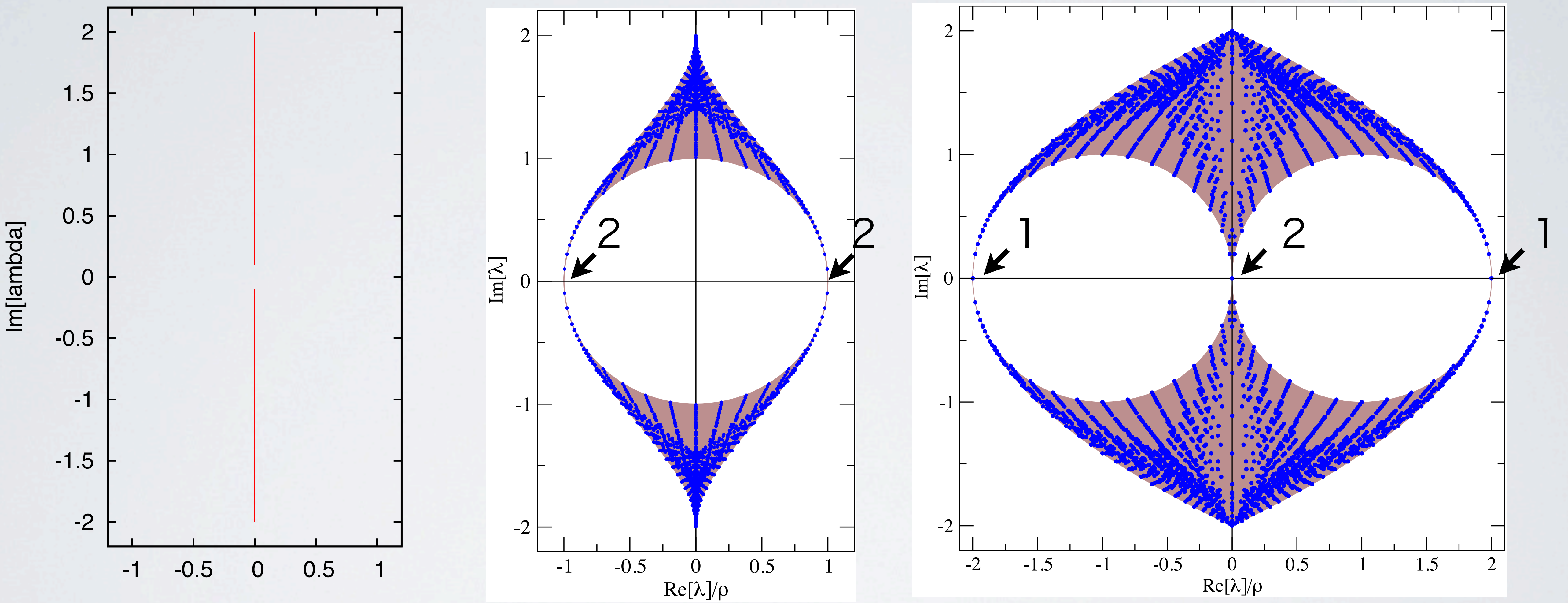
w/ Adams



w/ Hoelbling

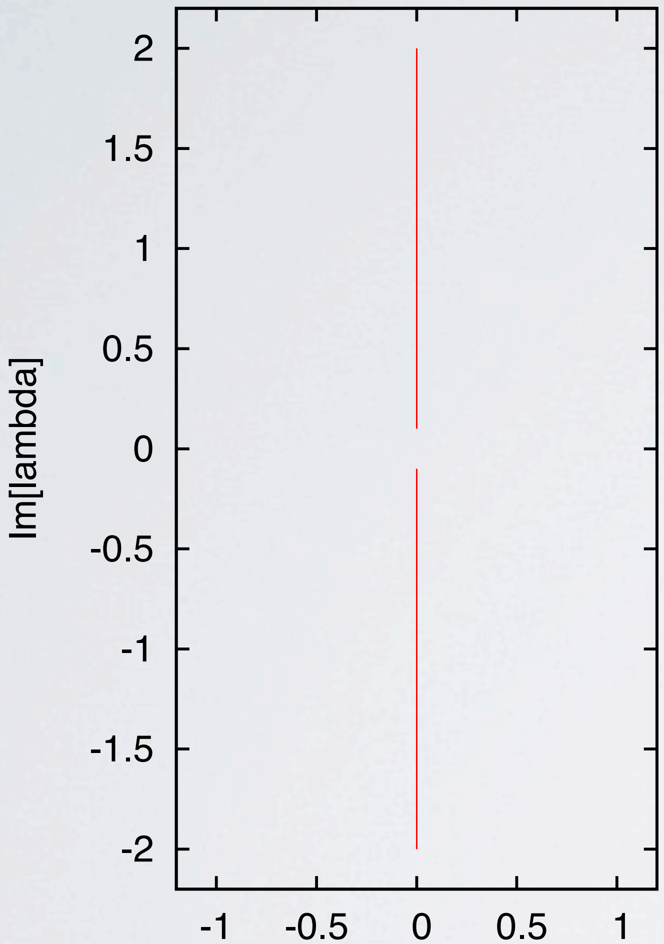
Dirac Spectrum

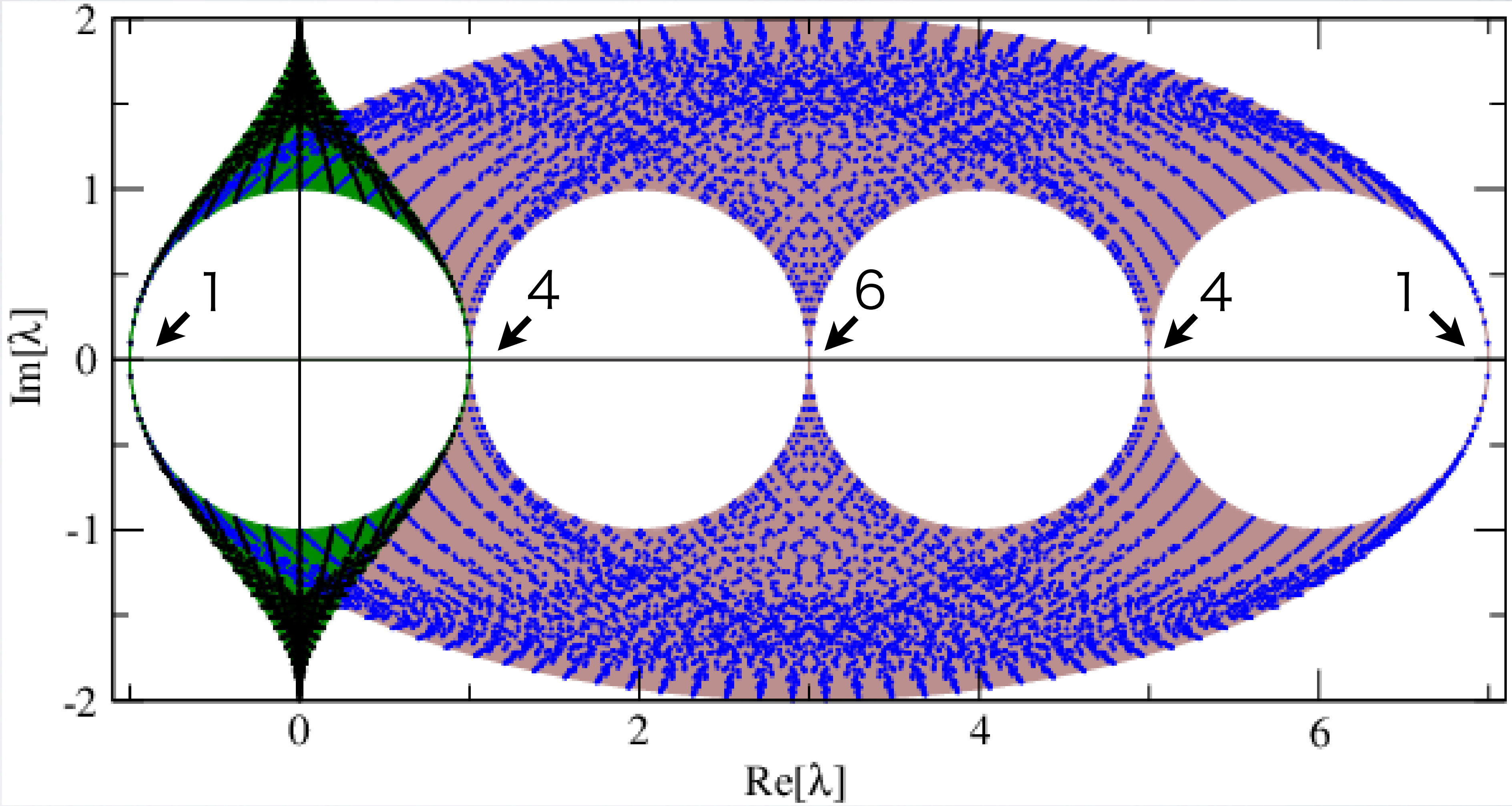
[de Forcrand-Kurkela-Panero '12]



Staggered w/ Adams w/ Hoelbling

Dirac Spectrum





Naive

Wilson

Lattice Fermions

	#species	chiral	spinor
Naive	16	○	4
Wilson	1	×	4
Staggered	4	○	1
St. Wilson	2 or 1	×	1
Minimal dbl.	2	○	4

HOW TO USE ST. WILSON ?

How to Use Wilson

1. Chiral symmetry broken explicitly by Wilson term
2. Mass renormalization
3. Fine tuning required for quark mass

Chiral limit

Existence of the parity-broken phase

(Aoki phase)

[Aoki '84]

How to Use St. Wilson?

1. Chiral symmetry broken explicitly by Wilson term

2. Mass renormalization

3. Fine tuning required for quark mass

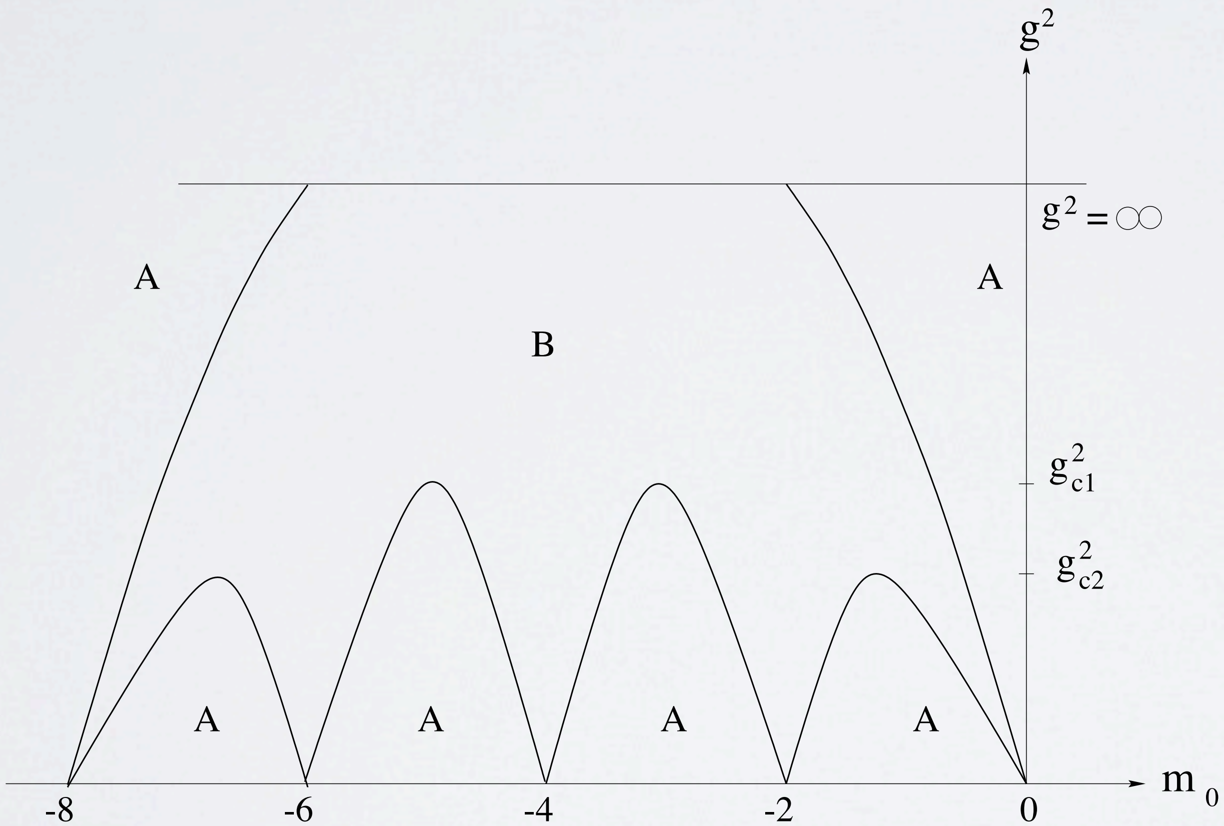
Chiral limit

Existence of the parity-broken phase?

(Aoki phase)

Wilson Phase Structure

- Phase diagram



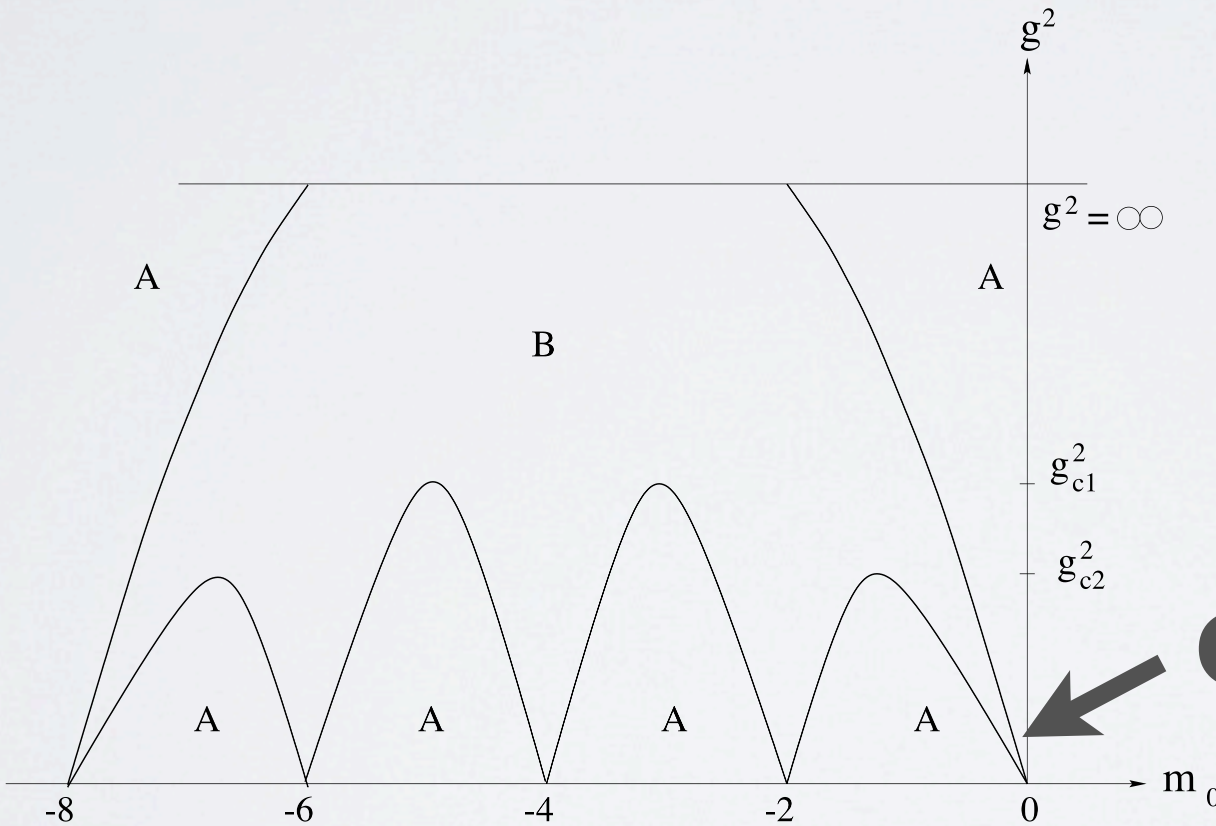
$$A : \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle = 0$$

$$B : \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle \neq 0$$

(Aoki phase)

Wilson Phase Structure

- Phase diagram



$$\mathrm{A}: \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle = 0$$

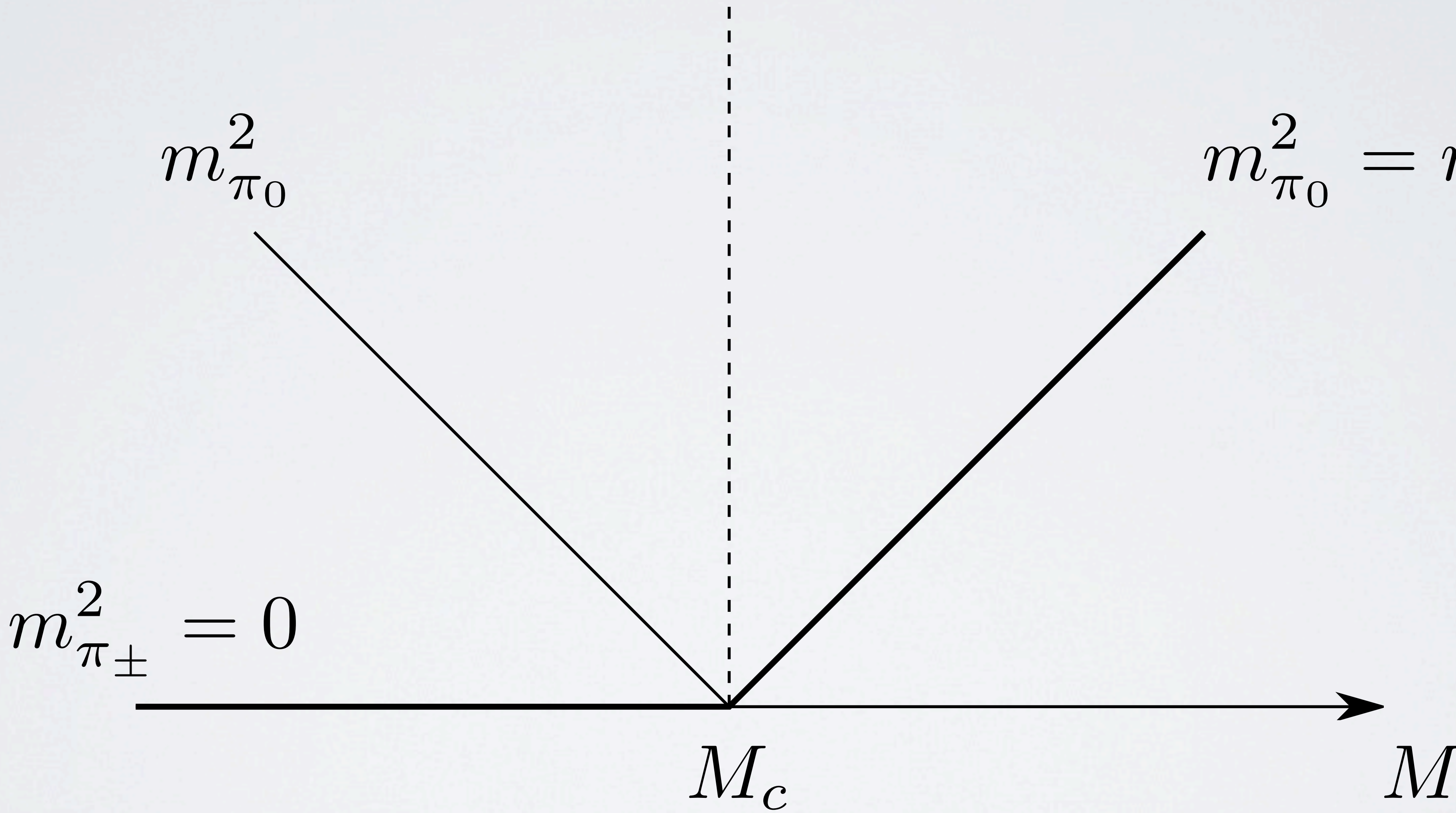
$$\mathrm{B}: \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle \neq 0$$

(Aoki phase)

Chiral limit

Pion Mass

$$\mathrm{B} : \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle \neq 0 \qquad\qquad \mathrm{A} : \langle \bar{\psi}(i\gamma_5 \otimes \tau_3)\psi \rangle = 0$$



2nd order phase transition

cf. [Sharpe-Singleton '98]

St. Wilson Phase Structure

1. QCD-like theory
   - Gross-Neveu model [Creutz-TK-Misumi '11]
2. Strong-coupling lattice QCD [Misumi-Nakano-TK-Ohnishi '12]
   - Hopping parameter expansion
   - Effective potential

STRONG-COUPLING LATTICE QCD

Strong-coupling Analysis

1. Hopping parameter expansion
   - Diagrammatic method
   - Difficulty in treating vacua
2. Effective potential method
   - Link variable integral

Hopping Parameter Expansion

- Staggered Wilson with 2-link hopping
  - Hoelbling type : $M_{\mathrm{H}} = \mathrm{diag}(+2, 0, 0, -2)$

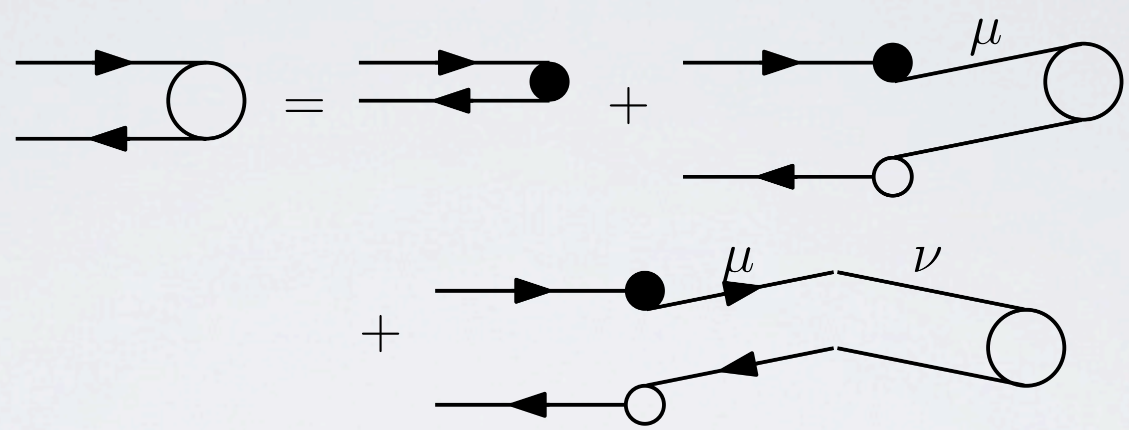
$$D_{\mathrm{stW}} = D_{\mathrm{st}} + r(2 + M_{\mathrm{H}}) + m_0$$

- Hopping parameter

$$K^{-1} = 2(m_0 + 2r)$$

Hopping Parameter Expansion

- 1-pt. function :



  - Parity symmetric : $\sigma = 1, \quad \pi = 0$

  - Parity broken :

$$\sigma = \frac{1}{16K^2}, \quad \pi = \pm\sqrt{\frac{1}{16K^2}\left(1 - \frac{1}{16K^2}\right)}$$

Hopping Parameter Expansion

- 2-pt. function : $\langle \bar{\chi}_0^a \chi_0^a \bar{\chi}_x^a \chi_x^a \rangle, \quad \langle \bar{\chi}_0^a i\epsilon_0 \chi_0^a \bar{\chi}_x^a i\epsilon_x \chi_x^a \rangle$
- Pion mass : $\cosh m_\pi = 1 + \dfrac{1 - 16K^2}{6K^2}$

$$|K| > 1/4 \quad \longrightarrow \quad m_\pi^2 < 0$$

Existence of Aoki phase

↓

Analysis with effective potential

Summary

- Staggered Wilson fermion phase structure
  - Gross-Neveu model [Creutz-TK-Misumi '11]
  - Strong-coupling lattice QCD [Misumi-Nakano-TK-Ohnishi '12]

Existence of Aoki phase as well as Wilson

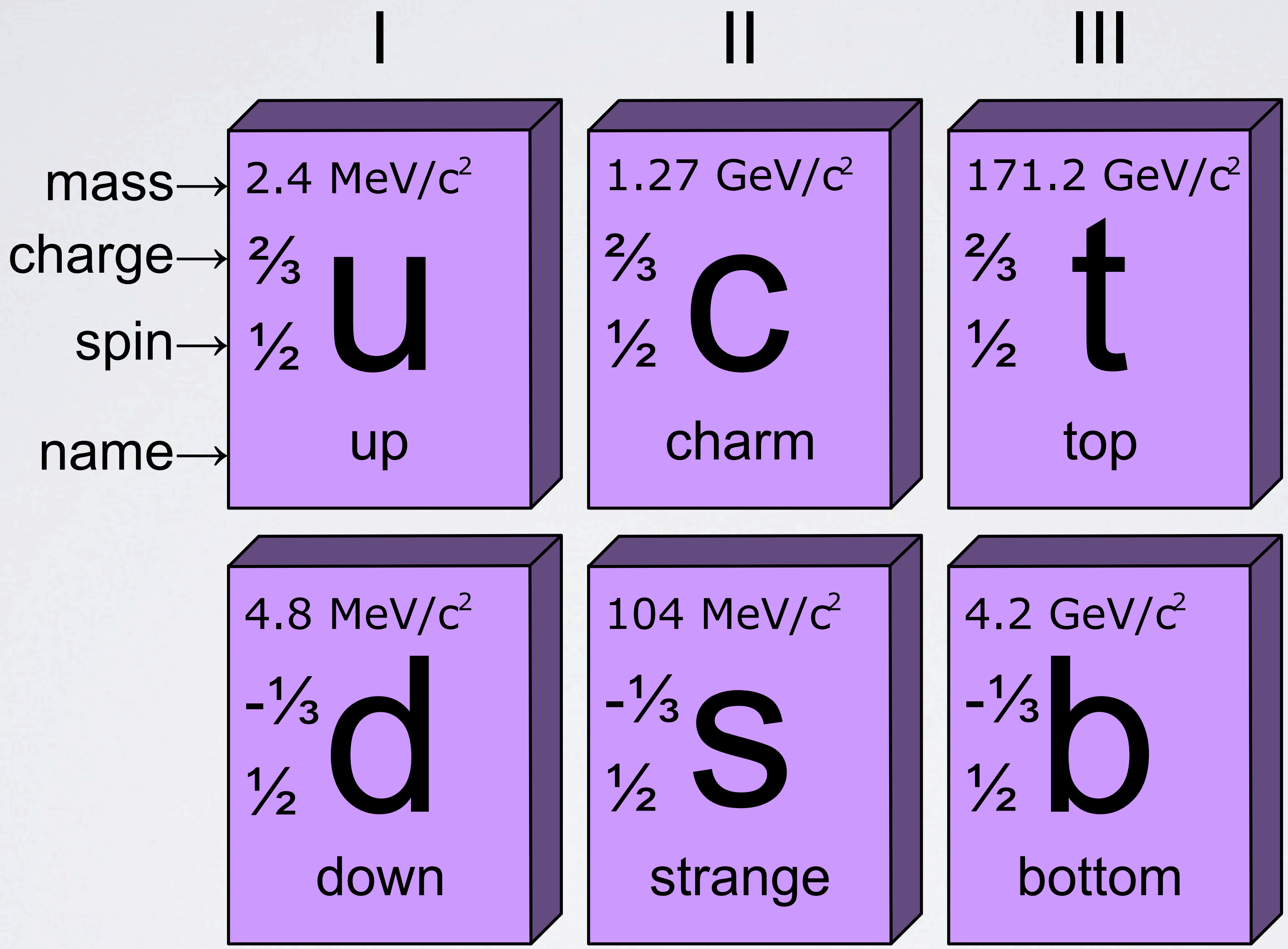
↓

Chiral limit

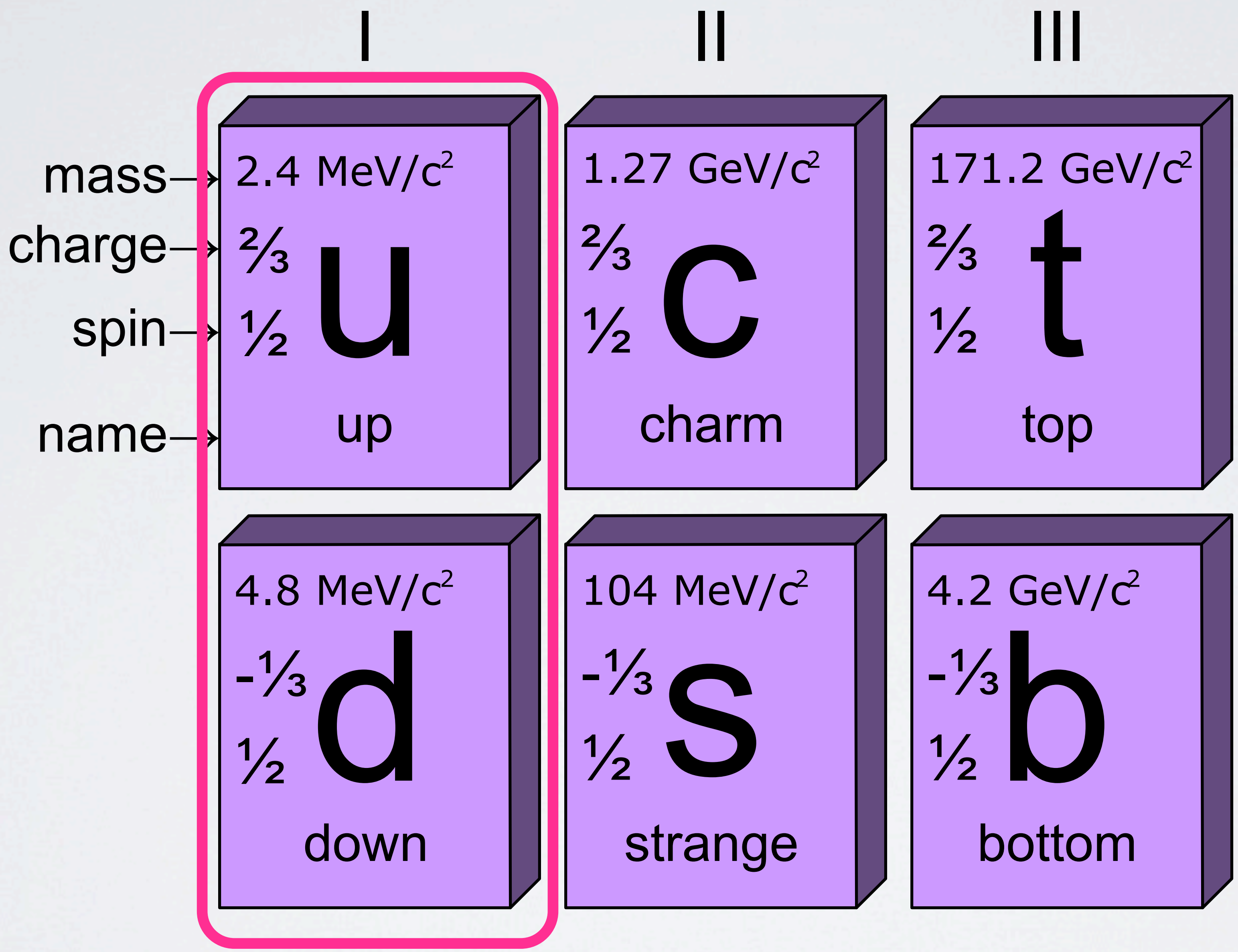
MINIMALLY-DOUBLED FERMION

WHY 2-FLAVOR QCD?

Quarks


I
II
III
mass
charge
spin
name
2.4 MeV/c²
⅔
½
u
up
1.27 GeV/c²
⅔
½
c
charm
171.2 GeV/c²
⅔
½
t
top
4.8 MeV/c²
-⅓
½
d
down
104 MeV/c²
-⅓
½
s
strange
4.2 GeV/c²
-⅓
½
b
bottom

Quarks



Effectively two massless quarks

Nielsen-Ninomiya's Theorem

- Chirality of the lattice fermion has to be canceled out
- There should be doublers, but how many?

#doublers ≧ 2

Lattice Fermions

	#species	chiral	spinor
Naive	16	○	4
Wilson	1	×	4
Staggered	4	○	1
St. Wilson	2 or 1	×	1
Minimal dbl.	2	○	4

Minimally-doubled Fermion

- #doublers = 2
- Exact chiral symmetry
- Ultra-locality

[Karsten '81] [Wilczek '87]
[Creutz '08] [Borici '08]
[Creutz-Misumi '10]

$$D_{\mathrm{KW}}(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i r \gamma_4 \underbrace{\sum_{j=1}^{3} (1 - \cos p_j)}_{\text{Wilson-like term}}$$

not mass, but (imaginary) chemical potential

[Misumi '12]

Minimally-doubled Fermion

- Weak point :

Symmetry is not enough to restore Lorentz symmetry in the continuum limit.

- KW fermion :
    - P & CT
    - Cubic symmetry

Minimally-doubled Fermion

- Weak point :

Symmetry is not enough to restore Lorentz symmetry in the continuum limit.

- KW fermion :
  - P & CT
  - Cubic symmetry

What's the meaning of this symmetry?

Symmetry of KW Fermion

- P & CT
- Cubic symmetry

$$S_{\mathrm{KW}} = \sum_x \Bigg[\frac{1}{2} \sum_{\mu=1}^{4} \bar{\psi}_x \gamma_\mu (U_{x,x+\hat{\mu}} \psi_{x+\hat{\mu}} - U_{x,x-\hat{\mu}} \psi_{x-\hat{\mu}})$$
$$+ \frac{r}{2} \sum_{j=1}^{3} \bar{\psi}_x i\gamma_4 (2\psi_x - U_{x,x+\hat{j}} \psi_{x+\hat{j}} - U_{x,x-\hat{j}} \psi_{x-\hat{j}}) \Bigg]$$

Symmetry of KW Fermion

- P & CT
- Cubic symmetry

$$S_{\mathrm{KW}} = \sum_x \Bigg[\frac{1}{2} \sum_{\mu=1}^{4} \bar{\psi}_x \gamma_\mu (U_{x,x+\hat{\mu}} \psi_{x+\hat{\mu}} - U_{x,x-\hat{\mu}} \psi_{x-\hat{\mu}})$$

$$+ \frac{r}{2} \sum_{j=1}^{3} \underline{\bar{\psi}_x i\gamma_4 (2\psi_x - U_{x,x+\hat{j}} \psi_{x+\hat{j}} - U_{x,x-\hat{j}} \psi_{x-\hat{j}})} \Bigg]$$

Specifying temporal direction

Symmetry of Finite Density

- P & CT
- Cubic symmetry

$$S_{\mathrm{naive}} = \sum_x \Bigg[\sum_{j=1}^{3} \bar{\psi}_x \gamma_j (U_{x,x+\hat{j}} \psi_{x+\hat{j}} - U_{x,x-\hat{j}} \psi_{x-\hat{j}}) \\ + \bar{\psi}_x \gamma_4 (e^{\mu} U_{x,x+\hat{4}} \psi_{x+\hat{4}} - e^{-\mu} U_{x,x-\hat{4}} \psi_{x-\hat{4}}) \Bigg]$$

Symmetry of Finite Density

- P & CT
- Cubic symmetry

$$S_{\mathrm{naive}} = \sum_x \Bigg[\sum_{j=1}^{3} \bar{\psi}_x \gamma_j (U_{x,x+\hat{j}} \psi_{x+\hat{j}} - U_{x,x-\hat{j}} \psi_{x-\hat{j}})$$

$$+ \underline{\bar{\psi}_x \gamma_4 (e^{\mu} U_{x,x+\hat{4}} \psi_{x+\hat{4}} - e^{-\mu} U_{x,x-\hat{4}} \psi_{x-\hat{4}})} \Bigg]$$

Specifying temporal direction

KW fermion

Finite density

Same symmetry

KW fermion

Finite density

Same symmetry

Same universality class in continuum limit

Renormalization Effect

- KW term → flavored chemical potential

$$S_{\mathrm{KW}} = i\gamma_4 \sum_{j=1}^{3} (1 - \cos p_j)$$

  - additive (imaginary) chemical potential renormalization

cf. Wilson fermion

- Counter term : $\mu_3 \, \bar{\psi}_x i\gamma_4 \psi_x$

Tuning this μ₃

STRONG-COUPLING ANALYSIS IN FINITE DENSITY

[Misumi-TK-Ohnishi '12]

Strong-coupling Analysis

1. Link variable integral

2. Bosonization & fermion integral

3. Determine the vacuum from the effective potential

Applied to finite temperature & density

Meson Fields

- Chiral : $\langle \bar{\psi}\psi \rangle = \sigma$
- Vector (imaginary chemical potential) : $\langle \bar{\psi} i\gamma_4 \psi \rangle = \pi_4$

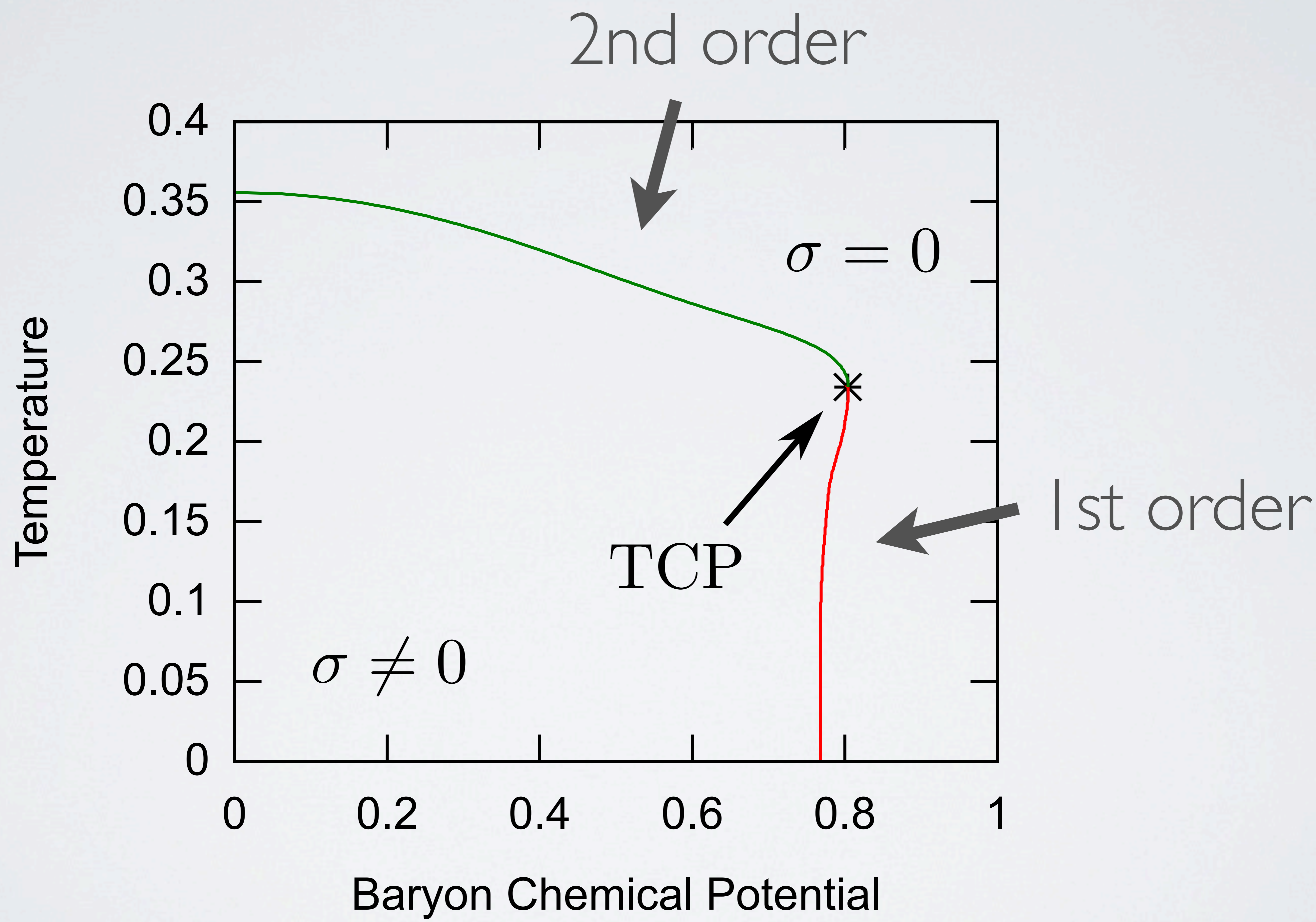
$$\rightarrow \langle i\psi^\dagger \psi \rangle$$

Effective potential for these mesons

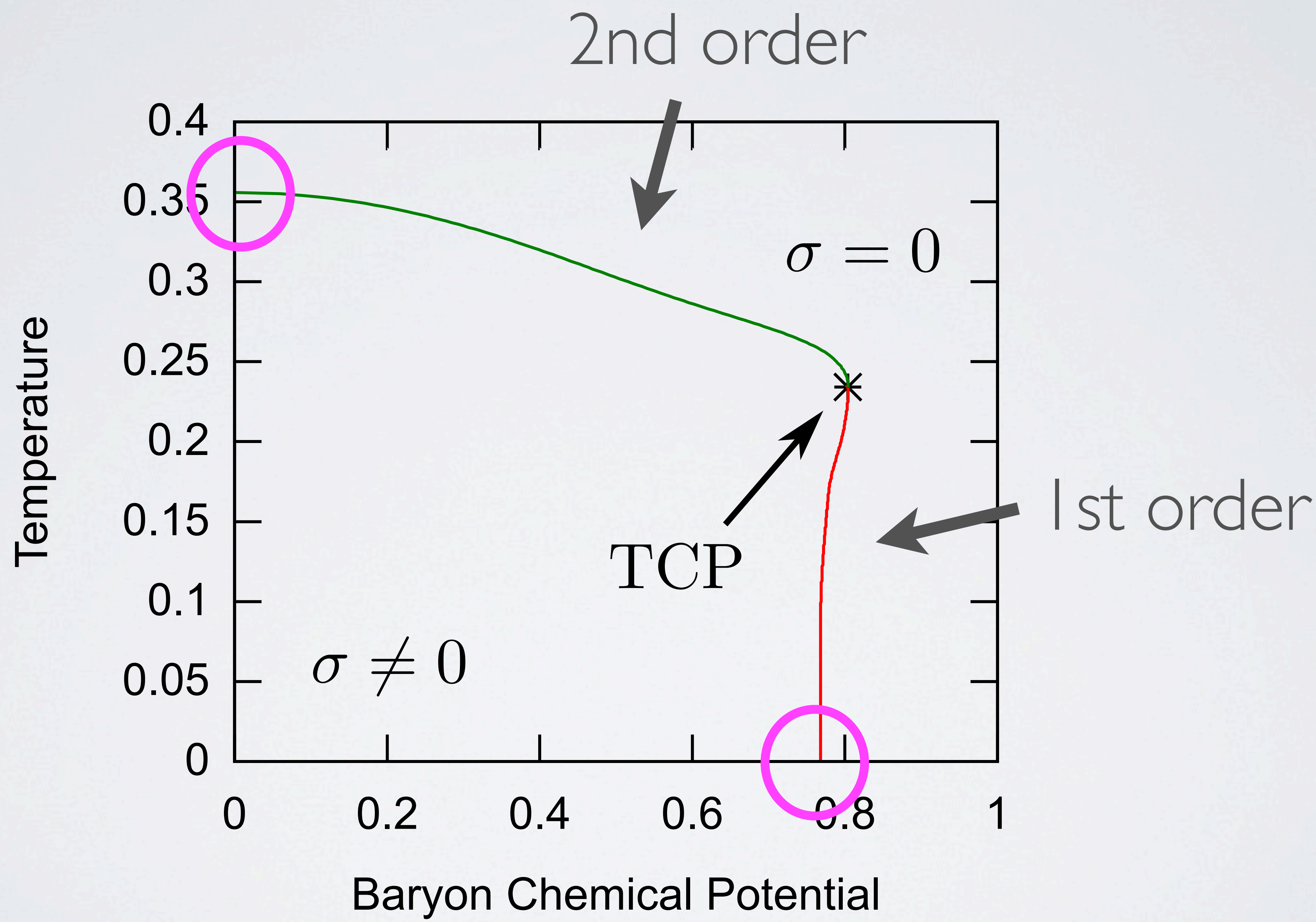
$$\mathcal{F}_{\mathrm{eff}}(\sigma, \pi_4; T, \mu, \mu_3) = \ldots$$

Please see [Misumi-TK-Ohnishi '12]

Chiral Phase Diagram



Chiral Phase Diagram



Chiral Phase Diagram

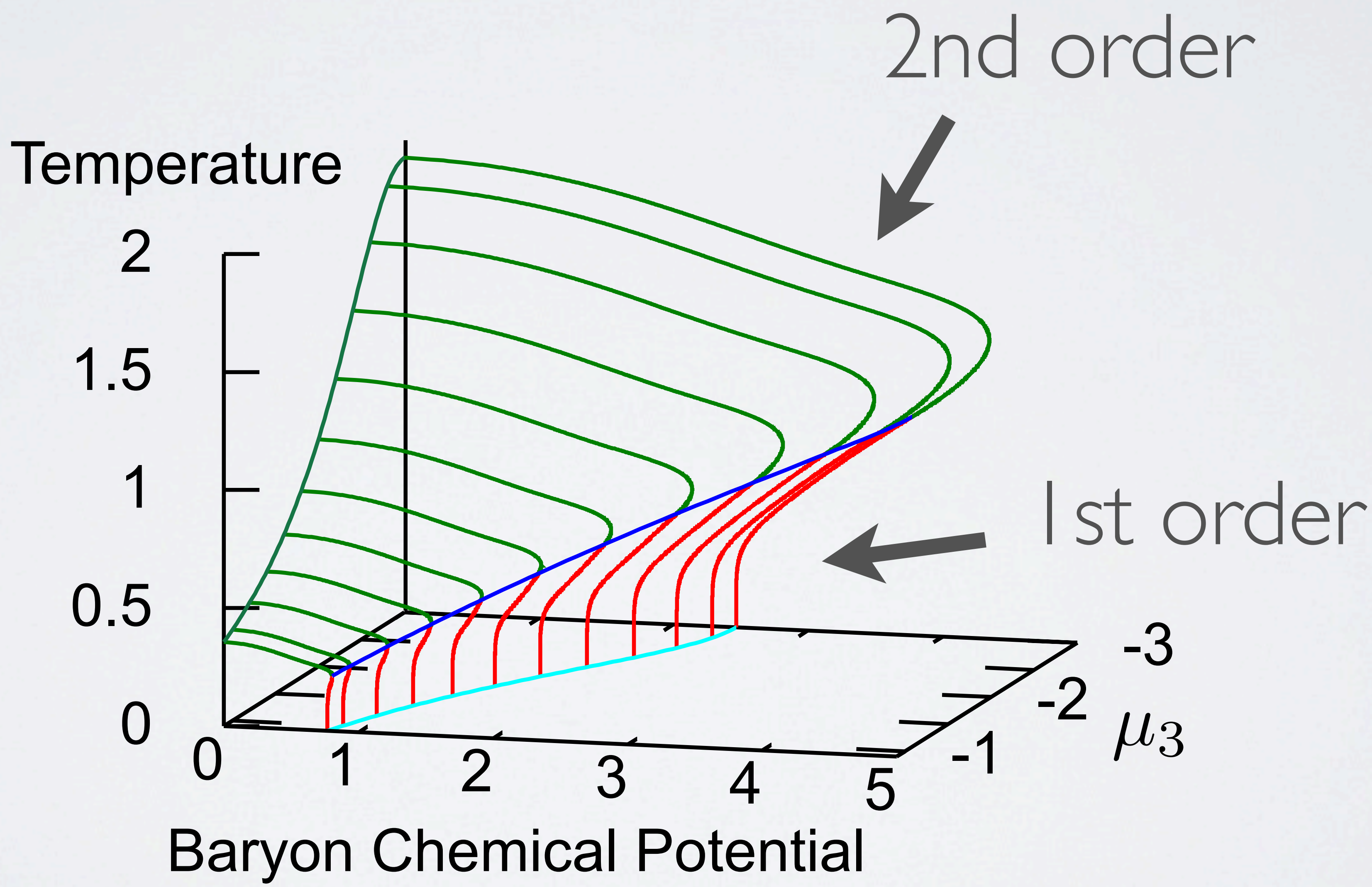
- Critical density/temperature ratio
  - KW fermion : $R^0_{\mathrm{KW}} = \dfrac{\mu_c(T=0)}{T_c(\mu_B=0)} \sim 2.3$
  - Staggered : $R^0_{\mathrm{st}} \sim 1$
  - Phenomenology : $R^0_{\mathrm{ph}} \gtrsim 5.5$

Chiral Phase Diagram

- Tricritical point ratio
  - KW fermion : $R_{\mathrm{KW}}^{\mathrm{tri}} = \dfrac{\mu_B^{\mathrm{tri}}}{T^{\mathrm{tri}}} \simeq 3.4$
  - Staggered : $R_{\mathrm{st}}^{\mathrm{tri}} \simeq 2.0$
  - Monte-Carlo simulation : $R_{\mathrm{MC}}^{\mathrm{tri}} \gtrsim 3$

Chiral Phase Diagram

- 3-dimensional diagram : (μ_B, T, μ_3)



Summary

- KW-type minimally-doubled fermion
    - Finite density 2-flavor QCD with exact chiral symmetry
- QCD phase diagram
    - close to phenomenological result

Summary

- Staggered Wilson fermion
    - Chiral limit and Aoki phase
- Minimally-doubled fermion
    - Applicability to finite density