Twisted volume reduction in large N QCD

M. Okawa with A. Gonzalez-Arroyo

We study large N QCD using twisted volume reduced model.

Plan of the talk

1） Twisted Eguchi Kawai model

2） Large N QCD with $N_f = 2$ adjoint quark

3） Large N QCD with $N_f = 1$ adjoint quark

4） Relation to non-commutative field theory

Plan of the talk

- Twisted Eguchi-Kawai model
  for pure SU(N) gauge theory

- large N QCD with two adjoint fermions

- large N QCD with one adjoint fermion

- Relation with non-commutative field theory

- Eguchi-Kawai model, PRL 48 (1982) 1063.

Eguchi-Kawai model is obtained from the usual SU(N) lattice gauge theory

$$Z_W = \int \prod_{x,\mu} dU_{x,\mu} \exp\left\{ bN \sum_x \sum_{\mu\neq\nu=1}^{d} Tr\left(I - U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu} U^{\dagger}_{x,\nu} \right) \right\}$$

by neglecting the space-time dependence of the link variables

$$U_{x,\mu} \to U_{\mu}$$

$$Z_{EK} = \int \prod_{\mu} dU_{\mu} \exp\left\{ bN \sum_{\mu\neq\nu=1}^{d} Tr\left(I - U_{\mu} U_{\nu} U^{\dagger}_{\mu} U^{\dagger}_{\nu} \right) \right\}, \qquad b = \frac{1}{g^2 N}$$

In the same way, Wilson loop is defined by

$$W_W(C) = \left\langle Tr\left(U_{x,\mu} U_{x+\mu,\nu} \cdots U_{x-\rho,\rho} \right) \right\rangle$$

$$W_{EK}(C) = \left\langle Tr\left(U_{\mu} U_{\nu} \cdots U_{\rho} \right) \right\rangle$$

Eguchi and Kawai show that in the large N limit,
the Schwinger-Dyson eqs. satisfied by Wilson loops are identical
in both theories provided that the $Z(N)^d$ symmetry

$$U_\mu \to e^{i\theta_\mu} U_\mu, \qquad e^{i\theta_\mu} \in Z(N)$$

of the EK model is not spontaneously broken.

SD eqs. are identical except for the contributions from open loops

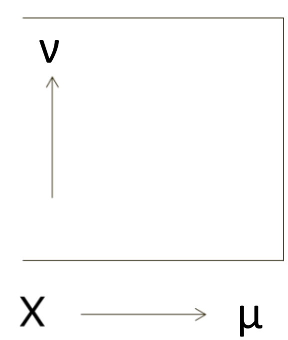
In the usual theory, expectation values of open loops are zero due to
Local gauge invariance

$$\left\langle \mathrm{Tr}(U_{x,\mu} U_{x+\mu,\nu} U^\dagger_{x+\nu,\mu}) \right\rangle = 0$$

In the EK model, if the $Z(N)^d$ symmetry is not spontaneously broken
expectation values of open loops are zero

$$\left\langle Tr U_\mu U_\nu U_\mu^\dagger \right\rangle = e^{i\theta_\nu} \left\langle Tr U_\mu U_\nu U_\mu^\dagger \right\rangle$$

$$\therefore \quad \langle \mathrm{Tr}(U_\mu U_\nu U^\dagger_\mu) \rangle = 0$$



Bhanot, Heller and Neuberger found, however, that this symmetry is broken spontaneously in the weak coupling region (1982).

The vacuum configurations $U_\mu^{(0)}$ having zero action satisfy

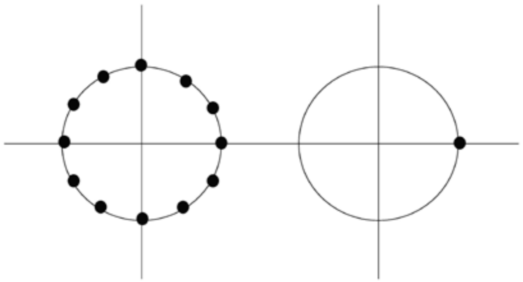
$$\mathrm{Tr}(I - U_\mu^{(0)} U_\nu^{(0)} U_\mu^{(0)\dagger} U_\nu^{(0)\dagger}) = 0$$

$$\therefore \quad U_\mu^{(0)} U_\nu^{(0)} = U_\nu^{(0)} U_\mu^{(0)}$$

Thus $U_\mu^{(0)}$ are diagonal matrices. Let $e^{i\theta_\alpha}$ (**α**=1,N) be their eigenvalues. Bhanot, Heller and Neuberger found that, in the week coupling region, there exist an attractive forces between different θ_α. As a result, θ_α tend to have same value.

Thus $U_\mu^{(0)}$ are proportional to the unit matrix, then

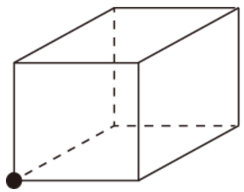
$$\mathrm{Tr}\, U_\mu^{(0)} \neq 0$$



$Z(N)^d$ symmetry is broken in the weak coupling

$\mathrm{Tr}\, U_\mu^{(0)} = \sum_\alpha e^{i\theta_\alpha}$ is the simplest order parameter

- Twisted Eguchi-Kawai model, PRD 27 (1983) 2397.



EK model can be viewed as the usual Wilson gauge theory having only one site with the periodic boundary conditions.

We introduce twisted boundary conditions in $SU(N)$, $N = L^2$ theory

$$S_{TEK} = bN \sum_{\mu \neq \nu = 1}^{d} Tr\left(I - Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right)$$

$$Z_{\mu\nu} = \begin{pmatrix} 1 & z^{*k} & z^{*k} & z^{*k} \\ z^k & 1 & z^{*k} & z^{*k} \\ z^k & z^k & 1 & z^{*k} \\ z^k & z^k & z^k & 1 \end{pmatrix}$$

$$Z_{\mu\nu} = \exp\left(k \frac{2\pi i}{L}\right), \quad Z_{\nu\mu} = Z_{\mu\nu}^*, \quad \mu > \nu$$

$$z = \exp\left(\frac{2\pi i}{L}\right)$$

k , L : co-prime, k/L fixed as we go $N = L^2 \to \infty$

$$Tr(I - z_{\mu\nu} U_\mu^{(0)} U_\nu^{(0)} U_\mu^{(0)\dagger} U_\nu^{(0)\dagger}) = 0$$

$$\therefore U_\nu^{(0)} U_\mu^{(0)} = z_{\mu\nu} U_\mu^{(0)} U_\nu^{(0)}$$

We can construct Γ_μ from 't Hooft matrices P_L , Q_L as

$$P_L = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdot & & \\ & & & \cdot & 1 \\ 1 & & & & 0 \end{pmatrix} \quad , \quad Q_L = \begin{pmatrix} 1 & & & & \\ & z & & & \\ & & z^2 & & \\ & & & \cdot & \\ & & & & z^{L-1} \end{pmatrix} \quad , \quad z = \exp\left(\frac{2\pi i}{L}\right)$$

$$P_L Q_L = z Q_L P_L \qquad P_L, Q_L : L \times L \text{ matrices}$$

$$\Gamma_1 = P_L \otimes I_L$$

$$\Gamma_2 = Q_L^k \otimes P_L$$

$$\Gamma_3 = Q_L^k \otimes P_L Q_L^k$$

$$\Gamma_4 = Q_L^k \otimes Q_L^k$$

$$\Gamma_\mu : N \times N \text{ matrices, } N = L^2$$

$$\Gamma_\nu \Gamma_\mu = z^k \Gamma_\mu \Gamma_\nu \, , \quad \mu > \nu$$

The order parameters of $Z(L)^4$ symmetry $U_\mu \to zU_\mu$ of the TEK action are

$$\left\langle Tr(U_\mu^\ell)\right\rangle, \quad \ell = 1 \sim (L-1)$$

For the classical vacuum $U_\mu^{(0)} = \Gamma_\mu$, it is straightforward to show

$$Tr(U_\mu^{(0)\ell}) = Tr(\Gamma_\mu^\ell) = 0, \quad \ell = 1 \sim (L-1) \qquad ; \ \frac{1}{N} Tr(\Gamma_\mu^L) = 1$$

Then the Schwinger-Dyson eqs. satisfied by the TEK model and the corresponding lattice theory are identical in the large L limit.

We naturally expect that both theories are equivalent even nonperturbatively.

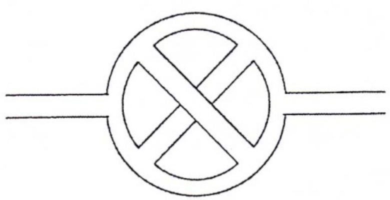
A. Gonzalez-Arroyo and M. O. 1983 $k = 1$

Propagator is identical to that of the lattice theory on $V = L^4 = N^2$.
$L^4 = N^2$ is the number of degree of freedom of SU(N) matrix !

If we introduce interactions, there appears phase factor in each vertex.
However, they cancel completely in planar diagram.

For non-planar diagrams, phase factor survives, which oscillates very rapidly in the large N limit, and suppressing the contribution of non-planar diagram.

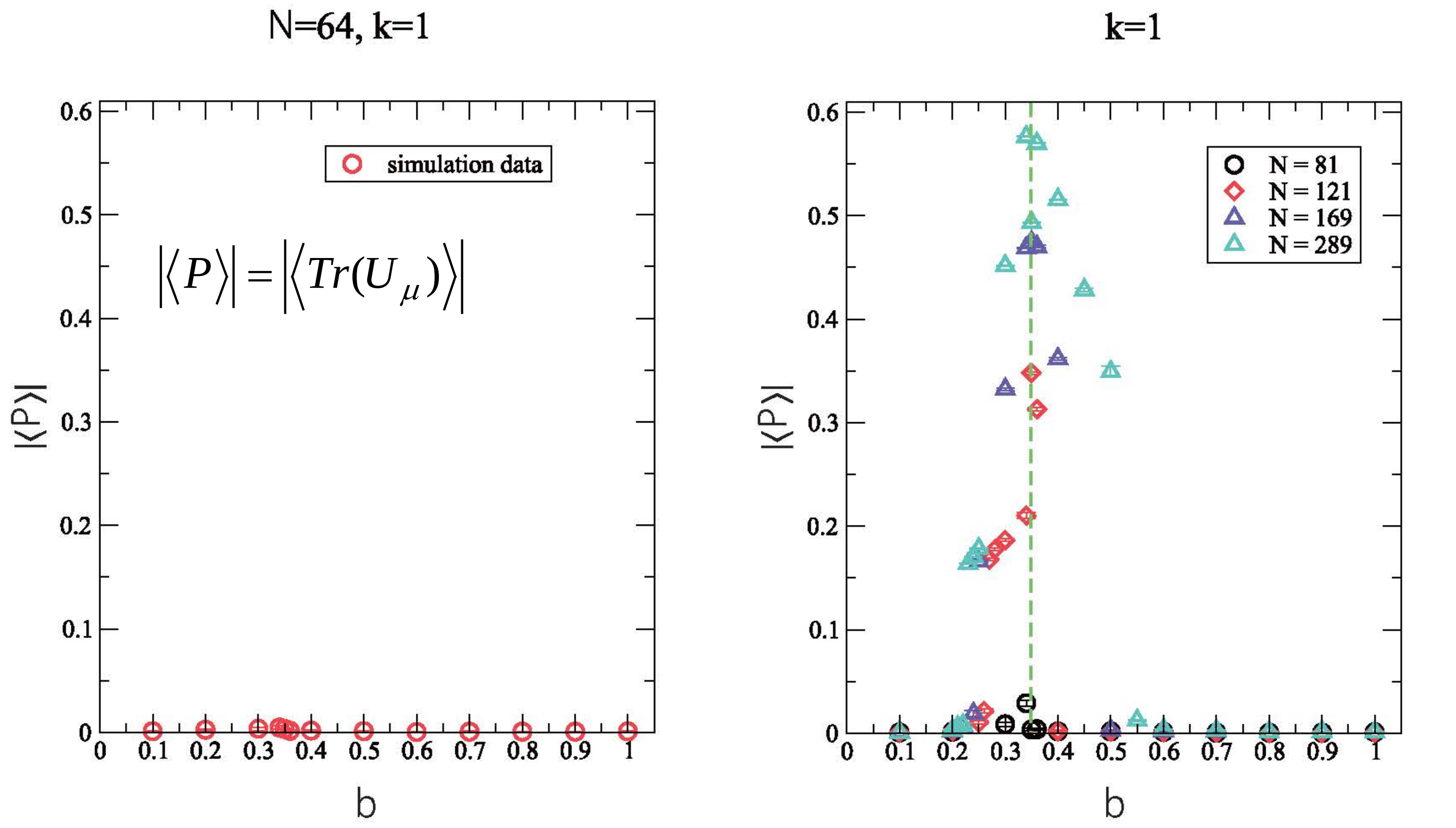
no phase factor remains



$$\int dp \exp\left[i\bar{k} L f(p) \right] \xrightarrow{L\to\infty} 0$$

$$k\bar{k} = 1 \quad \mathrm{mod}(L)$$

In 2003, Tomomi Ishikawa and M. O. found that in the intermediate coupling region, $\langle Tr(U_\mu) \rangle \neq 0$ for N>100 with *k*=1.



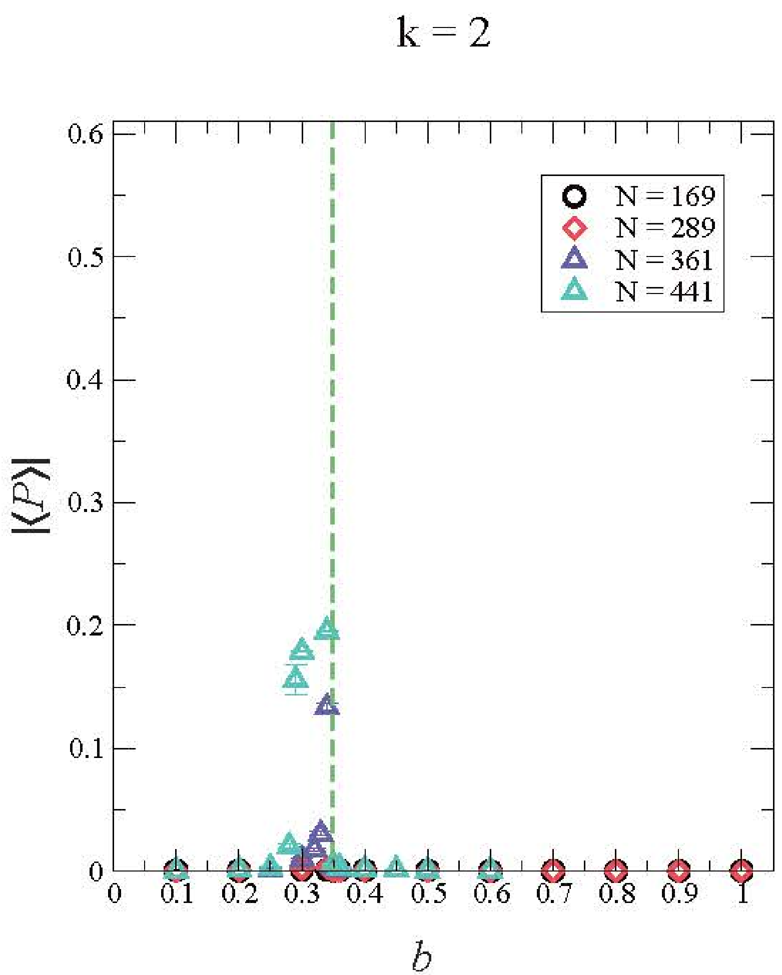
Z_L^4 symmery is not broken all over the coupling region.

Z_L^4 symmetry is broken in the intermediate region for N> 100

We also found that

$Z(L)^4$ symmetry is broken

for $N \geq 360$ with k=2.



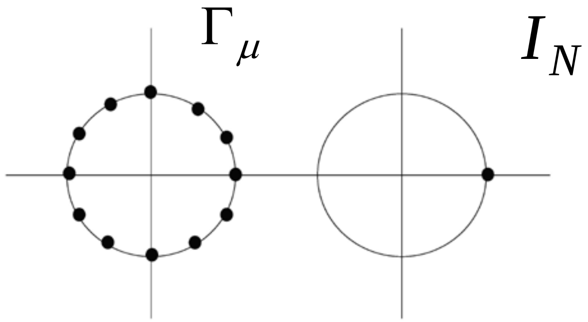
Z_L^4 **symmetry is broken in the intermediate region for** $N \gtrsim 360$.

Why $Z(L)^4$ symmetry is broken ?

k=1 M. Teper, H. Vairinhops (2007)

general k A. Gonzalez-Arroyo, M. O. JHEP 07 (2010) 043.

In the $Z(L)^4$ symmetry broken phase, the eigenvalues of U_μ attract each other in the complex plane, thus $U_\mu^{(0)} \sim I_N$



It is then the competition of energy gap and entropy between two configurations

$$E(U_\mu^{(0)} = \Gamma_\mu) = 0$$

$$E(U_\mu^{(0)} = I_N) = 12bN^2\left(1-\cos\left(\frac{2\pi k}{L}\right)\right) \sim O(bN), \quad for\ k=1$$

$$\sim O(bN^2), \quad for\ \ finite\ k/L$$

Entropy of $U_\mu^{(0)} \sim I_N$ is always larger than that of $U_\mu^{(0)} \sim \Gamma_\mu$ and difference is of order $O(N^2)$

So far, the arguments are perturbative, which we should not trust so much. We need non-perturbative study.

$k=0:$ $Z(L)$ symmetry is broken for $L>0$

$k=1:$ $Z(L)$ symmetry is broken for $L>10$

$k=2:$ $Z(L)$ symmetry is broken for $L>18$

$k=3:$ $Z(L)$ symmetry is broken for $L>28$

$k=4:$ $Z(L)$ symmetry is broken for $L>37$

The above numerical results strongly suggest that

$Z(L)^4$ symmetry is not broken for $\dfrac{k}{L} > \dfrac{1}{9}$

We also found that *k* should not be chosen too large .

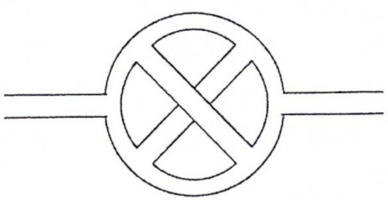
In fact, for $k=\frac{L-1}{2}$, we observe at $L=17, 19, 21, 23$

$$\left\langle Tr(U_\mu)\right\rangle = 0 \quad \text{but} \quad \left\langle Tr(U_\mu^2)\right\rangle \neq 0$$

We notice $2k = L-1 = 1\,(\text{mod } L)$, then $\bar{k} = 2$ $\quad \left[k\bar{k} = 1 \text{ mod } L\right]$

Related to the tachyonic instability of
the non-commutative field theory !

In any case, large value of $\bar{k}$ is desirable
to suppress non planer diagrams



$$\int dp \exp\left[i\bar{k} L f(p)\right] \xrightarrow{L\to\infty} 0$$

Take large L keeping $k/L > 1/9$ with large $\bar{k}$. L and k co-prime.

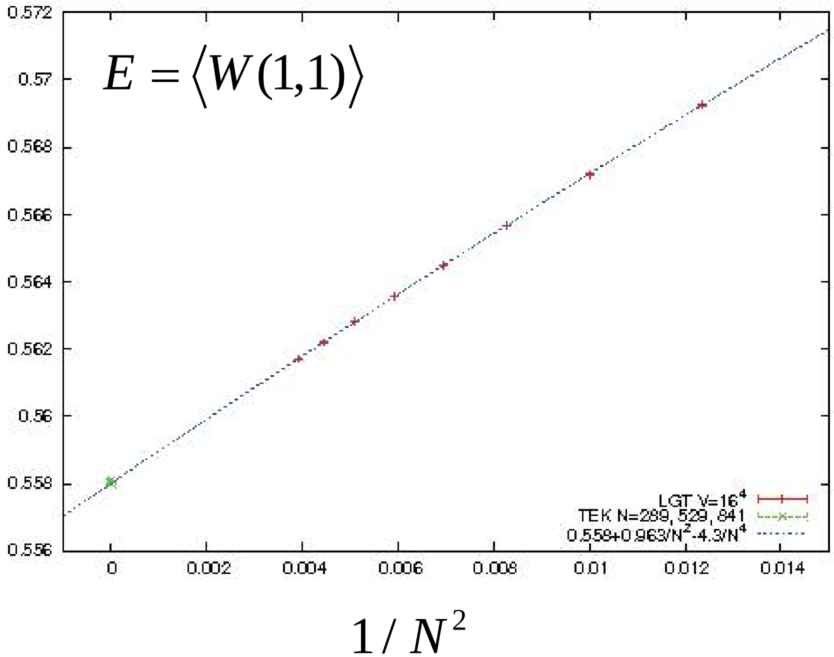
We mainly make numerical simulations for the following four parameter sets

N	L	k	$\bar{k}$
289	17	5	7
529	23	7	10
841	29	9	13
1369	37	11	10

We note $\frac{k}{L} \sim \frac{\bar{k}}{L} \sim 0.3-0.4$ for all cases !

N dependence of W(1,1)

Detailed comparison at b=0.36



For $V = 16^4$ $N = 9$ -16

$$E = 0.55800(2) + \frac{0.963(6)}{N^2} - \frac{4.3(4)}{N^4}$$

E=0.557998(5) (N=841, k=9)

E=0.557999(19) (N=289, k=5)

E=0.557991(13) (N=529, k=7)

If the TEK model is correct nonperturbatively,

we should be able to calculate the string tension.

In our reduced model, the Wilson loop $W(R,T)$ is defined by

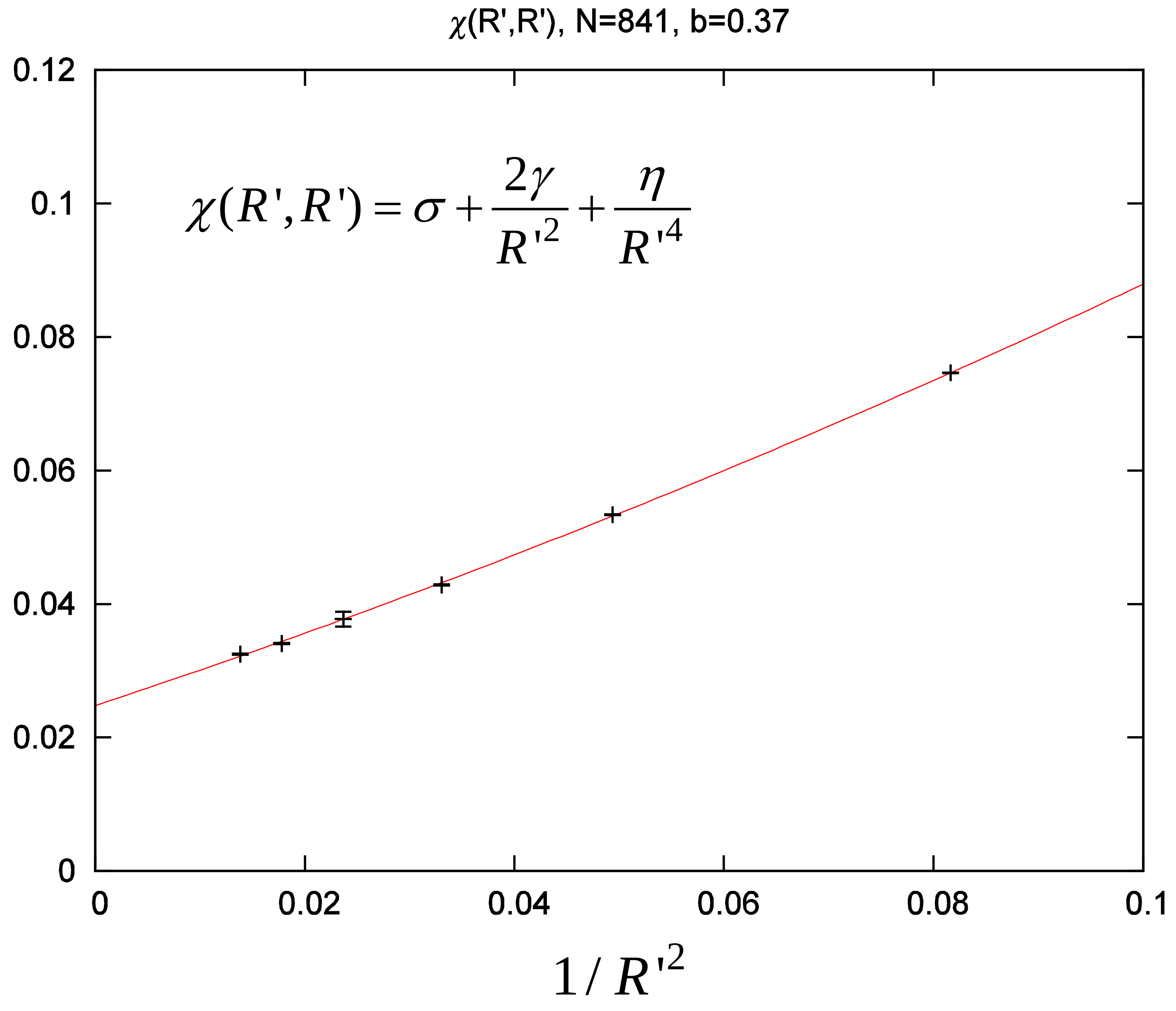
$$W(R,T) = Z_{\mu\nu}^{RT} \left\langle Tr\left(U_\mu^R U_\nu^T U_\mu^{\dagger R} U_\nu^{\dagger T}\right)\right\rangle \sim \sigma RT + \cdots$$

Then the string tension σ is obtained from Creutz ratio as

$$\chi(R',T') = -\log \frac{W(R'+0.5,T'+0.5)W(R'-0.5,T'-0.5)}{W(R'+0.5,T'-0.5)W(R'-0.5,T'+0.5)}$$

$$\chi(R',R') = \sigma + \frac{2\gamma}{R'^2} + \frac{\eta}{R'^4}$$

with half-integer R',T' .


χ(R',R'), N=841, b=0.37
$\chi(R',R') = \sigma + \frac{2\gamma}{R'^2} + \frac{\eta}{R'^4}$
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.02
0.04
0.06
0.08
0.1
$1/R'^2$

We calculate the continuum string tension by extrapolating the TEK data with $N = 841 = 29^2,\ k = 9$ at 6 values of b

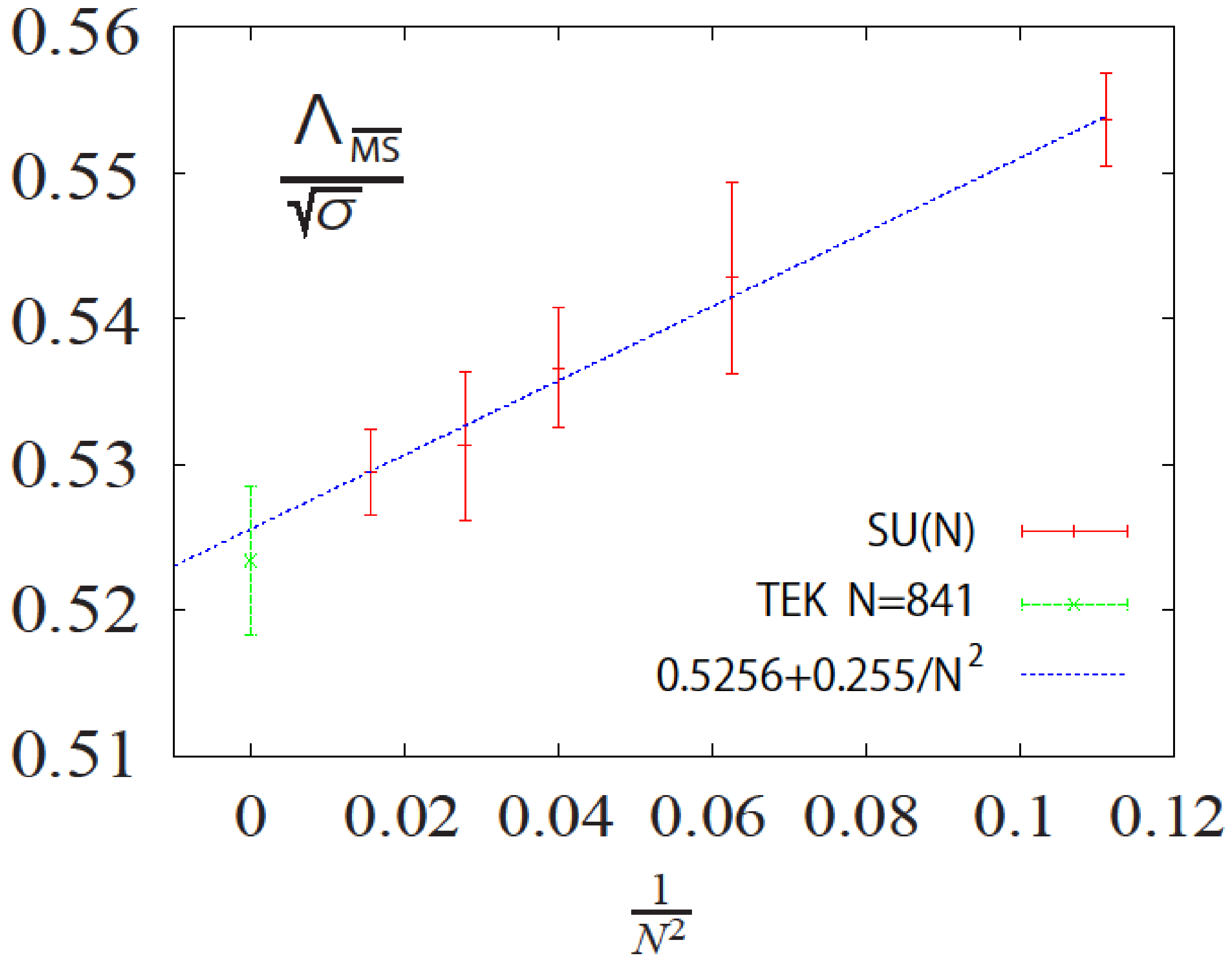
$$b = 0.36,\ 0.365,\ 0.37,\ 0.375,\ 0.38,\ 0.385$$

Our system should be related to the lattice theory with $V = 29^4$

For comparison, we also calculate the continuum string tension using ordinary SU(N) lattice gauge theory with $N = 3, 4, 5, 6, 8$ on a $V = 32^4$ lattice

Comparison of the continuum string tension $\Lambda_{\overline{MS}} / \sqrt{\sigma}$

TEK model with $N = 841 = 29^2$ and LGT with $N = 3, 4, 5, 6, 8$



Plan of the talk

- Twisted Eguchi-Kawai model
  for pure SU(N) gauge theory
- large N QCD with two adjoint fermions
- large N QCD with one adjoint fermion
- Relation to non-commutative field theory

Motivation for $N_f = 2$ adjoint fermions

SU(N) LGT with two adjoint fermions is thought to be conformal or nearly conformal for any value of N since the first two coefficient of beta functions expressed in term of ‘t Hooft coupling is independent of N.

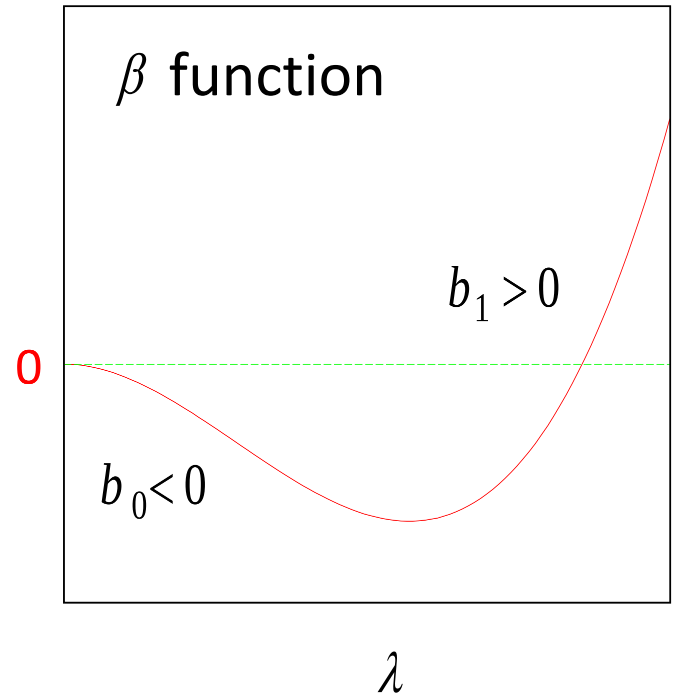
$$b_0 = \frac{4N_f - 11}{24\pi^2}, \quad b_1 = \frac{16N_f - 17}{192\pi^4}$$

- asymptotic free

$$b_0 < 0 \rightarrow N_f < \frac{11}{4} = 2.75$$

- infrared fixed point

$$b_1 > 0 \rightarrow N_f > \frac{17}{16} = 1.08$$



● Twisted reduced model of large N QCD with two adjoint Wilson fermions

We consider gauge group $SU(N),\ N = L^2$

$$S = bN \sum_{\mu\neq\nu=1}^{d} Tr\left(Z_{\mu\nu}U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right) + \sum_{j=1}^{N_f} \bar{\psi}_j D_W \psi_j$$

$$Z_{\mu\nu} = \exp\left(k\frac{2\pi i}{L}\right), \quad Z_{\nu\mu} = Z_{\mu\nu}^*, \qquad \mu > \nu$$

$$D_W = 1 - \kappa\sum_{\mu=1}^{4}\left[(1-\gamma_\mu)U_\mu^{adj} + (1+\gamma_\mu)U_\mu^{\dagger adj}\right], \quad U_\mu^{adj}\psi_j = U_\mu \psi_j U_\mu^\dagger$$

k , L : co-prime, $\quad m_q = (1/\kappa - 1/\kappa_c)/2$

$k = 0$ corresponds to periodic boundary condition

$k \neq 0$ corresponds to twisted boundary condition

We calculate the string tension with $N = 289 = 17^2, \ k = 5$
at 2 values of $b = 0.35, \ 0.36$ for various values of κ

Our system should be related to the lattice theory with $V = 17^4$

- For $N_f = 2$, we use the Hybrid Monte Carlo method.

Simulations have been done on Hitachi SR16000 at KEK

One node: 32 cores power 7,
peak speed 980 GFlops
256 GB shared memory

Sustained speed of our code in one node is
600 Gflops at N=289

We thank to Hitachi system engineers !

If the theory is governed by an infrared fixed point with the relevant mass term $m_q\bar{\psi}\psi$, all physical quantity having mass dimension should vanish as $m_q \to 0.$

In particular, the string tension having mass square dimension should behave as

$$\sigma \sim m_q^{2/(1+\gamma_*)}$$

with γ_* the mass anomalous dimension at infrared fixed point.

Simple derivation

$$\mu \frac{dm(\mu)}{d\mu} = -\gamma_* m(\mu)$$

$$\therefore \; m(\mu) \propto \mu^{-\gamma_*} \mu_0^{\gamma_*} m(\mu_0)$$

Define RG invariant mass M as $m(M) = M$.

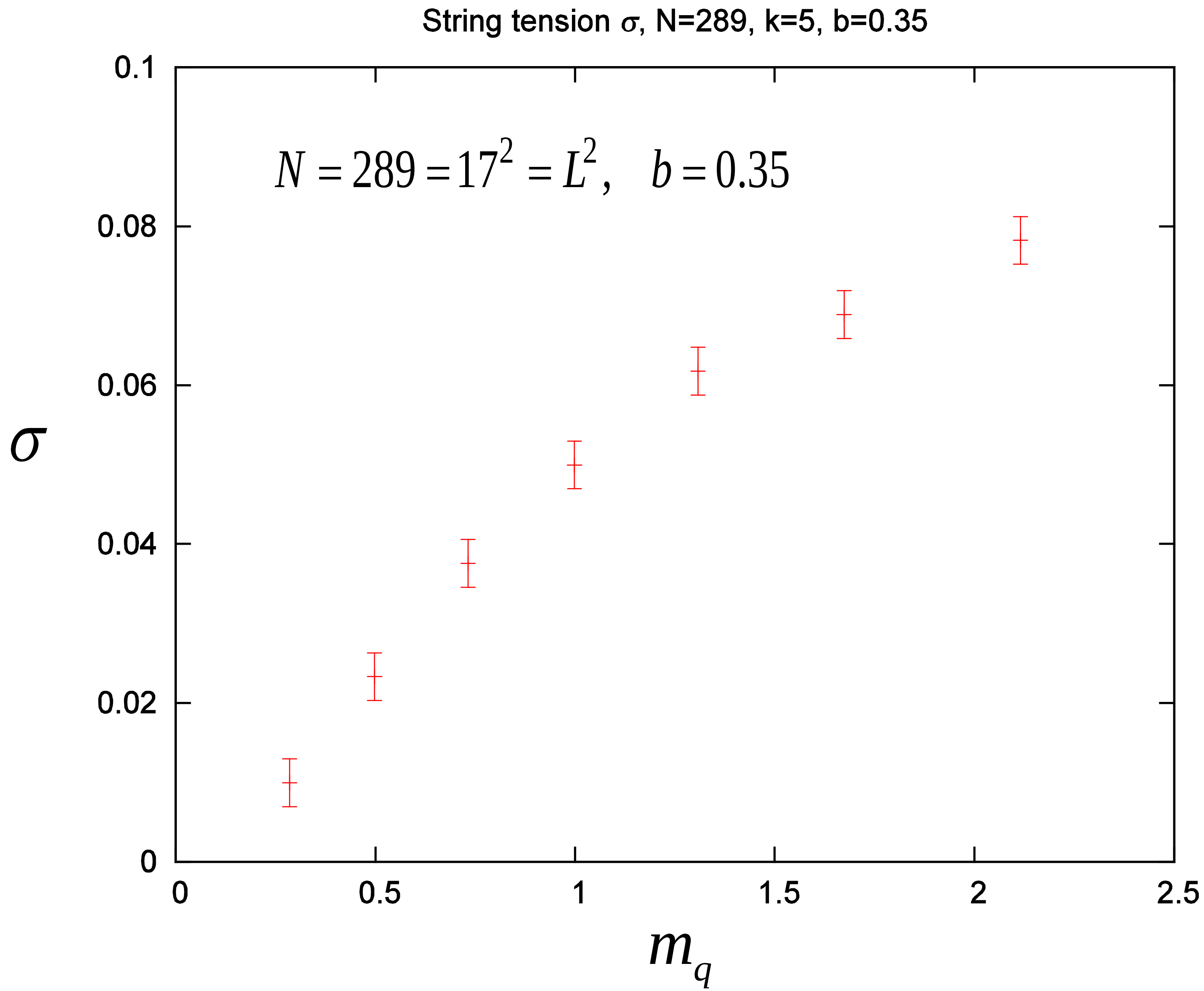
Then setting $\mu_0 = M$ and $\mu = a^{-1}$, we have

$$m(a^{-1}) = m_q \propto a^{-\gamma_*} M^{1+\gamma_*}$$

$$\therefore \; aM \propto (am_q)^{1/(1+\gamma_*)}$$

We can show that all physical quantity M_X having mass dimension is proportional to M , then

$$aM_X \propto (am_q)^{1/(1+\gamma_*)}$$



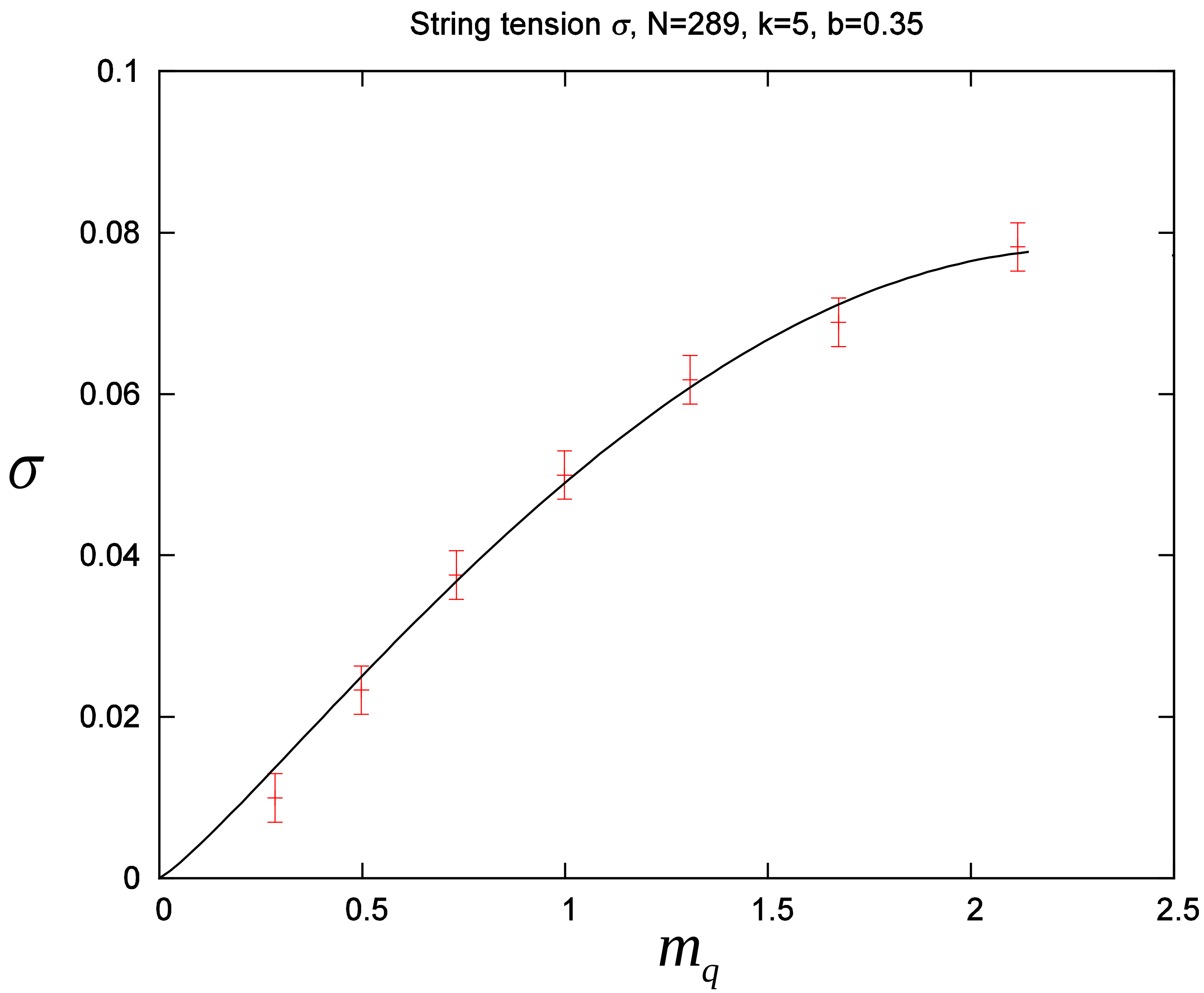
String tension σ, N=289, k=5, b=0.35
$N = 289 = 17^2 = L^2, \quad b = 0.35$
σ
m_q
0
0.02
0.04
0.06
0.08
0.1
0
0.5
1
1.5
2
2.5

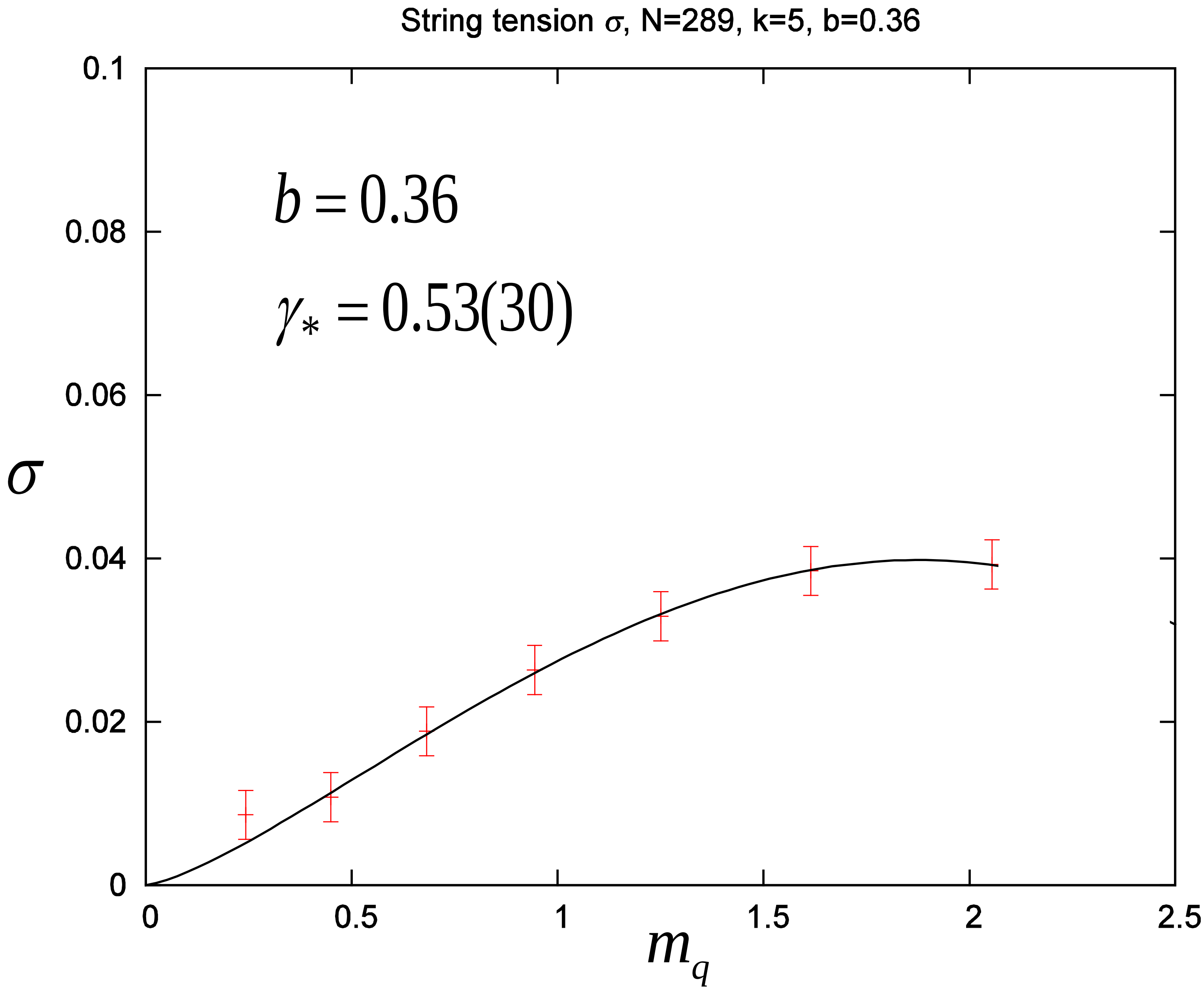
Fit the string tension σ with the fitting function

$$\sigma = A m_q^{2/(1+\gamma_*)} (1 + B m_q)$$

The result is

$$\gamma_* = 0.87(40)$$


String tension σ, N=289, k=5, b=0.35
0.1
0.08
0.06
0.04
0.02
0
σ
0
0.5
1
1.5
2
2.5
m_q


String tension σ, N=289, k=5, b=0.36
b = 0.36
γ* = 0.53(30)
σ
m_q
0.1
0.08
0.06
0.04
0.02
0
0
0.5
1
1.5
2
2.5

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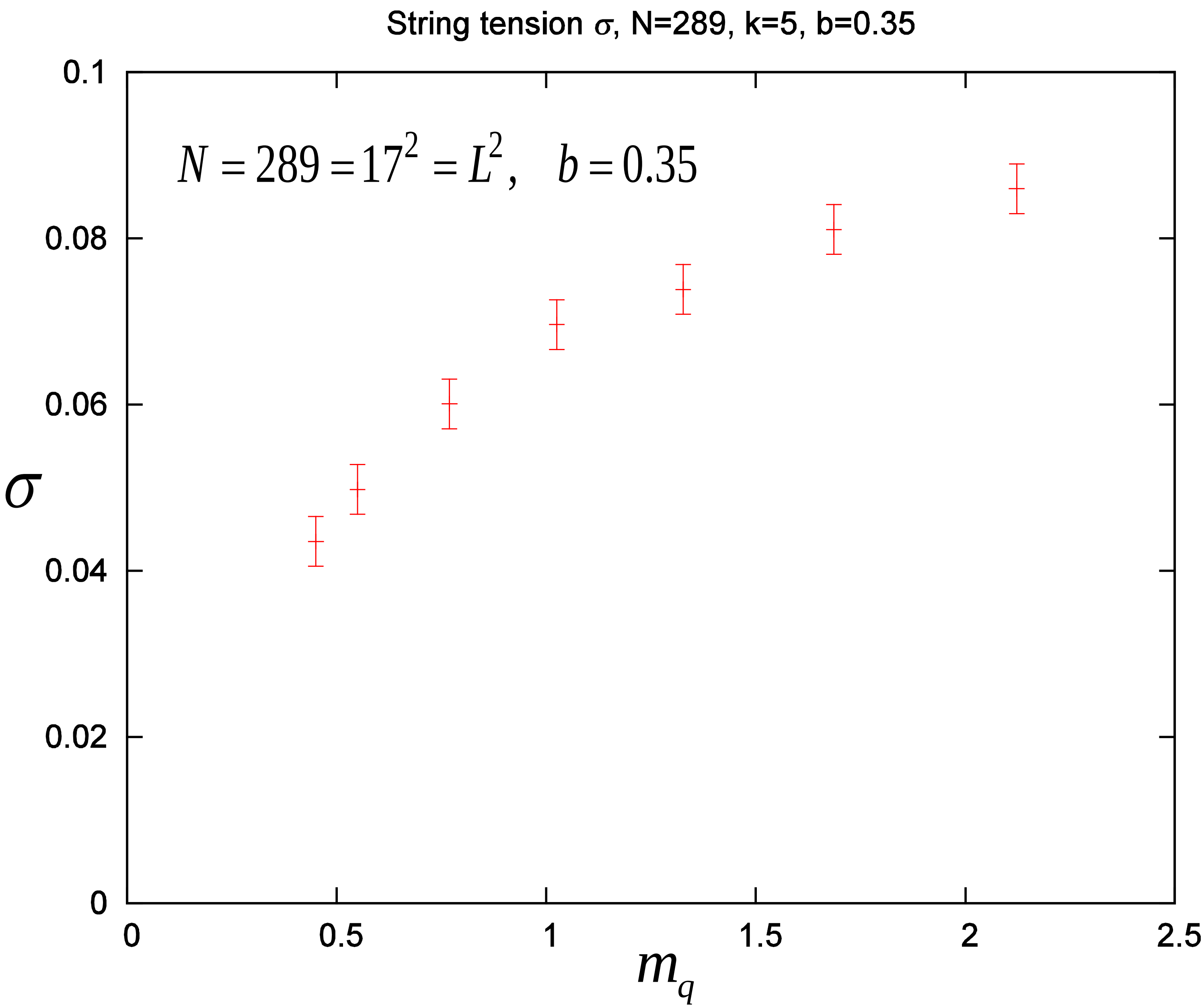
Motivation for $N_f = 1$ adjoint fermion

In the large N limit, $N_f = 1$ adjoint fermion is equivalent to $N_f = 2$ fundamental fermion in rank two anti-symmetric rep. (Armoni, Shifman, Veneziano, Kovtun, Unsal, Yaffe)

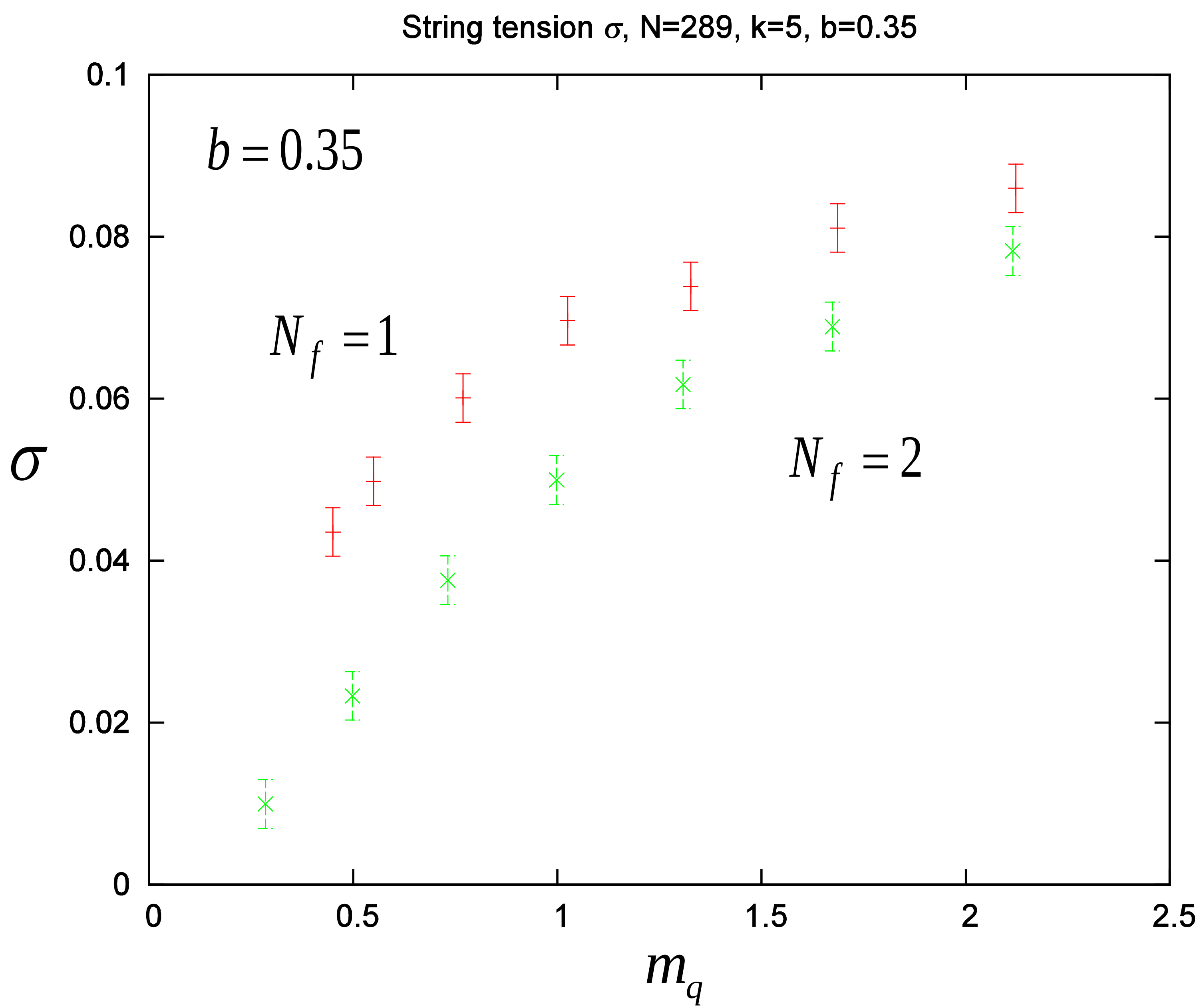
For N=3, the latter theory is just two flavor QCD and our model corresponds to Corrigan-Ramond large-N limit.

We then expect the reduced model of $N_f = 1$ adjoint fermion as

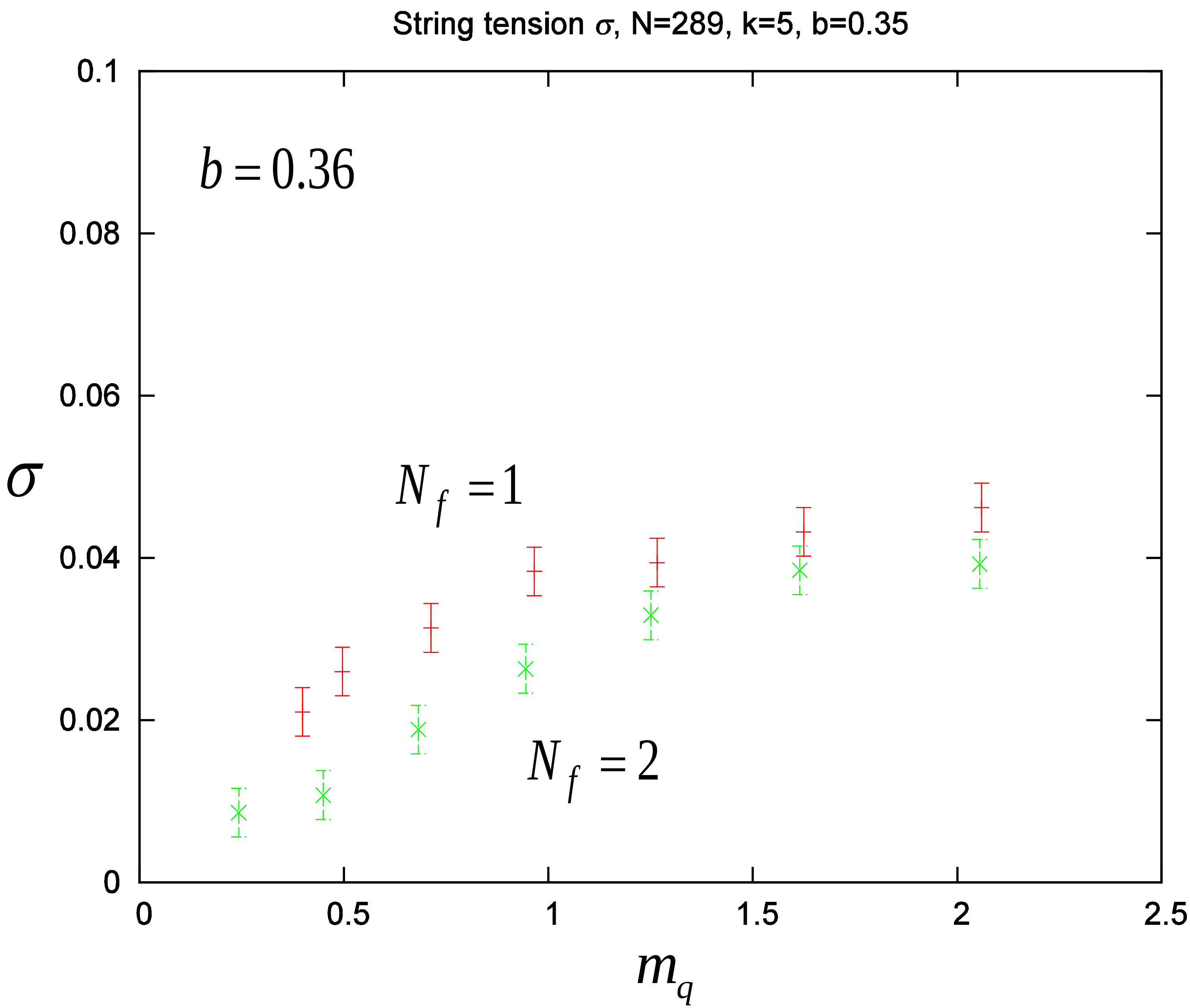
confining theory



String tension σ, N=289, k=5, b=0.35
$N = 289 = 17^2 = L^2, \quad b = 0.35$
σ
m_q
0.1
0.08
0.06
0.04
0.02
0
0
0.5
1
1.5
2
2.5



String tension σ, N=289, k=5, b=0.35
b = 0.35
N_f = 1
N_f = 2
σ
m_q
0
0.02
0.04
0.06
0.08
0.1
0
0.5
1
1.5
2
2.5



String tension σ, N=289, k=5, b=0.35
b = 0.36
σ
N_f = 1
N_f = 2
m_q
0
0.02
0.04
0.06
0.08
0.1
0
0.5
1
1.5
2
2.5

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Relation to non-commutative field theory

Consider SU(N) gauge theory on $V = \ell^4$ volume with the following twisted boundary condition with $N = L^2$

$$A_\mu(x + \ell \vec{e}_\nu) = \Gamma_\nu A_\mu(x) \Gamma_\nu^\dagger$$

TEK model corresponds to $\ell = a$. This theory is closely related to the non-commutative field theory on $V = (\ell L)^4$ even for finite N .

$$S_{NC} = \int_{V=(\ell L)^2} d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} \right\}$$

$$f(x) \star g(x) \equiv f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta_{\mu\nu} \vec{\partial}_\nu \right) g(x)$$

$$\theta_{\mu\nu} = \frac{(\ell L)^2}{4\pi^2} \tilde{\varepsilon}_{\mu\nu} \frac{2\pi \bar{k}}{L}, \qquad \varepsilon_{\mu\nu} \tilde{\varepsilon}_{\nu\sigma} = \delta_{\mu\sigma}$$

Let consider the perturbative expansion in gauge theory.

Momentum zero state is excluded in the twisted boundary condition.
Thus the perturbative expansion generally has the following structure.

$$E^2 \sim \frac{A}{(\ell L)^2} - \lambda \frac{B}{(\ell L)^2} \frac{1}{\left(\bar{k} / L\right)^2}, \qquad \lambda = g^2 N$$

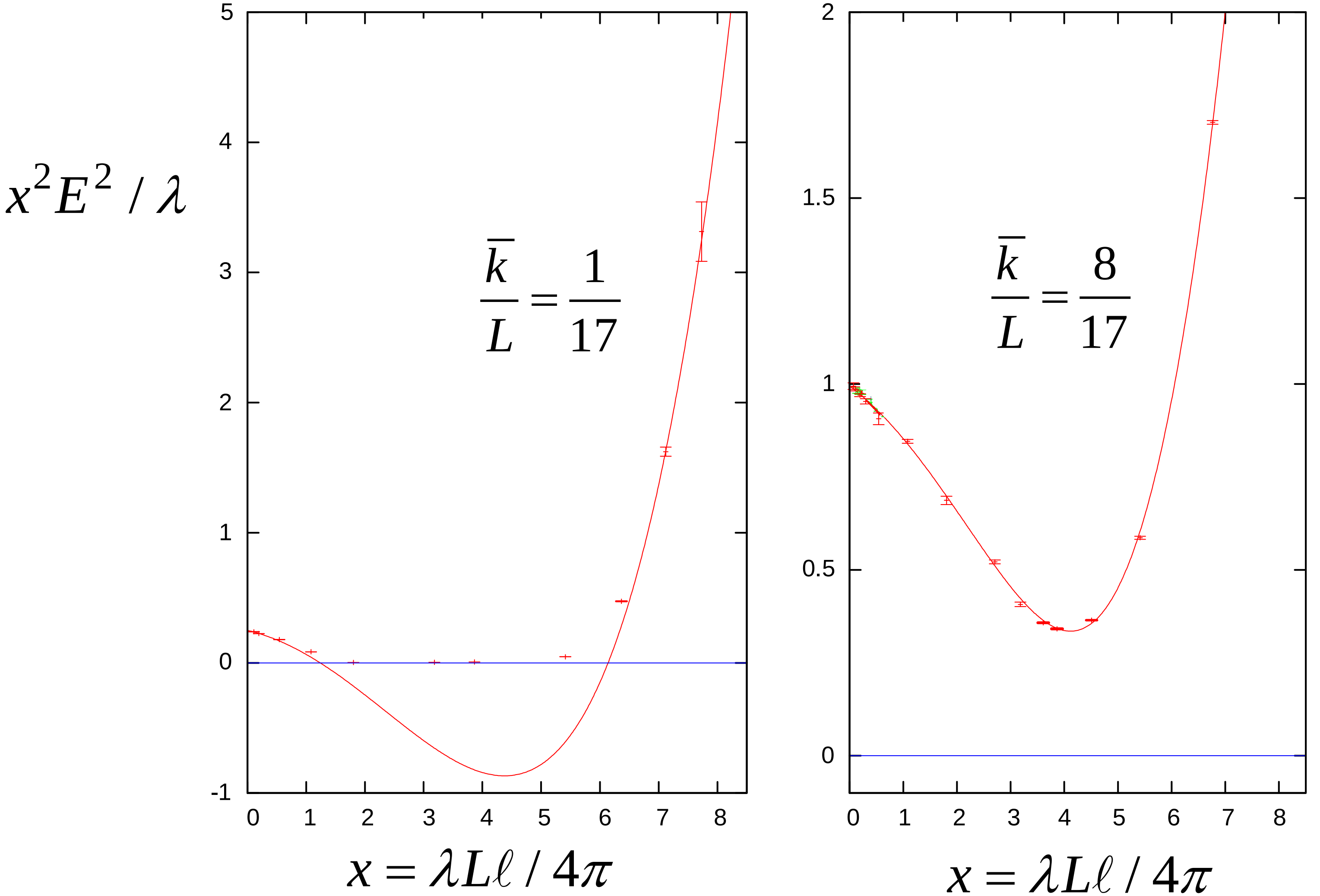
Tachyonic instability ($E^2 < 0$) might occure at

$$\lambda_c > \frac{A}{B} \times \left(\frac{\bar{k}}{L} \right)^2$$

However, if λ_c is too large, we can not trust perturbative arguments since we expect non-negligible non-perturbative effect !

Result of (2+1) dimensional simulation (in this case $N = L$)

M. Garcia Perez, A. Gonzalez-Arroyo, M. O. JHEP 1309 (2013) 003.



Conclusion

We demonstrated that the twisted reduced model of large N QCD works quite well with appropriately chosen twist tensor.

$N_f = 2$

String tension is calculated at N=289, which clearly decreases as we decrease m_q and seems to vanish at $m_q = 0$, in a way consistent with the theory governed by an infrared fixed point .

$N_f = 1$

String tension is calculated at N=289, which seems to remain finite as we decrease m_q, strongly suggesting this is the confining theory.

Remaining important problems

We need to understand the finite N (finite volume) effects
to make reliable calculation of mass amomalous dimension γ_* .

We need to make non-perturbative understanding about
the relation with non-commutative field theory.

Study half-integer N_f corresponding to Majorana fermions.
We need to estimate the Pfaffian of anti-symmetric matrix CD_W .