The Interpolating Function

Masazumi Honda

Weizmann Institute of Science

References: MH,JHEP1412 019 (1408.2960),
MH-DPJ,NPB900 (2015) 533 (1504.02276),
AC-MH-ST and AC-MH, to appear,
MH, work in progress

based on collaborations with

Abhishek Chowdhury (HRI), Dileep P. Jatkar (HRI),
Somyadip Thakur (TIFR)

24th, Nov.

Seminar @Osaka University

Perturbative expansion

- ubiquitous
- does not often give satisfactory understanding of physics…
  (unless it has nice properties)
- even if it has nice property,
  higher order computation is usually hard task

Today I consider somewhat limited situation:

We know

perturbative expansions around 2-points

Today I consider somewhat limited situation:

We know

perturbative expansions around 2-points

e.g. theories with S-duality, theories with gravity dual,
lattice field theory with weak & strong coupling expansions,
statistical systems with high & low temperature expansions, etc...

Today I consider somewhat limited situation:

We know

perturbative expansions around 2-points

e.g. theories with S-duality, theories with gravity dual,
lattice field theory with weak & strong coupling expansions,
statistical systems with high & low temperature expansions, etc...

⇩

Approximation at finite values of parameters

How do we interpolate these two expansions?

Tool : Interpolating function

Single function consistent with the 2 pertubative expansions

Tool : Interpolating function

Ex.) $$F(g) = \frac{g}{1+g} + e^{-\frac{1}{g}}$$

Tool : Interpolating function

Ex.) $$F(g) = \frac{g}{1+g} + e^{-\frac{1}{g}}$$

- Small-g exp.: $F(g) = g + \mathcal{O}(g^2)$
- Large-g exp.: $F(g) = 2 + \mathcal{O}(g^{-1})$

Tool : Interpolating function

Ex.) $$F(g) = \frac{g}{1+g} + e^{-\frac{1}{g}}$$

- Small-g exp.: $F(g) = g + \mathcal{O}(g^2)$
- Large-g exp.: $F(g) = 2 + \mathcal{O}(g^{-1})$

Interpolation ⇒ $$\frac{2g}{2+g}$$

(2-point Pade approximant)

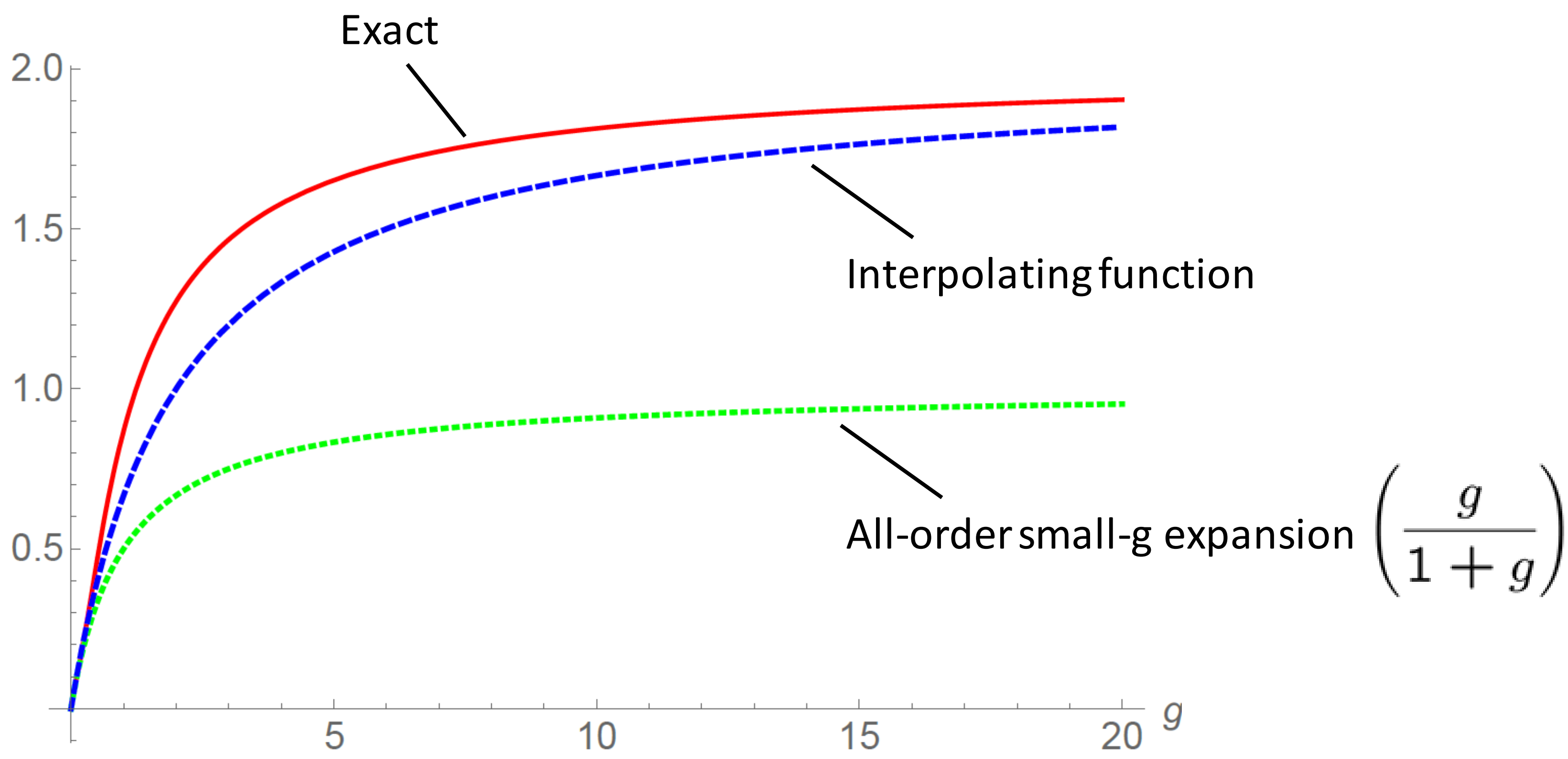
Tool : Interpolating function

Ex.) $F(g) = \frac{g}{1+g} + e^{-\frac{1}{g}}$

$$\begin{cases} \text{Small-g exp.:} \quad F(g) = g + \mathcal{O}(g^2) \\ \text{Large-g exp.:} \quad F(g) = 2 + \mathcal{O}(g^{-1}) \end{cases}$$

Interpolation ⟹ $\frac{2g}{2+g}$

(2-point Pade approximant)



Outline

- introduce a class of interpolating functions [Sen '13, MH '14]

  ——— generalization of Pade and Sen's interpolating function

- problem of this approach [MH '14]

  ——— $\exists$ many interpolating functions. Which is best?

- Criterion to choose the best interpolating function [MH '14]
  (for a class of problems)

- Implications of analytic property of interpolating function

  [MH-Jatkar '15]

Introduction to Interpolating function

Setup

Suppose that we know small-g and large-g expansions of a function F(g):

$$F(g) = \begin{cases} g^a(s_0 + s_1 g + s_2 g^2 + \cdots), \\ g^b(l_0 + l_1 g^{-1} + l_2 g^{-2} + \cdots), \end{cases}$$

Then we would like to find approximation of F(g) at finite g.

(When we have expansions around $g=g_1$ and $g=g_2$,
changing the variable as $x=(g-g_1)/(g-g_2)$ gives small-x and large-x expansions)

(Two-point) Pade approximant

$$\mathcal{P}_{m,n}(g) = s_0 g^a \frac{1 + \sum_{k=1}^{p} c_k g^k}{1 + \sum_{k=1}^{q} d_k g^k},$$

$$\left(\quad p = \frac{m+n+1+(b-a)}{2} \in \mathbf{Z}, \quad q = \frac{m+n+1-(b-a)}{2} \in \mathbf{Z} \quad \right)$$

The coefficients are determined to reproduce the small-g exp. up to $\mathcal{O}(g^{a+m+1})$ and large-g exp. up to $\mathcal{O}(g^{b-n-1})$

(Two-point) Pade approximant

$$\mathcal{P}_{m,n}(g) = s_0 g^a \frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k},$$

$$\left(p = \frac{m+n+1+(b-a)}{2} \in \mathbf{Z}, \quad q = \frac{m+n+1-(b-a)}{2} \in \mathbf{Z} \right)$$

The coefficients are determined to reproduce the small-g exp. up to $\mathcal{O}(g^{a+m+1})$ and large-g exp. up to $\mathcal{O}(g^{b-n-1})$

Some properties:

- can construct only for (b-a)∈**Z** (although avoidable by a change of variable)
  - ((b-a) :even → (m+n): odd, (b-a) :odd → (m+n): even)
- has poles. No branch cut

Fractional Power of Polynomial (FPP)

[Sen '13]

$$F_{m,n}(g) = s_0 g^a \left[1 + \sum_{k=1}^{m} c_k g^k + \sum_{k=0}^{n} d_k g^{m+n+1-k} \right]^{\frac{b-a}{m+n+1}}$$

The coefficients are determined to reproduce the small-g exp. up to $\mathcal{O}(g^{a+m+1})$ and large-g exp. up to $\mathcal{O}(g^{b-n-1})$

Fractional Power of Polynomial (FPP)

[Sen '13]

$$F_{m,n}(g) = s_0 g^a \left[1 + \sum_{k=1}^{m} c_k g^k + \sum_{k=0}^{n} d_k g^{m+n+1-k} \right]^{\frac{b-a}{m+n+1}}$$

The coefficients are determined to reproduce the small-g exp. up to $\mathcal{O}(g^{a+m+1})$ and large-g exp. up to $\mathcal{O}(g^{b-n-1})$

Some properties:

- can construct for arbitrary (a,b,m,n)
- Type of branch cut is uniquely determined by (a,b,m,n)

Fractional Power of Rational function (FPR)

$$F^{(\alpha)}_{m,n}(g) = s_0 g^a \left[\frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k} \right]^{\alpha},$$

[MH '14]

$$\left(p = \frac{1}{2}\left(m+n+1+\frac{b-a}{\alpha}\right) \in \mathbf{Z}, \quad q = \frac{1}{2}\left(m+n+1-\frac{b-a}{\alpha}\right) \in \mathbf{Z} \right)$$

$$\alpha = \begin{cases} \frac{a-b}{2\ell+1} & \text{for } m+n : \text{even} \\ \frac{a-b}{2\ell} & \text{for } m+n : \text{odd} \end{cases}, \quad \text{with } \ell \in \mathbb{Z}.$$

Fractional Power of Rational function (FPR)

$$F^{(\alpha)}_{m,n}(g) = s_0 g^a \left[\frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k} \right]^{\alpha},$$

[MH '14]

$$\left(p = \frac{1}{2}\left(m+n+1+\frac{b-a}{\alpha}\right) \in \mathbf{Z}, \quad q = \frac{1}{2}\left(m+n+1-\frac{b-a}{\alpha}\right) \in \mathbf{Z} \right)$$

$$\alpha = \begin{cases} \frac{a-b}{2\ell+1} & \text{for } m+n : \text{even} \\ \frac{a-b}{2\ell} & \text{for } m+n : \text{odd} \end{cases}, \quad \text{with } \ell \in \mathbb{Z}.$$

Some properties:

- includes the Pade and FPP as the special cases
- can construct for arbitrary (a,b,m,n)
- can control type of branch cut

Fractional Power of Rational function (FPR)

$$F_{m,n}^{(\alpha)}(g) = s_0 g^a \left[\frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k} \right]^\alpha ,$$

[MH '14]

$$\left(\quad p = \frac{1}{2}\left(m+n+1+\frac{b-a}{\alpha}\right) \in \mathbf{Z}, \qquad q = \frac{1}{2}\left(m+n+1-\frac{b-a}{\alpha}\right) \in \mathbf{Z} \quad \right)$$

$$\alpha = \begin{cases} \frac{a-b}{2\ell+1} & \text{for } m+n : \text{even} \\ \frac{a-b}{2\ell} & \text{for } m+n : \text{odd} \end{cases}, \qquad \text{with } \ell \in \mathbb{Z}.$$

Some properties:

- includes the Pade and FPP as the special cases
- can construct for arbitrary (a,b,m,n)
- can control type of branch cut

There are many cases where FPR gives very precise approximation.

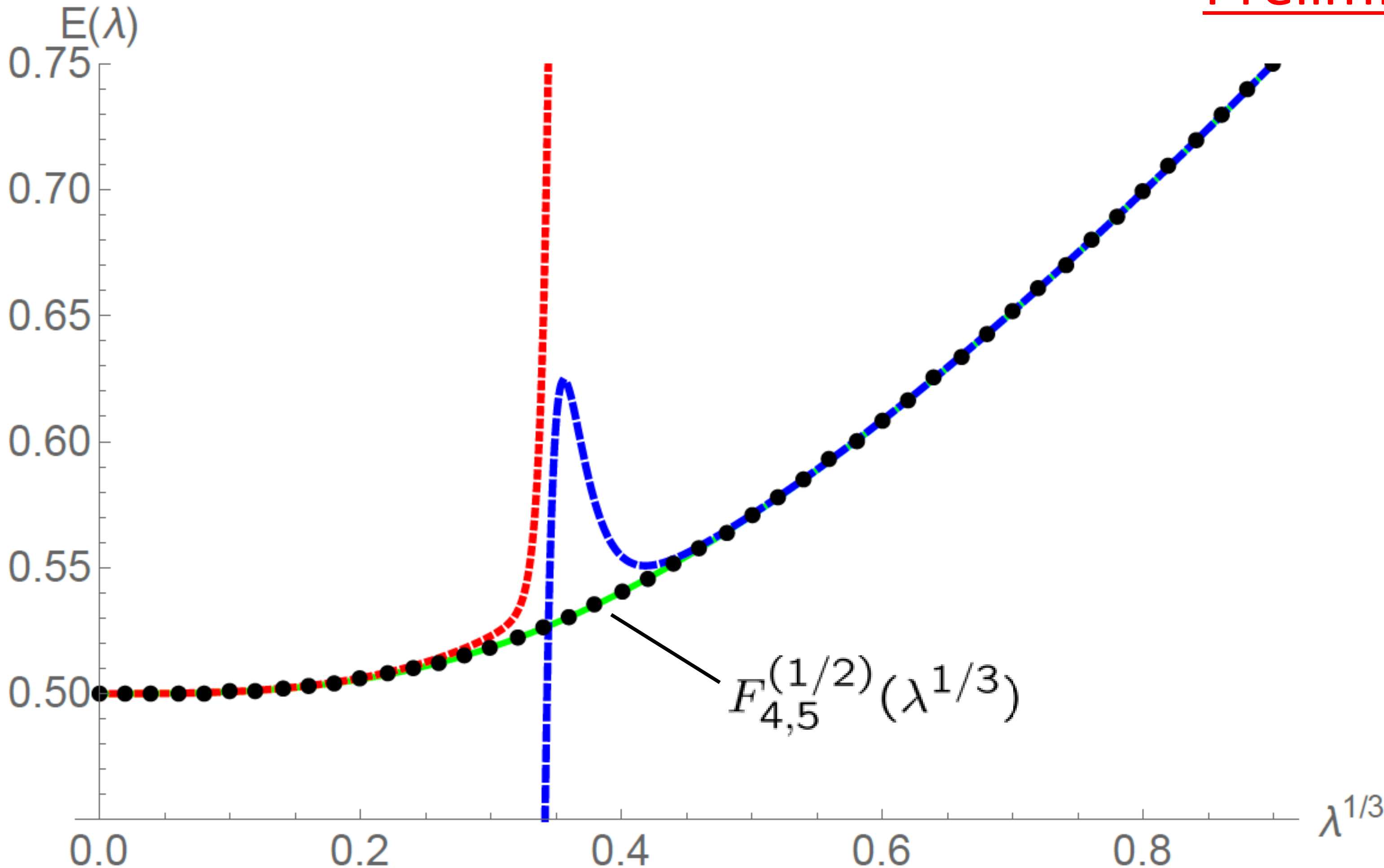
(although there are also many unsuccessful cases)

Ex.) Grand state Energy in anharmonic oscillator

[MH, work in progress]

$$\left[-\frac{d^2}{dx^2}+\frac{1}{4}x^2+\frac{1}{4}\lambda x^4\right]\psi(x)=E(\lambda)\psi(x)$$

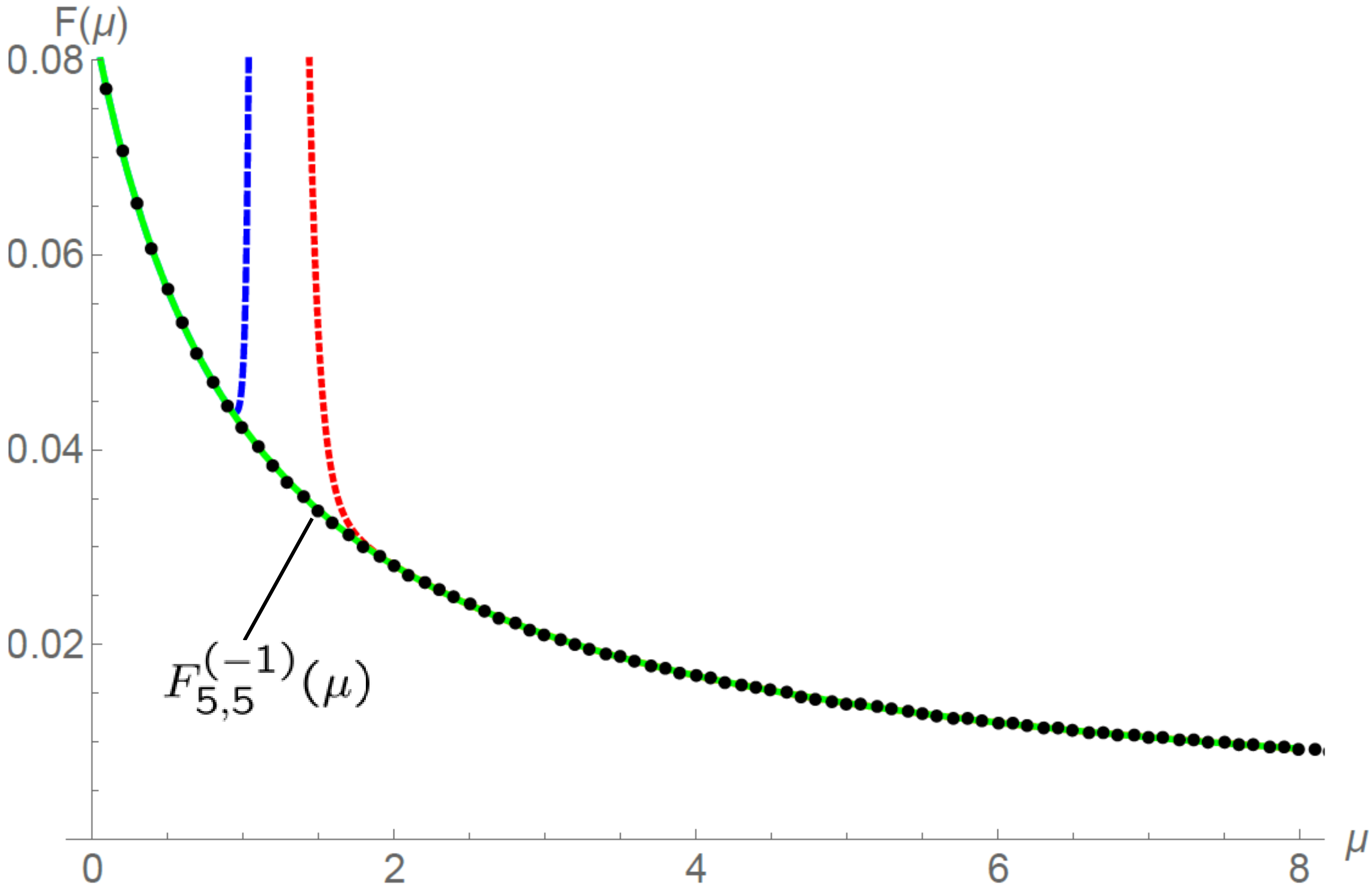
Preliminary



Ex.) Free energy of c=1 non-critical string

[MH '14]

μ: cosmological constant

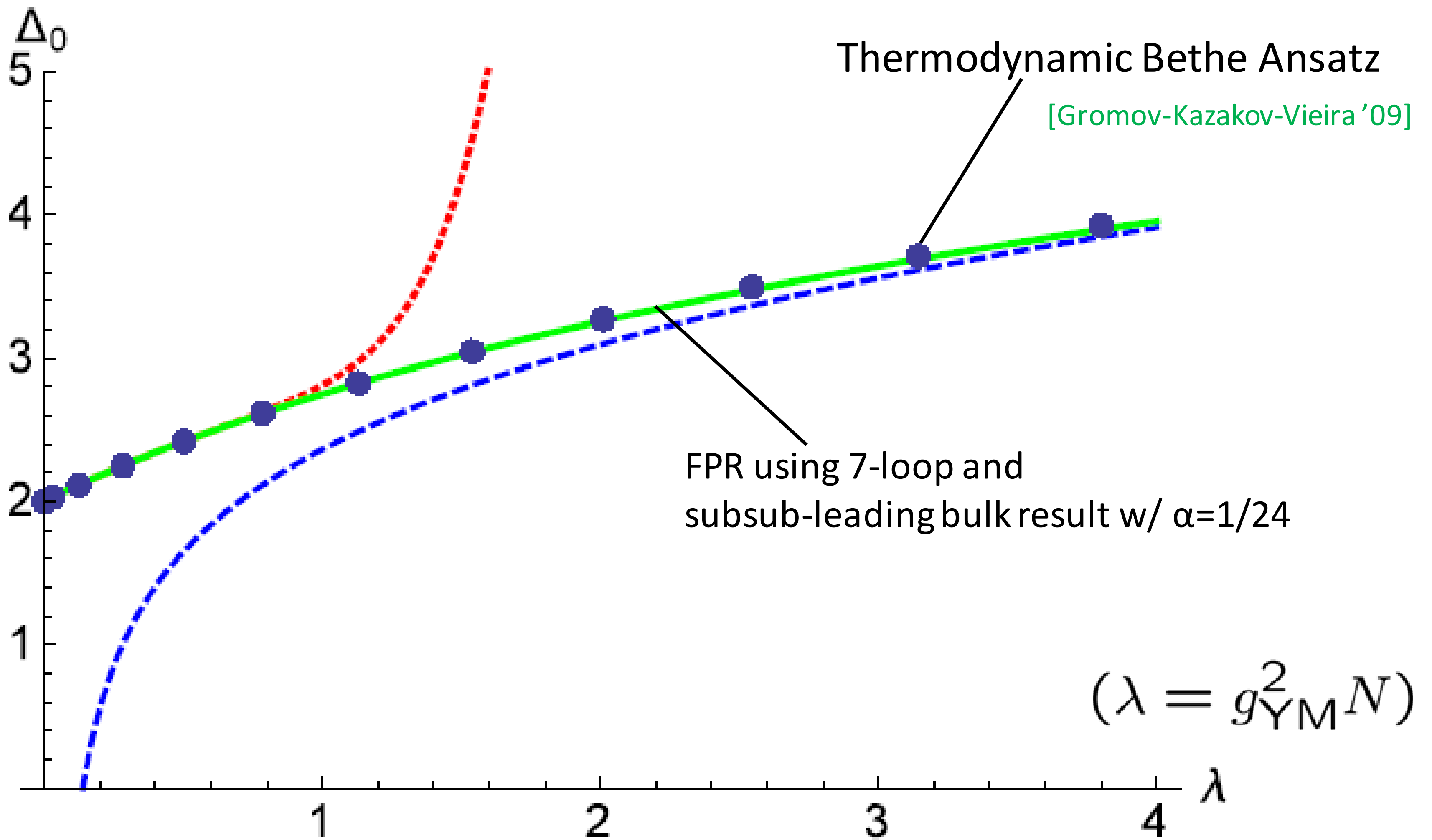


Ex.)Dimension of Konishi op. in planar $\mathcal{N}=4$ SYM

[Chowdhury-MH-Thakur, to appear]

$$\mathcal{O}_{\text{Konishi}} = \text{Tr}\phi^I\phi^I$$

Preliminary



Problem of this approach (via 0d ϕ^4 theory)

[MH ’14]

Partition function of 0d ϕ^4 theory

$$F(g) = \frac{1}{\sqrt{g}} \int_{-\infty}^{\infty} dx\; e^{-\frac{x^2}{2g} - x^4} = \frac{\pi e^{\frac{1}{32g^2}}}{4g} \left[I_{-\frac{1}{4}}\left(\frac{1}{32g^2}\right) - I_{\frac{1}{4}}\left(\frac{1}{32g^2}\right) \right]$$

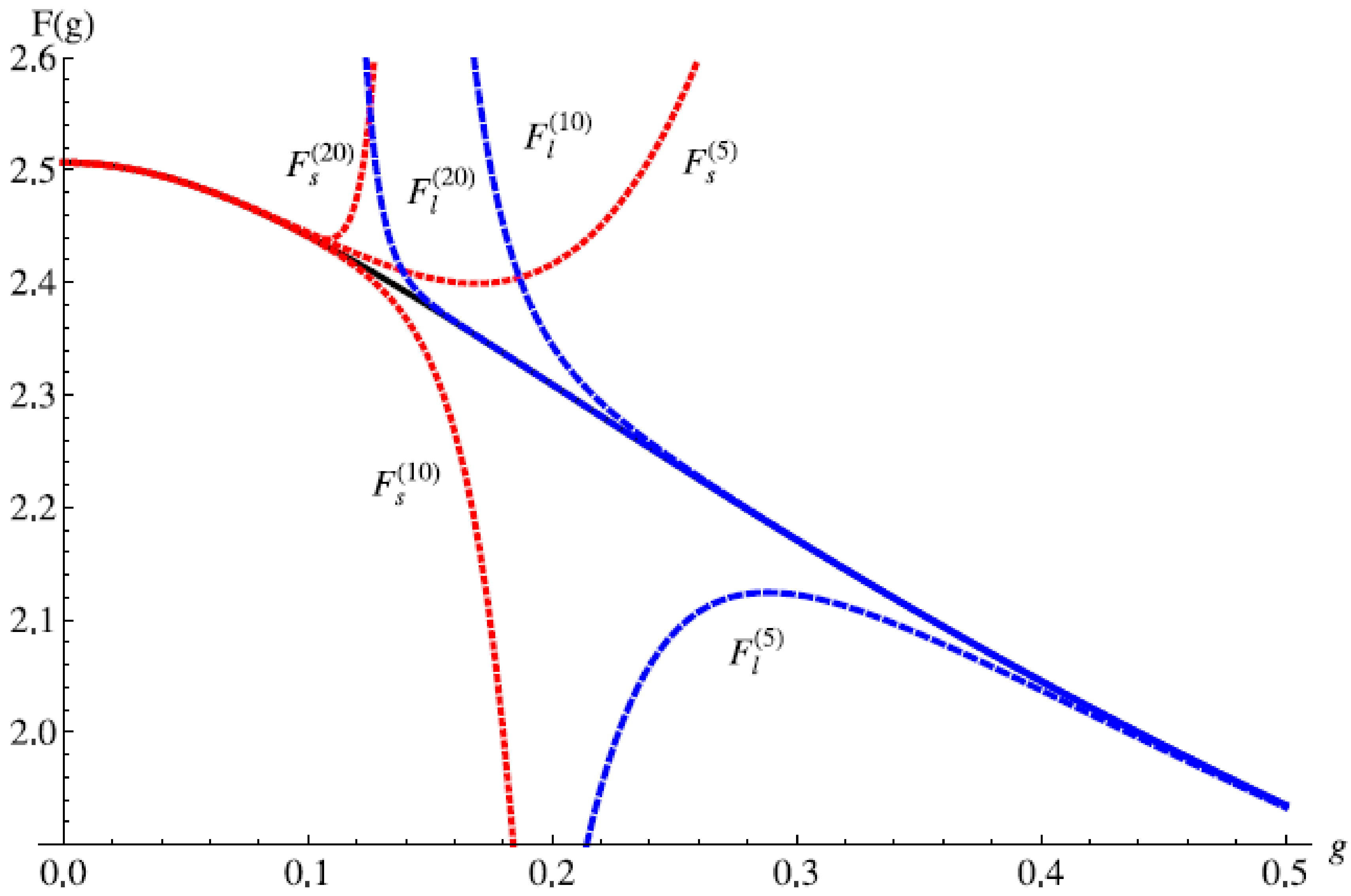
Small-g expansion:

$$F(g) = \sqrt{2\pi} - 3\sqrt{2\pi} g^2 + 105\sqrt{\frac{\pi}{2}} g^4 + \mathcal{O}(g^6) \qquad \text{(a=0)}$$

Large-g expansion:

$$F(g) = g^{-1/2}\left(\frac{1}{2}\Gamma(1/4) + \frac{1}{16}\Gamma(-1/4)g^{-1} + \frac{1}{64}\Gamma(1/4)g^{-2} + \mathcal{O}(g^{-3})\right) \qquad \text{(b=-1/2)}$$

We can construct FPR-type interpolating functions $F_{m,n}^{(\alpha)}$



F(g)
2.6
2.5
2.4
2.3
2.2
2.1
2.0
0.0
0.1
0.2
0.3
0.4
0.5
g
$F_s^{(20)}$
$F_l^{(20)}$
$F_l^{(10)}$
$F_s^{(5)}$
$F_s^{(10)}$
$F_l^{(5)}$

Some explicit forms of FPRs… [MH'14]

$$F(g) = F_{m,n}^{(\alpha)}(g) + \mathcal{O}(g^{a+m+1}, g^{b-n-1})$$

$$F_{0,0}^{(1/2)}(g)=\sqrt{2\pi}\left(\frac{8\pi g}{\Gamma(1/4)^2}+1\right)^{-1/2}, \quad F_{1,1}^{(1/2)}(g)=\sqrt{2\pi\Gamma(1/4)}\left(\frac{8\pi g\Gamma(1/4)+\Gamma(1/4)^3+2\pi\Gamma(-1/4)}{64\pi^2g^2+8\pi g\Gamma(1/4)^2+\Gamma(1/4)^4+2\pi\Gamma(-1/4)\Gamma(1/4)}\right)^{1/2},$$

$$F_{1,1}^{(1/6)}(g)=2.50663\left(\frac{1}{6.98929g^3+7.08691g^2+1}\right)^{1/6}, \quad F_{2,2}^{(1/2)}(g)=2.50663\sqrt{\frac{37.9117g^2+10.1532g+1}{72.4854g^3+43.9117g^2+10.1532g+1}},$$

$$F_{2,2}^{(1/10)}(g)=2.50663\left(25.5499g^5+43.1779g^4+32.1482g^3+30g^2+1\right)^{-1/10},$$

$$F_{3,3}^{(1/2)}(g)=2.50663\sqrt{\frac{324.019g^3+110.261g^2+16.0304g+1}{619.509g^4+420.201g^3+116.261g^2+16.0304g+1}},$$

$$F_{3,3}^{(1/6)}(g)=2.50663\left(\frac{28.2525g^2+8.0997g+1}{197.465g^5+256.834g^4+145.795g^3+46.2525g^2+8.0997g+1}\right)^{1/6},$$

$$F_{3,3}^{(1/14)}(g)=2.50663\left(93.3994g^7+220.976g^6+239.216g^5+155.758g^4+42g^2+1\right)^{-1/14},$$

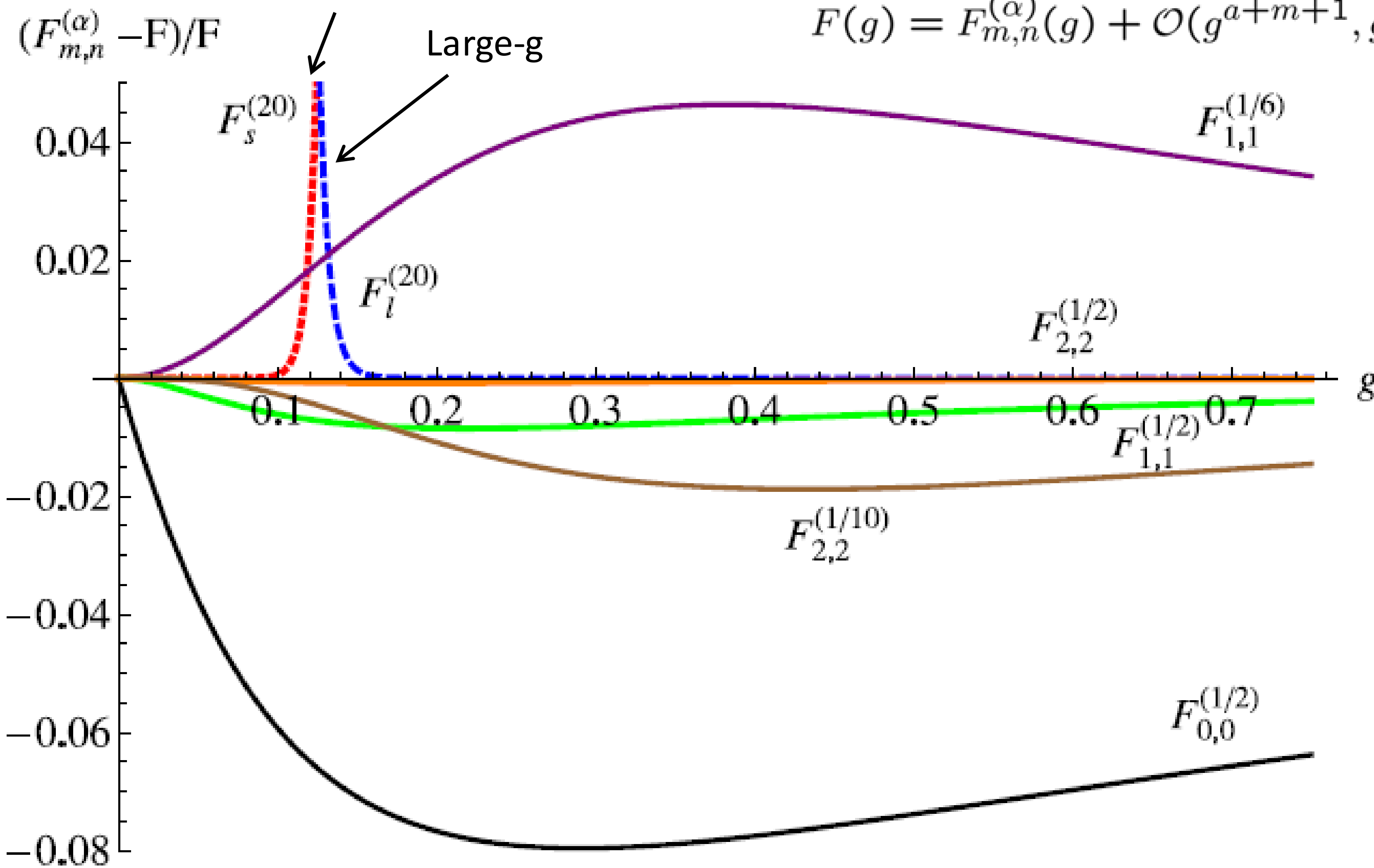
$$F_{4,4}^{(1/2)}(g)=2.50663\sqrt{\frac{3224.56g^4+1303.49g^3+238.239g^2+22.8745g+1}{6165.22g^5+4576.g^4+1440.74g^3+244.239g^2+22.8745g+1}},$$

$$F_{4,4}^{(1/10)}(g)=2.50663\left[\frac{14.4369g^2+5.07251g+1}{368.86g^7+752.954g^6+708.689g^5+403.106g^4+152.175g^3+44.4369g^2+5.07251g+1}\right]^{1/10},$$

$$F_{4,4}^{(1/18)}(g)=2.50663\left(341.428g^9+1038.59g^8+1475.35g^7+1294.34g^6+780.788g^5+594g^4+54g^2+1\right)^{-1/18}.$$

[Sen '13, MH '14]

$$F(g) = F^{(\alpha)}_{m,n}(g) + \mathcal{O}(g^{a+m+1}, g^{b-n-1})$$



- Roughly, larger (m,n) seems better
- The same (m,n) but different α sometimes give different precisions

Predictability of interpolating function?

We can construct many interpolating functions.

It is very unclear which interpolating function gives the best approximation when we don't know exact or numerical results.

("Landscape problem" of interpolating functions)

Predictability of interpolating function?

We can construct many interpolating functions.

It is very unclear which interpolating function gives the best approximation when we don't know exact or numerical results.

("Landscape problem" of interpolating functions)

Possible solutions(?):

1. Constrain by consistency from other information

   (e.g. such as reality, positivity, absence of phase transition, etc…)

—— Useful but usually we still have multiple consistent interpolating functions

Predictability of interpolating function?

We can construct many interpolating functions.

It is very unclear which interpolating function gives the best approximation when we don't know exact or numerical results.

("Landscape problem" of interpolating functions)

Possible solutions(?):

1. Constrain by consistency from other information

   (e.g. such as reality, positivity, absence of phase transition, etc…)

—— Useful but usually we still have multiple consistent interpolating functions

2. Sometimes many interpolating functions give almost the same results

—— Natural to think this as the prediction.

Predictability of interpolating function?

We can construct many interpolating functions.

It is very unclear which interpolating function gives the best approximation when we don't know exact or numerical results.

("Landscape problem" of interpolating functions)

Possible solutions(?):

1. Constrain by consistency from other information

   (e.g. such as reality, positivity, absence of phase transition, etc…)

—— Useful but usually we still have multiple consistent interpolating functions

2. Sometimes many interpolating functions give almost the same results

—— Natural to think this as the prediction.

3. Choose the best interpolating function in some ways (ambitious)

—— Criterion for the best interpolating function [MH '14]

Criterion for the best interpolating function

[MH '14]

Restriction of problems

We assume that physical quantity F(g) satisfies

1. We know asymptotic large order behaviors of
   2 perturbative exapansions

2. We know rough weights of non-perturbative correction

   (e.g. we can often compute instanton actions in QFT and string theory)

3. Non-perturbative corrections in both sense are not large

   (e.g. exp(-g-1/g) does not contribute to the 2 expansions)

Conjecture:

[MH '14]

Given set of possible interpolating functions {G(g)},

the best interpolating function minimizes the quantity

$$I_s[G_{\mathrm{best}}] + I_l[G_{\mathrm{best}}] = \min\{I_s[G] + I_l[G]\}.$$

Conjecture:

[MH '14]

Given set of possible interpolating functions {G(g)},

the best interpolating function minimizes the quantity

$$I_s[G_{\rm best}] + I_l[G_{\rm best}] = \min\{I_s[G] + I_l[G]\}.$$

where

cutoff, Λ>>1

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right|, \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right|,$$

$$F_s^{(N_s)} = g^a \sum_{k=0}^{N_s} s_k g^k, \ \ F_l^{(N_l)} = g^b \sum_{k=0}^{N_l} l_k g^{-k},$$

Conjecture:

[MH '14]

Given set of possible interpolating functions {G(g)},

the best interpolating function minimizes the quantity

$$I_s[G_{\rm best}] + I_l[G_{\rm best}] = \min \{ I_s[G] + I_l[G] \}.$$

where

cutoff, Λ>>1

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right|, \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right|,$$

$$F_s^{(N_s)} = g^a \sum_{k=0}^{N_s} s_k g^k, \ F_l^{(N_l)} = g^b \sum_{k=0}^{N_l} l_k g^{-k},$$

$(g_s^*, N_s^*, g_l^*, N_l^*)$ are determined such that

$(F_s^{(N_s^*)}, F_l^{(N_l^*)})$ are almost F(g) for $(0 \le g \le g_s^*, g \ge g_l^*)$

I have tested the conjecture for partition function of 0d ϕ^4 theory, average plaquette in SU(3) YM on lattice, specific heat in 2d Ising, free energy of c=1 non-critical string, etc…

Determination of $(g_s^*, N_s^*, g_l^*, N_l^*)$

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right|, \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right|,$$

- $(g_s^*, N_s^*, g_l^*, N_l^*)$ are determined such that $(F_s^{(N_s^*)}, F_l^{(N_l^*)})$ are almost F(g) for $(0 \le g \le g_s^*, g \ge g_l^*)$
- These depend on large order behaviors of the 2 expansions and weights of non-perturbative effects.

Determination of $(g_s^*, N_s^*, g_l^*, N_l^*)$

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right| , \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right| ,$$

- $(g_s^*, N_s^*, g_l^*, N_l^*)$ are determined such that
  $(F_s^{(N_s^*)}, F_l^{(N_l^*)})$ are almost F(g) for $(0 \leq g \leq g_s^*, g \geq g_l^*)$
- These depend on large order behaviors of the 2 expansions and weights of non-perturbative effects.

Case1: Two expansions are convergent and there are no non-perturbative effects

$$N_s^*, N_l^* \gg 1, \qquad g_s^*, g_l^* \sim \text{ (radius of convergence)}$$

Determination of $(g_s^*, N_s^*, g_l^*, N_l^*)$

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right|, \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right|,$$

- $(g_s^*, N_s^*, g_l^*, N_l^*)$ are determined such that $(F_s^{(N_s^*)}, F_l^{(N_l^*)})$ are almost F(g) for $(0 \le g \le g_s^*, g \ge g_l^*)$
- These depend on large order behaviors of the 2 expansions and weights of non-perturbative effects.

Case1: Two expansions are convergent and there are no non-perturbative effects

$$N_s^*, N_l^* \gg 1, \qquad g_s^*, g_l^* \sim \text{(radius of convergence)}$$

Case2: Two expansions are convergent for $(g < g_s^c, g > g_l^c)$
but non-perturbative effects become important for $(g \ge g_s^I, g \le g_l^I)$

$$N_s^*, N_l^* \gg 1, \quad g_s^*, g_l^* \sim \min(g_s^c, g_s^I), \min(g_l^c, g_l^I)$$

Determination of $(g_s^*, N_s^*, g_l^*, N_l^*)$

$$I_s[G] = \int_0^{g_s^*} dg \left| G(g) - F_s^{(N_s^*)}(g) \right|, \quad I_l[G] = \int_{g_l^*}^{\Lambda} dg \left| G(g) - F_l^{(N_l^*)}(g) \right|,$$

- $(g_s^*, N_s^*, g_l^*, N_l^*)$ are determined such that $(F_s^{(N_s^*)}, F_l^{(N_l^*)})$ are almost F(g) for $(0 \le g \le g_s^*, g \ge g_l^*)$
- These depend on large order behaviors of the 2 expansions and weights of non-perturbative effects.

Case1: Two expansions are convergent and there are no non-perturbative effects

$$N_s^*, N_l^* \gg 1, \quad g_s^*, g_l^* \sim \text{(radius of convergence)}$$

Case2: Two expansions are convergent for $(g < g_s^c, g > g_l^c)$ but non-perturbative effects become important for $(g \ge g_s^I, g \le g_l^I)$

$$N_s^*, N_l^* \gg 1, \quad g_s^*, g_l^* \sim \min(g_s^c, g_s^I), \min(g_l^c, g_l^I)$$

Case3: Some of expansions are asymptotic

⟶ use optimization of asymptotic series (details are skipped)

Test of the conjecture in 0d ϕ^4 theory

	$\Lambda^{-1}\int dg \left\lvert \frac{(F_{m,n}^{(\alpha)}-F}{F} \right\rvert$	$I_s[F_{m,n}^{(\alpha)}]$	$I_l[F_{m,n}^{(\alpha)}]$	$I_s + I_l$
$F_{0,0}^{(1/2)}$	0.000659728	0.00446072	0.381344	0.385805
$F_{1,1}^{(1/2)}$	9.27801×10^{-6}	0.000297906	0.0142222	0.0145201
$F_{1,1}^{(1/6)}$	0.0000760393	0.000432581	0.106287	0.106720
$F_{2,2}^{(1/2)}$	4.61177×10^{-7}	0.0000230059	0.000849124	0.000872130
$F_{2,2}^{(1/6)}$	5.24010×10^{-6}	0.0000450000	0.00905419	0.00909919
$F_{2,2}^{(1/10)}$	0.0000235129	0.0000430012	0.0373156	0.0373586
$F_{3,3}^{(1/2)}$	2.96944×10^{-8}	1.94617×10^{-6}	0.0000576043	0.0000595505
$F_{3,3}^{(1/6)}$	3.84001×10^{-7}	5.09656×10^{-6}	0.000738006	0.000743103
$F_{3,3}^{(1/14)}$	8.84054×10^{-6}	2.59016×10^{-6}	0.0148826	0.0148852
$\mathbf{F_{4,4}^{(1/2)}}$	$\mathbf{2.17241 \times 10^{-9}}$	$\mathbf{1.78480 \times 10^{-7}}$	$\mathbf{4.25411 \times 10^{-6}}$	$\mathbf{4.43259 \times 10^{-6}}$
$F_{4,4}^{(1/6)}$	2.85852×10^{-8}	5.50786×10^{-7}	0.0000577750	0.0000583258
$F_{4,4}^{(1/10)}$	5.77057×10^{-7}	1.52640×10^{-6}	0.00111431	0.00111584
$F_{4,4}^{(1/18)}$	3.17581×10^{-6}	8.72352×10^{-7}	0.00549043	0.00549131

Average plaquette in SU(3) YM on lattice

Average plaquette in SU(3) YM on lattice

Wilson action:

$$S = \beta \sum_{\mu<\nu} \sum_{\mathrm{x}} \left[1 - \frac{1}{3} \mathrm{ReTr} U_{\mathrm{x},\mu} U_{\mathrm{x}+\hat{\mu},\nu} U^{\dagger}_{\mathrm{x}+\hat{\nu},\mu} U^{\dagger}_{\mathrm{x},\nu} \right],$$

Average plaquette:

$$P(\beta) = \left\langle 1 - \frac{1}{3} \mathrm{Tr} U_{\mathrm{x},\mu} U_{\mathrm{x}+\hat{\mu},\nu} U^{\dagger}_{\mathrm{x}+\hat{\nu},\mu} U^{\dagger}_{\mathrm{x},\nu} \right\rangle.$$

Weak coupling expansion:

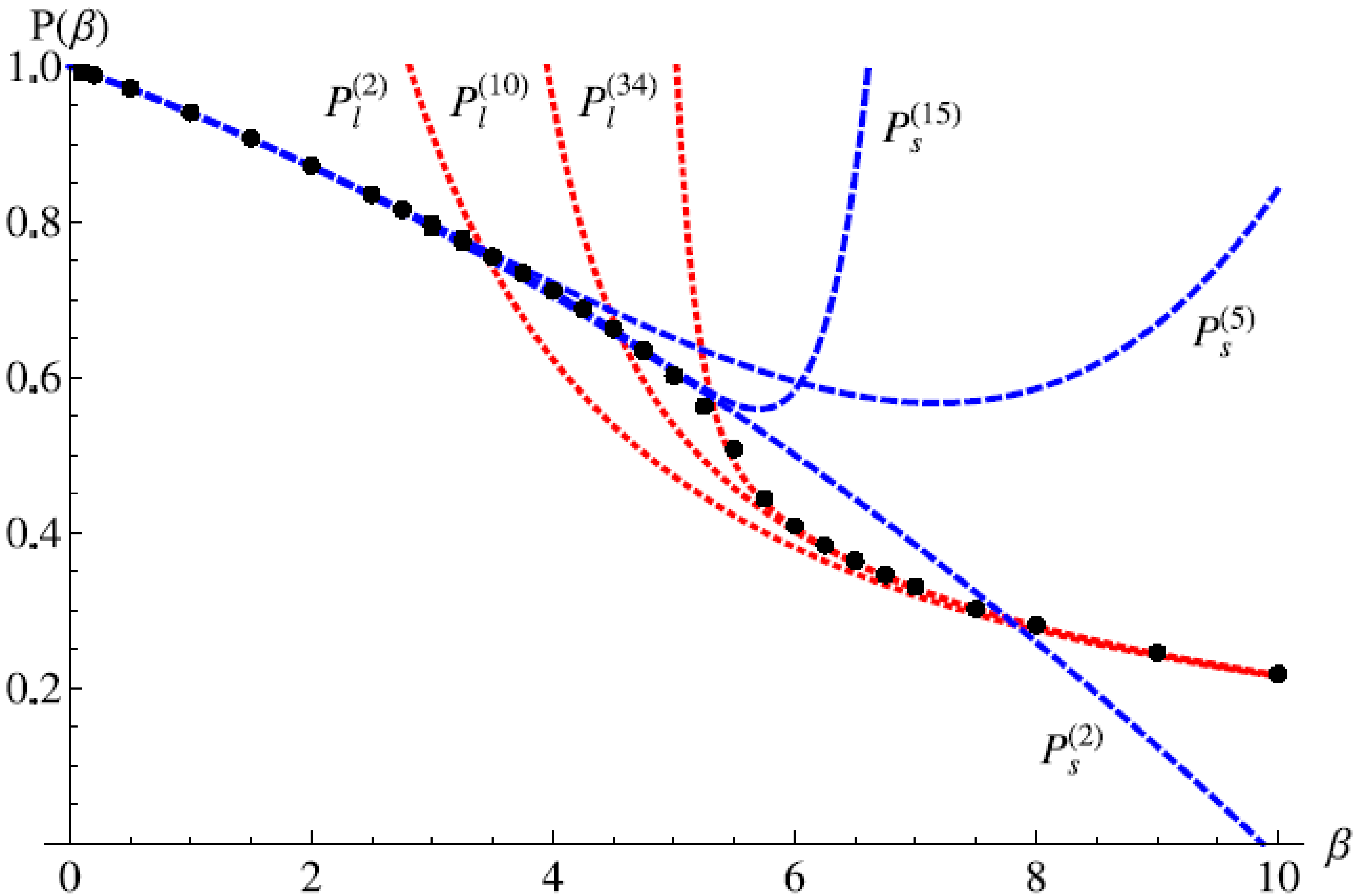
[O(β^{-35}) computation: Bali-Bauer-Pineda '14]

$$P(\beta) = \beta^{-1} \left(2 + 1.22084 \beta^{-1} + 2.96043 \beta^{-2} + \mathcal{O}(\beta^{-3}) \right)$$

Strong coupling expansion:

[O(β^{15}) computation: Wilson, unpublished]

$$P(\beta) = 1 - \frac{1}{18}\beta - \frac{1}{216}\beta^2 + \mathcal{O}(\beta^3)$$



(data points: reference values from Monte Carlo on 10^4 lattice)

Some explicit forms…

$$P_{1,1}^{(-1)}(\beta)=\frac{1600000\pi^2(4\beta+9)+43377282}{800000\pi^2(4\beta^2+9\beta+18)+2409849(\beta+18)}, \qquad P_{1,1}^{(-1/3)}(\beta)=\left(\frac{\beta^3}{8}-\frac{7229547\beta^2}{3200000\pi^2}+\frac{\beta}{6}+1\right)^{-1/3},$$

$$P_{2,2}^{(-1)}(\beta)=\frac{2\beta^2+6.23458\beta+20.8014}{\beta^3+2.50687\beta^2+7.39021\beta+20.8014}, \qquad P_{3,3}^{(-1)}(\beta)=\frac{2(\beta+3.3955)\left(\beta^2-0.198893\beta+12.0174\right)}{\left(\beta^2-2.09682\beta+9.61319\right)\left(\beta^2+4.683\beta+8.48936\right)},$$

$$P_{4,4}^{(-1)}(\beta)=\frac{2\left(\beta^2-2.50678\beta+13.4054\right)\left(\beta^2+5.7244\beta+12.6824\right)}{(\beta+3.09342)\left(\beta^2-3.54558\beta+11.2267\right)\left(\beta^2+3.05935\beta+9.79082\right)},$$

$$P_{5,5}^{(-1)}(\beta)=\frac{2(\beta+3.68567)\left(\beta^2-4.08079\beta+14.5847\right)\left(\beta^2+3.61475\beta+13.7196\right)}{\left(\beta^2-4.63748\beta+12.6358\right)\left(\beta^2+1.38581\beta+10.9814\right)\left(\beta^2+5.86088\beta+10.6298\right)},$$

$$P_{6,6}^{(-1)}(\beta)=\frac{2\left(\beta^2-5.18648\beta+15.5937\right)\left(\beta^2+1.581\beta+14.6612\right)\left(\beta^2+6.83275\beta+14.4495\right)}{(\beta+3.39479)\left(\beta^2-5.47262\beta+13.8545\right)\left(\beta^2-0.0916253\beta+12.0825\right)\left(\beta^2+4.78629\beta+11.6262\right)},$$

$$P_{7,7}^{(-1)}(\beta)=\frac{2(\beta+3.89835)\left(\beta^2-5.9913\beta+16.4715\right)\left(\beta^2-0.140455\beta+15.5324\right)\left(\beta^2+5.47891\beta+15.258\right)}{\left(\beta^2-6.12185\beta+14.911\right)\left(\beta^2-1.34364\beta+13.1167\right)\left(\beta^2+3.49201\beta+12.5578\right)\left(\beta^2+6.60856\beta+12.3918\right)},$$

$$P_{8,8}^{(-1)}(\beta)=\frac{2\left(\beta^2-6.59846\beta+17.2502\right)\left(\beta^2-1.55339\beta+16.3433\right)\left(\beta^2+3.93023\beta+16.0062\right)\left(\beta^2+7.48658\beta+15.9063\right)}{(\beta+3.62668)\left(\beta^2-6.63649\beta+15.8369\right)\left(\beta^2-2.39994\beta+14.0937\right)\left(\beta^2+2.20409\beta+13.4255\right)\left(\beta^2+5.86018\beta+13.2095\right)},$$

$$P_{9,9}^{(-1)}(\beta)=\frac{2(\beta+4.0654)\left(\beta^2-7.07259\beta+17.9526\right)\left(\beta^2-2.71136\beta+17.0902\right)\left(\beta^2+2.43467\beta+16.6901\right)\left(\beta^2+6.55819\beta+16.5611\right)}{\left(\beta^2-7.05373\beta+16.6601\right)\left(\beta^2-3.29763\beta+15.0097\right)\left(\beta^2+1.00968\beta+14.2272\right)\left(\beta^2+4.8619\beta+13.9689\right)\left(\beta^2+7.14368\beta+13.8747\right)},$$

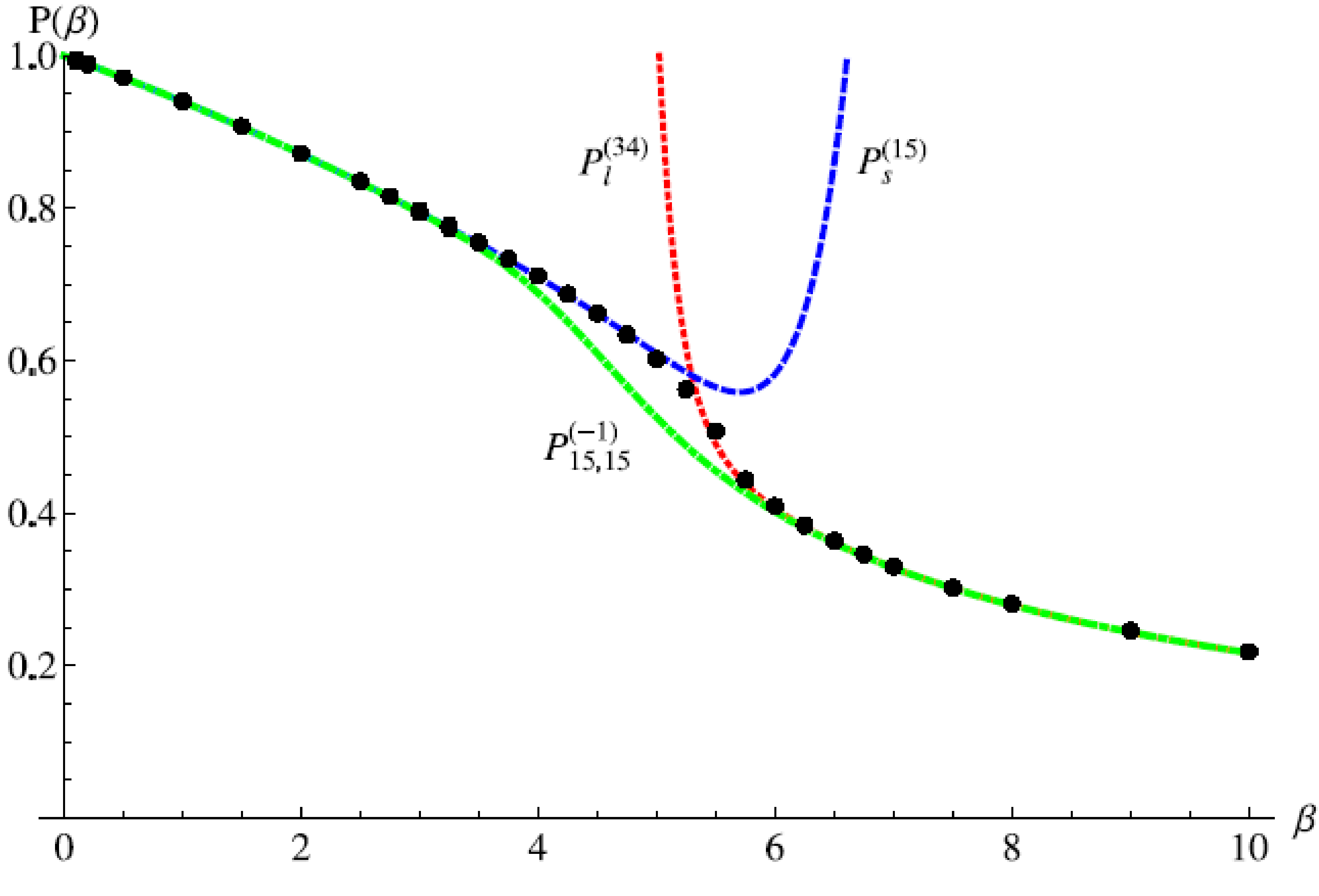
$$\begin{aligned}P_{10,10}^{(-1)}(\beta)=\,&2\left(\beta^2-7.45432\beta+18.593\right)\left(\beta^2-3.66556\beta+17.7643\right)\left(\beta^2+1.08036\beta+17.3105\right)\left(\beta^2+5.38063\beta+17.1591\right)\\&\left(\beta^2+7.9308\beta+17.1034\right)\times\Big[(\beta+3.80954)\left(\beta^2-7.39967\beta+17.4015\right)\left(\beta^2-4.06586\beta+15.8524\right)\left(\beta^2-0.0631074\beta+14.9646\right)\\&\left(\beta^2+3.78748\beta+14.6692\right)\left(\beta^2+6.5931\beta+14.5475\right)\Big]^{-1},\end{aligned}$$

$$\begin{aligned}P_{11,11}^{(-1)}(\beta)=\,&2(\beta+4.19731)\left(\beta^2-7.76923\beta+19.1795\right)\left(\beta^2-4.45532\beta+18.3621\right)\left(\beta^2-0.111216\beta+17.8786\right)\left(\beta^2+4.14811\beta+17.709\right)\\&\left(\beta^2+7.25733\beta+17.6376\right)\times\Big[\left(\beta^2-7.69195\beta+18.0747\right)\left(\beta^2-4.72437\beta+16.6111\right)\left(\beta^2-1.01157\beta+15.6487\right)\\&\left(\beta^2+2.72805\beta+15.3187\right)\left(\beta^2+5.80926\beta+15.1747\right)\left(\beta^2+7.54715\beta+15.1161\right)\Big]^{-1},\end{aligned}$$

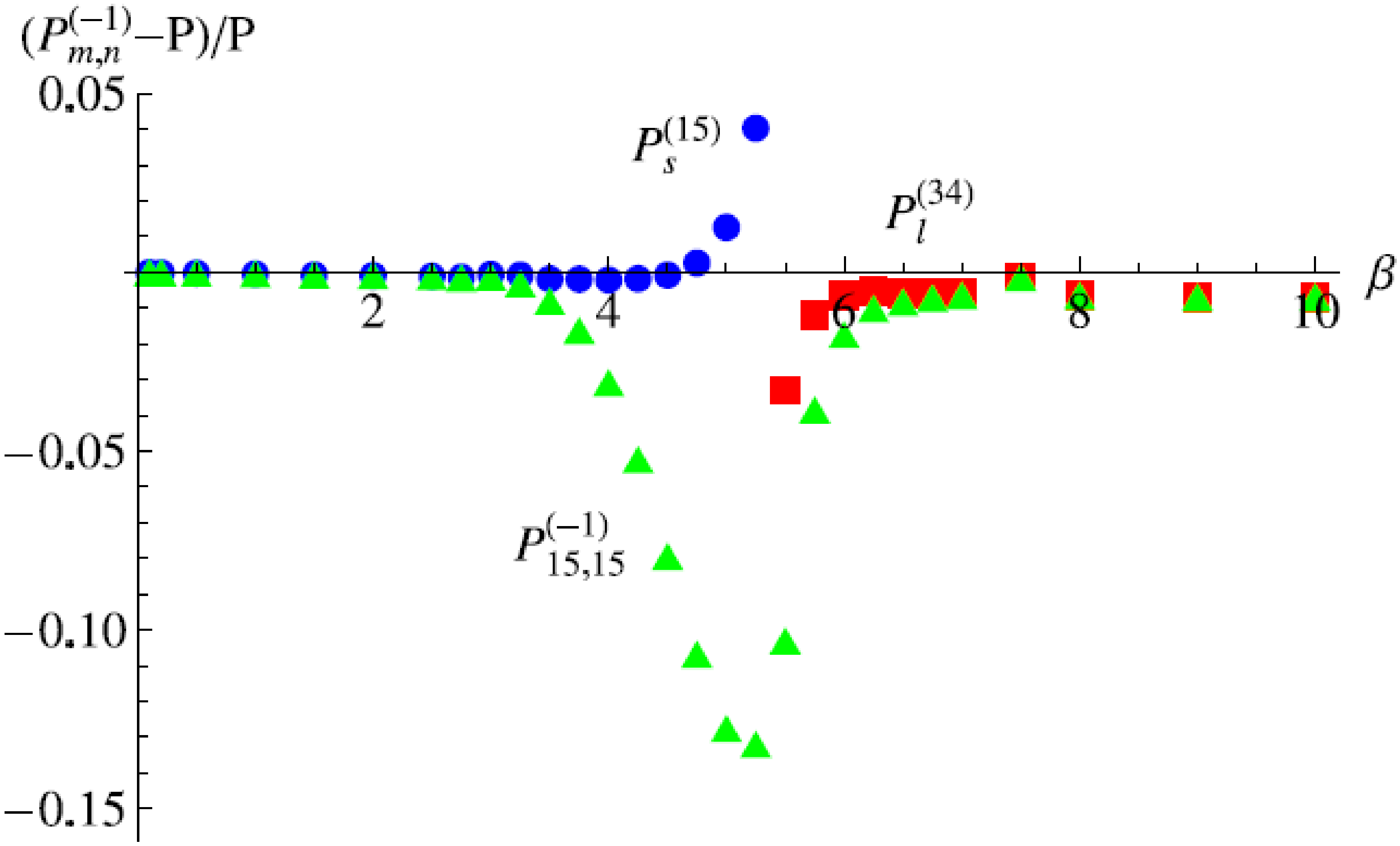
Test of the conjecture in SU(3) YM on lattice

	$\frac{1}{29}\sum_i \left\|\frac{P^{(\alpha)}_{m,n}-P}{P}\right\|$	$I_s[P^{(\alpha)}_{m,n}]$	$I_l[P^{(\alpha)}_{m,n}]$	I_s+I_l
$P^{(-1)}_{1,1}$	0.228616	0.634296	0.222215	0.856510
$P^{(-1/3)}_{1,1}$	0.115055	0.206451	0.070088	0.276539
$P^{(-1)}_{2,2}$	0.158456	0.380170	0.0924484	0.472619
$P^{(-1)}_{3,3}$	0.119927	0.247194	0.0472852	0.294479
$P^{(-1)}_{4,4}$	0.0956988	0.168693	0.0272632	0.195956
$P^{(-1)}_{5,5}$	0.0790835	0.118552	0.0169992	0.135551
$P^{(-1)}_{6,6}$	0.0670207	0.0848353	0.0112119	0.0960472
$P^{(-1)}_{7,7}$	0.0579211	0.0614099	0.00772215	0.0691320
$P^{(-1)}_{8,8}$	0.0508609	0.0447886	0.00550651	0.0502934
$P^{(-1)}_{9,9}$	0.0452512	0.0328091	0.00403859	0.0368477
$P^{(-1)}_{10,10}$	0.0406960	0.0240752	0.00303056	0.0271057
$P^{(-1)}_{11,11}$	0.0369267	0.0176544	0.00231792	0.0199723
$P^{(-1)}_{12,12}$	0.0337611	0.0129187	0.00180261	0.0147214
$P^{(-1)}_{13,13}$	0.0310727	0.00942950	0.00142323	0.0108527
$P^{(-1)}_{14,14}$	0.0287673	0.00686572	0.00113935	0.00800507
$\mathbf{P}^{(-1)}_{\mathbf{15,15}}$	**0.0267697**	**0.00498586**	**0.000923484**	**0.00590935**

Result of the best interpolating function



Result of the best interpolating function (Cont'd)



Specific heat in 2d Ising model

- Interpolating functions fail to approximate behavior around phase transition point

- Nevertheless, interpolating functions seem to give non-trivial information on the phase transition

Specific heat in 2d Ising model

Standard 2d Ising on (L x L) square lattice:

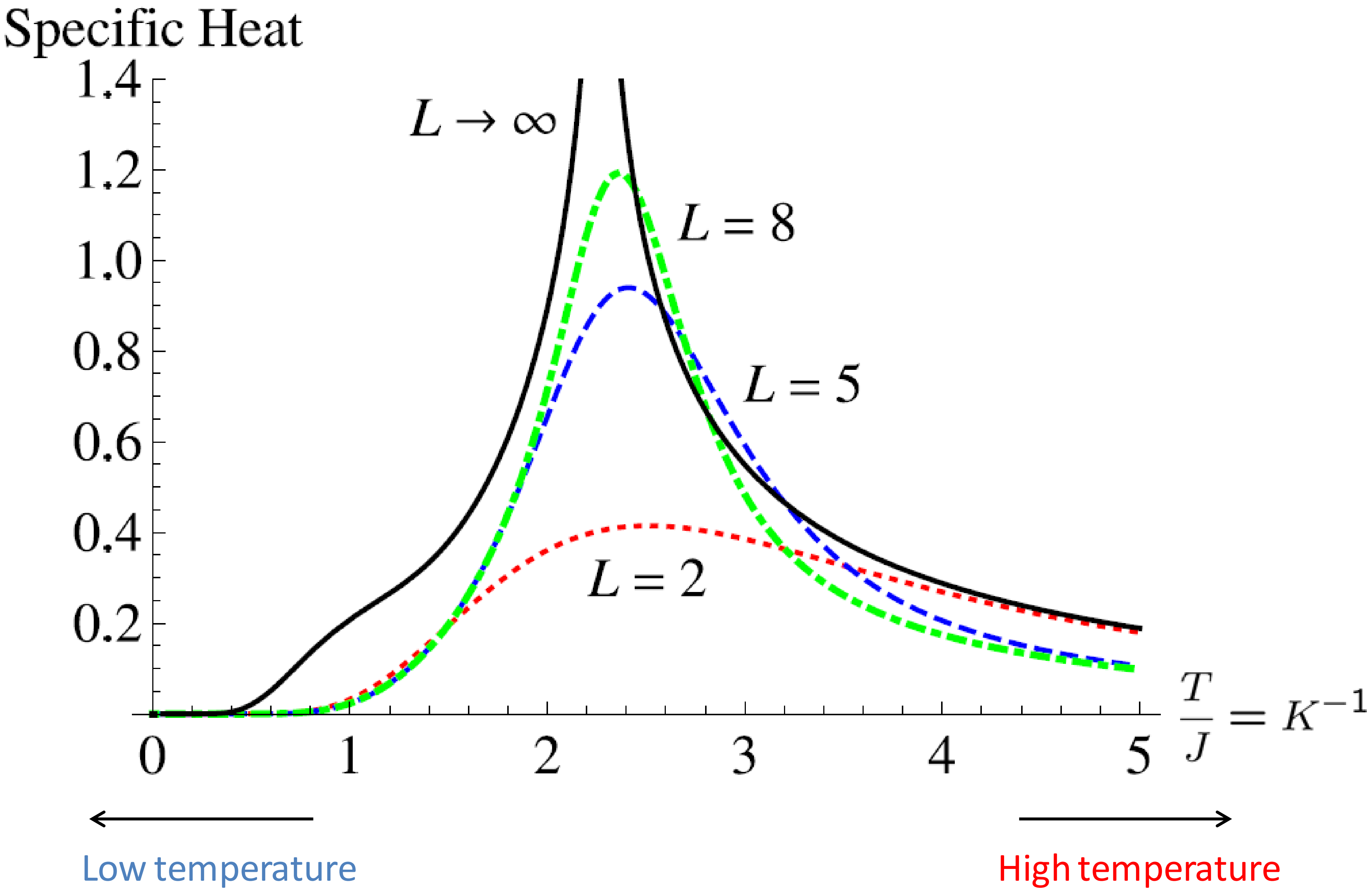
$$Z_L(K) = \sum_{\{\sigma_x\}=\pm 1} e^{K\sum_{(x,y)}\sigma_x\sigma_y} \qquad \left(K = \frac{J}{T}\right)$$
$$= \frac{1}{2}\left(S_{11}(K) + 2S_{10}(K) - S_{00}(K)\right)$$

$$\left(S_{\sigma_1\sigma_2}(K) = 2^{L^2} \prod_{p,q=0}^{L-1} \left[\cosh^2(2K) - \sinh(2K)(\cos\frac{(2p+\sigma_1)\pi}{L} + \cos\frac{(2q+\sigma_2)\pi}{L})\right]^{\frac{1}{2}} \right)$$

Specific heat:

$$K^2 C_L(K) \qquad C_L(K) = \frac{1}{L^2}\frac{\partial^2}{\partial K^2}\log Z_L(K),$$

We can also compute low temperature (large K) & high temperature (small K) expansions (even if we didn't know the exact result)



Specific Heat
1.4
1.2
1.0
0.8
0.6
0.4
0.2
$L \to \infty$
$L = 8$
$L = 5$
$L = 2$
0
1
2
3
4
5
$\frac{T}{J} = K^{-1}$
Low temperature
High temperature

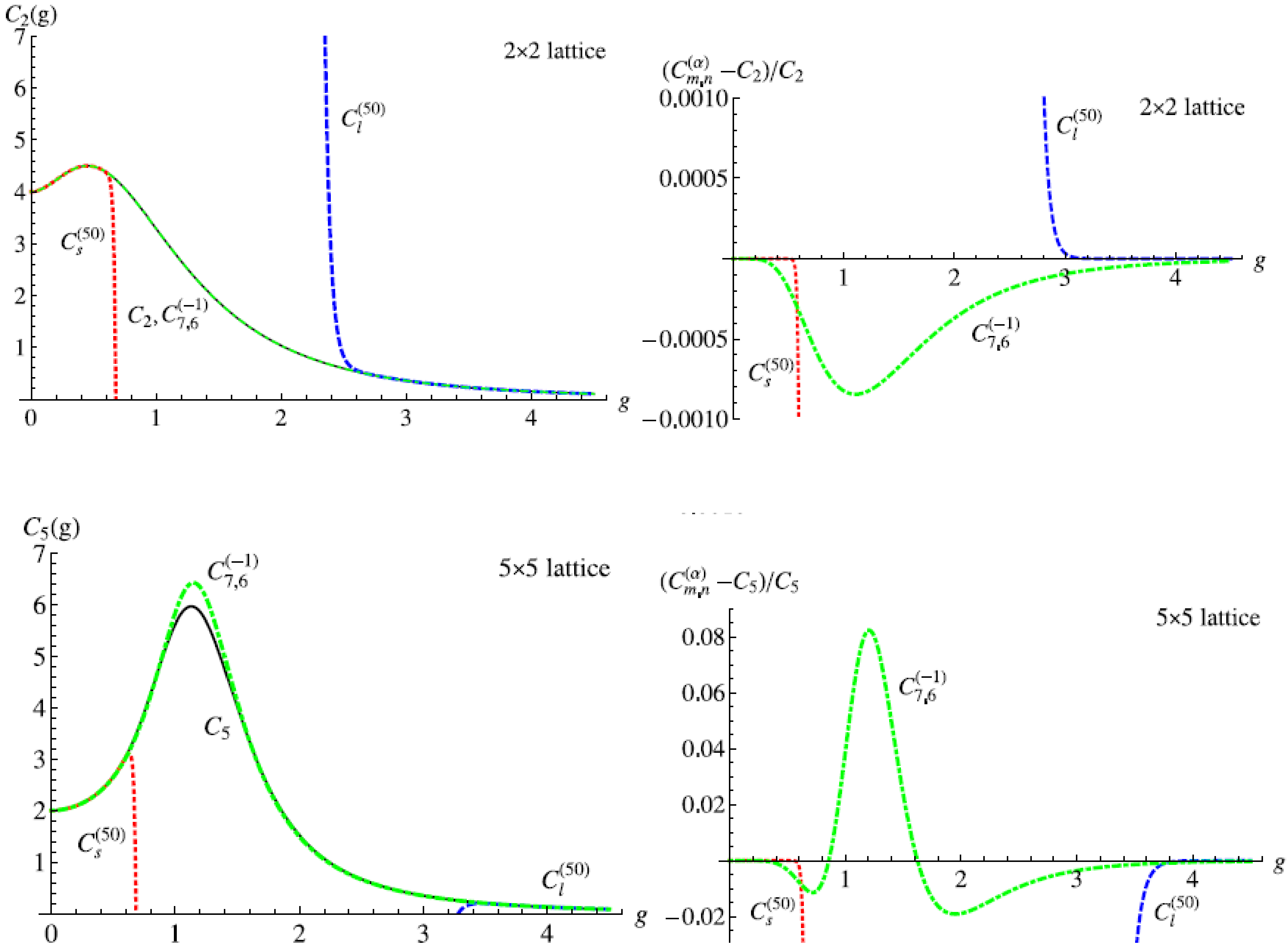
Construction of interpolating function

(Small-K expansion) = (power series of K)
(Large-K expansion) = (power series of e^{-K})

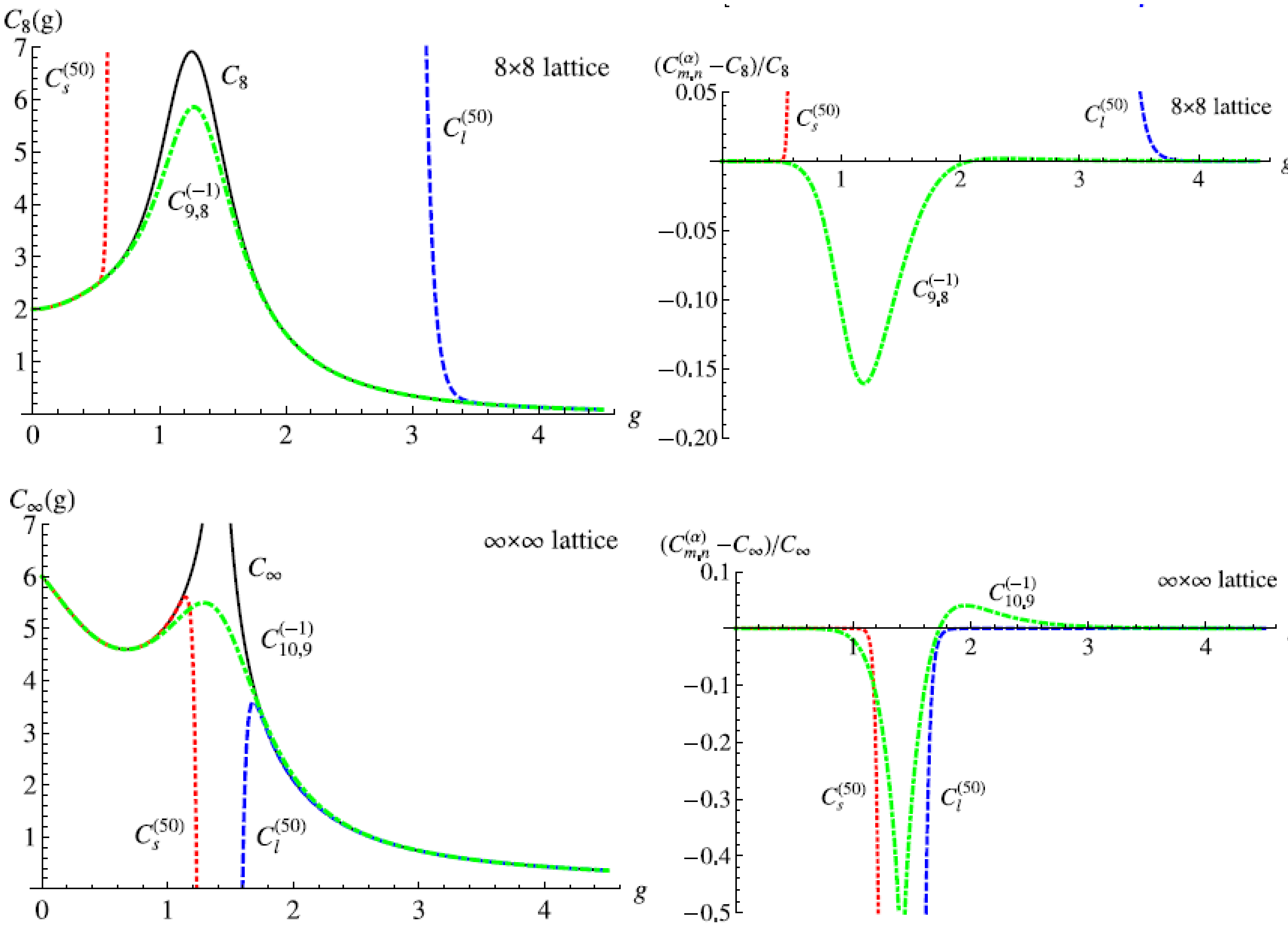
⇓ $e^{2K} = 1 + g$

(High temperature expansion) = (power series of g)
(Low temperature expansion) = (power series of g^{-1})

We construct interpolating functions in terms of g



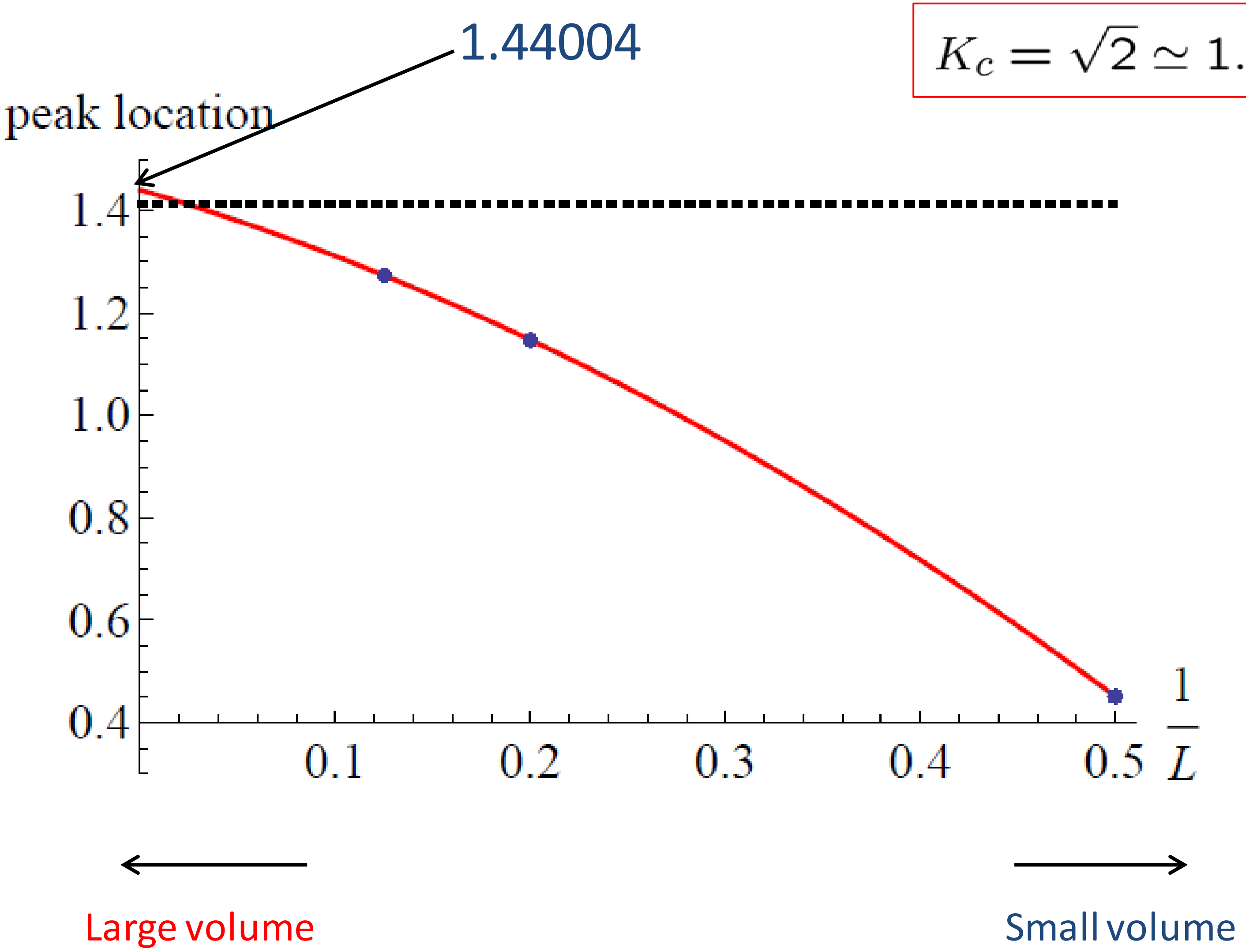
$C_2(g)$
2×2 lattice
$C_l^{(50)}$
$C_s^{(50)}$
$C_2, C_{7,6}^{(-1)}$
g
$(C_{m,n}^{(\alpha)} - C_2)/C_2$
2×2 lattice
$C_l^{(50)}$
$C_{7,6}^{(-1)}$
$C_s^{(50)}$
g
$C_5(g)$
5×5 lattice
$C_{7,6}^{(-1)}$
C_5
$C_s^{(50)}$
$C_l^{(50)}$
g
$(C_{m,n}^{(\alpha)} - C_5)/C_5$
5×5 lattice
$C_{7,6}^{(-1)}$
$C_s^{(50)}$
$C_l^{(50)}$
g



$C_8(g)$
8×8 lattice
C_8
$C_s^{(50)}$
$C_l^{(50)}$
$C_{9,8}^{(-1)}$
g
$(C_{m,n}^{(\alpha)} - C_8)/C_8$
$C_\infty(g)$
∞×∞ lattice
C_∞
$C_{10,9}^{(-1)}$
$(C_{m,n}^{(\alpha)} - C_\infty)/C_\infty$

Critical point from interpolating function?

$$K_c = \sqrt{2} \simeq 1.4142$$



Implications of analytic structures of interpolating function

[MH-Jatkar '15]

Partition function of 0d ϕ^4 theory

$$F(g) = \frac{1}{\sqrt{g}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2g} - x^4},$$

$$= \begin{cases} \frac{\pi e^{\frac{1}{32g^2}}}{4g} \left[I_{-\frac{1}{4}}\left(\frac{1}{32g^2}\right) - I_{\frac{1}{4}}\left(\frac{1}{32g^2}\right) \right] & \text{for } \mathrm{Re}(g) > 0 \\ \frac{\pi e^{\frac{1}{32g^2}}}{4\sqrt{-g^2}} \left[I_{-\frac{1}{4}}\left(\frac{1}{32g^2}\right) + I_{\frac{1}{4}}\left(\frac{1}{32g^2}\right) \right] & \text{for } \mathrm{Re}(g) \le 0 \end{cases},$$

Small-g expansion:

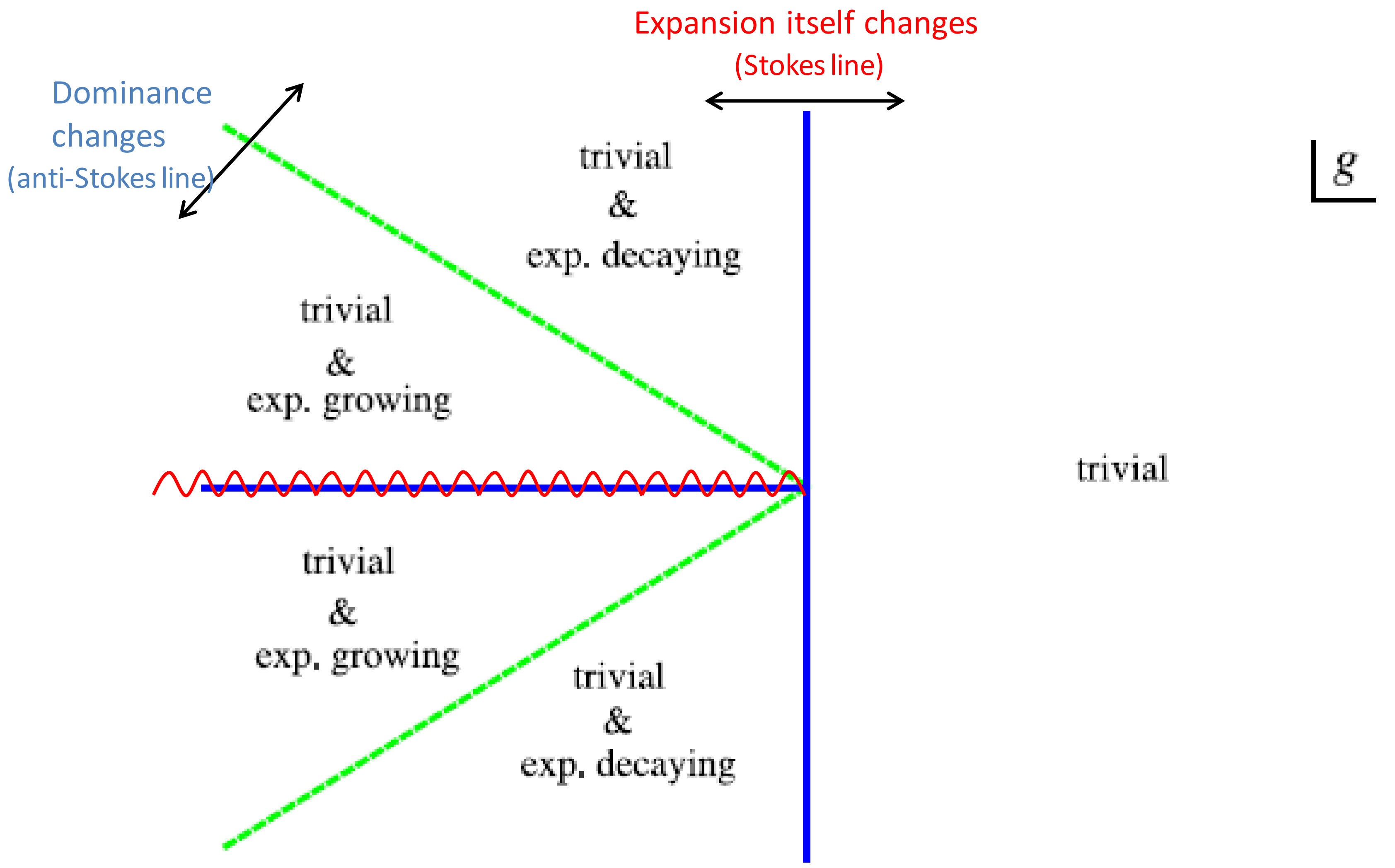
$$F(g) = \begin{cases} \sqrt{2\pi} - 3\sqrt{2\pi} g^2 + 105\sqrt{\frac{\pi}{2}} g^4 + \mathcal{O}(g^6) & \text{for } \arg(g) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \sqrt{\frac{\pi}{2}}\left(105g^4 - 6g^2 + 2 + \mathcal{O}(g^6)\right) - \sqrt{\pi} i e^{\frac{1}{16g^2}}\left(105g^4 + 6g^2 + 2 + \mathcal{O}(g^6)\right) & \text{for } \arg(g) \in (\frac{\pi}{2}, \pi) \\ \sqrt{\frac{\pi}{2}}\left(105g^4 - 6g^2 + 2 + \mathcal{O}(g^6)\right) + \sqrt{\pi} i e^{\frac{1}{16g^2}}\left(105g^4 + 6g^2 + 2 + \mathcal{O}(g^6)\right) & \text{for } \arg(g) \in (-\pi, -\frac{\pi}{2}) \end{cases}$$

Stokes Phenomena

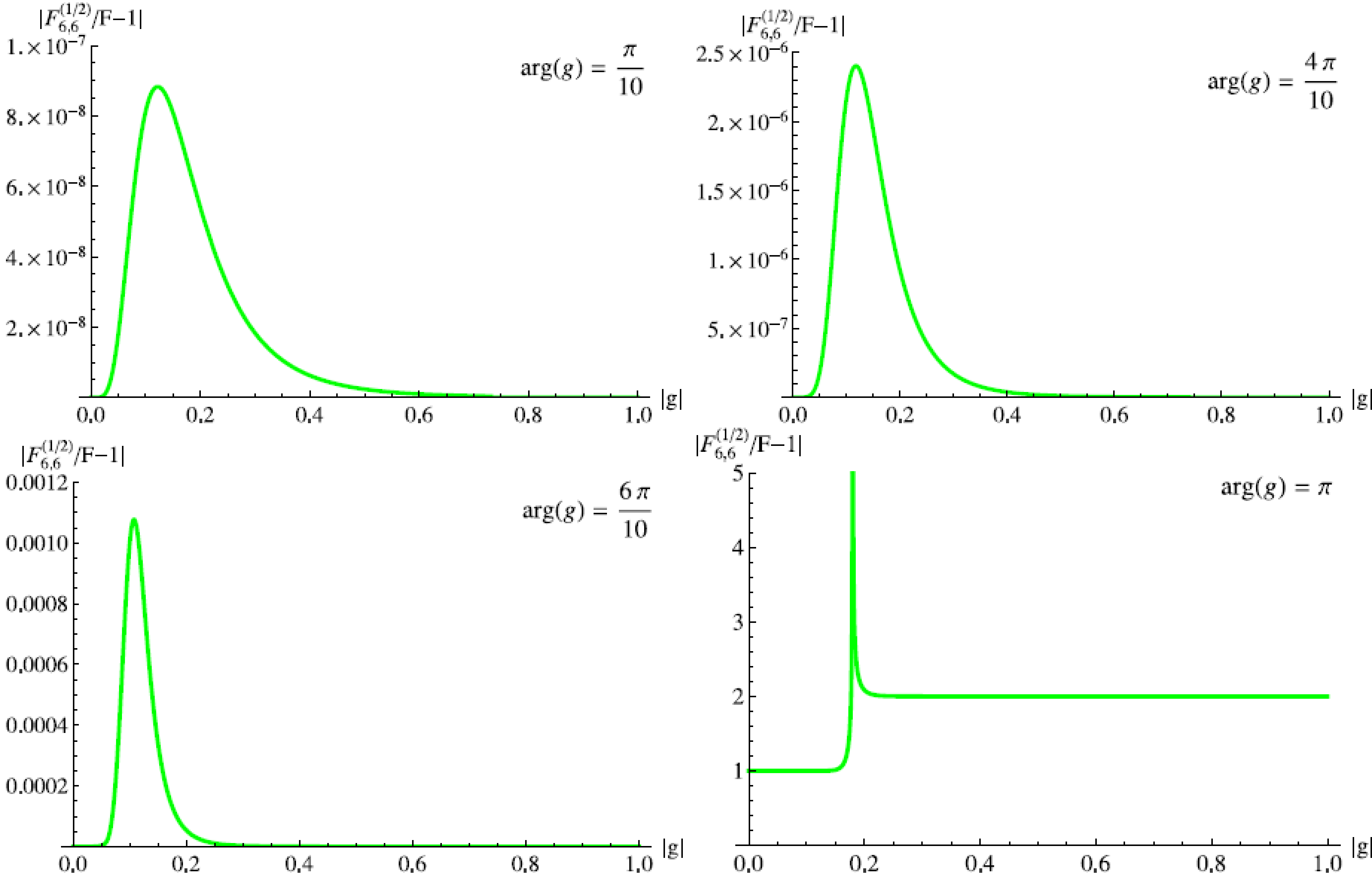
Large-g expansion:

$$F(g) = g^{-1/2}\left(\frac{1}{2}\Gamma(1/4) + \frac{1}{16}\Gamma(-1/4) g^{-1} + \frac{1}{64}\Gamma(1/4) g^{-2} + \mathcal{O}(g^{-3})\right)$$

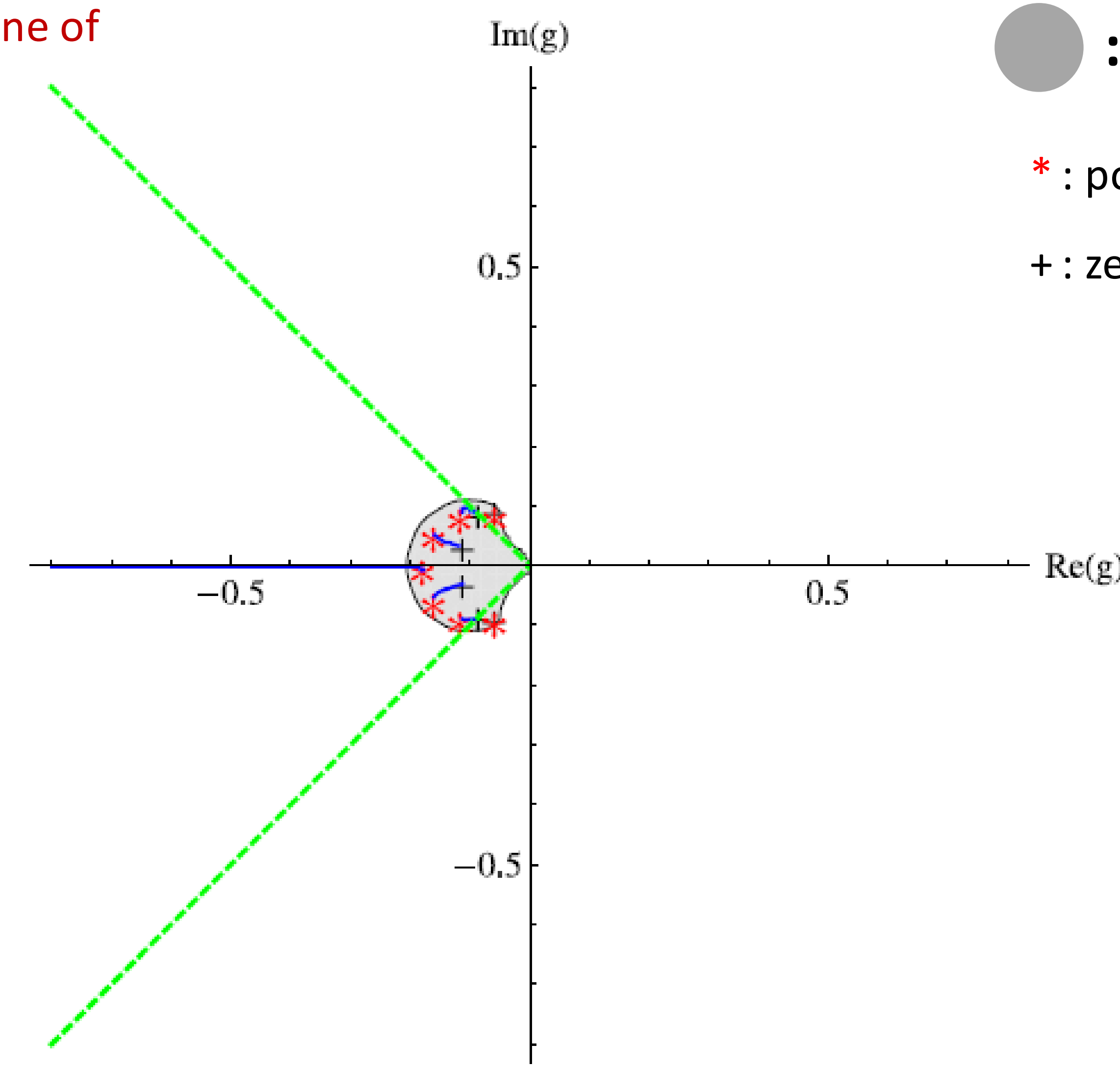
$$F(g) = \begin{cases} \sqrt{2\pi} - 3\sqrt{2\pi}g^2 + 105\sqrt{\frac{\pi}{2}}g^4 + \mathcal{O}(g^6) & \text{for } \arg(g) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \sqrt{\frac{\pi}{2}}\left(105g^4 - 6g^2 + 2 + \mathcal{O}(g^6)\right) - \sqrt{\pi}ie^{\frac{1}{16g^2}}\left(105g^4 + 6g^2 + 2 + \mathcal{O}(g^6)\right) & \text{for } \arg(g) \in (\frac{\pi}{2}, \pi) \\ \sqrt{\frac{\pi}{2}}\left(105g^4 - 6g^2 + 2 + \mathcal{O}(g^6)\right) + \sqrt{\pi}ie^{\frac{1}{16g^2}}\left(105g^4 + 6g^2 + 2 + \mathcal{O}(g^6)\right) & \text{for } \arg(g) \in (-\pi, -\frac{\pi}{2}) \end{cases}$$



$\left(F_{6,6}^{1/2}(g) : \text{ the best interpolating function along positive real axis} \right)$



Anti-Stokes line of small-g exp.



: (relative error) > 10%

* : poles of $(\mathrm{FPR})^{1/\alpha}$

+ : zeros of $(\mathrm{FPR})^{1/\alpha}$

Similar results hold also on 0d Sine-Gordon mode, anharmonic oscillator, circular Wilson loop in N=4 SYM, dimensions of operators in ABJM

Summary

Summary

- We have introduced a class of interpolating functions (FPR)

$$F_{m,n}^{(\alpha)}(g) = s_0 g^a \left[\frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k} \right]^{\alpha},$$

which includes Pade and FPP as the special cases

- "Landscape problem" of interpolating function

- Criterion to choose the best interpolating function
  (for a class of problems)

- Implications of analytic property of interpolating function

Results on which I didn't talk (due to time)

- Interpolating function in Borel plane? [MH-Jatkar '15]
  —— Naïve idea is failed.

- Comparison with resurgence approach [MH-Jatkar '15]

- Analytic property of FPR gives physical information on dimensions of twist operators in the planar ABJM [Chowdhury-MH, to appear]

- S-duality invariant interpolating function for twist op. in N=4 SYM
  [Chowdhury-MH-Thakur, to appear] [Generalization of Beem- Rastelli-Sen-van Rees, Alday-Bissi]

$$F_m^{(s,\alpha)}(\tau) = \left[\frac{\sum_{k=1}^{p} c_k E_{s+k}(\tau)}{\sum_{k=1}^{q} d_k E_{s+k}(\tau)} \right]^\alpha \quad E_s(\tau) = \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \frac{1}{|m+n\tau|^{2s}} (\mathrm{Im}\tau)^s. \quad F_m^{(s,\alpha)}\left(\frac{a\tau+b}{c\tau+d}\right) = F_m^{(s,\alpha)}(\tau)$$

⟶ Compare with conformal bootstrap and draw conformal manifold

- Applications to W-loop in N=4 SYM, free energy of ABJM theory, etc…

Thanks!

Appendix

Analytic property of interpolating function
&
Twist operators in planar ABJM

[Chowdhury-MH, to appear]

Twist-operators in ABJM

ABJM theory: [Aharony-Bergman-Jafferis-Maldacena '08]

3d $\mathcal{N}=6$ U(N)$_k$ x U(N)$_{-k}$ (k: CS level)

superconformal Chern-Simons theory

$$\mathcal{O}_{L,S} = \mathrm{Tr}\left[D_+^S (Y^1 Y_4^\dagger)^L\right]$$

$Y^1, Y_4^\dagger$: (anti-)bi-fundamental scalar

The dimension of this operator is anomalous (unless S=0):

$$\Delta_{L,S}(k,N) = L + S + \gamma_{L,S}(k,N)$$

Here we focus on the planar limit:

$$\Delta_{L,S}(k,N) = \Delta^{(0)}_{L,S}(\lambda) + \mathcal{O}(N^{-2}) \qquad \lambda = \frac{N}{k}$$

Dressed coupling constant h(λ)

In the context of integrability analysis,

the dimension is described in terms of an unknown function h(λ).

[Giombi-Gaiotto-Yin]

$h(\lambda) \propto$ (Central charge of SU(2|2) sub-superconfomal algebra)

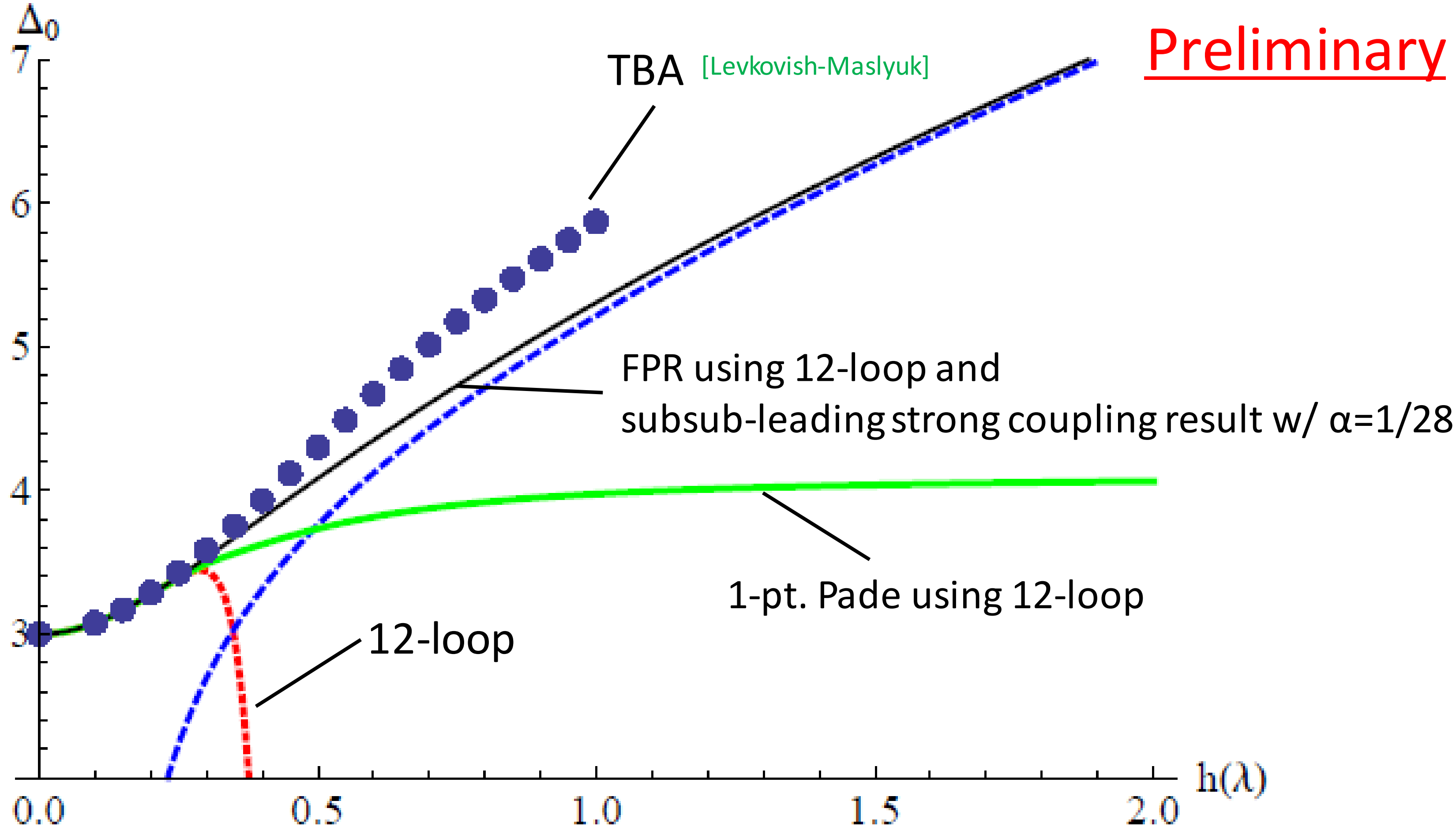
$$h(\lambda) = \lambda + \mathcal{O}(\lambda^3) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(\lambda^0)$$

Recently, exact form of h(λ) has been conjectured as [Gromov-Sizov]

$$\lambda = \frac{\sinh(2\pi h)}{2\pi}\, {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right).$$

Comparison with TBA for (L,S)=(1,2)

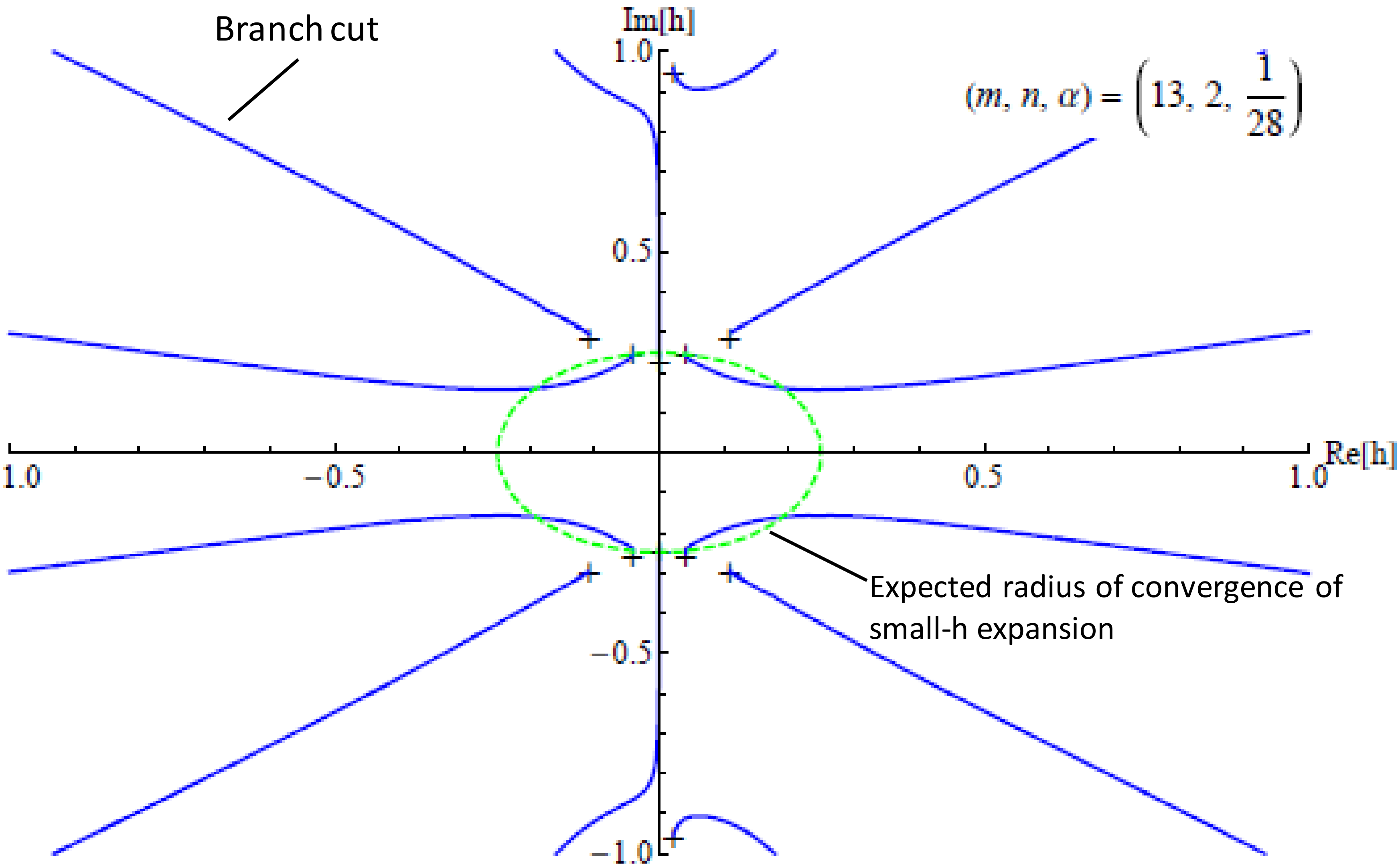
For both weak and strong coupling expansions, we use results obtained by integrability technique → Work in h(λ) rather than λ

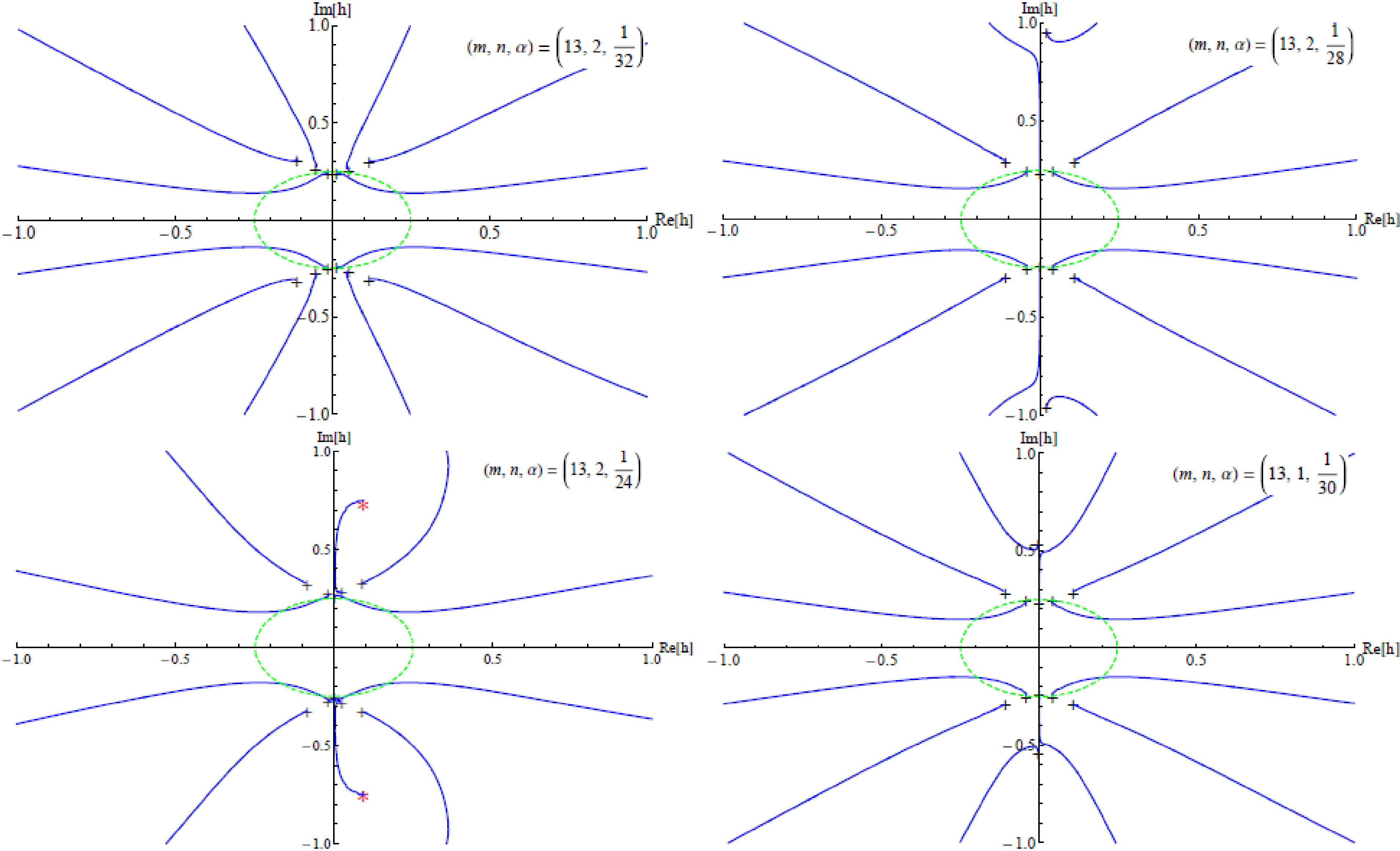


(Similar results hold for many other interpolating functions)

Analytic property of FPR for (L,S)=(1,2)

Preliminary





Many interpolating functions have singularities around h=±i/4 !!

Similar results hold also for other (L,S).

Physical Interpretation

Many interpolating functions have singularity around h=±i/4.

⟶ Natural to think this as the prediction.

If the conjecture $\lambda = \frac{\sinh(2\pi h)}{2\pi}\, {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right).$ is correct,

then the singularity is around $\lambda = -\frac{2iK}{\pi^2} \simeq -0.186i$

Surprisingly,

this is exactly the same as the critical point of S^3 free energy of ABJM

(where ABJM free energy behaves as the one of c=1 non-critical string.) [Drukker-Marino-Putrov]

⇩

Indirect evidence for the conjecture on h(λ)