A Systematic Study on Matrix Models for Chern-Simons-matter Theories

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Introduction

Chern-Simons-matter theories (CSM) have been studied for 5 years, in relation to

- **boundary conditions for $\mathcal{N} = 4$ SYM$_4$,** [Gaiotto,Witten]
- **worldvolume theory on M2-branes,** [BLG][ABJM]
- **M-theory on AdS$_4 \times M$.** [ABJM]

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The question we would like to discuss today is

> **What is the possible set of exponents of planar $\mathcal{N} = 3$ CSMs?**

Here the exponent γ is defined as

$$\gamma \ := \ \lim_{\lambda \to \infty} \log \Big[\log |\langle W \rangle| \Big] / \log \lambda.$$

This exponent has a geometric meaning in the context of AdS/CFT correspondence.

E.g.1) ABJM theory

This is $\mathcal{N} = 6$ CS theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, believed to be dual to Type IIA on $\mathbf{AdS}_4 \times \mathbb{CP}^3$ with flux.

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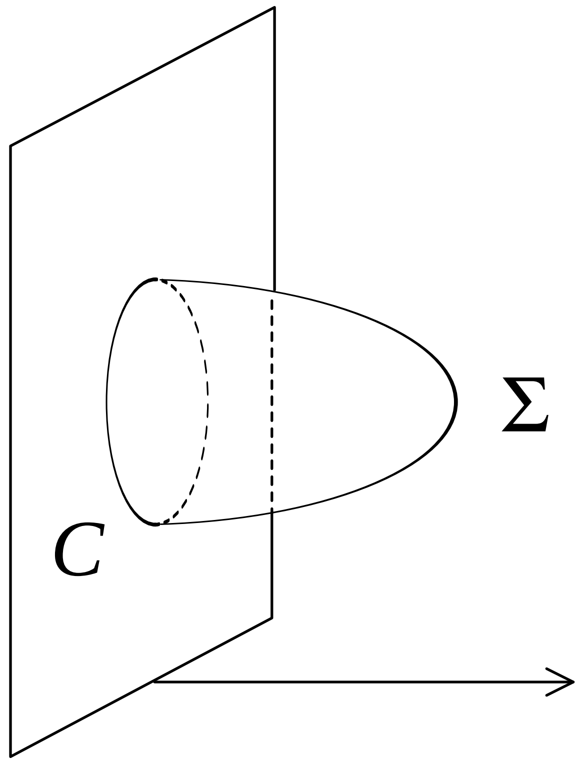
$$\langle W[C]\rangle \;\sim\; e^{-\mathrm{Area}(\Sigma)}, \qquad \partial\Sigma \;=\; C.$$


C
Σ

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$\mathrm{Area}(\Sigma) \propto R^2$ **(R is the radius of AdS_4) and $R^4 \propto \lambda$ ($\lambda \to \infty$) is one of the proposals in [ABJM]. Therefore one should obtain**

$$\underline{\gamma_{\mathrm{ABJM}} = \frac{1}{2}}$$

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For ABJM, this was confirmed by showing

$$\langle W[C] \rangle \ \sim \ e^{\pi\sqrt{2\lambda}}, \qquad (\lambda \to \infty)$$

where $\lambda = N/k$ **is the 't Hooft coupling.** [Marino,Putrov]

Note: The coefficient in the exponent also matches.

E.g.2) Gaiotto-Tomasiello theory [Gaiotto,Tomasiello]

This is an $\mathcal{N} = 3$ CSM with gauge group $\mathrm{U}(N)_{k_1} \times \mathrm{U}(N)_{k_2}$, dual to massive Type IIA.

It was found that **[TS]**

$$\log |\langle W \rangle| \sim \frac{\sqrt{3}}{2} \left(\frac{6\pi^3}{k_1 + k_2} N \right)^{\frac{1}{3}} \quad \Rightarrow \quad \underline{\gamma_{\mathrm{GT}} = \frac{1}{3}}.$$

This is consistent with massive IIA result since **[AJTZ]**

$$R \sim N^{\frac{1}{6}}.$$

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E.g.3) Pure CS theory [Witten]

For this case, the exact expression for $\langle W\rangle$ is simply

$$\langle W\rangle = \frac{1}{N}\frac{\sin(\pi N/k)}{\sin(\pi/k)} \rightarrow \frac{\sin\pi\lambda}{\pi\lambda}.$$

This implies $\underline{\gamma_{\mathrm{pure}} = 0}$.

Note: A gravity dual was discussed in [Maldacena,Nastase].

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Questions:

- **Do they exhaust all possible values?**
- **If not, what are the other possibilities?**
- **Do they have geometric interpritations?**
- **Which CSMs correspond to which values?**

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In the following, we will show that

- **Most of $\mathcal{N} = 3$ CSMs have $\gamma = 0$.**
- **$\exists$ a necessary condition for $\gamma \neq 0$.**
  $\Rightarrow$ A hint for the principle underlying AdS/CFT??

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1. Localization **[KWY]**

The path-integral may be reduced to ordinary integral in SUSY QFT. ⇒ One may compute exactly quantities, e.g.

- **Free energy** ⇒ $N^{\frac{1}{3}}$ **behavior** **[DMP]**
- **Wilson loop. For ABJM theory,**

$$\langle W \rangle \;=\; Z^{-1} \int d^N u d^N \tilde{u}\, e^{\frac{ik}{4\pi}\sum_i (u_i^2 - \tilde{u}_i^2)} \frac{\prod_{i<j} \sinh^2 \frac{u_i - u_j}{2} \sinh^2 \frac{\tilde{u}_i - \tilde{u}_j}{2}}{\prod_{ij} \cosh^2 \frac{u_i - \tilde{u}_j}{2}} \cdot \frac{1}{N} \sum_i e^{u_i}.$$

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2. Matrix model

The above integral can be calculated in the planar limit.
$\Leftrightarrow$ Saddle-point approximation.

All information is encoded in $\rho(x), \tilde{\rho}(x)$. For example,

$$\langle W \rangle = \int dx \, \rho(x) e^x.$$

$\rho(x), \tilde{\rho}(x)$ **are then encoded in analytic functions $v(z), \tilde{v}(z)$.**

$\mathcal{N}=3$ planar CSM

We focus on this family of CSMs since

- $\mathcal{N}=3$ **SUSY** $\Rightarrow$ **Localization formula is simple.**
- $\exists$ **'t Hooft limit** $\Rightarrow$ **Matrix model techenique is available.**

Def. $\mathcal{N}=3$ **planar CSM is a theory specified by**

- **gauge group:** $\prod_a \mathrm{U}(N_a)_{k_a}$ **for simplicity,**
- **matter reps.:** $R=\oplus_i R_i$ **where** $R_i =$ **f, adj, sym, asym, bf, ff.**

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For each theory, we obtain S^3 **partition function**

$$Z = \int \prod_{a=1}^{n_g} d^{N_a}u^a \exp\left[-\sum_a S^a_{\rm tree}[u]-\sum_a S^a_{\rm v}[u]-\sum_i S_i[u]\right].$$

The saddle-point equations are

$$\frac{k_a}{2\pi}u^a_{i_a} = \sum_{j_a\neq i_a}\coth\frac{u^a_{i_a}-u^a_{j_a}}{2}-\sum_i\frac{\partial S_i}{\partial u^a_{i_a}}.$$

Gaussian matrix model

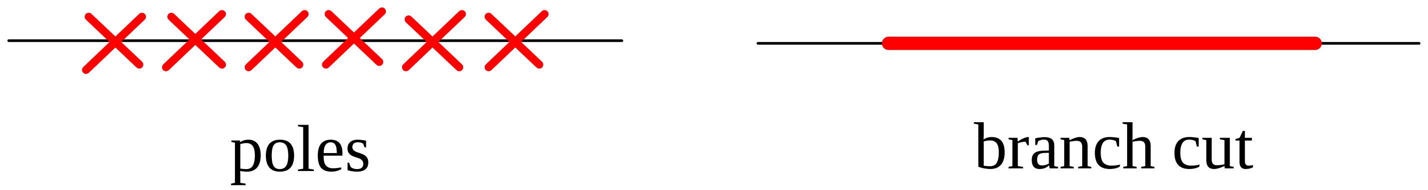
A simple toy model:

$$\frac{k}{2\pi}u_i = \sum_{j\neq i}\frac{1}{u_i - u_j}.$$

Define **resolvent**

$$v(z) := \frac{1}{N}\sum_{j=1}^{N}\frac{1}{z-u_j}, \qquad (z\in\mathbb{C}\backslash\{u_j\}).$$

In the 't Hooft limit, $v(z)$ is assumed to have a branch cut on $[a,b]$.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$$\frac{1}{\pi\lambda}u = v(u+i0) + v(u-i0), \quad (u \in [a,b])$$

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$$\frac{1}{\pi\lambda} u \;=\; v(u + i0) + v(u - i0), \quad (u \in [a, b])$$

with the boundary condition $z\, v(z) \;\to\; 1$. The solution is

$$v(z) \;=\; \frac{1}{2\pi\lambda} \left[z - \sqrt{(z-a)(z-b)} \right], \quad -a = b = \sqrt{4\pi\lambda}.$$

The distribution of u_i is described by

$$\rho(u) \;:=\; \frac{1}{\pi} \mathrm{Im}\, v(u - i0) \;=\; \frac{1}{2\pi^2\lambda} \sqrt{b^2 - u^2}.$$

Saddle-point eqs. for CSM

We focus on a CSM with n^a funds. and n^{ab} bi-funds. Then

$$2\kappa^a \log(\epsilon_a y_a) + \nu^a \frac{\epsilon_a y_a - 1}{\epsilon_a y_a + 1} \;=\; v^a(y_a^+) + v^a(y_a^-) - \sum_b n^{ab} v^b(y_a),$$

where $y_a \in [p_a, q_a]$**,** $\epsilon_a = \pm 1$**,** $\kappa^a = k^a/k$**,** $\nu^a = n^a/k$**.**

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where $y_a \in [p_a, q_a]$, $\epsilon_a = \pm 1$, $\kappa^a = k^a/k$, $\nu^a = n^a/k$. Assuming

$$v^a(z) = r^a(z) + \tilde{v}^a(z),$$

the 2nd term in LHS can be eliminated if

$$\nu^a \frac{\epsilon_a z - 1}{\epsilon_a z + 1} = \sum_b (2\delta^{ab} - n^{ab}) r^b(z).$$

$\exists$ a solution if $C^{ab} := 2\delta^{ab} - n^{ab}$ is non-degenerate.

Note: $\tilde{v}^a(z)$ have poles at $z = \pm 1$. The residues contain information of ν^a.

Similarly, log-term can be eliminated, assuming

$$\tilde{v}^a(z) \;=\; \int_0^\infty d\xi\, v^a(z,\xi),$$

since

$$\log(\epsilon_a z) \;=\; -\int_0^\infty d\xi \left[\frac{1}{\xi-\epsilon_a z} - \frac{1}{\xi - 1}\right].$$

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$\Rightarrow$ **For non-degenerate cases, the task is to solve**

$$\boxed{\omega^a(y_a^+) + \omega^a(y_a^-) - \sum_b n^{ab}\omega^b(y_a) \;=\; 0.}$$

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Note: ξ-dependence enters through the conditions of poles at $z = \pm\xi$.

Note2: All CSMs with known gravity duals are degenerate.

Monodromy

The homogeneous eqs. can be written as

$$\boxed{\omega(y_a^+) = \omega(y_a^-)M_a,} \quad \omega(z) := (\omega^1(z), \cdots, \omega^{n_g}(z)),$$

where

$$M_a = \begin{bmatrix} 1 & & & n^{a1} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & n^{a,a-1} & & & \\ & & & -1 & & & \\ & & & n^{a,a+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & n^{a,n_g} & & & 1 \end{bmatrix}, \quad M_a^2 = I.$$

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These eqs. define monodromies of $\omega(z)$ at $z = p_a, q_a$:

$$\omega(p_a+\epsilon) \;=\; \omega(p_a+\epsilon e^{2\pi i})M_a, \qquad \omega(q_a-\epsilon) \;=\; \omega(q_a-\epsilon e^{-2\pi i})M_a.$$


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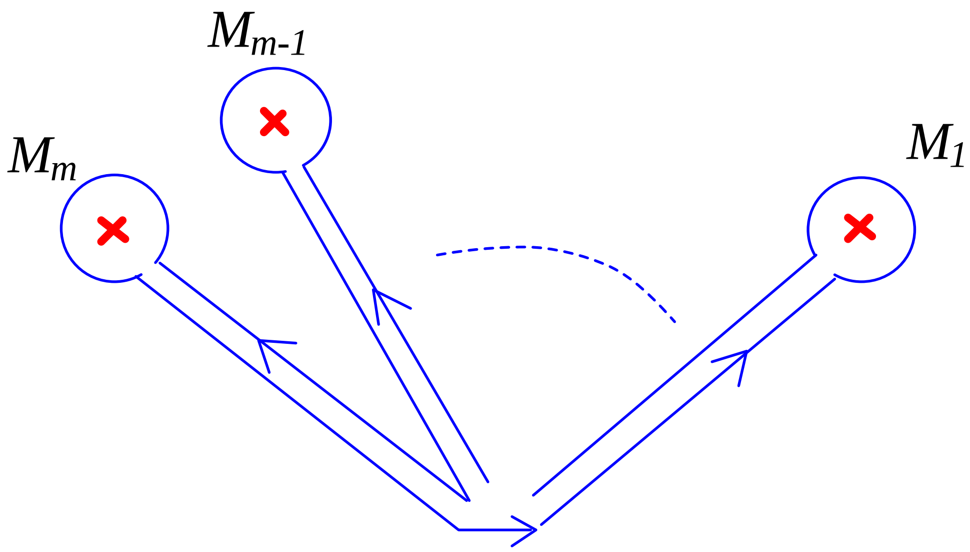
$$M_m M_{m-1} \cdots M_1 \;=\; I.$$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$$M_m M_{m-1} \cdots M_1 = I.$$

In our case, this is trivially satisfied.
⇒ Find a vector $\omega(z)$ with the prescribed monodromies.

cf) The hypergeometric fun. : 2-vector with 3 singularities.

Riemann-Hilbert problem

Known result **[Plemelj]**

RH has a solution if one of M_i is diagonalizable.

This means $\exists$ a Fuchsian system

$$\frac{dy}{dz} = \sum_i \frac{A_i}{z - z_i} y$$

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In our case, any M_a is diagonalizable. $\Rightarrow \exists\, y_1, \cdots, y_{n_g}$.
Define $Y(z)$ from a matrix $(y_1, \cdots, y_{n_g})$. This satisfies

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$Y(z)$ is holomorphic and non-degenerate on $\mathbb{C} \backslash \{p_a, q_a\}$, and

$$Y(z) \sim z\, V(z) \quad (z \to \infty)$$

under a certain continuity assumption.

Planar resolvents

$\omega(z,\xi)$ **is given as**

$$\omega(z,\xi) \;=\; r(z,\xi)Y(z),$$

where $r(x,\xi)$ **is a row vector of rational functions s.t.**

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Let $\rho_\pm$ **be vectors of redisues at** $z=\pm\xi$**. Then**

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Finally, the resolvent vector is given as

$$v(z) \;=\; r_1(z) + \int_0^\infty d\xi \left[r_2(z,\xi) + \omega(z,\xi)\right].$$

This contains all information of the matrix model.

Note: Explicit form of $Y(z)$ **is known for** $n_g \le 2$**.**

Exponent of non-degenerate theories

The vector of 't Hooft couplings is given as

$$t \;=\; -v(0) \;=\; -r_1(0) - \int_0^\infty d\xi\ \left[r_2(0,\xi) + \omega(0,\xi)\right].$$

This is a function of coordinates of branch points.

t **may diverge when**

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Case 2: Lengths of branch cuts are finite in the limit.

$$\Rightarrow \quad |\langle W\rangle| \leq {}^{\exists}C, \qquad \boxed{\gamma = 0.}$$

cf. Isomonodromic deformations of Fuchsian system [Schlesinger]

Degenerate theory: an example

Gaiotto-Tomasiello theory ($n_g = 2$, 2 bi-funds) was solved. [TS]
The resolvent is

$$v(z) \;=\; \kappa_1 \int_{a_1}^{b_1} \frac{dx}{2\pi} \frac{\log(e^{(t_2-t_1)/\kappa_1} x)}{z-x} \frac{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}}{\sqrt{|(x-a_1)(x-b_1)(x-a_2)(x-b_2)|}} + (1 \leftrightarrow 2).$$

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When 't Hooft couplings are large, it behaves as

$$t_1 \;\sim\; \frac{\kappa_1+\kappa_2}{3\pi^2}\alpha^3 + O(\alpha^2), \qquad -\log a_1, -\log a_2 \;\sim\; \alpha.$$

This implies $\alpha \;\sim\; t_1^{1/3}$. Since $\langle W \rangle \sim e^{\alpha}$, one obtains

$$\boxed{\gamma \;=\; \frac{1}{3}.}$$

Note: When $\kappa_1 + \kappa_2 = 0$ (ABJM), then $t_1 = O(\alpha^2)$ and $\gamma = \frac{1}{2}$.

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$$v(z) \;=\; \kappa_1 \int_{a_1}^{b_1} \frac{dx}{2\pi} \frac{\log(e^{(t_2-t_1)/\kappa_1} x)}{z-x} \frac{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}}{\sqrt{|(x-a_1)(x-b_1)(x-a_2)(x-b_2)|}} + (1 \leftrightarrow 2).$$

When 't Hooft couplings are large, it behaves as

$$t_1 \;\sim\; \frac{\kappa_1+\kappa_2}{3\pi^2}\alpha^3 + O(\alpha^2), \qquad -\log a_1, -\log a_2 \;\sim\; \alpha.$$

This implies $\alpha \;\sim\; t_1^{1/3}$. Since $\langle W\rangle \sim e^{\alpha}$, one obtains

$$\boxed{\gamma \;=\; \frac{1}{3}.}$$

Note: When $\kappa_1+\kappa_2 = 0$ (ABJM), then $t_1 = O(\alpha^2)$ and $\gamma = \frac{1}{2}$.

Conjecture: $\gamma = 0,\ \frac{1}{2},\ \frac{1}{3}$ are all possible exponents.

Summary

- **The exponents of $\mathcal{N}=3$ planar CSMs are investigated.**
- **Most of such theories has $\gamma=0$.**
- $\gamma\neq 0 \Rightarrow \det C^{ab}=0$**.**
- **CSM with generic matter contents can be reduced to CSM with bi-funds.**

Open issues

- **Analysis of degenerate theories (a limit of non-degenerate theories)**
- **Meaning of** $\det C^{ab}=0$
  $\Rightarrow$ **The condition for the presence of AdS_4 dual**
- **Implications to interacting Fermi gas**
- **etc.**