Entanglement Entropy and Modular Invariance

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[Modular Invariance and Entanglement Entropy",
Sagar Lokhande and Sunil Mukhi, arXiv: 1504.01921]

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Outline

Introduction

- We will be interested in a quantity called entanglement entropy.
- Consider a Hilbert space divided into two parts. $\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$. If ρ is any density matrix on $\mathcal{H}$, then let

$$\rho_A = \mathrm{tr}_B\, \rho$$

  This is the reduced density matrix on subsystem A.
- The entanglement entropy is defined as:

$$S_A = -\mathrm{tr}\rho_A \log \rho_A$$

- If $\rho_A = \mathrm{diag}\left(\lambda_A^{(1)}, \lambda_A^{(2)}, \cdots, \lambda_A^{(N)}\right)$ then:

$$S_A = -\sum_{i=1}^{N} \left(\lambda_A^{(i)} \log \lambda_A^{(i)}\right)$$

  so $\lambda = 0, 1$ do not contribute – as desired.

- Entanglement entropy can be hard to compute, partly because of the log in the definition.
- A related measure called the Rényi entropy is defined as:

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{tr} \, (\rho_A)^n$$

where n is an integer ≥ 2. This is easier to compute by taking n copies of the theory ("replica trick") that works for free fields.

- If we can analytically continue to arbitrary real values of n then we can obtain the entanglement entropy from this:

$$S_A = \lim_{n \to 1} S_A^{(n)}$$

- The Rényi entropy can be computed by expressing the trace as:

$$\operatorname{tr}(\rho_A)^n = \frac{Z_n}{(Z_1)^n}$$

- Here, Z_1 is the ordinary partition function of the theory and Z_n, called the "replica partition function", is obtained via a "replica trick" as we will shortly discuss.
- We will study entanglement in conformal field theory (CFT) in two dimensions.
- We work at finite temperature and finite size. Then the two dimensions form a (Euclidean) torus: one axis is the size of the system L and the other is the inverse temperature β.

Outline

Basic results on entanglement and CFT

- We will consider real-space entanglement in a CFT of central charge c. Partition the 1d space into an interval of length ℓ and the rest, called respectively A and B.
- If the total space is infinite and we work at zero temperature, it is a now-celebrated result that:

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + c'$$

where c is the central charge, a is a UV cutoff and c' is a non-universal constant. Thus in this case, the entropy only depends on the central charge.

- At finite temperature $T = (\beta)^{-1}$ the original density matrix is thermal (rather than a pure state) and the entanglement entropy changes to:

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + c'$$

- At zero temperature but in a finite spatial region of size L,

$$S_A = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c'$$

- Notice that the above formulae are interchanged under the modular transformation $\beta \leftrightarrow L, \ell \to i\ell$.
- Calculations are much more difficult when there are several entangling intervals. The case of finite space and finite temperature is difficult even for a single interval. Here, the answer is expressed in terms of Jacobi θ-functions.

- Now we consider free fermion CFT.
- Boundary conditions on a torus of sides L, β:

$$\psi(z+L) = \pm\psi(z)$$
$$\psi(z+i\beta) = \pm\psi(z)$$

- With these boundary conditions, denote the path integral by $Z_{\pm\pm}(L,\beta)$ and the Hamiltonian by $H_\pm(L)$. Then:

$$Z_{--} = \mathrm{tr} e^{-\beta H_-} \qquad Z_{+-} = \mathrm{tr} e^{-\beta H_+}$$
$$Z_{-+} = \mathrm{tr}(-1)^F e^{-\beta H_-} \qquad Z_{++} = \mathrm{tr}(-1)^F e^{-\beta H_+}$$

- Let $\tau = i\frac{\beta}{L}$. Then only Z_{++} is invariant under modular transformations:

$$\tau \to \tau + 1, \quad \tau \to -\frac{1}{\tau}$$

while the other three are permuted. However, Z_{++} is not a physical thermal ensemble (periodic in time). Also it vanishes.

- As shown long ago by Seiberg and Witten, the combination:

$$Z(L,\beta) = \tfrac{1}{2}(Z_{--} + Z_{-+} + Z_{+-} + Z_{++})$$
$$= \mathrm{tr}\left(\frac{1+(-1)^F}{2}\right) e^{-\beta H_-} + \mathrm{tr}\left(\frac{1+(-1)^F}{2}\right) e^{-\beta H_+}$$

  is modular-invariant. It is a physical thermal ensemble, being a sum over the projected spectra of two Hamiltonians H_+, H_-.

- For a Dirac fermion ($c = 1$), by direct computation we find:

$$Z_{--} = \left|\frac{\theta_3(0|\tau)}{\eta(\tau)}\right|^2 \qquad Z_{+-} = \left|\frac{\theta_2(0|\tau)}{\eta(\tau)}\right|^2$$
$$Z_{-+} = \left|\frac{\theta_4(0|\tau)}{\eta(\tau)}\right|^2 \qquad Z_{++} = \left|\frac{\theta_1(0|\tau)}{\eta(\tau)}\right|^2 = 0$$

- The modular-invariant partition function of the free Dirac fermion is therefore:

$$Z_{\mathrm{Dirac}} = \frac{1}{2}\sum_{\nu=2,3,4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^2$$

- Next consider a free boson $\phi(z, \bar{z})$ that takes a compact set of values:

$$\phi(z, \bar{z}) \sim \phi(z, \bar{z}) + 2\pi R$$

  This also has $c = 1$.

- Its partition function is easily computed:

$$Z_{\text{boson}}(R) = \sum_{e,m \in \mathbb{Z}} q^{\left(\frac{e}{R} + \frac{mR}{2}\right)^2} \bar{q}^{\left(\frac{e}{R} - \frac{mR}{2}\right)^2}$$

  where $q = e^{i\pi\tau}$.

- The statement of Bose-Fermi duality at $c = 1$ is then:

$$Z_{\text{Dirac}} = Z_{\text{boson}}(R = 1)$$

  Notice that this holds only with the spin-structure-summed partition function on the LHS.

- With multiple fermions one can have multiple theories depending on whether the spin structures are mutually correlated or not.
- For example with 2 Dirac fermions having uncorrelated spin structures, the partition function is:

$$Z^{\mathsf{u}}_{\text{Two Dirac}} = \left(\frac{1}{2} \sum_{\nu=2,3,4} \left| \frac{\theta_\nu(0|\tau)}{\eta(\tau)} \right|^2 \right)^2$$

- However if the spin structures of the two fermions are correlated then the partition function is:

$$Z^{\mathsf{c}}_{\text{Two Dirac}} = \frac{1}{2} \sum_{\nu=2,3,4} \left| \frac{\theta_\nu(0|\tau)}{\eta(\tau)} \right|^4$$

- The two theories have very different spectra and correlation functions. In particular the latter theory is not the direct sum of two CFT's.

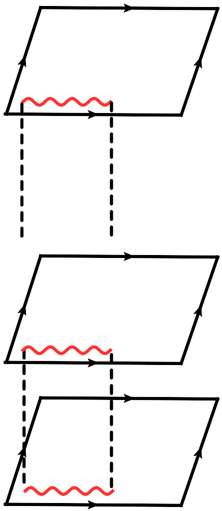
Outline

Replica partition function

- The Rényi entropy can be expressed in terms of a quantity called the "replica partition function":

$$\mathrm{tr}(\rho_A)^n = \frac{Z_n}{(Z_1)^n}$$

  where Z_1 is the ordinary partition function.
- To compute Z_n one extends the original torus to an n-fold cover joined at branch cuts along spatial intervals from 0 to ℓ.



- The quantity $(\rho_A)^n$ is created by gluing the copies together.
- Let $\tilde{\psi}_k$ be the field on the kth replica. An operator called the twist field sends each field to the next replica:

$$\sigma_k : \tilde{\psi}_k \to \tilde{\psi}_{k+1}$$

- By a suitable diagonalisation of the problem, one reduces the problem to a set of fields ψ_k on a single copy of the space. The twist field acts on each one by a phase:

$$\sigma_k : \psi_k \to \omega^k \psi_k$$

where $\omega = e^{2\pi i/n}$ and $k = -\frac{n-1}{2}, -\frac{n-1}{2} + 1, \cdots, \frac{n-1}{2}$.

- This is achieved if the OPE of the twist field and the fundamental fermion is of the form:

$$\sigma_k(z, \bar{z})\psi(w) \sim (z - w)^{\frac{k}{n}} \tag{1}$$

- The conformal dimensions Δ_k of the twist fields can be shown to satisfy:

$$\sum_k \Delta_k = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

- One can then show that:

$$\mathrm{tr}\, \rho_A^n = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \sigma_k(\ell, \ell)\, \sigma_{-k}(0, 0)\rangle$$

- It is convenient to use the un-normalised correlators to define the "replica partition function":

$$Z_n = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} Z_1 \langle \sigma_k(\ell,\ell)\, \sigma_{-k}(0,0) \rangle = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle\langle \sigma_k(\ell,\ell)\, \sigma_{-k}(0,0) \rangle\rangle$$

where Z_1 is the ordinary partition function. Then:

$$\mathrm{tr}\, \rho_A^n = \frac{Z_n}{Z_1^n}$$

from which the Rényi entropies are easily obtained.

- Notice that $Z_{n=1} = \langle\langle 1 \rangle\rangle = Z_1$ so our notation is consistent.

Outline

Free fermion and spin structures

- Consider a Dirac fermion with a single entangling interval of length ℓ.
- This theory has $c = 1$. Denote the complex Dirac fermion as $D(z)$.
- The first calculation of a finite-size, finite-temperature replica partition function was performed by [Azeyanagi, Nishioka, Takayanagi]. They identified the twist field by bosonisation.
- At $R = 1$ the physical vertex operators are:

$$\mathcal{O}_{e,m} = e^{i\left(e+\frac{m}{2}\right)\phi(z)}e^{i\left(e-\frac{m}{2}\right)\bar{\phi}(\bar{z})}$$

with $(\Delta_{e,m}, \bar{\Delta}_{e,m}) = \left(\frac{1}{2}\left(e+\frac{m}{2}\right)^2, \frac{1}{2}\left(e-\frac{m}{2}\right)^2\right)$.

- The fermion $D(z) \ \sim \ e^{i\phi(z)}$.

- The fermionic twist field is identified as:

$$\sigma_k = \mathcal{O}_{0,\frac{2k}{n}}, \qquad k = -\tfrac{n-1}{2}, \dots \tfrac{n-1}{2}$$

- These operators have $(\Delta, \bar{\Delta}) = (\frac{k^2}{2n^2}, \frac{k^2}{2n^2})$. They are nonlocal operators with the OPE:

$$\mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\, D(w) \sim (z-w)^{\frac{k}{n}}$$

as desired.

- A standard computation now gives:

$$\langle\langle \mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z}) \mathcal{O}_{0,-\frac{2k}{n}}(0) \rangle\rangle = \left| \frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)} \right|^{\frac{2k^2}{n^2}} \times \frac{1}{2} \frac{\sum_{\nu=1}^{4} |\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- [Azeyanagi et al] restricted to a specific spin structure, to get:

$$\langle\langle \mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\mathcal{O}_{0,-\frac{2k}{n}}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \times \frac{|\theta_3(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

(recall that θ_3 corresponds to $(--)$ boundary conditions).

- Taking the product over replicas they got:

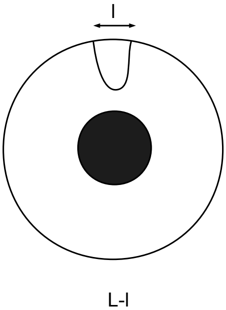
$$Z_n(L,\beta;\ell) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{|\theta_3(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- To get $\mathrm{tr}\rho_A^n$, this has to be divided by:

$$(Z_1)^n = \left|\frac{\theta_3(0|\tau)}{\eta(\tau)}\right|^{2n}$$

- The result satisfies some important consistency conditions (as we will see), but is clearly not modular invariant.

- In the [Ryu-Takayanagi] proposal, entanglement entropy is dual to the length of a minimal line coming in from the boundary of a 3d bulk spacetime whose boundary is the CFT torus.



- After a modular transformation, the CFT entanglement changes (because the spin structure changes). However, in general the bulk spacetime also changes. So perhaps this is a consistent picture.

- On the other hand, studies of the Euclidean AdS_3/CFT_2 correspondence ([Dijkgraaf-Maldacena-Moore-Verlinde], [Manschot-Moore]) indicate the relation:

$$Z_{CFT}(\tau) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)} Z_{grav}\left(\frac{a\tau + b}{c\tau + d}\right)$$

  where on the LHS we have the modular-invariant partition function of the CFT. Due to the sum, the RHS is also modular-invariant.

- In the same spirit we may expect replica partition functions to be modular invariant.
- Also, only modular-invariant entanglement can satisfy the Bose-Fermi correspondence as stressed by [Headrick, Lawrence,Roberts].

- With this motivation, we return to the spin-structure summed expression:

$$\langle\langle \mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\mathcal{O}_{0,-\frac{2k}{n}}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \times \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- Now we must decide how to take the product over replicas.
- One way would be to just take the product of the above result over all k.
- Thus, the spin structures are summed over before we carry out replication, leading to the "uncorrelated replica partition function":

$$Z_n^{\mathrm{u}}(L,\beta;\ell) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- There is another way to take the product, which is to take the product over replicas before summing over spin structures.
- This leads to the "correlated replica partition function":

$$Z_n^{\mathsf{c}}(L,\beta;\ell) = \frac{1}{2}\left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \sum_{\nu=1}^{4} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- Notice that the two types of replica partition functions coincide at $n=1$:

$$Z_1^{\mathsf{u}} = Z_1^{\mathsf{c}} = Z_1 = \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(0|\tau)|^2}{|\eta(\tau)|^2}$$

which is the ordinary modular-invariant partition function.

- We also observe that as $\ell \to 0$ the two types of partition function are quite different:

$$Z_n^{\mathsf{U}}(L, \beta; \ell \to 0) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \left(\frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(0|\tau)|^2}{|\eta(\tau)|^2}\right)^n$$

$$Z_n^{\mathsf{C}}(L, \beta; \ell \to 0) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(0|\tau)|^{2n}}{|\eta(\tau)|^{2n}}$$

- The second factors in the two cases are the ordinary partition functions of n Dirac fermions with, respectively, uncorrelated and correlated spin structures.

- Corresponding to two possible replica partitions, we can define two possible Rényi entropies:

$$S_n^{\mathrm{u}} = \frac{1}{1-n} \log \frac{Z_n^{\mathrm{u}}}{(Z_1)^n}$$
$$S_n^{\mathrm{c}} = \frac{1}{1-n} \log \frac{Z_n^{\mathrm{c}}}{(Z_1)^n}$$

- The denominators are the same because, as we pointed out earlier, the two types of partition functions coincide at $n = 1$.
- Before deciding which one is right, let us check the modular transformation properties of the two quantities Z_n^{u} and Z_n^{c}.

- The modular transformation $\tau \to \tau + 1$ permutes $\theta_3 \leftrightarrow \theta_4$ and $\theta_1 \leftrightarrow \theta_2$. It also induces phases, but there are modulus signs everywhere. Thus both expressions are invariant under it.
- The other transformation $\tau \to -\frac{1}{\tau}$ acts as $\beta \leftrightarrow L$ and $\ell \to i\ell$ (we have used the identification $\tau = i\tau_2 = i\frac{\beta}{L}$ and $z = \frac{\ell}{L}$).
- For this we use:

$$\theta_{\alpha\beta}\left(\frac{z}{\tau}\middle| -\frac{1}{\tau}\right) = (-i)^{\alpha\beta}(-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_{\beta\alpha}(z,\tau)$$

- Applying this to Z_n^{u} or Z_n^{c}, one finds that they pick up the same multiplicative factor:

$$Z_n^{\mathsf{u,c}}(\beta, L; i\ell) = \left(\frac{\beta}{L}\right)^{\frac{1}{6}\left(n-\frac{1}{n}\right)} Z_n^{\mathsf{u,c}}(L, \beta; \ell)$$

- We see that even after summing over spin structures, the replica partitions acquire a multiplicative pre-factor under modular transformations.
- This factor vanishes at $n = 1$, so Z_1 is indeed modular invariant as it must be.
- We propose that for every CFT of central charge c, the following result holds:

$$Z_n(\beta, L; i\ell) = \left(\frac{\beta}{L}\right)^{\frac{c}{6}\left(n-\frac{1}{n}\right)} Z_n(L, \beta; \ell)$$

  and will verify this in all known cases.
- As a result the Rényi and entanglement entropies shift by an additive term. Notice that the term is independent of the entangling interval ℓ.

- We can make the replica partition functions invariant by multiplying them by a factor:

$$\tilde{Z}_n = \left(\frac{\beta}{L}\right)^{\frac{c}{12}\left(n-\frac{1}{n}\right)} Z_n$$

- Our conjecture implies that the above quantity is modular invariant for every CFT.
- Returning now to the Dirac fermion, we have two possible modular-invariant Rényi entropies:

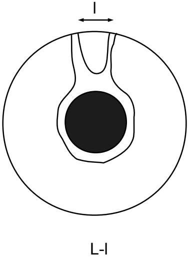
$$\tilde{S}_n^{\mathrm{u,c}} = \frac{1}{1-n} \log \frac{\tilde{Z}_n^{\mathrm{u,c}}}{(Z_1)^n}$$

The corresponding entanglement entropies obtained by taking $n \to 1$ will also be modular invariant.

Outline

Thermal entropy relation

- To decide which replica partition function is correct, we use the thermal entropy relation [Azeyanagi, Nishioka, Takayanagi].



- This arises because, with a black hole in the bulk, the minimal surfaces with boundary ℓ and $L - \ell$ are not the same.
- As $\ell \to 0$ the difference is the surface wrapping the black hole, which gives the thermal entropy of the CFT state. Hence:

$$\lim_{\ell \to 0} \left(S_A(L-\ell) - S_A(\ell) \right) = S_{\text{thermal}}(\beta)$$

- Indeed, within CFT it has been argued [Cardy, Herzog], [Chen,Wu] that one must have:

$$\lim_{\ell\to 0} Z_n(L,\beta;\ell) = \left(\frac{\ell}{L}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} (Z_1(L,\beta))^n$$

$$\lim_{\ell\to 0} Z_n(L,\beta;L-\ell) = \left(\frac{\ell}{L}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} Z_1(L,n\beta)$$

- The intuition for this is that the replicas are connected through the branch cut of the entangling interval.
- For a small interval the replicas are effectively decoupled, so one finds n independent copies of the ordinary partition function. On the other hand for a large entangling interval one always goes from one replica to the next so the replicas are effectively "joined" into a single torus of n times the height.
- However the above are not just intuitive statements, but have been justified by formal manipulations in CFT.

- The above statements, if true, immediately imply the thermal entropy relation:

$$
\begin{aligned}
\lim_{\ell\to 0}\left(S_A(L-\ell)-S_A(\ell)\right) &= \lim_{\ell\to 0}\lim_{n\to 1}\frac{1}{1-n}\log\frac{Z_n(L,\beta;L-\ell)}{Z_n(L,\beta;\ell)} \\
&= \lim_{n\to 1}\frac{1}{1-n}\log\frac{Z_1(L,n\beta)}{(Z_1(L,\beta))^n} \\
&= \log Z_1\left(\frac{\beta}{L}\right)-\frac{\beta}{L}\frac{Z_1'(\frac{\beta}{L})}{Z_1(\frac{\beta}{L})} \\
&= Z_{\text{thermal}}
\end{aligned}
$$

- An extra assumption is that the limits $\ell \to 0$ and $n \to 1$ can be commuted.
- We will subject our two candidate replica partition functions to these conditions and find a surprising result.

- First consider the limit $\ell \to 0$. We have already seen that in this limit:

$$Z_n^{\mathsf{u}}(L,\beta;\ell) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \left(\frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(0|\tau)|^2}{|\eta(\tau)|^2}\right)^n$$

$$Z_n^{\mathsf{c}}(L,\beta;\ell) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(0|\tau)|^{2n}}{|\eta(\tau)|^{2n}}$$

- As $n \to 1$ the first factor becomes 1. The second factor becomes $(Z_1)^n$ only for Z_n^{u} and not for Z_n^{c}.
- Recall that the former ("uncorrelated") replica partition function involved summing over spin structures before taking the product over replicas.

- Now we consider these quantities as functions of $L-\ell$ in the limit $\ell \to 0$. This time we find:

$$Z_n^{\mathsf{u}}(L,\beta;L-\ell) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(\frac{k}{n}|\tau)|^2}{|\eta(\tau)|^2}$$

$$Z_n^{\mathsf{c}}(L,\beta;L-\ell) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \frac{1}{2}\sum_{\nu=1}^{4}\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{|\theta_\nu(\frac{k}{n}|\tau)|^2}{|\eta(\tau)|^2}$$

- Focusing on the second factor, there is a beautiful θ-identity that allows us to evaluate the correlated case:

$$\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left|\theta_\nu\left(\frac{k}{n}-z\middle|\tau\right)\right| = \left(\prod_{p=1}^{\infty}\left|\frac{(1-q^{2p})^n}{1-q^{2pn}}\right|\right)\left|\theta_\nu(nz|n\tau)\right|$$

- It follows easily that:

$$Z_n^{\mathsf{C}}(\ell \to L) = \frac{1}{2}\left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \sum_{\nu=1}^{4} \frac{|\theta_\nu(0|n\tau)|^2}{|\eta(n\tau)|^2}$$
$$= \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} Z_1(L, n\beta)$$

- This time it is the "correlated" replica partition function, where the sum over spin structures is taken after the product over replicas, that satisfies the desired relation.
- It is easy to check that, due to cross terms, the uncorrelated one does not satisfy any similar relation.
- To summarise: as $\ell \to 0$ the sum over spin structures must be performed before the product over replicas. As $\ell \to L$ it must be performed after the product over replicas.
- For intermediate values of ℓ it is not (yet) clear what is the prescription.

Outline

Free boson CFT

- For the free boson replica partition function, one considers a complex boson ($c = 2$) and twist fields $\mathcal{T}_k$ satisfying:

$$\mathcal{T}_k(z,\bar{z})\phi(w) \sim (z-w)^{\frac{k}{n}}$$

  and one has:

$$Z_n = \prod_{k=0}^{n-1} \langle\langle \mathcal{T}_k(z,\bar{z})\mathcal{T}_{-k}(0,0)\rangle\rangle$$

  At the end one can take a square root to get the $c = 1$ theory.
- This is more difficult than the fermion case. There, the twist field for fermions was explicit in the bosonic representation. Here it is implicit.
- This problem was studied by [Datta,David] and [Chen, Wu] using techniques developed many years ago for orbifold compactifications.

- There have been contradictory results in the literature, but the most convincing one is of the form:

$$Z_n(R) = Z_n^{(1)} Z_n^{(2)} Z_n^{(3)}(R)\, Z_n^{(3)}\left(\frac{2}{R}\right)$$

where:

$$Z^{(1)} = \frac{1}{|\eta(\tau)|^{2n}} \prod_{k=0}^{n-1} \frac{1}{|W_1^1(k,n;\frac{\ell}{L}|\tau)|}$$

$$Z^{(2)} = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)}$$

$$Z^{(3)}(R) = \sum_{m_j \in \mathbb{Z}} \exp\left(-\frac{\pi R^2}{2n}\sum_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right| \times \sum_{j,j'=0}^{n-1}\left[\cos 2\pi(j-j')\frac{k}{n}\right] m_j m_{j'}\right)$$

- Here $W_1^1(k,n;\frac{\ell}{L}|\tau)$ and $W_2^2(k,n;\frac{\ell}{L}|\tau)$ are integrals of the cut differentials over the different periods of the torus:

$$W_1^1 = \int_0^1 dz\ \theta_1(z|\tau)^{-\left(1-\frac{k}{n}\right)}\theta_1\left(z-\frac{\ell}{L}|\tau\right)^{-\frac{k}{n}}\theta_1\left(z-\frac{k\ell}{nL}|\tau\right)$$

$$W_2^2 = \int_0^{\bar{\tau}} d\bar{z}\ \bar{\theta}_1(\bar{z}|\tau)^{-\frac{k}{n}}\bar{\theta}_1\left(\bar{z}-\frac{\ell}{L}|\tau\right)^{-\left(1-\frac{k}{n}\right)}\bar{\theta}_1\left(\bar{z}-\left(1-\frac{k}{n}\right)\frac{\ell}{L}|\tau\right)$$

- We would now like to investigate the modular transformation of this expression. To this end, we note the following results:

$$\eta\Big(-\frac{1}{\tau}\Big) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$$

$$W_1^1\Big(k,n;\frac{i\ell}{\beta}|-\frac{1}{\tau}\Big) = \frac{1}{\tau}e^{-\frac{i\pi\ell^2}{L^2\tau}\frac{k}{n}\left(1-\frac{k}{n}\right)}W_2^2(k,n;\frac{\ell}{L}|\tau)$$

$$\frac{\theta_1'\left(0|-\frac{1}{\tau}\right)}{\theta_1\left(\frac{z}{\tau}|-\frac{1}{\tau}\right)} = i\tau e^{-\frac{i\pi z^2}{\tau}}\frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)}$$

- Next, performing a multi-variable Poisson resummation following [Chen,Wu], we find that:

$$Z^{(3)}\left(R;\frac{z}{\tau}\Big|-\frac{1}{\tau}\right)=\frac{2^{\frac{n}{2}}}{R^n}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right)Z^{(3)}\left(\frac{2}{R};z\Big|\tau\right)$$

$$Z^{(3)}\left(\frac{2}{R};\frac{z}{\tau}\Big|-\frac{1}{\tau}\right)=\frac{R^n}{2^{\frac{n}{2}}}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right)Z^{(3)}\left(R;z\Big|\tau\right)$$

- Thus the product transforms as:

$$Z^{(3)}(R)\,Z^{(3)}\left(\frac{2}{R}\right)\to\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|\right)Z^{(3)}(R)\,Z^{(3)}\left(\frac{2}{R}\right)$$

- Putting everything together, we find that:

$$Z_n\left(R;\frac{z}{\tau}\Big|-\frac{1}{\tau}\right)=|\tau|^{\frac{1}{6}\left(n-\frac{1}{n}\right)}Z_n(R;z|\tau)$$

Thus, it will become modular invariant precisely upon multiplying by the factor given in our proposal.

- Ideally one would like to compare the above with the free fermion result at $c = 1$ to verify Bose-Fermi duality.
- However the above result is extremely implicit and hard to compute. And on the fermion side, we don't know the replica partition function at intermediate values of ℓ.
- However, as $\ell \to 0$ and $\ell \to L$ the above expression has been evaluated by [Chen, Wu] and found to agree with the predictions $(Z_1(\tau))^n$ and $Z_1(n\tau)$ respectively.
- Since at $R = 1$, the function Z_1 is equal to the free Dirac fermion partition function, this means our results and theirs are in full agreement in the regions where they can be compared.

Outline

Multiple fermions and lattice bosons

- The theory of d free Dirac fermions with correlated spin structures is dual to a specific compactification of d free bosons on a target-space torus:

$$T^c = R^d / \Gamma_d$$

  where Γ_d is the root lattice of Spin$(2d)$.
- This can be achieved by starting with a rectangular torus and choosing a suitable constant metric and B-field.
- In this case the d different bosons are not orthogonal to each other, while the fermions have correlated spin structures, so on both sides of the Bose-Fermi duality we are dealing with CFT's that are not direct sums of simpler ones.

- In the free boson theory, let Λ_R be the root lattice and Λ_W be the dual weight lattice.
- Then the vertex operators are:

$$\mathcal{O}_{w^i,\bar{w}^i} = e^{iw^i\phi_i} e^{i\bar{w}^i\bar{\phi}_i}$$

where $w^i, \bar{w}^i \in \Lambda_W$ and $w^i - \bar{w}^i \in \Lambda_R$.

- Elements of the weight lattice can be parametrised as:

$$w^i = \frac{1}{\sqrt{2}} g^{ij} v_j, \qquad \bar{w}^i = \frac{1}{\sqrt{2}} g^{ij} \bar{v}_j$$

where $v_i, \bar{v}_i$ are integers and g^{ij} is the inverse of g_{ij} which is the half the Cartan matrix of $\mathrm{Spin(2d)}$.

- We have $\frac{1}{\sqrt{2}}(v_i - \bar{v}_i) = \sqrt{2} n_i$ where n_i are integers.

- To reconstruct the fermion operators, we must look for pairs of points of unit length in the weight lattice that differ by an element of the root lattice.
- If $\vec{\alpha}_i$ are the d simple roots of Spin(2d) and $\vec{\lambda}^i$ are the fundamental weights then one finds:

$$D_p(z) \sim e^{iw^{(p)i}\phi_i(z)}$$

where $w^{(p)i} = \sqrt{2}(\vec{\lambda}^i)_p$.

- We can now look for the twist field, which induces a monodromy:

$$\sigma_k : D_p(z) \to e^{\frac{2\pi ik}{n}} D_p(z)$$

corresponding to a shift:

$$w^{(p)i}\phi_i(z) \to w^{(p)i}\phi_i(z) + \frac{2\pi k}{n}$$

- This will be induced by a shift $\phi_i \to \phi_i + 2\pi\zeta_i^{(k)}$ where $\zeta_i^{(k)}$ is a constant vector satisfying:

$$w^{(p)i}\zeta_i^{(k)} = \frac{k}{n}$$

  for all p.

- As the last weight of Spin(2d) is $\lambda^{(d)} = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, the shift is given by:

$$\zeta_i^{(k)} = \frac{\sqrt{2}k}{n}(0, 0, \cdots, 0, 1)$$

  Thus the twist field only acts on the last scalar ϕ_d.

- It takes the form:

$$\sigma_k = \mathcal{O}_{\zeta^{(k)i}, -\zeta^{(k)i}} = e^{i\zeta^{(k)i}\phi_i(z)} e^{-i\zeta^{(k)i}\bar{\phi}_i(\bar{z})}$$

  and has the desired conformal dimension $\sum_k \Delta_k = \frac{d}{24}\left(n - \frac{1}{n}\right)$.

- Now we can calculate the two-point function of each σ_k and thereby the replica partition function.
- Recall that the ordinary partition function for these theories is:

$$Z_1 = \frac{1}{|\eta(\tau)|^{2d}} \sum_{\substack{w,\bar{w}\in\Lambda_W \\ w-\bar{w}\in\Lambda_R}} q^{w^2}\bar{q}^{\,\bar{w}^2}$$

$$= \frac{1}{2}\frac{1}{|\eta(\tau)|^{2d}} \sum_{\nu=2,3,4} |\theta_\nu(0|\tau)|^{2d}$$

- The un-normalised two-point function of twist fields is:

$$\langle\langle\sigma_k(z,\bar{z})\sigma_{-k}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2dk^2}{n^2}} \frac{1}{|\eta(\tau)|^{2d}} \times \sum_{\substack{w,\bar{w}\in\Lambda_W \\ w-\bar{w}\in\Lambda_R}} q^{w^2}\bar{q}^{\,\bar{w}^2} e^{2\pi i \frac{\ell}{L} g_{ij}(w^i+\bar{w}^i)\zeta^{(k)j}}$$

- Now we have:

$$g_{ij}(w^i + \bar{w}^i)\zeta^{(k)j} = \frac{k}{n}\sum_{p=1}^{d}(n_p + m_p), \quad w, \bar{w} \in \Lambda_R \cup \Lambda_V$$
$$= \frac{k}{n}\sum_{p=1}^{d}(n_p + m_p - 1), \quad w, \bar{w} \in \Lambda_S \cup \Lambda_C$$

- It follows that:

$$\langle\langle \sigma_k(z,\bar{z})\sigma_{-k}(0)\rangle\rangle = \frac{1}{2}\left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2dk^2}{n^2}} \frac{\sum_{\nu=1}^{4}|\theta(\frac{k\ell}{nL}|\tau)|^{2d}}{|\eta(\tau)|^{2d}}$$

- Taking the product over k after/before the sum over spin structures gives us the uncorrelated/correlated Z_n.
- As before, we choose the former as $\ell \to 0$ and the latter as $\ell \to L$, and the thermal entropy relation follows.
- The replica partition function can be rendered modular invariant after multiplying with our proposed prefactor.

Outline

- We have shown that modular-invariant Rényi and entanglement entropies do exist for free fermions.
- There were two surprises:
  - We could only find the answer in the limiting regions $\ell \to 0$ and $\ell \to L$. In the first case the spin structures are uncorrelated across replicas and in the second they are correlated.
  - Modular invariance is achieved only upto an overall ℓ-independent factor which can be removed by a suitable modification of Z_n.
- We verified that answers in the literature for compact boson CFT are also modular-invariant in the same way. However here there is a (very implicit) form at all ℓ.
- We extended the free-fermion computation to multiple correlated fermions, dual to free bosons on a Spin(2d) lattice and it agrees with everything above.

- For the future, many directions are suggested:
  - Can one write the replica partition function for fermions at intermediate values of ℓ as a linear combination of correlated/uncorrelated quantities?
  - Can this computation be extended to other CFT's?
  - For free bosons, there is a result but it is very implicit. Can its form be simplified?
  - What is the bulk analogue of these results in AdS/CFT? Is there a "Farey tail" extension of the [Ryu-Takayanagi] proposal?

Thank you!

ありがとうございました