Thermodynamics of SU(3) gauge theory from gradient flow

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Noether current / generator of space-time translation

\[ T_{\mu \nu} \]

\[
\begin{bmatrix}
T_{00} & T_{01} & T_{02} & T_{03} \\
T_{10} & T_{11} & T_{12} & T_{13} \\
T_{20} & T_{21} & T_{22} & T_{23} \\
T_{30} & T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]

- energy
- momentum
- stress
- pressure
Noether current / generator of space-time translation

Einstein Equation

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \]

Hydrodynamic Eq.

\[ \partial_\mu T^{\mu\nu} = 0 \]
The definition of $T_{\mu\nu}$ on the lattice is nontrivial...

...because of the lack of translational symmetry

cf) Caracciolo+ (1989)
Rough Idea

no translational invariance  \rightarrow\text{coarse graining}  \rightarrow\text{translational symmetry is recovered!}
lattice regularized
 gauge theory

YM gradient flow

$T^{R}_{\mu \nu}$
continuum theory
( with dimensional regularization )

YM gradient flow

continuum theory
( with dimensional regularization )

Luescher, Weiss (2012)
Suzuki (2013)
lattice regularized gauge theory

YM gradient flow

continuum theory (with dimensional regularization)

\( T^{R}_{\mu\nu} \)

analytic relation (in perturbation)

\( \tilde{O}_{\mu\nu} \)

continuum theory (with dimensional regularization)

\( \tilde{O}_{\mu\nu} \)

YM gradient flow

UV finite

equivalent!

Luescher, Weiss (2012)
Suzuki (2013)
What we can measure with $T_{\mu\nu}$:

- $\langle T_{\mu\nu} \rangle$:
  - bulk thermodynamics (energy density, pressure)

- $\langle T_{\mu\nu}(x)T_{\mu\nu}(0) \rangle$:
  - correlation functions
  - viscosity, thermal excitation
  - vacuum structure?

- $\langle (\delta T_{\mu\nu})^n \rangle$:
  - fluctuations, specific heat
  - non-Gaussian fluctuations, etc.

Asakawa, Ejiri, MK (2009)

Pink chars: T>0 physics
QCD EoS
(Energy Density, Pressure)

- Rapid increase of $\varepsilon/T^4$ around $T=150-200$ MeV
- Crossover transition
- Rapid but smooth change of medium from hadronic to QGP-like
QCD Thermodynamics

\[ Z(T) = \text{Tr} \left[ e^{-H/T} \right] \]

\[ = \int D A \exp \left[ - \int_0^{1/T} d\tau \int_V d^3 x \mathcal{L}_E \right] \]

Thermodynamic relations

\[ \varepsilon = \frac{T^2}{V} \frac{\partial \ln Z}{\partial T} \]

\[ p = T \frac{\partial \ln Z}{\partial V} \]

How do we take T and V derivatives?
Lattice Spacing Derivative

Changing lattice spacing $\alpha$ $\rightarrow$ $1/T$ and $V$ change

$$\frac{\partial \ln Z}{\partial \alpha} \sim \varepsilon - 3p$$

$$\frac{\partial \ln Z}{\partial \alpha} = \frac{\partial \beta}{\partial \alpha} \frac{\partial \ln Z}{\partial \beta} \sim \frac{\partial \beta}{\partial \alpha} \langle S \rangle$$

$$\beta = 2N_c/g^2$$

\[
\frac{\partial \beta}{\partial \alpha}, \langle S \rangle \rightarrow [\varepsilon - 3p]_{\text{thermodyn.}} = [\varepsilon - 3p]_T - [\varepsilon - 3p]_{\text{vac}}
\]
Differential Method

- 2 independent “beta functions”
- perturbative result Karsch (1982)
- negative pressure with Karsch coeffs.
- vacuum subtraction
Integral Method

\[ T \frac{\partial (p/T^4)}{\partial T} = \frac{\varepsilon - 3p}{T^4} \]

\[ \frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\varepsilon - 3p}{T^5} \]

- measurements of \( e-3p \) for many \( T \)
- vacuum subtraction for each \( T \)
- information on beta function

Boyd+ 1996
Gradient Flow Method

$\langle T_{\mu\nu} \rangle$
Gradient Flow

\[ \partial_t B_\mu(t, x) = D_\nu G_{\mu\nu} \]

where:
- \( t \): “flow time”
- dim: [length\(^2\)]
- \( B_\mu(0, x) = A_\mu(x) \)
- \( G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \)

The steepest descent direction of the action.
Gradient Flow

\[ \partial_t B_\mu(t, x) = D_\nu G_{\mu\nu} \]

- \( B_\mu(0, x) = A_\mu(x) \)
- \( G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \)

- modify gauge field toward the stationary point of the action
- smoothing similarly to diffusion equation:
  \[ \partial_t B_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu B_\mu + \cdots \]
- diffusion length \( d \sim \sqrt{8t} \)
- All composite operators at \( t > 0 \) are UV finite \( \text{Luescher,Weisz,2011} \)
\[ \tilde{\mathcal{O}}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- an operator at \( t > 0 \)
- remormalized operators of original theory
Constructing EMT

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

- gauge-invariant dimension 4 operators

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4} \delta_{\mu\nu} G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4} \delta_{\mu\nu} G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^{R\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^{R\rho\rho}(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T^{R\rho\rho}(x) + \mathcal{O}(t) \]

Suzuki coefficients

\[ \begin{align*}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + O(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + O(g^4) \right]
\end{align*} \]

\[ g = g(1/\sqrt{8t}) \]

\[ s_1 = 0.03296 \ldots \]

\[ s_2 = 0.19783 \ldots \]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^R_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + \mathcal{O}(t) \]

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Remormalized EMT

\[
T^R_{\mu\nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{1}{4\alpha_E(t)} \delta_{\mu\nu} E(t, x)_{\text{subt.}} \right]
\]
Numerical Simulation on the Lattice
Gauge field has to be sufficiently smeared!

\[ a \ll \sqrt{8t} \]

Perturbative relation has to be applicable!

\[ \sqrt{8t} \ll \Lambda^{-1}, T^{-1} \]
① generate gauge configurations

② solve gradient flow for each confs.

③ measure U and E

④ obtain $T_{\mu\nu}$ with Suzuki formula

⑤ take a $\rightarrow 0$, t $\rightarrow 0$ limit
Numerical Simulation 1

- SU(3) YM theory
- Wilson gauge action
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10$
- configurations: 100-300

<table>
<thead>
<tr>
<th>$N_T$</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>$T/T_c$</th>
</tr>
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<tbody>
<tr>
<td>$\beta$</td>
<td>6.20</td>
<td>6.40</td>
<td>6.56</td>
<td>1.65</td>
</tr>
<tr>
<td></td>
<td>6.02</td>
<td>6.20</td>
<td>6.36</td>
<td>1.24</td>
</tr>
<tr>
<td></td>
<td>5.89</td>
<td>6.06</td>
<td>6.20</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Scale setting:
alpha Collab., NPB538,669(1999)
There exists a wide range of $t$ at which the Suzuki formula is safely used with $Nt=10$. The emergent plateau is given by:

$$2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1}$$
There exists a wide range of $t$ at which the Suzuki formula is safely used with $Nt=10$. The emergent plateau is given by $2a \lesssim 8t \lesssim 0.4T^{-1}$. The systematic error indicates that $\sqrt{8t}$ is a critical point in the system.
Continuum Limit

- Statistical error of $e-3p$ is significantly smaller than Boyd+1996 which used $\sim 10000$ confs.
- No integral! Direct measurement of $e$ and $p$ at a given $T$
- No vacuum subtraction for $e+p$
Comparison with Integral Method
Numerical Simulation 2

- SU(3) YM theory
- Wilson gauge action
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10, 32$
- configurations: 100-300
- $\beta = 5.89 - 6.56$

- lattice size: $64^3 \times N_t$
- $N_t = 8, 10, \ldots, 16, 18, 64$
- configurations: ~2000
- $\beta = 6.4 - 7.4$

On BlueGene/Q @ KEK
Efficiency of our code:
- Gauge update (HB+OR): ~25%
- Gradient flow (RK$^4$): ~40%
Dependence of $e+p$

$N_x = 32, T \sim 1.65T_c$

FlowQCD,1312.7492

Plateau region extends toward small $t$!

NEW!!

$\sqrt{8t} = 2a$

$p$-dependence of $e+p$

preliminary

$N_x = 64, T \sim 2T_c$

$\beta=7.2$, $N_t=18$

$\beta=7.0$, $N_t=14$
Summary

$T_{\mu\nu}^{R}(x)$
Summary

EMT formula from gradient flow

\[ T_{\mu\nu}^{R}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{1}{4\alpha_E(t)} \delta_{\mu\nu} E(t, x)_{\text{subt.}} \right] \]

Our strategy can successfully define the EMT on the lattice in practical simulations.

This operator provides us with novel approaches to measure observables on the lattice!

They are direct, intuitive and less noisy.
Many Future Works

- precision measurement of YM thermodynamics
- EMT correlation functions → measurement of viscosity
- specific heat, non-Gaussian fluctuations, etc.
- scale setting

- taking double limit $a \to 0$, $t \to 0$
- full QCD Makino, Suzuki, 1403.4772

from Gradient flow to Hydrodynamic flow
Two Point Functions

\[ \langle T_{\mu\nu}(x, t)T_{\mu\nu}(0, 0) \rangle \]
EMT Correlator

- **Kubo Formula:** $T_{12}$ correlator $\leftrightarrow$ shear viscosity

\[ \eta = \int_0^\infty dt \int_0^{1/T} d\tau \int d^3x \langle T_{12}(x, -i\tau) T_{12}(0, t) \rangle \]

- Hydrodynamics describes long range behavior of $T_{\mu\nu}$

- **Energy fluctuation** $\leftrightarrow$ specific heat

\[ c_V = \frac{\langle \delta E^2 \rangle}{VT^2} \]
EMT Correlator : Noisy...

With naïve EMT operators

$$\langle T_{12}(\tau)T_{12}(0) \rangle$$

$$\langle T_{\mu\nu}(\tau)T_{\mu\nu}(0) \rangle$$

Nakamura, Sakai, PRL, 2005

\( N_t = 8 \)

improved action

\( \sim 10^6 \) configurations

\( N_t = 16 \)

standard action

\( 5 \times 10^4 \) configuration

... no signal
\[ \int d^3 x \langle T_{12}(x, \tau) T_{12}(0, 0) \rangle \]

- **Smearing length**: \( \sqrt{8t} \)
- **64^3 \times 16**
- **\( \beta = 7.2 \) (\( T \sim 2.2T_c \))**
- **1200 confs**

- **Converge at** \( \tau > \sqrt{8t} \)
- **Improvement of the statistics at large \( t \)**
Correlation Function

\[ C_{\mu\nu}(\tau) = \int d^3 x \langle T_{\mu\nu}(x, \tau) T_{\mu\nu}(0, 0) \rangle \]

64^3 \times 16
\beta = 7.2 \ (T \sim 2.2 T_c)
1200 confs
t/a^2 = 1.9

\[ C_{44}(\tau) : \text{constant} \leftarrow \text{conservation law!} \]
\[ \partial_\tau \langle \delta E(\tau) \delta E(0) \rangle = 0 \]
(for \( \tau \neq 0 \))

\[ C_{12}(\tau) \]
\[ C_{41}(\tau) \]
negative \leftarrow i^2 = -1
Energy Fluctuation and Specific Heat

Specific Heat

\[ c_V = \frac{1}{V} \frac{\partial E}{\partial T} \bigg|_V \]

\[ = \frac{\langle \delta E^2 \rangle}{VT^2} \]

\[ = \frac{\langle \delta E(\tau)\delta E(0) \rangle}{VT^2} \]

\[ \frac{c_V}{T^3} = \frac{\langle \delta E(\beta/2)\delta E(0) \rangle}{VT^5} \]

\[ \beta = 7.2 \ (T \sim 2.2T_c) \]

\[ 64^3 \times 16 \]

\[ 1200 \text{ confs} \]

\[ \sqrt{8tT} \]
Energy Fluctuation and Specific Heat

Specific Heat

\[
\frac{c_V}{T^3} = \frac{\langle \delta E(\beta/2)\delta E(0) \rangle}{VT^5}
\]

Gavai, et al., 2005 differential method for $T = 2T_c$

64$^3$x16
\[\beta = 7.2 \ (T \sim 2.2T_c)\]
1200 confs

Novel approach to measure $c_V$