Holomorphic Blocks for 3d Gauge Theories

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based on [M.T, arXiv:1303.5915]

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@ Osaka Univ

Localization opens new avenue

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[Vasily Pestun, 07]

solves gauge theory on 4-sphere

→ AGT, etc

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solves gauge theory on 4-sphere

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[Kapsutin-Willett-Yaakov, 09]
[Marino-Putrov, 09]

gauge theory on 3-sphere

→ quantum nature of M2-branes

Localization opens new avenue

Way to compute supersymmetric quantities **exactly**

Localization **opens new avenue**

Way to compute supersymmetric quantities exactly

$$\int_a^b dx \, \frac{df(x)}{dx}$$

Localization opens new avenue

Way to compute supersymmetric quantities exactly

$$\int_a^b dx \, \frac{df(x)}{dx} = f(b) - f(a)$$

integration → **summation**

1. 3d Partition Function & Superconformal Index : reviev

3d gauge theory

IIA IIB

3d gauge theory

$E_8 \times E_8$ Het

SO(32) Het

IIA

IIB

3d gauge theory

$E_8 \times E_8$ Het

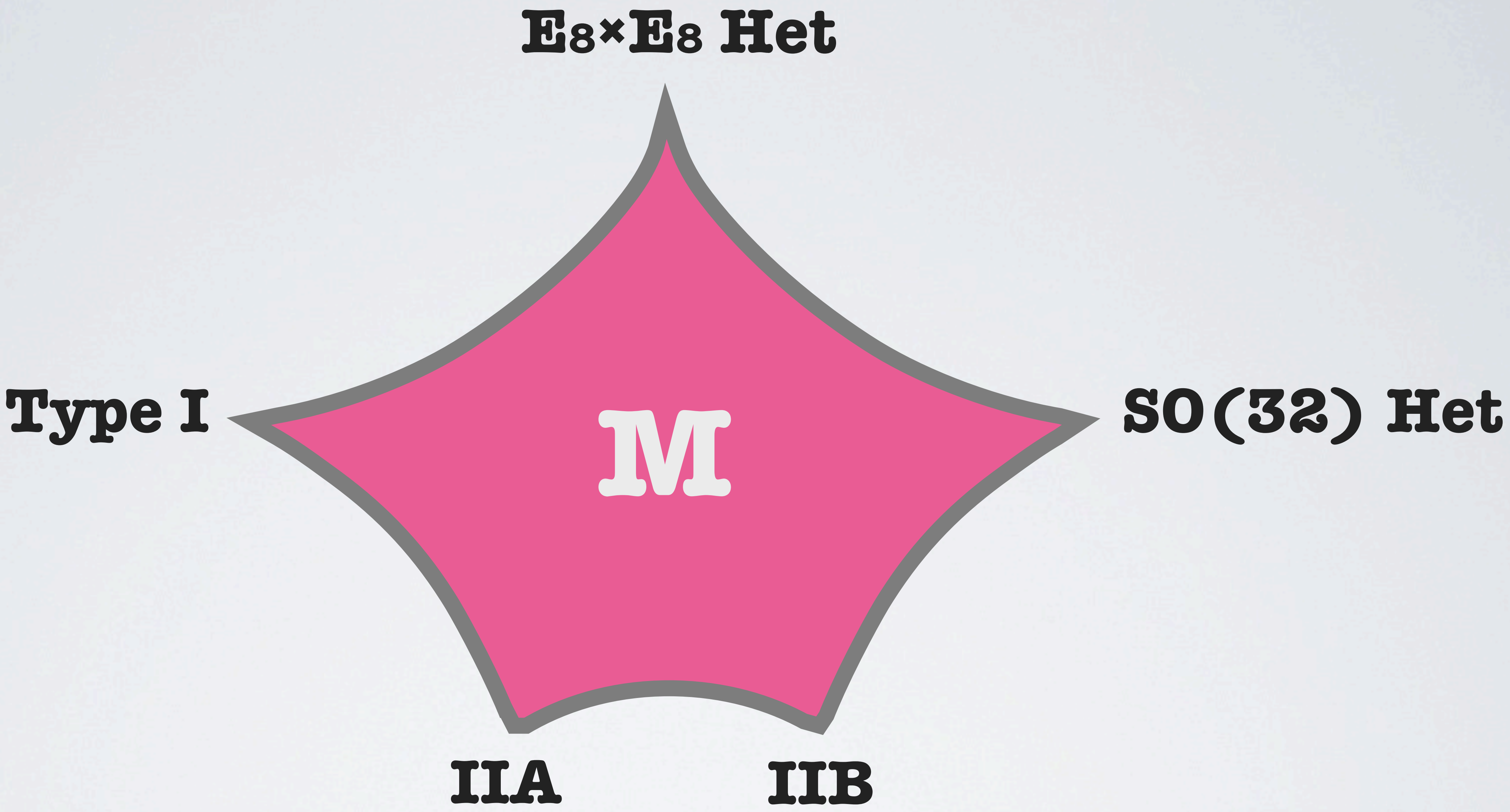
Type I

SO(32) Het

IIA

IIB

3d gauge theory



3d gauge theory

M2-branes

3d gauge theory

M2-branes

M5-branes

3d gauge theory

M2-branes → **3d Chern-Simons theory**

[Bagger-Lambert], [ABJM]

M5-branes

3d gauge theory

M2-branes → 3d Chern-Simons theory

M5-branes → still-mysterious 6d CFT

3d gauge theory

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3d $\mathcal{N}$=2 gauge theory

- vector mult.

$$A_\mu, \ \sigma, \ \lambda, \ \bar{\lambda}, \ D$$

3d $\mathcal{N}$=2 gauge theory

- vector mult.

$$A_\mu,\ \sigma,\ \lambda,\ \bar{\lambda},\ D$$

$\sigma \leftarrow A_4$

3d $\mathcal{N}$=2 gauge theory

- vector mult.

$$A_\mu,\ \sigma,\ \lambda,\ \bar{\lambda},\ D$$

$$\downarrow$$

SUSY Chern-Simons term

$$\int d^3x \mathrm{Tr}\left(A \wedge dA + \frac{2i}{3} A \wedge A \wedge A - \lambda\bar{\lambda} + 2D\sigma \right)$$

3d $\mathcal{N}$=2 gauge theory

- vector mult.

$$A_\mu, \; \sigma, \; \lambda, \; \bar{\lambda}, \; D$$

- chiral mult.

$$\phi, \; \bar{\phi}, \; \psi, \; \bar{\psi}, \; F, \; \bar{F}$$

Localization

YM term on 3-sphere

$$t\int d^3x \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\sigma D^\mu\sigma+\frac{1}{2}\left(D+\frac{\sigma}{r}\right)^2+\cdots\right)$$

Localization

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$$=t\int d^3x\,\delta_{\bar{\epsilon}}\delta_\epsilon \mathrm{Tr}\left(\frac{1}{2}\bar{\lambda}\lambda-2D\sigma\right)$$

is SUSY exact !

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is SUSY exact ! : t-independent

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we can evaluate in $t\longrightarrow\infty$

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is SUSY exact ! : t-independent

we can evaluate in $t\to\infty$

the YM term is positive definite

Localization

YM term on 3-sphere

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path integral reduces to

$$\begin{cases} F_{\mu\nu}=0 \\ D_\mu\sigma=0 \\ D+\dfrac{\sigma}{r}=0 \end{cases}$$

Localization

YM term on 3-sphere

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path integral reduces to

$$\begin{cases} F_{\mu\nu}=0 & \Longrightarrow \quad A_\mu=0 \\ D_\mu\sigma=0 & \Longrightarrow \quad \sigma=\sigma_0 \\ D+\dfrac{\sigma}{r}=0 & \end{cases}$$

const. Hermite matrix

Localization

YM term on 3-sphere

$$t\int d^3x \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\sigma D^\mu\sigma+\frac{1}{2}\left(D+\frac{\sigma}{r}\right)^2+\cdots\right)$$

path integral reduces to

conventional integral

over Hermite matrices : $\sigma=\sigma_0$

calculable examples

Localization of superconformal index

$$SO(2)_j \times SO(3)_\epsilon \times SO(2)_R \ \subset SO(3,2) \times SO(2)$$

$$I(q,z) = \mathrm{Tr}(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{S}\}} q^{\epsilon+j} \prod_i z_i^{F_i}$$

Localization of superconformal index

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$$I(q,z) = \mathrm{Tr}(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{S}\}} q^{\epsilon+j} \prod_i z_i^{F_i}$$

localization via path integral rep.

$$\longrightarrow \quad I = \int \mathcal{D}\Phi \, e^{-S[S^1 \times S^2] - QV}$$

[S.Kim, 09] [Imamura, Yokoyama, 09] etc

Factorization [Krattenthaler-Spiridonov-Vartanov, 11]

[Dimofte-Gaiotto-Gukov, 11] [Hwang-Kim-Park, 12]

the result exhibits the **factorization**

$$I = \sum_i Z_V^{(i)}(q, z)\overline{Z}_V^{(i)}(\bar{q}, \bar{z})$$

holomorphic blocks

Localization of partition function

[Kapustin, 11] [Jaferis, 11]

$$Z = \int \mathcal{D}\Phi \, e^{-S[S^3] - tQV}$$

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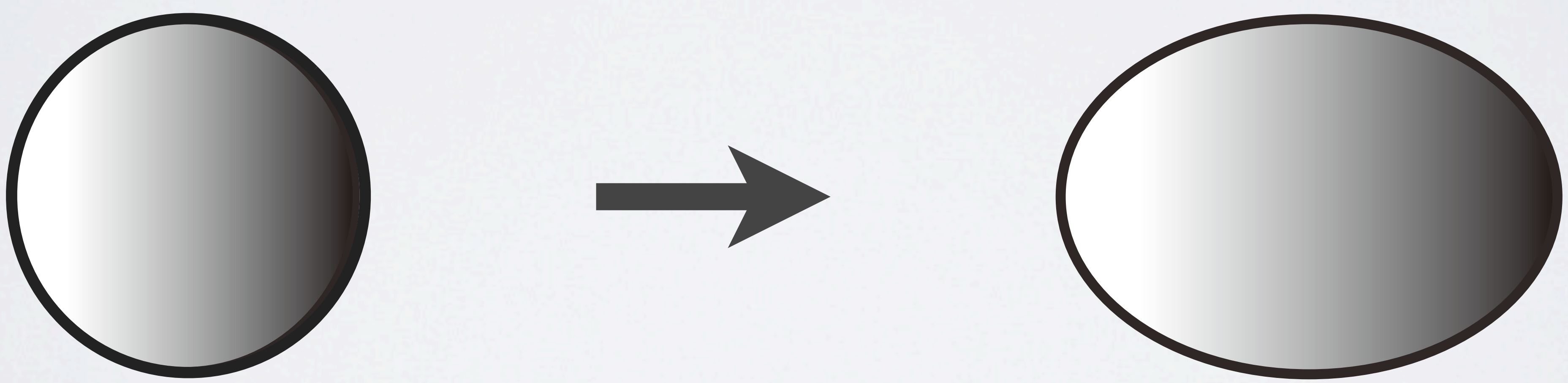
ellipsoid S_b^3 **: [Hama-Hosomichi-Lee, 11]**



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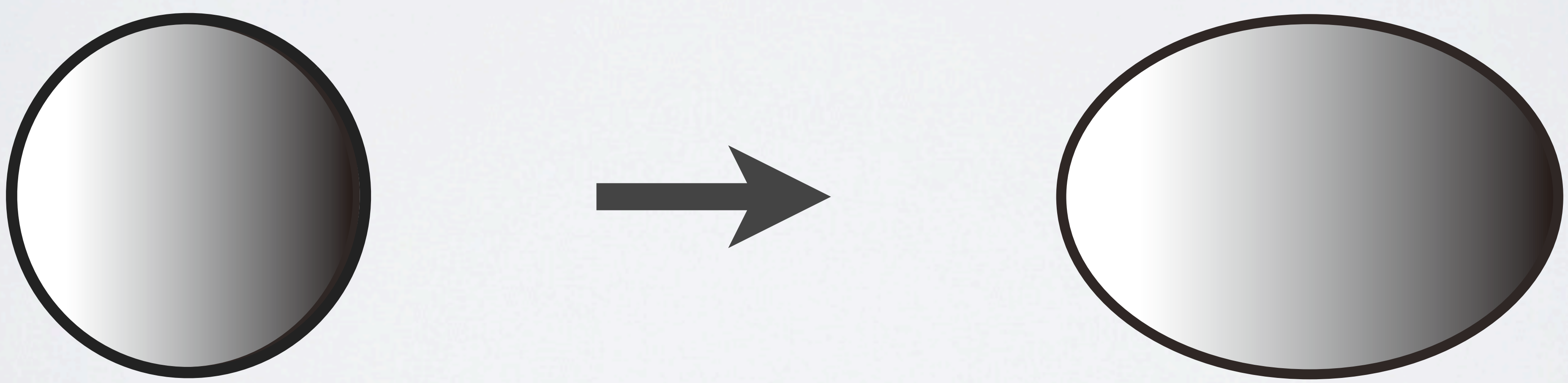

$$ds^2 = \tilde{\ell}^2(dx_1^2 + dx_2^2) + \ell^2(dx_3^2 + dx_4^2)$$

$$\sum_i x_i^2 = 1$$

Localization of partition function

[Kapustin, 11] [Jaferis, 11]

$$Z = \int \mathcal{D}\Phi\, e^{-S[S^3] - tQV}$$

ellipsoid S_b^3 **:** [Hama-Hosomichi-Lee, 11]

$$\boxed{b = \sqrt{\frac{\tilde{\ell}}{\ell}}}$$

$$ds^2 = \tilde{\ell}^2(dx_1^2 + dx_2^2) + \ell^2(dx_3^2 + dx_4^2)$$

$$\sum_i x_i^2 = 1$$

Factorization [Pasquetti, 11]

the partition function also exhibits the factorization

$$Z = \sum_i Z_V^{(i)}(q,z)\widetilde{Z}_V^{(i)}(\tilde{q},\tilde{z})$$

BUT it is checked only for U(1) theories.

Factorization [Pasquetti, 11]

$$Z = \sum_i Z_V^{(i)}(q, z)\widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

$$q = e^{-2\pi i b^2}$$

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Factorization [Pasquetti, 11]

perturbative

$$Z = \sum_i Z_V^{(i)}(q, z)\,\widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

$$q = e^{-2\pi i b^2} \xrightarrow[\text{S}]{} \tilde{q} = e^{-\frac{2\pi i}{b^2}}$$

Factorization [Pasquetti, 11]

perturbative

non-perturbative

$$Z = \sum_i Z_V^{(i)}(q, z)\, \widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

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Factorization [Pasquetti, 11]

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Factorization [Pasquetti, 11]

non-perturbative → perturbative

$$Z = \sum_i Z_V^{(i)}(q, z)\,\widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

$$q = e^{-2\pi i b^2} \xrightarrow[\mathrm{S}]{} \tilde{q} = e^{-\frac{2\pi i}{b^2}}$$

Factorization [Pasquetti, 11]

non-perturbative

perturbative

$$Z = \sum_i Z_V^{(i)}(q, z)\widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

$$q = e^{-2\pi i b^2} \xrightarrow{S} \tilde{q} = e^{-\frac{2\pi i}{b^2}}$$

Z is completed non-perturbatively

Factorization conjecture [Been, Dimofte, Pasquetti, 12]

3d partition function

$$Z = \sum_i Z_V^{(i)}(q,z)\widetilde{Z}_V^{(i)}(\tilde{q},\tilde{z})$$

3d superconformal index

$$I = \sum_i Z_V^{(i)}(q,z)\overline{Z}_V^{(i)}(\bar{q},\bar{z})$$

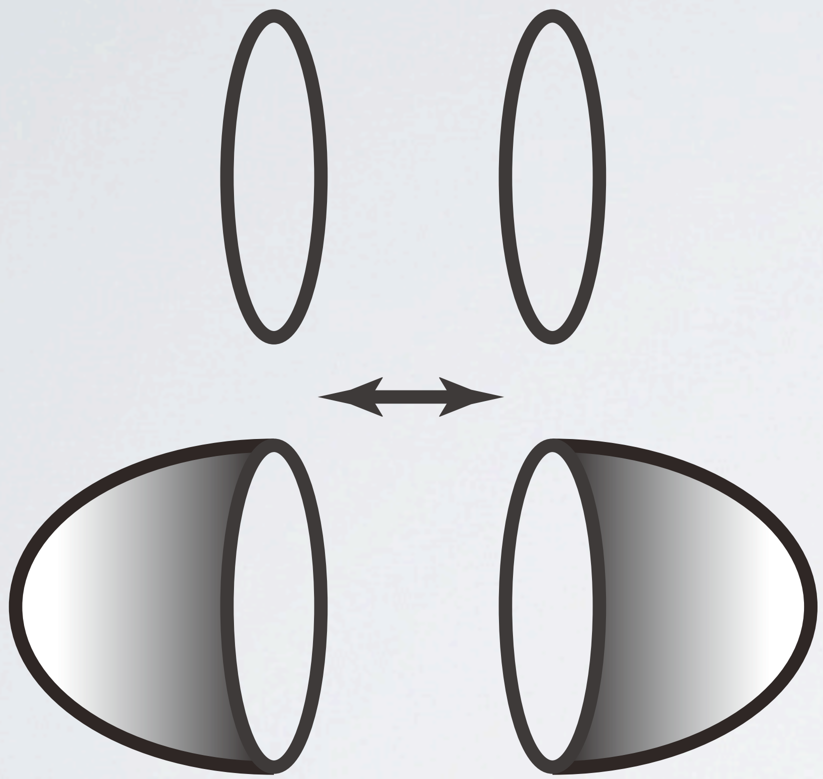
"It holds for generic 3d N=2 theories with gapped vacua"

Path integral representations

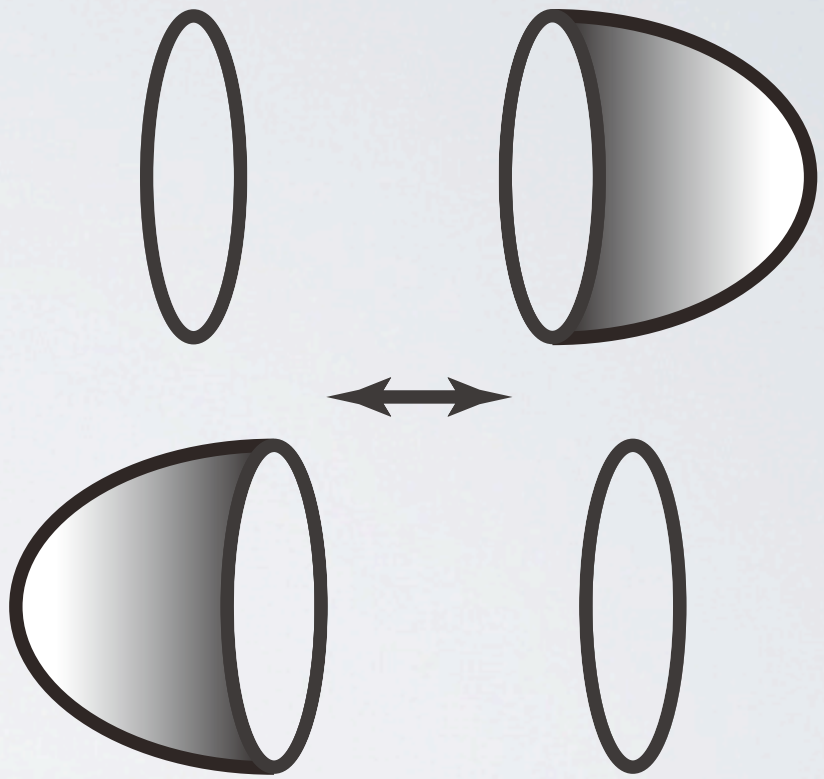
$$Z = \int \mathcal{D}\Phi \, e^{-S[S^3]}$$

$$I = \int \mathcal{D}\Phi \, e^{-S[S^1 \times S^2]}$$

Heegaard decompositions of $S^1 \times S^2$ & S^3

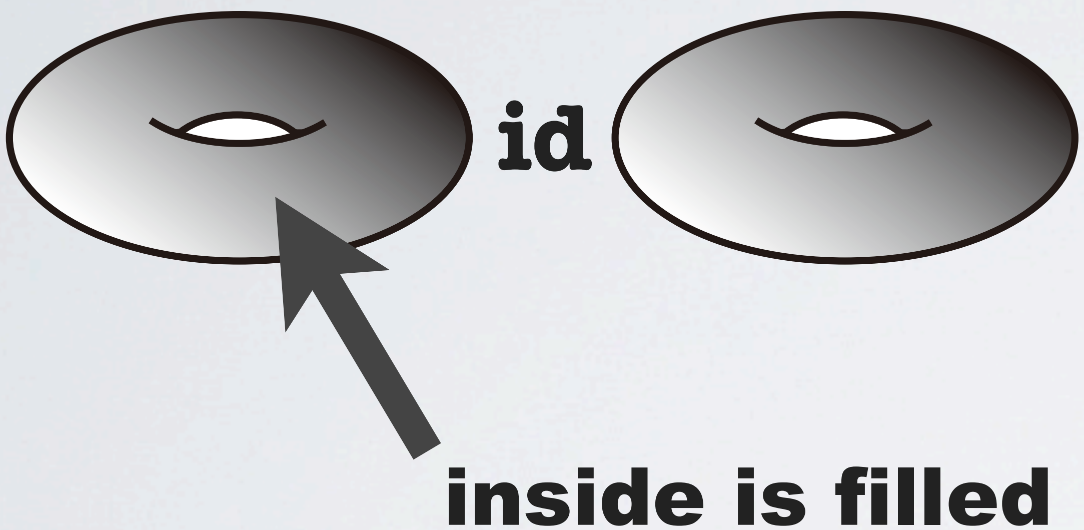


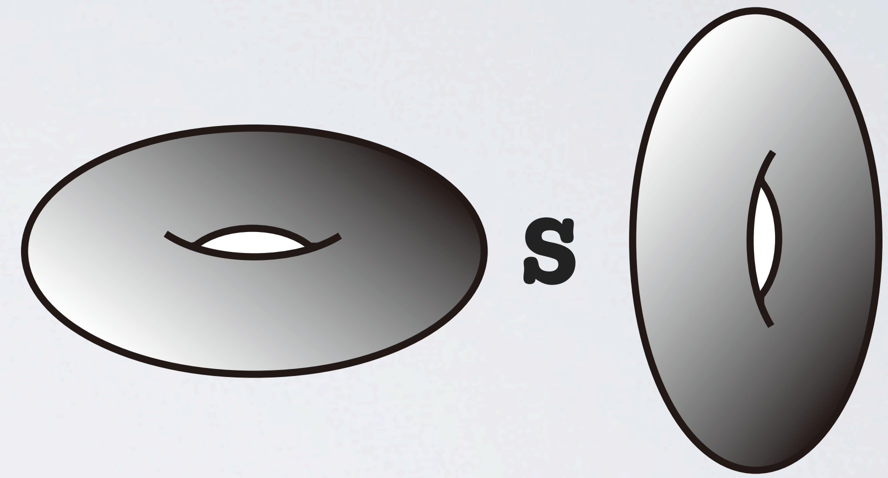
(a)



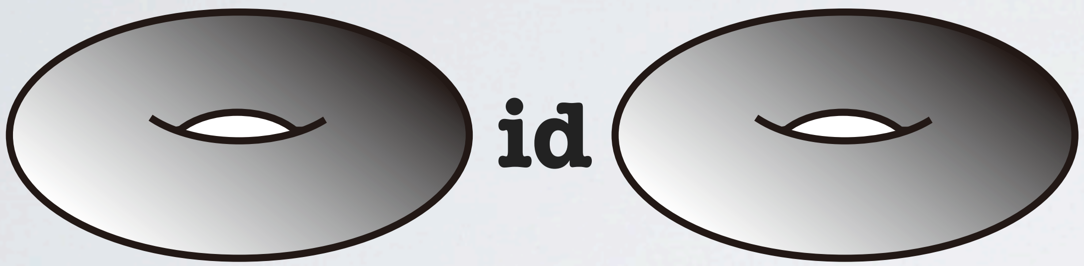
(b)

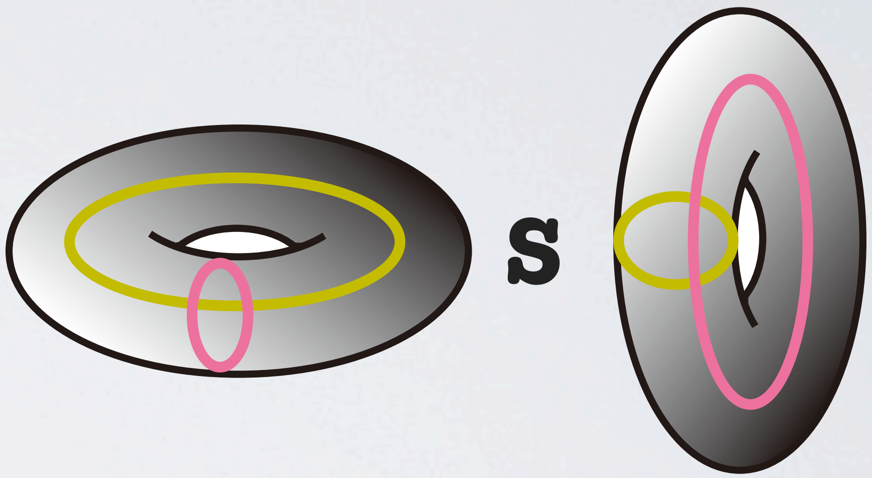
Heegaard decompositions of $S^1 \times S^2$ & S^3





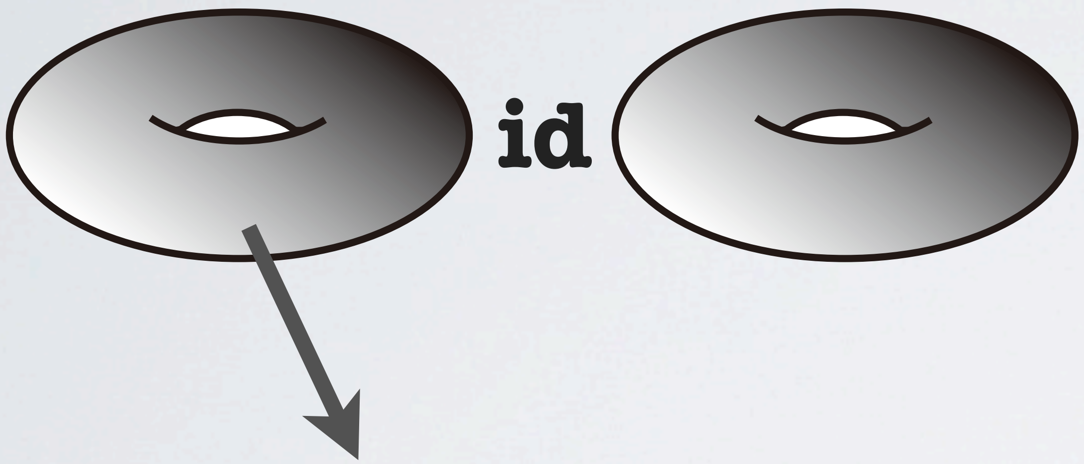
Heegaard decompositions of $S^1 \times S^2$ & S^3

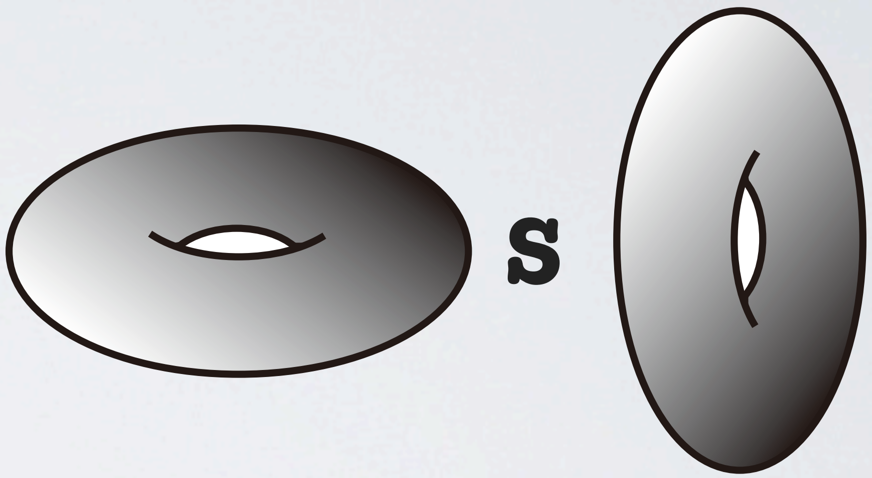




$$S \in SL(2, \mathbb{Z})$$

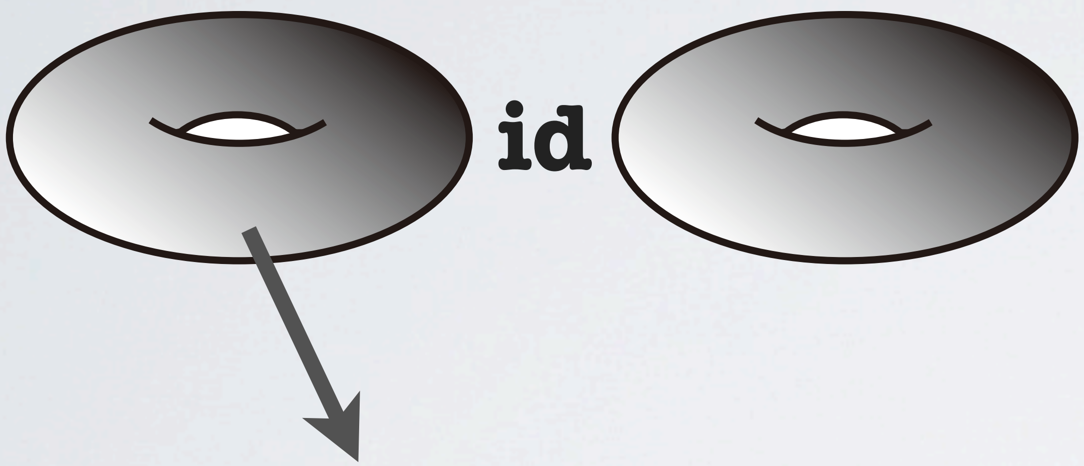
Heegaard decompositions of $S^1 \times S^2$ & S^3

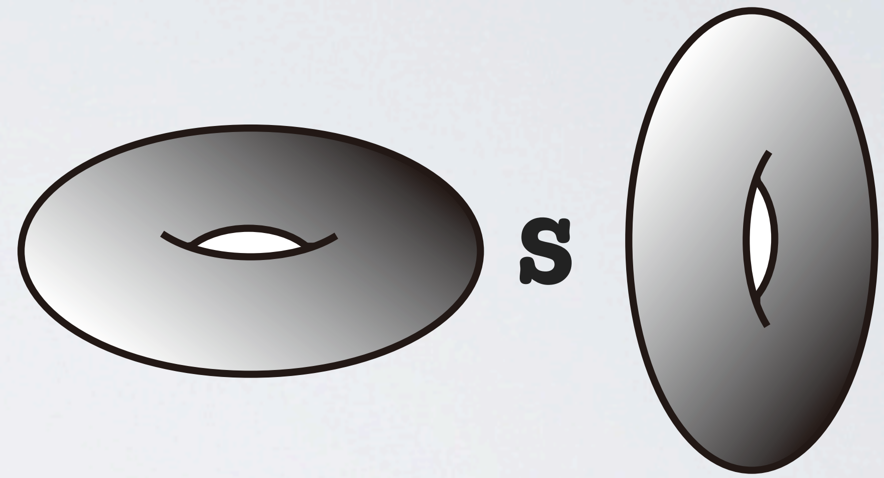




$$\Psi = \langle i|q\rangle$$

Heegaard decompositions of $S^1 \times S^2$ & S^3





$$\Psi = \langle i | q \rangle$$

$$\longrightarrow \quad I = \sum \langle i | q \rangle \langle \bar{q} | i \rangle$$

2. Factorization of 3d Partition Function

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!}\int d^N x \ e^{-i\pi k \sum x_\alpha^2 + 2\pi i \xi \sum x_\alpha}$$

$$\times \prod_{1 \le \alpha < \beta \le N} 4 \sinh \pi b (x_\alpha - x_\beta) \ \sinh \pi b^{-1} (x_\alpha - x_\beta)$$

$$\times \prod_{\alpha=1}^{N} \prod_{i=1}^{N_f} \frac{s_b(x_\alpha + m_i + \mu_i/2 + iQ/2)}{s_b(x_\alpha + m_i - \mu_i/2 - iQ/2)}$$

[Hama-Hosomichi-Lee, 11]

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

double sine function

$$s_b(x) = \prod_{m,n\geq 0} \frac{mb + nb^{-1} - ix}{mb + nb^{-1} + ix}$$

all the poles & zeros are known

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!}\int d^N x \; e^{-i\pi k \sum x_\alpha^2 + 2\pi i \xi \sum x_\alpha}$$

$$\times \prod_{1\leq\alpha<\beta\leq N} 4\sinh \pi b(x_\alpha - x_\beta) \; \sinh \pi b^{-1}(x_\alpha - x_\beta)$$

$$\times \prod_{\alpha=1}^{N} \prod_{i=1}^{N_f} \frac{s_b(x_\alpha + m_i + \mu_i/2 + iQ/2)}{s_b(x_\alpha + m_i - \mu_i/2 - iQ/2)}$$

pole structure is known

Cauchy formula

$$\prod_{1\leq\alpha<\beta\leq N} 2\sinh(x_\alpha - x_\beta)$$

$$= \frac{1}{\prod_{1\leq\alpha<\beta\leq N} 2\sinh(\chi_\alpha - \chi_\beta)}$$

$$\times \sum_{\sigma\in S^N} (-1)^\sigma \prod_\alpha \prod_{\beta\neq\sigma(\alpha)} 2\cosh(x_\alpha - \chi_\beta)$$

Cauchy formula

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[Marino-Putrov,11] "ABJM theory as a Fermi gas"

→ $N^{3/2}$ **behavior**

Toy model

$$Z = \oint dx_1\, dx_2\, 2\sinh(x_1 - x_2) \frac{f_1(x_1) f_2(x_2)}{(x_1 - m_1)(x_2 - m_2)}$$

Toy model

$$Z = \oint dx_1\, dx_2\, \boxed{2\sinh(x_1 - x_2)}\, \frac{f_1(x_1) f_2(x_2)}{(x_1 - m_1)(x_2 - m_2)}$$

$$\frac{1}{2\sinh(\chi_1 - \chi_2)}$$

$$\times \Big(4\sinh(x_1 - \chi_2)\sinh(x_2 - \chi_1) - 4\sinh(x_1 - \chi_1)\sinh(x_2 - \chi_2) \Big)$$

Toy model

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$$= \frac{f_1(m_1) f_2(m_2)}{2\sinh(\chi_1 - \chi_2)}$$

$$\times \Big(4\sinh(m_1 - \chi_2)\sinh(m_2 - \chi_1)$$

$$- 4\sinh(m_1 - \chi_1)\sinh(m_2 - \chi_2) \Big)$$

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Toy model

$$Z = \oint dx_1\, dx_2\, 2\sinh(x_1 - x_2) \frac{f_1(x_1) f_2(x_2)}{(x_1 - m_1)(x_2 - m_2)}$$

$$= f_1(m_1) f_2(m_2)\, 2\sinh(m_1 - m_2)$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!}\int d^N x\ e^{-i\pi k\sum x_\alpha^2 + 2\pi i\xi\sum x_\alpha}$$

$$\times \prod_{1\le\alpha<\beta\le N} 4\sinh\pi b(x_\alpha - x_\beta)\ \sinh\pi b^{-1}(x_\alpha - x_\beta)$$

$$\times \prod_{\alpha=1}^{N}\prod_{i=1}^{N_f}\frac{s_b(x_\alpha + m_i + \mu_i/2 + iQ/2)}{s_b(x_\alpha + m_i - \mu_i/2 - iQ/2)}$$

simple poles @

$$x_\alpha = -m_i - \mu_i + i\left(imb + inb^{-1} + \frac{Q}{2}\right)$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\rm cl}^{\{i_\alpha\}}(\xi_{\rm eff}) \left(Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \right) \left(\widetilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} \widetilde{Z}_V^{\{i_\alpha\}} \right)$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

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$$Z_{\text{cl}}^{\{i_\alpha\}}(m, \mu, \xi) = \prod_{\alpha=1}^{N} e^{-i\pi k (m_{i_\alpha} + \mu_{i_\alpha}/2)^2 - 2\pi i \xi (m_{i_\alpha} + \mu_{i_\alpha})/2}$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

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$$Z_{\text{1-loop}}^{\{i_\alpha\}} = \prod_{1 \le \alpha < \beta \le N} 4 \sinh(\pi b D_{i_\alpha i_\beta}) \prod_{\alpha} \prod_{\ell=1}^{\infty} \frac{\prod_{j \neq i_\alpha} (1 - q^\ell e^{-2\pi b D_{j i_\alpha}})}{\prod_j (1 - q^{\ell-1} e^{-2\pi b C_{j i_\alpha}})}$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\text{cl}}^{\{i_\alpha\}}(\xi_{\text{eff}}) \left(Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \right) \left(\widetilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} \widetilde{Z}_V^{\{i_\alpha\}} \right)$$

$$Z_V^{\{i_\alpha\}} = \sum_{m_1,\cdots,m_N=0}^{\infty} \prod_{\alpha=1}^{N} \left((-1)^N e^{\pi b \sum \mu_j} q^{N_f/2} z_\alpha \right)^{m_\alpha} q^{-k m_\alpha^2/2}$$

$$\times \prod_{\alpha=1}^{N} \prod_{l=1}^{m_\alpha} \frac{\prod_{j=1}^{N_f} 2\sinh \pi b (C_{j i_\alpha} + i(l-1)b)}{\prod_{\beta=1}^{N} 2\sinh \pi b (D_{i_\alpha i_\beta} + i(l-1-m_\alpha)b) \prod_{j=1,\notin\{i_\alpha\}}^{N_f} 2\sinh \pi b (D_{j i_\alpha} + ilb)}$$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\rm cl}^{\{i_\alpha\}}(\xi_{\rm eff}) \left(Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \right) \left(\widetilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} \widetilde{Z}_V^{\{i_\alpha\}} \right)$$

$$Z_V^{\{i_\alpha\}} = \sum_{m_1,\cdots,m_N=0}^{\infty} \prod_{\alpha=1}^{N} \left((-1)^N e^{\pi b \sum \mu_j} q^{N_f/2} z_\alpha \right)^{m_\alpha} q^{-k m_\alpha^2/2}$$

$$\times \prod_{\alpha=1}^{N} \prod_{l=1}^{m_\alpha} \frac{\prod_{j=1}^{N_f} 2\sinh \pi b (C_{j i_\alpha} + i(l-1)b)}{\prod_{\beta=1}^{N} 2\sinh \pi b (D_{i_\alpha i_\beta} + i(l-1-m_\alpha)b) \prod_{j=1,\notin\{i_\alpha\}}^{N_f} 2\sinh \pi b (D_{j i_\alpha} + ilb)}$$

▲ K-theoretic vortex partition function

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds

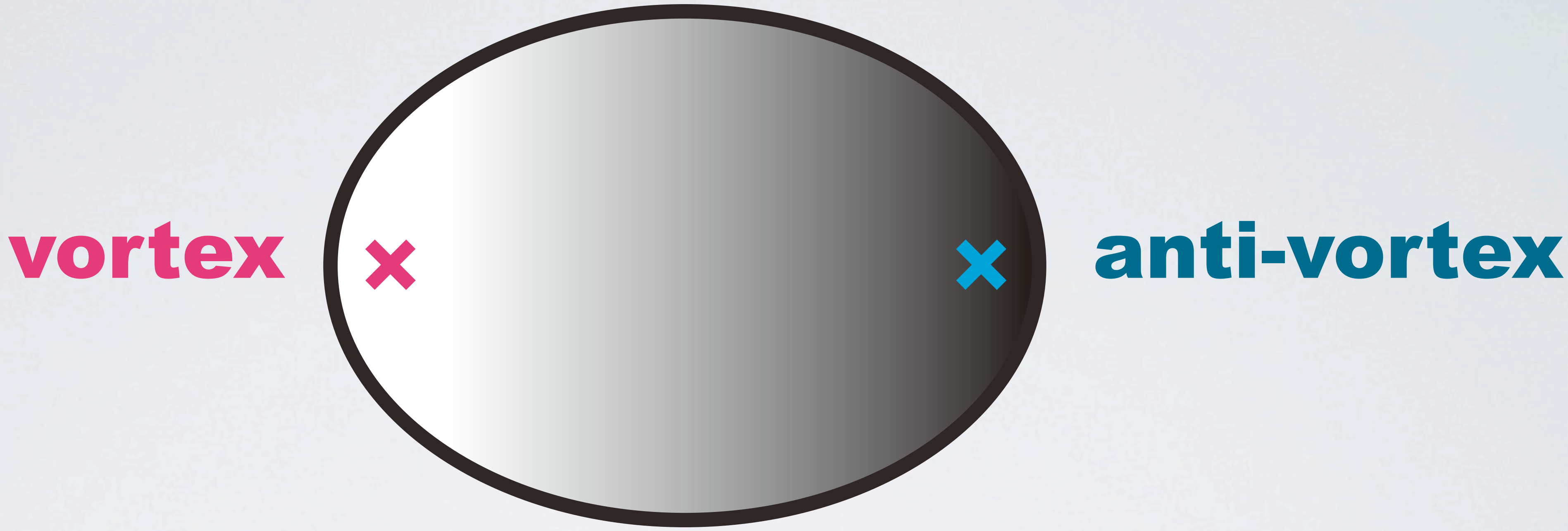
$$Z = \frac{1}{N!}\sum_{i_1=1}^{N_f}\cdots\sum_{i_N=1}^{N_f} Z_{\text{cl}}^{\{i_\alpha\}}(\xi_{\text{eff}})\left(Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}}\right)\left(\widetilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} \widetilde{Z}_V^{\{i_\alpha\}}\right)$$

$$Z_V^{\{i_\alpha\}} = \sum_{m_1,\cdots,m_N=0}^{\infty}\prod_{\alpha=1}^{N}\left((-1)^N e^{\pi b\sum \mu_j} q^{N_f/2} z_\alpha\right)^{m_\alpha} q^{-km_\alpha^2/2}$$

$$\times \prod_{\alpha=1}^{N}\prod_{l=1}^{m_\alpha}\frac{\prod_{j=1}^{N_f} 2\sinh \pi b(C_{ji_\alpha}+i(l-1)b)}{\prod_{\beta=1}^{N} 2\sinh\pi b(D_{i_\alpha i_\beta}+i(l-1-m_\alpha)b)\prod_{j=1,\notin\{i_\alpha\}}^{N_f} 2\sinh \pi b(D_{ji_\alpha}+ilb)}$$

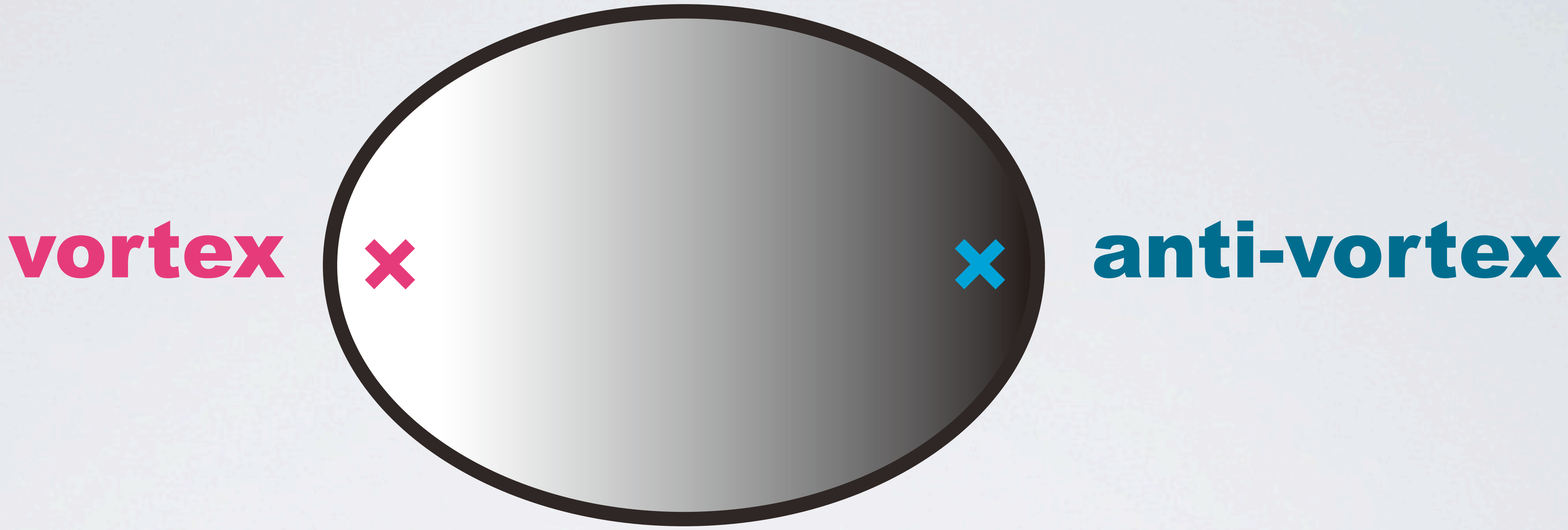
consistent with $I = \sum_i Z_V^{(i)}(q,z)\overline{Z}_V^{(i)}(\bar{q},\bar{z})$

3d $\mathcal{N}$=2 U(N) theory with N_f funds & anti-funds



$$Z = \sum_i Z_V^{(i)}(q, z)\widetilde{Z}_V^{(i)}(\tilde{q}, \tilde{z})$$

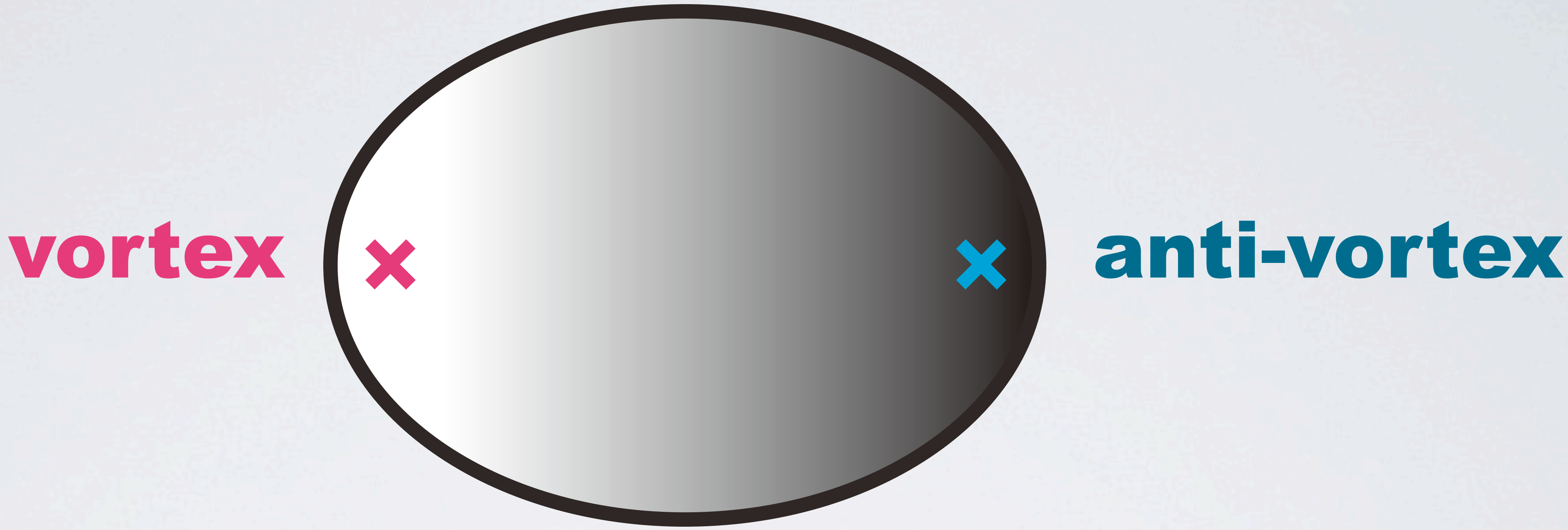
Expectation for direct proof



localization onto Coulomb branch : known

→ **integration rep.**

Expectation for direct proof



localization onto Coulomb branch : known

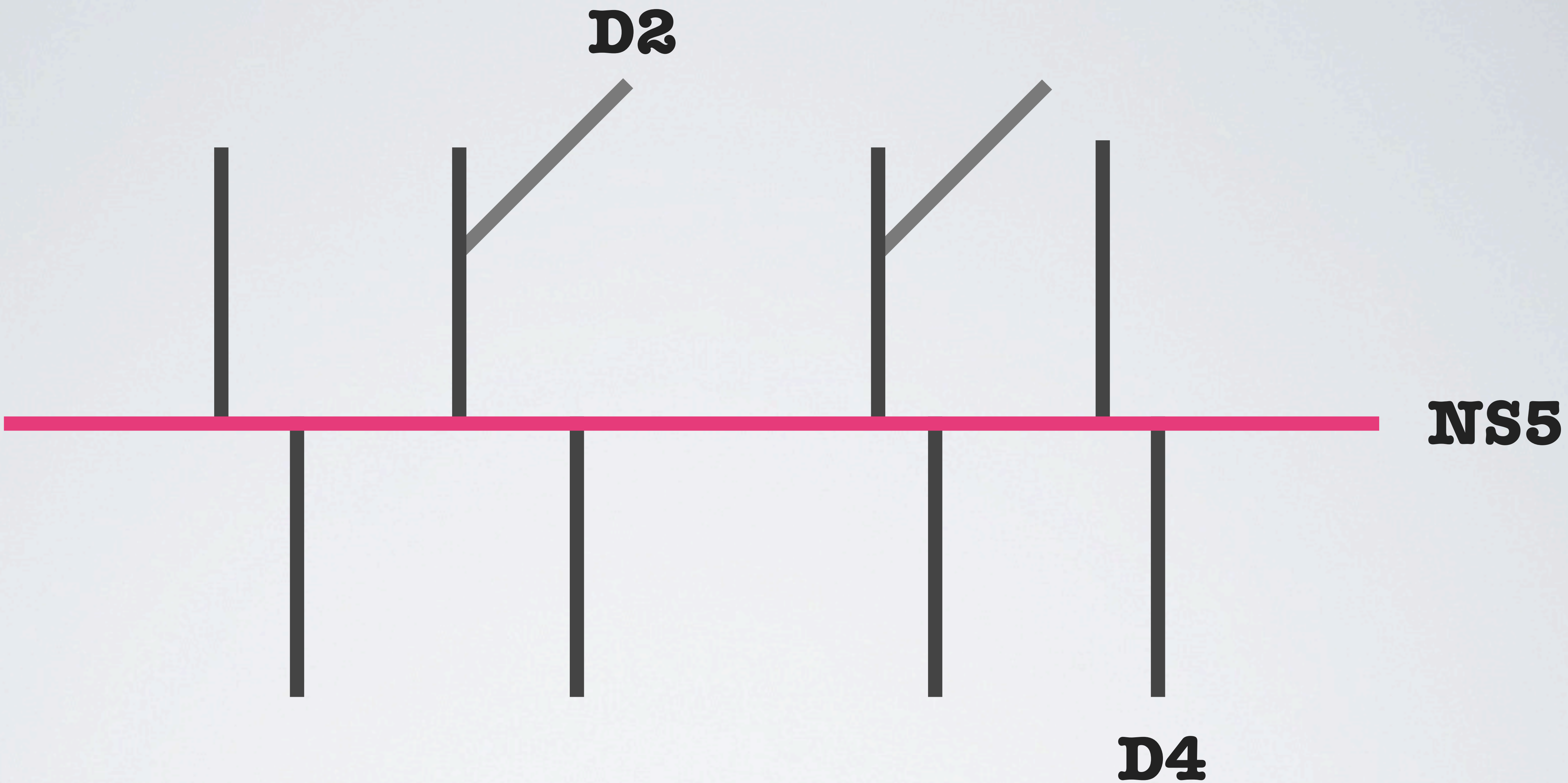
⟶ integration rep.

localization onto Higgs branch : unknown

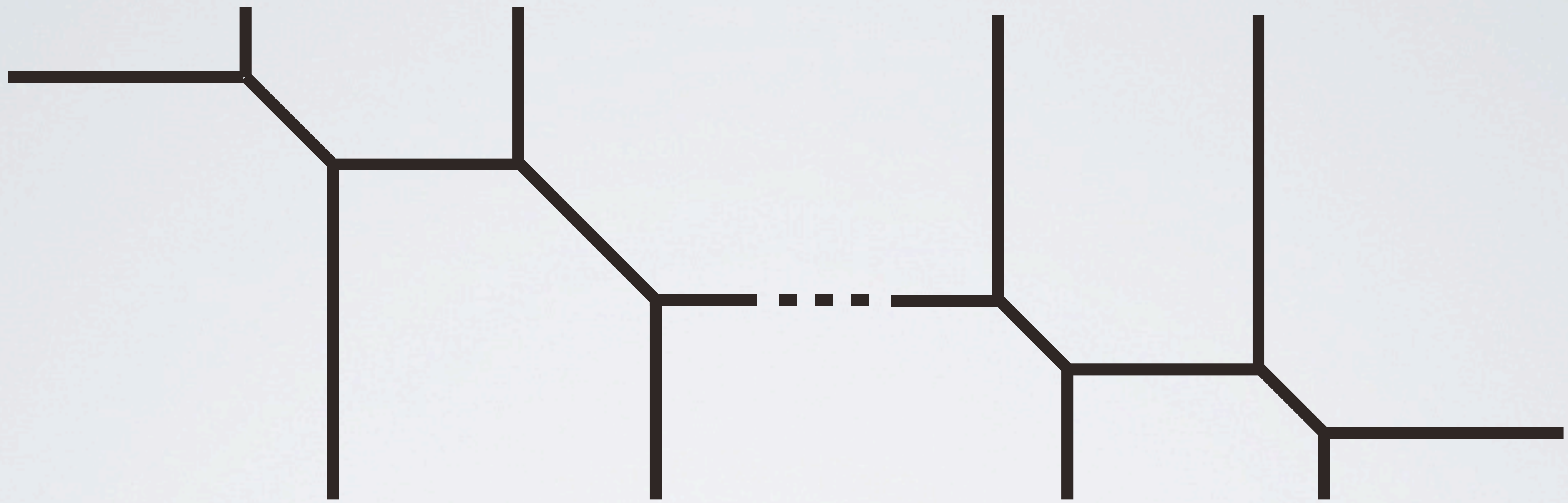
⟶ product rep.

3. Vortex and “The vertex on a strip”

Brane configuration

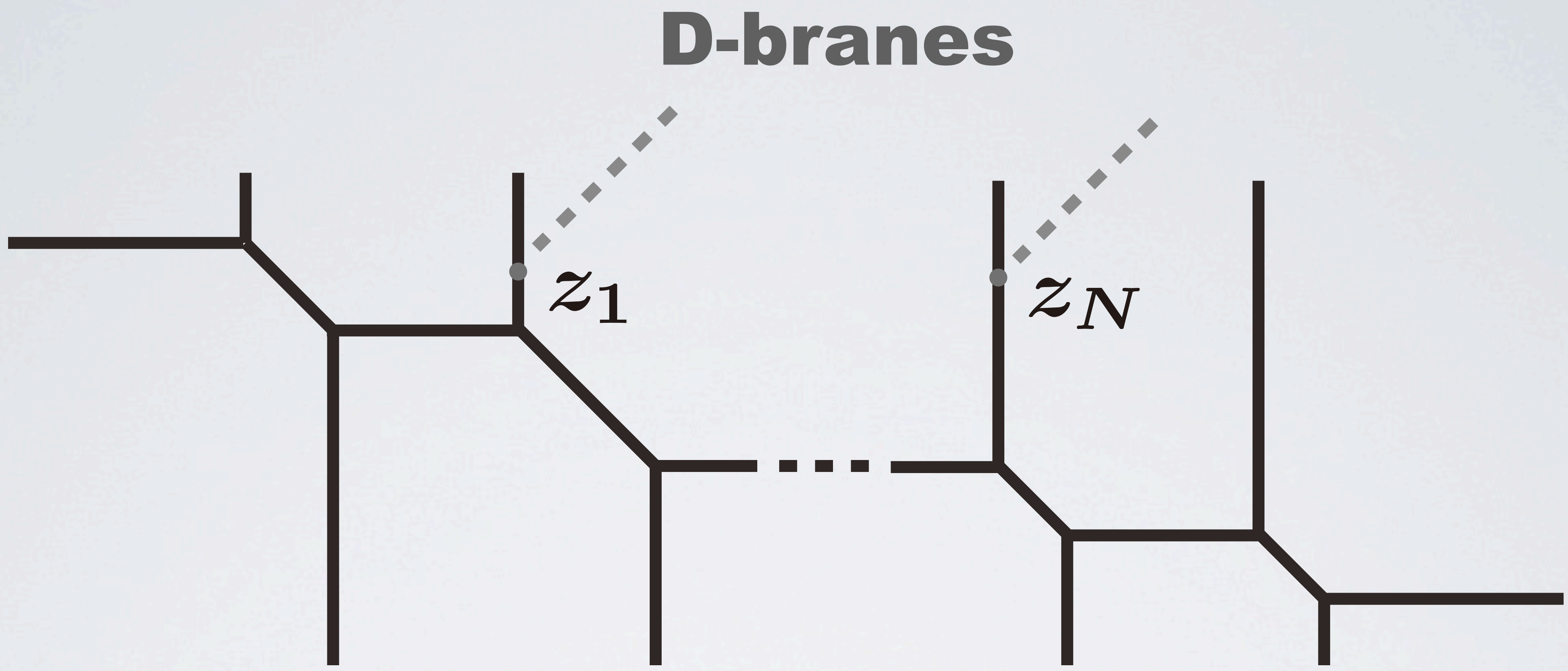


strip geometry

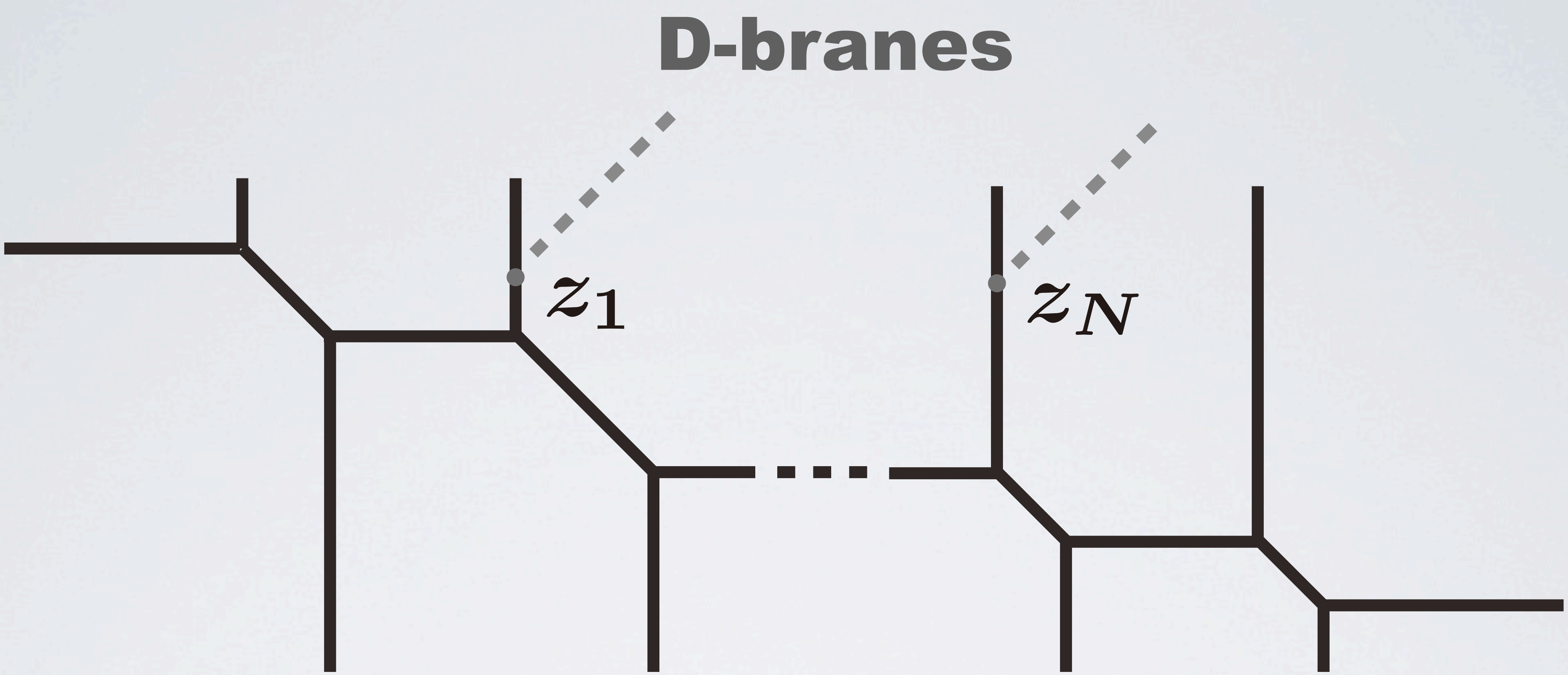


sketchy description of certain CY 3-fold

strip geometry



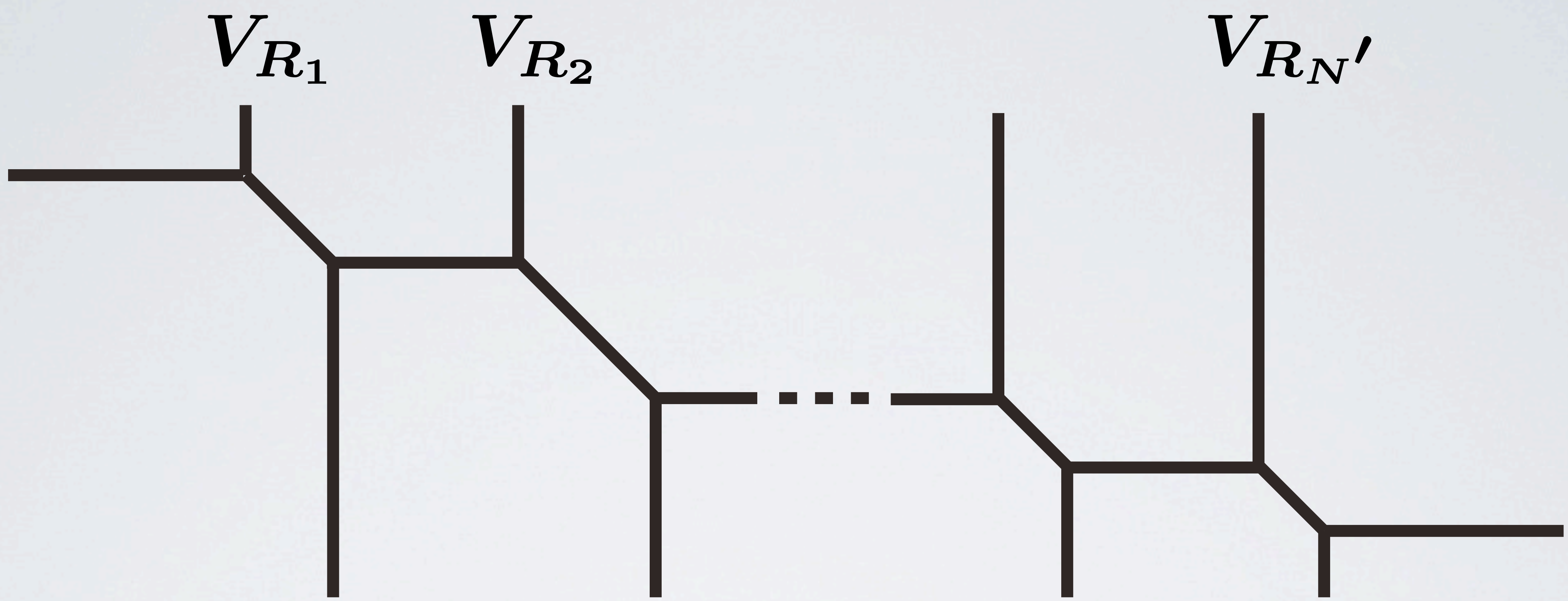
strip geometry



topological string amplitude (Veneziano-like)

$$\frac{\mathcal{K}^{R_1 R_2 \cdots R_{N'}}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}} = \prod_{i=1}^{N'} S_{R_i}(q^\rho) \prod_{l=1}^{\infty} \frac{\prod_{i\leq j}(1-q^l Q_{a_i b_j})^{C_l(R_i,\emptyset)} \prod_{j<i}(1-q^l Q_{b_j a_i})^{C_l(\emptyset,R_i^T)}}{\prod_{i<i'}(1-q^l Q_{a_i a_{i'}})^{C_l(R_i,R_{i'}^T)}}$$

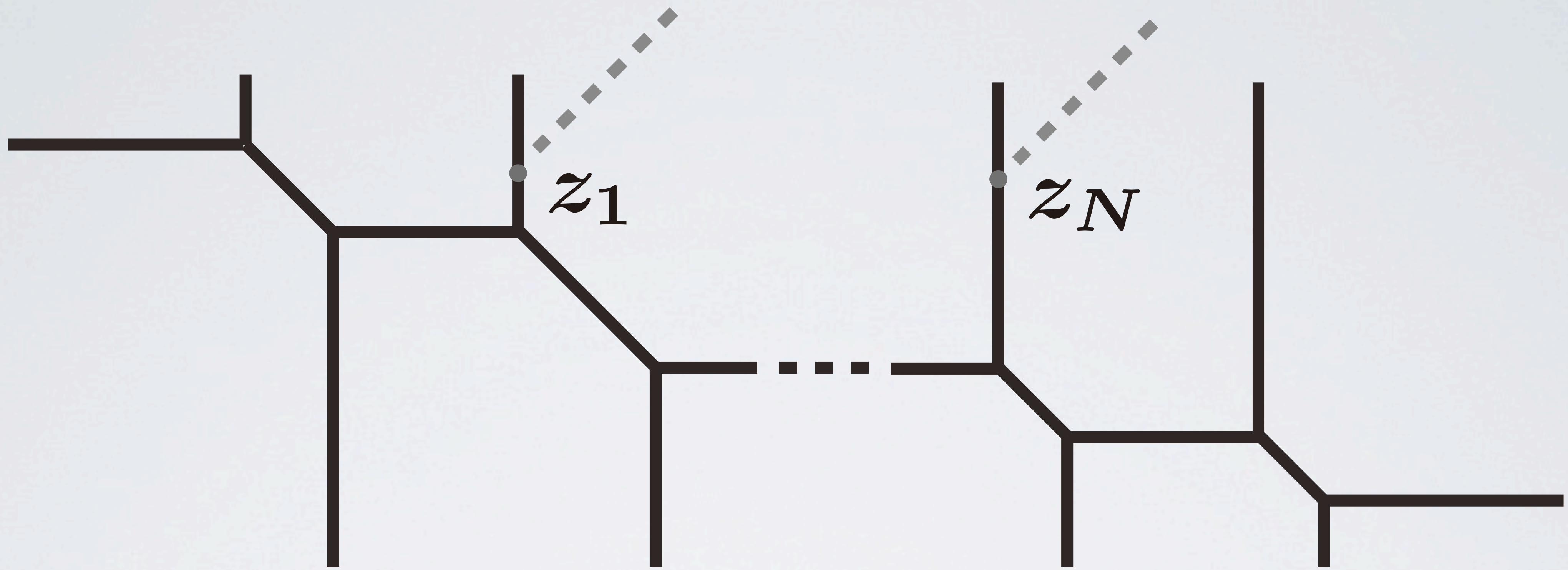
strip geometry



topological string amplitude (Veneziano-like)

$$\frac{\mathcal{K}^{R_1 R_2 \cdots R_{N'}}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}} = \prod_{i=1}^{N'} S_{R_i}(q^{\rho}) \prod_{l=1}^{\infty} \frac{\prod_{i\leq j}(1-q^l Q_{a_i b_j})^{C_l(R_i,\emptyset)} \prod_{j<i}(1-q^l Q_{b_j a_i})^{C_l(\emptyset,R_i^T)}}{\prod_{i<i'}(1-q^l Q_{a_i a_{i'}})^{C_l(R_i,R_{i'}^T)}}$$

strip geometry



$$R_{i_\alpha} = 1^{m_\alpha} \text{ for } i_{\alpha=1,2,\cdots,N}, \quad \text{otherwise } R_j = \emptyset$$

$$\frac{\mathcal{K}^{R_1 R_2 \cdots R_{N'}}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}} = \prod_{i=1}^{N'} S_{R_i}(q^\rho) \prod_{l=1}^{\infty} \frac{\prod_{i\le j}(1-q^l Q_{a_i b_j})^{C_l(R_i,\emptyset)} \prod_{j<i}(1-q^l Q_{b_j a_i})^{C_l(\emptyset,R_i^T)}}{\prod_{i<i'}(1-q^l Q_{a_i a_{i'}})^{C_l(R_i,R_{i'}^T)}}$$

strip geometry

$$\frac{\mathcal{K}^{\emptyset\cdots 1^{m_1}\cdots 1^{m_N}\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}} = \prod_{\alpha=1}^{N}\prod_{l=1}^{m_\alpha}\frac{1}{1-q^l}\prod_{l=1}^{m_\alpha}\frac{\prod_j(1-q^{l-1}Q_{a_{i_\alpha}b_j})}{\prod_{i_\alpha,i_\beta}(1-q^{-l}Q_{a_{i_\alpha}a_{i_\beta}})\prod_{j\not\in\{i_\alpha\}}(1-q^{1-l}Q_{a_{i_\alpha}b_j})}$$

$$Z_V = \sum_{\{i_\alpha\}} z_\alpha^{m_\alpha}\frac{\mathcal{K}^{\emptyset\cdots 1^{m_1}\cdots 1^{m_N}\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}$$

strip geometry

topological string → **holomorphic blocks**

$$Z_V = \sum_{\{i_\alpha\}} z_\alpha^{m_\alpha} \frac{\mathcal{K}^{\emptyset\cdots 1^{m_1}\cdots 1^{m_N}\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}$$

strip geometry

topological string → **holomorphic blocks**

geometric engineering

$$Z_V = \sum_{\{i_\alpha\}} z_\alpha^{m_\alpha} \frac{\mathcal{K}^{\emptyset\cdots 1^{m_1}\cdots 1^{m_N}\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}{\mathcal{K}^{\emptyset\emptyset\cdots\emptyset}_{\emptyset\emptyset\cdots\emptyset}}$$

4. Summary

review on localization of 3d gauge theory

review on localization of 3d gauge theory

partition function for non-abelian theory exhibits the factorization property

review on localization of 3d gauge theory

partition function for non-abelian theory exhibits the factorization property

topological string theory implies the holomorphic blocks: geometric engineering

FIN