A new look at instantons and Large N limit

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Introduction

We consider the instanton effect in the large N limit.

't Hooft limit: $g^2 N = \lambda$: fixed, $N \to \infty$

$$(\text{instanton effect}) = e^{-\frac{8\pi^2}{g^2}} = e^{-\frac{8\pi^2 N}{\lambda}} \to 0$$

Another large N limit in which instanton effects survives

Very strongly couples large N limit

⇒ g^2: fixed, $N \to \infty$ $(\sim \lambda = \mathcal{O}(N))$

- We calculate instanton effects in large N limit of $\mathcal{N} = 2 *$ theory.
- We consider the orbifold equivalence between $\mathcal{N} = 2 *$ theory and $\mathcal{N} = 2$ necklace quiver in the large N limit.

Large N limit

Large N expansion

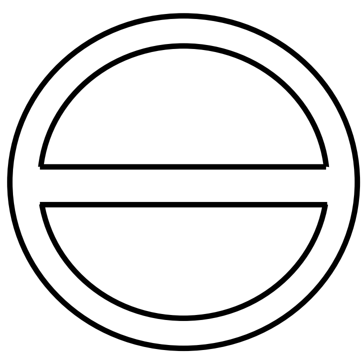
$$f(\lambda, N) = \sum_{n=0}^{\infty} f_n(\lambda) N^{2-2n}$$

't Hooft limit of large N gauge theories

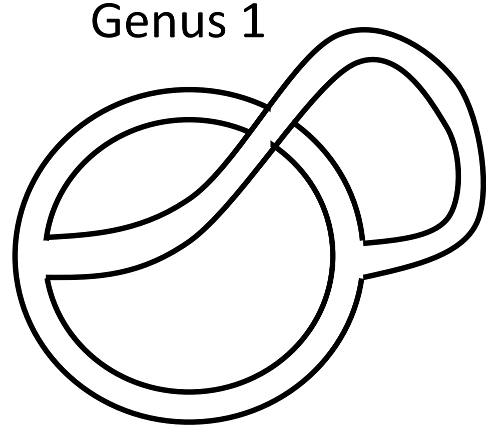
't Hooft coupling $\lambda = g^2 N$ is fixed to be finite

Large N expansion is the genus expansion of the Feynman diagram

Planar diagram



Genus 1



Very strongly coupled large N limit

[Azeyanagi-Fujita-Hanada '13]

We consider another large N limit.

$g^2 = \mathcal{O}(N^{-\alpha})$ $\alpha = 1$ is the 't Hooft limit

$\alpha < 1$: very strongly coupled large N limit

- We can analyze in the very strongly coupled large N limit, basically in the same way to the case of the 't Hooft limit.
- Large N limit and strong coupling limit are commute.
- As long as there are no phase transition, very strongly coupled large N limit can be analyzed by using the analytic continuation form the 't Hooft limit.

Example: AdS/CFT correspondence

$\lambda \gg 1$

Gauge theory | Supergravity (string theory)

Large N ⇒ Small $g_s = \lambda/N$ (perturbative thoery)

$\lambda \gg 1$ ⇒ $\lambda^{-1/2} = \dfrac{\alpha'}{R_{\mathrm{AdS}}^2} \ll 1$

The classical solution in the supergravity is obtained as the leading term of large N and large λ expansion.

The classical gravity description is valid when

$$1 \ll \lambda \ll N$$

where λ can be $\mathcal{O}(N)$ but satisfies $g_s = \lambda/N \ll 1$.

($\lambda \gg N$ breaks genus expansion ⇒ S-dual)

Example: M-branes

M-branes has non-standard N-dependence in the large N limit

M2-branes: $F \sim \mathcal{O}(N^{3/2})$ M5-branes: $F \sim \mathcal{O}(N^3)$

In the ABJM theory for M2-branes, the free energy is

$$F \sim N^{3/2}\sqrt{k}$$ k: Chern-Simons level

In the IIA regime, $k \gg 1$ but $N \gg k$, in terms of the 't Hooft coupling

$$F \sim \frac{N^2}{\sqrt{\lambda}} \quad \text{where} \quad \lambda = \frac{N}{k}$$

M5-branes are expected to be described by D4-branes,

$$F \sim N^3 R_{11}$$ For M5-branes

$$F \sim N^2 \lambda$$ For D4-branes

where $\lambda = N g_s = N R_{11}$

Example: Hermitian matrix model

Partition function of the matrix model

$$Z = \int d\Phi\, e^{-N \mathrm{tr} V(\Phi)}$$

Φ: $N \times N$ hermitian matrix

Large N limit of the matrix model is similar to the classical limit.

Diagonalizing the matrix Φ,

$$Z = \int \prod_i da_i \prod_{i<j} \left(a_i - a_j\right)^2 \, e^{-N \sum_i V(a_i)}$$

In the large N limit, configuration of the eigenvalues a_i are fixed by the saddle point equation.

Solution is given in terms of the eigenvalue density $\rho(a)$.

It is convenient to introduce the resolvent:

$$R(x) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{x-\Phi}\right\rangle = \frac{1}{N}\left\langle \sum_i \frac{1}{x-a_i}\right\rangle$$

The imaginary part of the resolvent is the eigenvalue density

$$\frac{1}{\pi}\mathrm{Im}\,R(x) = \frac{1}{N}\left\langle \sum_i \delta(x-a_i)\right\rangle = \rho(x)$$

where the eigenvalue density is normalized as

$$\int da\, \rho(a) = 1$$

In the large N limit, eigenvalues distributes continuously

$$R(x) = \int da \frac{\rho(a)}{x-a}$$

The Schwinger-Dyson equation in the large N limit (large N factorization)

$$R^2(x) - V'(x)R(x) = f_0(x)$$

The solution of the resolvent generally takes the following form:

$$R(x) = \frac{1}{2}\left(V'(x) - f(x)\sqrt{(x-a)(x-b)}\right)$$

if the eigenvalues are distributed in a region (1-cut) $a < x < b$.

If the eigenvalues are distributed in two (or more) disconnected regions (2-cut solution), the resolvent takes a different form.

The resolvent is given as an analytic function of coupling constants, as long as it is in the same phase (i.e. 1-cut phase, 2-cut phase,…)

In the large N limit, we can take the analytically continuation of the coupling constant, as long as there are no phase transition.

4D $\mathcal{N}=2*$ gauge theory

$\mathcal{N}=2$ theory which is $\mathcal{N}=4$ SYM with mass deformation with m .

Partition function is calculated by using the localization.

$$Z_{\mathcal{N}=2*}=\int d^{kN}a\left(\prod_{\substack{i,j=1\\i<j}}^{N}\left(a_i-a_j\right)^2\right)Z_{\mathcal{N}=2*}^{(\mathrm{part})}(a_i,m)\left|Z_{\mathcal{N}=2*}^{(\mathrm{inst})}(a_i,m)\right|^2\exp\left(-\frac{8\pi^2}{g_p^2}\sum_{i=1}^{N}a_i^2\right)$$

$$Z_{\mathcal{N}=2*}^{(\mathrm{part})}(a_i,m)=\prod_{\substack{i,j=1\\i\neq j}}^{kN}Z_{\mathrm{vec}}^{(\mathrm{part})}\left(a_i-a_j\right)\prod_{i,j=1}^{kN}Z_{\mathrm{mat}}^{(\mathrm{part})}\left(a_i-a_j,m\right)$$

$$Z_{\mathrm{vec}}^{(\mathrm{part})}\left(a_i-a_j\right)=H(i(a_i-a_j))$$

$$Z_{\mathrm{mat}}^{(\mathrm{part})}\left(a_i-a_j,m\right)=\left[H\left(i\left(a_i-a_j+m\right)\right)H\left(i\left(a_i-a_j-m\right)\right)\right]^{-1/2}$$

$$H(z)=e^{-(1+\gamma)z^2}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)^n e^{z^2/n}$$

0-instanton sector

The partition function takes similar form to diagonalized matrix model.

$$Z_{\mathcal{N}=2*} = \int d^{kN}a \left(\prod_{\substack{i,j=1 \\ i<j}}^{N} (a_i - a_j)^2 \right) Z_{\mathcal{N}=2*}^{(\mathrm{part})}(a_i, m) \exp\left(-\frac{8\pi^2}{g_p^2} \sum_{i=1}^{N} a_i^2 \right)$$

a_i are the “eigenvalues”

$Z^{(\mathrm{part})}$ is a very complicated function

We consider the case of small mass m .

$g_p^2 \ll 1$ and $\lambda \gg 4\pi^2 m^2$

Then, we have

$$Z_{\mathcal{N}=2*}^{(\mathrm{part})}(a_i, m) = \prod_{\substack{i,j=1 \\ i\neq j}}^{kN} (a_i - a_j)^{m^2}$$

Then, the partition function becomes

$$Z_{\mathcal{N}=2*} = \int d^{kN}a \left(\prod_{\substack{i,j=1 \\ i<j}}^{N} (a_i - a_j)^{2(1+m^2)} \right) \exp\left(-\frac{8\pi^2}{g_p^2} \sum_{i=1}^{N} a_i^2 \right)$$

In the large N limit, the eigenvalues are distributed continuously.

In terms of the eigenvalue density $\rho(a)$, the free energy $F = -\log Z$ is

$$\frac{F}{N^2} = \frac{8\pi^2}{\lambda} \int_{-\mu}^{\mu} dx\, \rho(x)\, x^2 - \frac{1}{2}(1+m^2) \int_{-\mu}^{\mu} dx\, dy\, \rho(x)\, \rho(y) \log(x-y)$$

$\pm\mu$: the ends of the eigenvalue support

The saddle point equation is

$$(1+m^2) \int da \frac{\rho(a)}{x-a} = \frac{8\pi^2}{\lambda} x$$

Eq. for the Gaussian matrix model with coupling $\lambda(1+m^2)$.

The eigenvalue distribution is that of the Gaussian matrix model.

$$\rho(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2} \qquad \mu = \frac{\sqrt{\lambda(1+m^2)}}{2\pi}$$

The free energy for the 0-instanton sector is

$$F^{(\text{part})} = -N^2(1+m^2)\left(\frac{1}{2}\log\frac{\lambda(1+m^2)}{16\pi^2} - \frac{3}{4}\right)$$

In this case, there are no phase transition as long as it satisfies $g_p^2 \ll 1$ and $\lambda \gg 4\pi^2 m^2$.

In this region, the free energy is given by analytic function and we can take the analytic continuation to g^2-fixed large N limit

1-instanton sector

Instanton partition function is given by the Nekrasov's formula

$$Z_{\mathcal{N}=2*} = \int d^{kN}a \left(\prod_{\substack{i,j=1 \\ i<j}}^{kN} (a_i - a_j)^2 \right) Z_{\mathcal{N}=2*}^{(\mathrm{part})}(a_i, m) \left| Z_{\mathcal{N}=2*}^{(\mathrm{inst})}(a_i\ \tilde{m}) \right|^2 \exp\left(-\frac{8\pi^2}{g_p^2} \sum_{i=1}^{kN} a_i^2 \right)$$

$$Z_{\mathcal{N}=2*}^{(\mathrm{inst})}(a_i, \tilde{m}) = \sum_{Y=\{Y_1,\cdots,Y_{kN}\}} e^{-\frac{8\pi^2|Y|}{g_p^2}} \prod_{i,j=1}^{N} Z_{\mathrm{vec}}^{(\mathrm{inst})}\left(a_i - a_j; Y_i, Y_j\right) Z_{\mathrm{mat}}^{(\mathrm{inst})}\left(a_i - a_j; Y_i, Y_j; \tilde{m}\right)$$

$$Z_{\mathrm{vec}}^{(\mathrm{inst})}\left(a_i - a_j; Y_i, Y_j\right) = \prod_{s\in Y_i} \left[E\left(a_i - a_j; Y_i, Y_j, s\right)\right]^{-1} \prod_{t\in Y_j} \left[2 - E\left(a_j - a_i; Y_j, Y_i, t\right)\right]^{-1}$$

$$Z_{\mathrm{mat}}^{(\mathrm{inst})}\left(a_i - a_j; Y_i, Y_j; \tilde{m}\right) = \prod_{s\in Y_i} \left(E\left(a_i - a_j; Y_i, Y_j, s\right) - \tilde{m}\right) \prod_{t\in Y_j} \left(2 - E\left(a_j - a_i; Y_j, Y_i, t\right) - \tilde{m}\right)$$

$$E\left(a_i - a_j; Y_i, Y_j, s\right) = -h_{Y_j}(s) + \left(v_{Y_i}(s) + 1\right) + i\left(a_j - a_i\right)$$

$$Z_{\mathrm{vec}}^{(\mathrm{inst})}(a_i - a_j; Y_i, Y_j) = \prod_{s \in Y_i} \left[E(a_i - a_j; Y_i, Y_j, s)\right]^{-1} \prod_{t \in Y_j} \left[2 - E(a_j - a_i; Y_j, Y_i, t)\right]^{-1}$$

In the Nekrasov's formula, Young tableau is associated to each eigenvalue.

$$a_i \quad \Longleftrightarrow \quad Y_i$$

The instanton number is the total number of boxes $|Y| = \sum_i |Y_i|$,

In the case of 1-instanton, only 1 eigenvalue has non-trivial Young tableau

$$a_i \quad \Longleftrightarrow \quad Y_i = \square$$

$$a_{j \neq i} \quad \Longleftrightarrow \quad Y_{j \neq i} = \emptyset$$

First, we consider the behavior of the instanton partition function in the large N limit: $\mathcal{O}(N^2)$ or $\mathcal{O}(N)$ or $\mathcal{O}(1)$.

In $Z^{(\text{inst})}$, product over i and j are taken

$$Z^{(\text{inst})}_{\mathcal{N}=2*}(a_i, \widetilde{m}) = \sum_{Y=\{Y_1,\cdots,Y_N\}} e^{-\frac{8\pi^2|Y|}{g_p^2}} \prod_{i,j=1}^{N} Z^{(\text{inst})}_{\text{vec}}\left(a_i - a_j; Y_i, Y_j\right) Z^{(\text{inst})}_{\text{mat}}\left(a_i - a_j; Y_i, Y_j; \widetilde{m}\right)$$

But in more detailed expression, we see that one of them should be instanton (s labels boxes in the Young tableaux)

$$Z^{(\text{inst})}_{\text{vec}}\left(a_i - a_j; Y_i, Y_j\right) = \prod_{s\in Y_i}\left[E\left(a_i - a_j; Y_i, Y_j, s\right)\right]^{-1} \prod_{t\in Y_j}\left[2 - E\left(a_j - a_i; Y_j, Y_i, t\right)\right]^{-1}$$

Only for $Y_i \neq \emptyset$ Only for $Y_j \neq \emptyset$

The instanton partition function has the form of

$$Z^{(\text{inst})}_{\mathcal{N}=2*}(a_i, \widetilde{m}) = \sum_{Y=\{Y_1,\cdots,Y_N\}} e^{-\frac{8\pi^2|Y|}{g_p^2}} \prod_{j=1}^{N} \prod_{i\in(\text{inst})} Z^{(\text{inst})}_{Y}\left(a_i, a_j, Y_i, Y_j\right)$$

and hence naively $\log Z^{(\text{inst})} = \mathcal{O}(N)$

Instantons in the large N limit

- The large N limit is a kind of the classical limit.
- The solution is determined by $\mathcal{O}(N^2)$ part in the Free energy.
- Instanton effects does not affect $\mathcal{O}(N^2)$ terms

  - Perturbative part does not depends on which eigenvalue is associated to the instanton.
  - Eigenvalue distribution is same to 0-instanton sector.
  - Saddle point equation determines where is the eigenvalue of the instanton.

In the large N limit, by using the eigenvalue distribution, instanton effects in the free energy can be expressed as

$$F_Y^{(\mathrm{inst})} = \frac{8\pi^2}{g_p^2}|Y| - N \sum_{b_i \in (\mathrm{inst})} \int da\, \rho(a) \log Z_Y^{(\mathrm{inst})}(b_i, a, Y_i, \emptyset)$$

where the eigenvalue density $\rho(a)$ is given by that in the 0-inst. sector.

$$\rho(x) = \frac{2}{\pi\mu^2}\sqrt{\mu^2 - x^2} \qquad \mu = \frac{\sqrt{\lambda(1+m^2)}}{2\pi}$$

From the Nekrasov's formula, $Z_Y^{(\mathrm{inst})}(b_i, a, Y_i, \emptyset)$ is

$$Z_Y^{(\mathrm{inst})}(b_i, a, Y_i, \emptyset) = \frac{m^2}{m^2+1} \frac{(2 - i(b-a) - \widetilde{m})(i(b-a) - \widetilde{m})}{\big(2 - i(b-a)\big) i(b-a)}$$

in 1-instanton case

In the case of small mass $4\pi^2 m^2 \ll \lambda$,

$$F_Y^{(\mathrm{inst})}(b) = \frac{8\pi^2}{g_p^2} - \frac{8\pi^2}{g_p^2}\left(1 - \frac{1}{\sqrt{1-\mu^2/b^2}}\right)$$

Here, b is the eigenvalue associated to the instanton.

Since the eigenvalue distribution is determined by the perturbative part, b must satisfy $-\mu < b < \mu$.

- In very strongly coupled large N limit, instanton effects in free energy is of $\mathcal{O}(1)$
- We cannot use the saddle point equation since the effective potential is not of $\mathcal{O}(N)$
- Even in the ’t Hooft limit, the saddle point equation does not determines the position of the eigenvalue, since the real part of $F_Y^{(\mathrm{inst})}(b)$ is constant.

Since integrant is an oscillating function, there is suppression.

$$\int_{-\mu}^{\mu} db\, e^{-\frac{8\pi^2}{g_p^2}+\frac{8\pi^2}{g_p^2}\left(1-\frac{1}{\sqrt{1-\mu^2/b^2}}\right)} = \frac{16\pi^2\mu}{g_p^2} K_1\left(\frac{8\pi^2}{g_p^2}\right)$$

where $K_1(x)$ is the modified Bessel function of the second kind

$$K_1(x) \sim e^{-x} \qquad\qquad (x \to \infty)$$

Then, the instanton partition function is $\quad Z^{(\text{inst})} \sim e^{-\frac{8\pi^2}{g_p^2}}$

In the very strongly coupled large N limit:

$$g_p = \mathcal{O}(1) \qquad \text{but} \qquad g_p \ll 1$$

Since the weight of the instanton is finite,
it has non-zero contribution even in the large N limit.

Orbifold equivalence

[Kachru-Silverstein ’98]

Parent theory	Z_k Orbiford	Daughter theory
4d $\mathcal{N} = 4$ $SU(kN)$ SYM	⇒	4d $\mathcal{N} = 2$ $[SU(N)]^k$ quiver gauge theory

In the ’t Hooft limit, the correlation functions of Z_k-invariant operators in the parent theory coincide with their counterpart in the daughter theory.

In the perturbative theory, the equivalence can be directly shown by calculating the planar diagram.

[Bershadsky-Johansen ’98]

Orbifold equivalence

[Kachru-Silverstein '98]

Parent theory	Z_k Orbiford	Daughter theory
4d $\mathcal{N} = 4$ $SU(kN)$ SYM	⇒	4d $\mathcal{N} = 2$ $[SU(N)]^k$ quiver gauge theory

Correspondence of the free energy

$$F_p(\lambda, N) = kF_d(\lambda, N)$$

Free energy in parent theory → F_p

Free energy in daughter theory → F_d

The 't Hooft coupling is same in both theory. In other word,

$$g_d^2 = kg_p^2$$

Gauge coupling in daughter theory → g_d^2

Gauge coupling in parent theory → g_p^2

Orbifold equivalence and AdS/CFT

[Kachru-Silverstein '98]

	Parent theory	Z_k Orbiford	Daughter theory
CFT	4d $\mathcal{N}=4$ $SU(kN)$ SYM	⇒	4d $\mathcal{N}=2$ $[SU(N)]^k$ quiver gauge theory
	⇕		⇕
AdS	Supergravity on $\mathrm{AdS}_5\times S^5$		Supergravity on $\mathrm{AdS}_5\times S^5/Z_k$

In the gravity side, we consider the classical solution.

Orbifolding acts to the extra dimension S^5, but locally same.

Anything expressed in terms of Z_k-invariant modes coincide with their counter part in daughter theory.

Orbifold equivalence and AdS/CFT

[Kachru-Silverstein '98]

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AdS	Supergravity on $\mathrm{AdS}_5 \times S^5$		Supergravity on $\mathrm{AdS}_5 \times S^5/Z_k$

Orbifold equivalence can be shown in the gravity side.

Analytic continuation to g^2-fixed limit is straightforward.

The equivalence holds in very strongly coupled large N limit in the gauge theory side.

Orbifold equivalence for $\mathcal{N} = 2*$

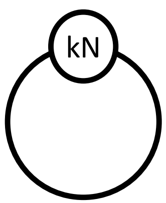
Parent theory

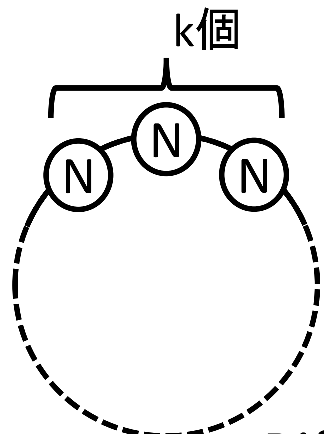
$\mathbb{Z}_k$ Orbiford

Daughter theory

$\mathcal{N} = 2 * SU(kN)$
gauge theory

$\mathcal{N} = 2\ [SU(N)]^k$
Necklace quiver
gauge theory





$k\ SU(N)$
gauge fields

1 $SU(kN)$ gauge field

Adjoint matter, which
connects $SU(kN)$ to itself

Bifund. matters which
connects nearest
neighbor $SU(N)$

Partition function of the necklace quiver

Partition function is calculated by using the localization.

$$Z_{[SU(N)]^k} = \int \prod_{\alpha}^{k} d^N a^{(\alpha)} \left(\prod_{\substack{i,j=1 \\ i<j}}^{N} \left(a_i^{(\alpha)} - a_j^{(\alpha)} \right)^2 \right) Z_{\mathcal{N}=2*}^{(\text{part})} \left| Z_{\mathcal{N}=2*}^{(\text{inst})} \right|^2 \exp\left(-\frac{8\pi^2}{g_d^2} \sum_{\alpha=1}^{k} \sum_{i=1}^{N} \left(a_i^{(\alpha)} \right)^2 \right)$$

where $\left(Z_{\text{vec}}^{(\text{part})}, Z_{\text{mat}}^{(\text{part})}, Z_{\text{vec}}^{(\text{inst})}, Z_{\text{mat}}^{(\text{inst})} \text{ are same to } \mathcal{N} = 2* \right)$

$$Z_{[SU(N)]^k}^{(\text{part})} = \left(\prod_{\alpha=1}^{k} \prod_{\substack{i,j=1 \\ i\neq j}}^{N} Z_{\text{vec}}^{(\text{part})} \left(a_i^{(\alpha)} - a_j^{(\alpha)} \right) \right) \left(\prod_{\alpha=1}^{k} \prod_{i,j=1}^{N} Z_{\text{mat}}^{(\text{part})} \left(a_i^{(\alpha)} - a_j^{(\alpha+1)}, m \right) \right)$$

$$Z_{[SU(N)]^k}^{(\text{inst})} = \sum_{Y^{(1)},\cdots,Y^{(k)}} e^{-\frac{8\pi^2}{g_d^2} \sum_\alpha |Y_\alpha|}$$

$$\times \prod_{\alpha=1}^{k} \prod_{i,j=1}^{N} Z_{\text{vec}}^{(\text{inst})} \left(a_i^{(\alpha)} - a_j^{(\alpha)}; Y_i^{(\alpha)}, Y_j^{(\alpha)} \right) Z_{\text{mat}}^{(\text{inst})} \left(a_i^{(\alpha)} - a_j^{(\alpha+1)}; Y_i^{(\alpha)}, Y_j^{(\alpha+1)}; \widetilde{m} \right)$$

Partition function of the necklace quiver

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kN eigenvalues in $\mathcal{N} = 2*$ is separated into k set of N eigenvalues which are associated to k vector multiplets of each $SU(N)$

$\mathcal{N} = 2*$ theory $\quad a_i \quad i = 1, \cdots, kN \quad$ kN eigenvalues of $SU(kN)$

Orbifold projection

$[SU(N)]^k$ quiver $\quad a_i^{(\alpha)} \quad \alpha = 1, \cdots, k$, $\; i = 1, \cdots, N \quad$ Eigenvalues of k of $SU(N)$

Orbifold equivalence at perturbative level

We know the equivalence of perturbaive theory in the 't Hooft limit.

How can we understand it in the localized partition function?

Perturbative parts in $\mathcal{N}=2*$ and $[SU(N)]^k$ quiver gauge theory

$$Z^{(\text{part})}_{\mathcal{N}=2*}(a_i,m)=\prod_{\substack{i,j=1\\ i\neq j}}^{kN} Z^{(\text{part})}_{\text{vec}}(a_i-a_j)\prod_{i,j=1}^{kN} Z^{(\text{part})}_{\text{mat}}(a_i-a_j,m)$$

$$Z^{(\text{part})}_{[SU(N)]^k}=\left(\prod_{\alpha=1}^{k}\prod_{\substack{i,j=1\\ i\neq j}}^{N} Z^{(\text{part})}_{\text{vec}}\left(a_i^{(\alpha)}-a_j^{(\alpha)}\right)\right)\left(\prod_{\alpha=1}^{k}\prod_{i,j=1}^{N} Z^{(\text{part})}_{\text{mat}}\left(a_i^{(\alpha)}-a_j^{(\alpha+1)},m\right)\right)$$

$\mathcal{N}=2*$	Interaction between eigenvalues in same $SU(kN)$	
$[SU(N)]^k$ quiver	Vector multiplets:	Interaction in same $SU(N)$
	Bifund. matters:	Interaction in different $SU(N)$

In the large N limit, eigenvalue distribution becomes continuous.

- The difference of $SU(kN)$ and $SU(N)$ gives factor k
- In $[SU(N)]^k$ quiver, there are k gauge group of $SU(N)$
- Relation between two free energies is $F_p = kF_d$

Difference of $SU(kN)$ and $SU(N)$ is not problem

Interaction of the bifund matters between different $SU(N)$ possibly break the equivalence.

Democratic ansatz $\rho^{(1)}(x) = \cdots = \rho^{(k)}(x)$

- Interaction between different $SU(N)$ and that in same $SU(N)$ gives same result.
- This is a solution of the saddle point equation

By using this ansatz, the saddle point equation becomes same to that for the parent theory. Then,

$$F_p(\lambda, N) = kF_d(\lambda, N)$$

Instanton part at classical level

Young tableaux are associated to each eigenvalue

$\mathcal{N} = 2 *$ theory

$a_i \quad i = 1, \cdots, kN$

kN eigenvalues of $SU(kN)$

$Y_i \quad i = 1, \cdots, kN$

Orbifold projection

$[SU(N)]^k$ quiver

$a_i^{(\alpha)} \quad \alpha = 1, \cdots, k \quad i = 1, \cdots, N$

Eigenvalues of k of $SU(N)$

$Y_i^{(\alpha)} \quad \alpha = 1, \cdots, k \quad i = 1, \cdots, N$

Young tableau associated to a_i

Young tableau associated to $a_i^{(\alpha)}$

- Each eivenvalue in parent has its counterpart in daughter
- Each Young tableaux in parent has its counterpart in daughter

Total instanton number agrees in parent and daughter

$$\sum_i |Y_i| = \sum_{\alpha,i} |Y_i^{(\alpha)}|$$

Classical free energy in the 1-instanton sector in parent theory

$$F_{p,\mathrm{cl}}^{(\mathrm{inst})} = \frac{8\pi^2}{g_p^2}\sum_i |Y_i| = \frac{8\pi^2 kN}{\lambda}\sum_i |Y_i|$$

Classical free energy in the 1-instanton sector in daughter theory

$$F_{d,\mathrm{cl}}^{(\mathrm{inst})} = \frac{8\pi^2}{g_d^2}\sum_{\alpha,i} |Y_i^{(\alpha)}| = \frac{8\pi^2 N}{\lambda}\sum_{\alpha,i} |Y_i^{(\alpha)}|$$

Here we do not sum up the instantons but only 1-instanton sector.

Since the total instanton number is same $\sum_i |Y_i| = \sum_{\alpha,i} |Y_i^{(\alpha)}|$

We obtain

$$F_{p,\mathrm{cl}}^{(\mathrm{inst})} = kF_{d,\mathrm{cl}}^{(\mathrm{inst})}$$

Orbifold equivalence for Nekrasov's formula

Nekrasov's formula of instanton partition function for $[SU(N)]^k$ quiver

$$Z^{(\mathrm{inst})}_{[SU(N)]^k} = \sum_{Y^{(1)},\cdots,Y^{(k)}} e^{-\frac{8\pi^2}{g_d^2}\sum_\alpha |Y_\alpha|}$$

$$\times \prod_{\alpha=1}^{k}\prod_{i,j=1}^{N} Z^{(\mathrm{inst})}_{\mathrm{vec}}\left(a_i^{(\alpha)} - a_j^{(\alpha)}; Y_i^{(\alpha)}, Y_j^{(\alpha)}\right) Z^{(\mathrm{inst})}_{\mathrm{mat}}\left(a_i^{(\alpha)} - a_j^{(\alpha+1)}; Y_i^{(\alpha)}, Y_j^{(\alpha+1)}; \widetilde{m}\right)$$

In the large N limit, the eigenvalue distribution becomes continuous

$$F^{(\mathrm{inst})}_{d,Y} = -N\sum_{\alpha=1}^{k}\sum_{b_i\in(\mathrm{inst})}\int da\, \rho^{(\alpha)}(a)\log Z^{(\mathrm{inst})}_{\mathrm{vec}}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$

$$-N\sum_{\alpha=1}^{k}\sum_{b_i\in(\mathrm{inst})}\int da\, \rho^{(\alpha+1)}(a)\log Z^{(\mathrm{inst})}_{\mathrm{vec}}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$

The eigenvalue distribution is determined by the perturbative part.

In the case of $\mathcal{N} = 2*$, the instanton effects in the free energy is

$$F_{p,Y}^{(\mathrm{inst})} = -kN \sum_{b_i \in (\mathrm{inst})} \int da\, \rho(a) \log Z_Y^{(\mathrm{inst})}(b_i, a, Y_i, \emptyset)$$

In the case of $[SU(N)]^k$ quiver,

$$F_{d,Y}^{(\mathrm{inst})} = -N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha)}(a) \log Z_{\mathrm{vec}}^{(\mathrm{inst})}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$
$$-N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha+1)}(a) \log Z_{\mathrm{mat}}^{(\mathrm{inst})}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$

Only for α for which $b^{(\alpha)}$ can be the instanton

The difference is that interaction is that between different $SU(N)$ in the contribution from bifund. matter

Since the eigenvalue distribution is determined by perturbative part, it is expected to satisfy

$$\rho^{(1)}(x) = \cdots = \rho^{(k)}(x)$$

Then, interaction equals to that with same $SU(N)$:

$$F_{d,Y}^{(\mathrm{inst})} = -N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha)}(a) \log Z_{\mathrm{vec}}^{(\mathrm{inst})}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$
$$-N \sum_{\alpha=1}^{k} \sum_{b_i \in (\mathrm{inst})} \int da\, \rho^{(\alpha)}(a) \log Z_{\mathrm{mat}}^{(\mathrm{inst})}\left(b_i^{(\alpha)}, a, Y_i, \emptyset\right)$$

As for the 0-instanton sector, the free energy becomes same but has an additional factor of k. Then,

$$F_p^{(\mathrm{inst})} = kF_d^{(\mathrm{inst})}$$

Conclusion

- We have calculated the instanton effects in the large N limit.
- By using the very strongly coupled large N limit (g^2-fixed limit), the instanton effect is finite.
- The orbifold equivalence is valid for instanton effects, too.
- Generalization to multi-instanton cases is straightforward.

Applications

- Application to M-theory
  - In the M-theory, $g_s \sim R_{11}$ becomes finite. (not 't Hooft limit)
  - 4d $\mathcal{N} = 2$ CFTs of M5-branes in the M-theory region.
  - Taking the instanton effects into account
  - And their relation to gravity dual
- Non-SUSY
  - Orbifold projection which breaks SUSY.
  - QCD. Instantons with radius of QCD scale.