Resurgence and Bions in Grassmann sigma models

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1 Borel sum of divergent series and Renormalon

Perturbation series

Partition function of ϕ^4 field theory in Euclidean d-dimension

$$Z(g^2) = \int D\phi(x)\, e^{-S_E}, \qquad S_E = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + m^2\frac{\phi^2}{2} + g^2\frac{\phi^4}{4}\right)$$

Perturbation series in g^2 $(m = 1)$: $Z(g^2) =$ sum of Feynman diagrams

$d \to 0$: **Number of Feynman diagrams** (with weight and sign)

$$Z(g^2) = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}}\, e^{-S_E}, \qquad S_E = \frac{1}{2}q^2 + g^2\frac{q^4}{4}$$

$Z(g^2)$ is well-defined for $g^2 > 0$, $(m = 1)$

(convergent integral $\longrightarrow$ real analytic function in the complex g^2 plane)

Perturbation : Formal power series defined by $Z(g^2) = \sum_{K=0}^{\infty}(g^2)^K Z_K$

$$Z_K = \frac{1}{K!}\int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}}\, e^{-\frac{q^2}{2}} \left(\frac{-q^4}{4}\right)^K$$

$$= \frac{1}{K!}\frac{(-1)^K}{\sqrt{\pi}}\Gamma\left(2K + \frac{1}{2}\right) \sim \frac{(-4)^K}{\sqrt{2\pi}}(K-1)!$$

Perturbation series is **Factorially divergent** and **Alternating**

Borel summation:

A method to make sense of the sum of **Factorially divergent series**

Factorially divergent series (Gevrey-I) is defined as

$$P(g^2) = \sum_{K=0}^{\infty} a_K (g^2)^K, \quad |a_K| \le C K! \left(\frac{1}{A}\right)^K, \quad C, A : \text{constants}$$

Def: **Borel transform** $BP(t)$ $\longrightarrow$ finite radius of convergence

$$BP(t) = \sum_{K=0}^{\infty} \frac{a_K}{K!} t^K$$

Def: **Borel resummation** $\mathbb{P}(g^2)$

$$\mathbb{P}(g^2) = \int_0^{\infty} dt e^{-t} BP(g^2 t)$$

If this integral is well-defined, the series is called **Borel-summable**

Singularities of $BP(g^2 t)$ in complex t-plane (Borel plane) is important

Alternating factorially divergent series ($A > 0$)

$$P(g^2) = C\sum_{K=0}^{\infty} K!\left(\frac{-g^2}{A}\right)^K$$

Borel transform becomes

$$BP(t) = C\sum_{K=0}^{\infty}\left(\frac{-t}{A}\right)^K = \frac{CA}{A+t},$$

Borel resummation becomes

$$\mathbb{P}(g^2) = \int_0^{\infty} dte^{-t}BP(g^2t) = \int_0^{\infty} dte^{-t}\frac{CA}{A+g^2t}$$

$BP(g^2t)$ has **Singularities** on the positive real axis of Borel plane

Alternating factorially divergent series is **Borel summable**

Non-Borel summable series → Imaginary ambiguity

Quantum mechanics with **degenerate minima**

$$H = \frac{p^2}{2} + V(q), \quad V(q) = \frac{q^2}{2}(1-gq)^2 = \frac{q^2}{2} - gq^3 + g^2\frac{q^4}{2}$$

Path-integral representation of ground state energy

$$E(g^2) = \lim_{\beta\to\infty} \frac{-1}{\beta} \log \mathrm{tr}(e^{-\beta H}), \quad \mathrm{tr}(e^{-\beta H}) = \int Dq(t) e^{-S_E}$$

$-gq^3$ is more important ($(gq^3)^2 \gg g^2 q^4$ for $|q| \gg 1$)

Large order behavior of perturbation series

$$E_{\mathrm{pert}}(g^2) = \sum_K^{\infty} (g^2)^K E_K, \quad E_K \sim -\frac{3}{\pi} 3^K K!$$

Borel transform becomes

$$BE_{\mathrm{pert}}(t) \sim -\frac{3}{\pi} \sum_{K=0}^{\infty} (3t)^K = -\frac{3}{\pi} \frac{1}{1-3t},$$

a **Pole** at $t = 1/(3g^2)$ on the positive real axis of Borel plane

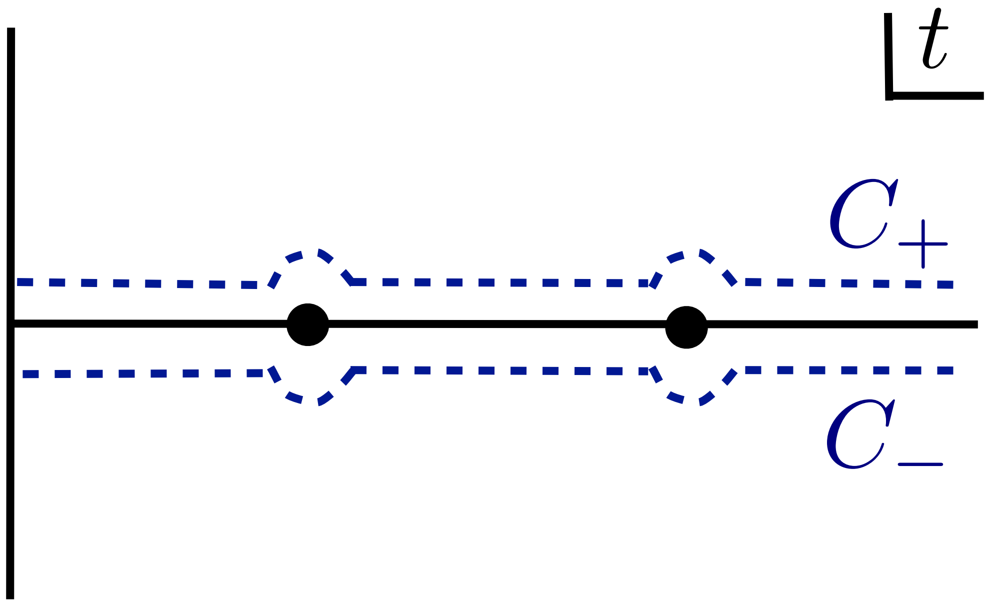
Borel resummation is ill-defined for $g^2 > 0$

$$\mathbb{E}_{\mathrm{pert}}(g^2) = \int_0^{\infty} dt e^{-t} BE_{\mathrm{pert}}(g^2 t) = -\frac{3}{\pi} \int_0^{\infty} dt e^{-t} \frac{1}{1-3g^2 t}$$

Non-alternating divergent series : **Borel non-summable**

Well-defined at $g^2 < 0 \rightarrow$ **Analytic** continuation to $g^2 > 0$ gives

$$\mathrm{Im}\mathbb{E}_{\mathrm{pert}}(g^2) \sim \mp 3 e^{\frac{-1}{3g^2}} \quad \text{imaginary ambiguity (path} - \text{depend.)}$$



t
C_+
C_-

Saddle points as solutions of Euclidean Action

$$\boldsymbol{S = \int dt \left[\frac{1}{2}\left(\frac{dq}{dt}\right)^2 + V(q)\right], \quad V(q) = \frac{q^2}{2}(1-gq)^2}$$

Instantons as nonperturbative saddle points (BPS solution)

$$\boldsymbol{q(t) = \frac{1}{g}\frac{1}{e^{-(t-t_0)}+1}, \quad S_{\mathrm{I}} = \frac{1}{6g^2}}$$

Instanton and antiinstanton attract : moduli integral requires $\boldsymbol{g^2 < 0}$

E.B.Bogomolny, Phys.Lett.**B91** 431 (1980); J.Zinn-Justin, Nucl.Phys.**B192** 125 (1981); $\cdots$

Analytic continuation $\longrightarrow$ (nonperturbative) imaginary ambiguity

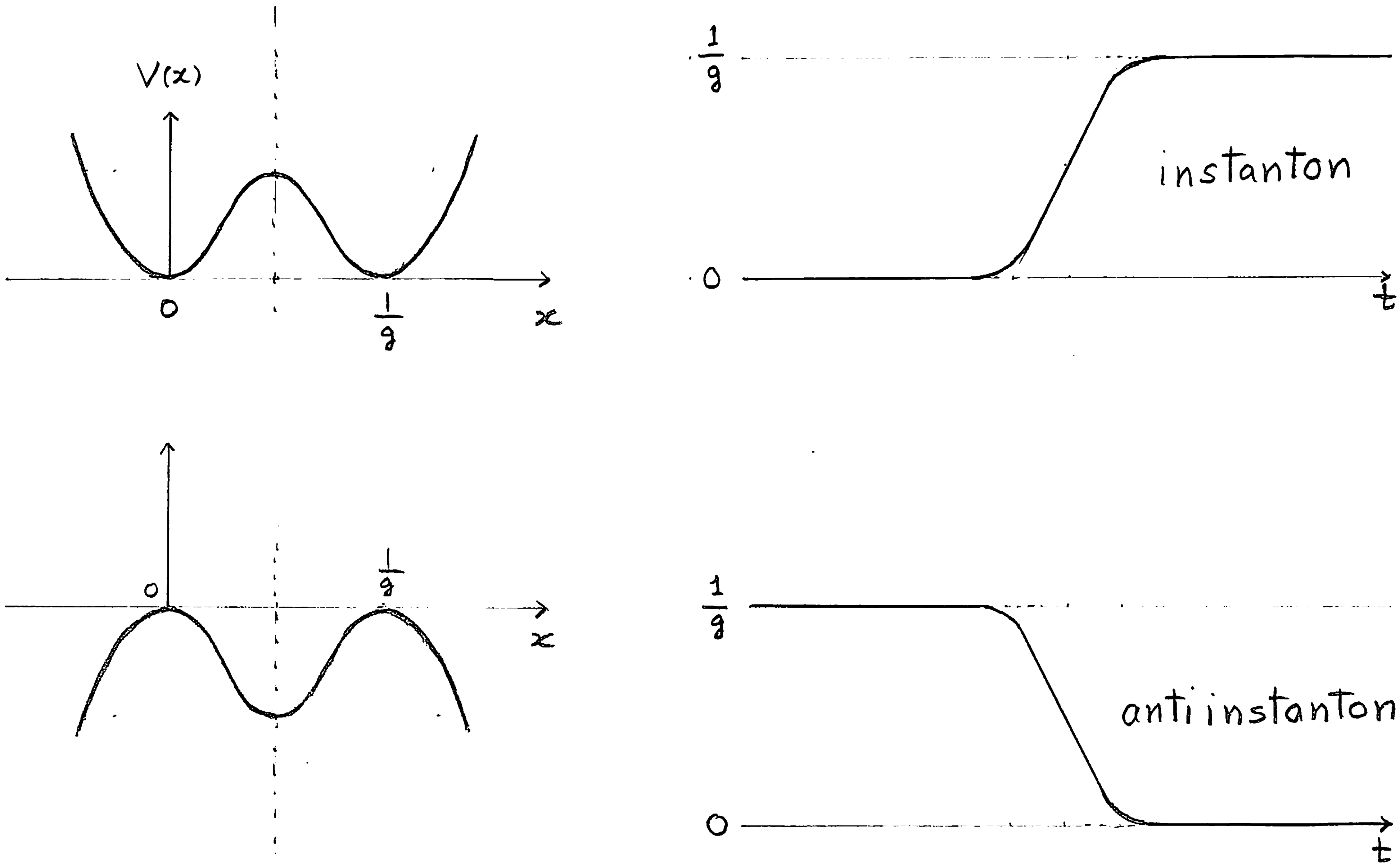
Cancellation of **perturbative** and **nonperturbative** contributions

$\longrightarrow$ **Resurgence** of perturbation series

(Divergent) Perturbation series knows (conspires) nonperturbative effects

J.Zinn-Justin, Phys.Rep.**70** 109 (1981); $\cdots$

Infrared (IR) Renormalon



V(x)
0
1/g
x
1/g
0
t
instanton
0
1/g
x
1/g
0
t
antiinstanton

Electric Current correlation function in QCD ($SU(N_{\rm C})$)

$$\int d^4x e^{-iqx} < 0|T(j_\mu(x)j_\nu(0))|0 >= i(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(Q^2)$$

$$D(Q^2) = 4\pi^2 \frac{d\Pi(Q^2)}{d(Q^2)}, \quad Q^2 = -q^2, \quad j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$$

n-loop gluon (momentum k) corrections to $\Pi(Q^2)$

Large logarithmic contributions from small momenta $|k^2| \ll |q^2|$:

$$n! \left[-\beta_0 \alpha_s \log\left(\frac{-k^2}{\mu^2}\right)\right]^n, \; \beta_0 = -\frac{1}{4\pi}\frac{11}{3}N_{\rm C} + \cdots < 0$$

β_0 : 1-loop beta func. coeff., μ : renormalization scale

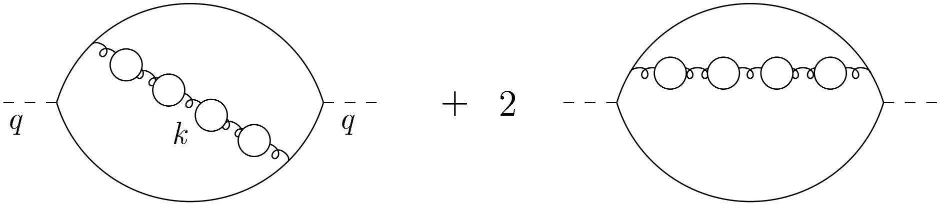
non-Borel summable : Borel singularity at $t \sim \frac{1}{-\beta_0 g^2} \sim \frac{1}{N_{\rm C} g^2}$

nearer to the origin by a factor $1/N_{\rm C}$

Asymptotically free ($\beta_0 < 0$) theory $\longrightarrow$ **IR renormalon** problem

Phenomenologically : (Nonperturbative) power corrections to perturb. QCD

M.Beneke, Phys.Rep.**317** 1 (1999); $\cdots$


q
k
q
+ 2

IR renormalon ambiguity is cancelled by Nonperturbative contributions from

Neutral Bions $\longrightarrow$ **Resurgence**

Divergence of Perturbation series knows nonperturbative effects

(Charged Biobs $\longrightarrow$ Confinement)

Compactified space (short-distance) :

Asymptotic freedom $\longrightarrow$ **weak coupling** at short-distances

P.C.Argyres and M.Ünsal, Phys.Rev.Lett.**109**, 121601 (2012); JHEP **08(2012)**, 063 (2012); G.V.Dunne, M.Ünsal, JHEP **11(2012)**, 170; Phys.Rev.D **87**, 025015 (2012); R.Dabrowski, G.V.Dunne, Phys.Rev.D **88**, 025020 (2013); G.V.Dunne, M.Ünsal, Phys.Rev.D **89**, 041701(R) (2014); A.Cherman, D.Dorigoni, G.V.Dunne, M.Ünsal, Phys.Rev.Lett.**112**, 021601 (2014); G.Basar, G.V.Dunne, M.Ünsal, JHEP **10(2013)** 041; G.V.Dunne, M.Ünsal, Phys.Rev.**D89** 105009(2014); A.Cherman, D.Dorigoni, M.Ünsal, [arXiv:1403.1277]; G.V.Dunne, M.Ünsal, [arXiv:1505.07803]; $\cdots$

2 $U(N_C)$ gauge theory and Grassmann sigma model

Gauge invariance as constraints on scalar fields $\longrightarrow$ Nonlinear sigma model

$U(N_C)$ gauge fields A_μ with N_F scalars H in fund. rep. (Euclidean 2D)

$$\mathcal{L}_{\rm gauge} = {\rm Tr}\left[\frac{1}{2g^2}F_{\mu\nu}F_{\mu\nu} + \mathcal{D}_\mu H\left(\mathcal{D}_\mu H\right)^\dagger\right] + {\rm Tr}\left[\frac{g^2}{4}\left(v^2 \mathbf{1}_{N_C} - HH^\dagger\right)^2\right]$$

$\mathcal{D}_\mu H = (\partial_\mu + iA_\mu)H,\quad F_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu],\quad g$: gauge coupling

Strong coupling limit $g^2 \longrightarrow \infty$: Nonlinear sigma model

$$\mathcal{L}_{\rm grassmann} = {\rm Tr}\left[\mathcal{D}_\mu H \mathcal{D}_\mu H^\dagger\right], \quad A_\mu = \frac{i}{v^2}\partial_\mu H H^\dagger, \quad HH^\dagger = v^2 \mathbf{1}_{N_C}$$

Grassmann (G_{N_F,N_C}) sigma model

$$H \in G_{N_F,N_C} = \frac{U(N_F)}{U(N_C)\times U(N_F - N_C)}$$

Coupling : $1/v$: **Asymptotic free** theory

Topological charge

$$Q = -\frac{1}{2\pi}\int d^2x\ {\rm Tr}\left(\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}\right) = \frac{i}{2\pi v^2}\int d^2x \epsilon_{\mu\nu}\partial_\mu {\rm Tr}\left(H\partial_\nu H^\dagger\right)$$

BPS bound : ($z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2$)

$$S = \int d^2x\, \mathcal{L}_{\rm grassmann} = \int d^2x\ {\rm Tr}\left(4\mathcal{D}_{\bar{z}} H \mathcal{D}_z H^\dagger\right) + 2\pi Q \geq 2\pi Q$$

BPS equation : $\mathcal{D}_{\bar{z}} H = 0$; **Anti-BPS equation** : $\mathcal{D}_z H = 0$

Defining S as $A_1 + iA_2 = -i2S^{-1}\bar{\partial}_z S$, BPS solution is given as

$$H = S^{-1} H_0(z), \quad H_0(z) : \text{ moduli matrix}$$

NLSM constraint $HH^\dagger = v^2 \mathbf{1}_{N_{\rm C}}$ is satisfied by $\Omega \equiv SS^\dagger = H_0 H_0^\dagger / v^2$

$U(N_{\rm C})$ gauge fixed (anti)BPS solutions, if $H_0(z)$ ($H_0(\bar{z})$)

$$H = U\Omega_{\rm d}^{-1/2} U^\dagger H_0, \text{ with } \Omega = U\Omega_{\rm d} U^\dagger, \quad UU^\dagger = \mathbf{1}_{N_{\rm C}}$$

(Hermitian matrix Ω can be diagonalized to nonnegative $\Omega_{\rm d}$)

$N_{\rm C} = 1$ case : $\mathbb{C}P^{N_{\rm F}-1}$ model

Ansatz for Bions : non-holomorphic Moduli Matrices $H_0(z, \bar{z})$

become solutions for well-separated constitutent (anti-)BPS solitons

3 Fractional instantons and neutral bions

$\mathbb{C}P^{N_\mathrm{F}-1}$ models in $\mathbb{R}\times S^1$: $z = x_1 + i x_2$, $0 \le x_2 < L$ for S^1

$\mathbb{Z}_{N_\mathrm{F}}$ as a **center** symmetry of the global symmetry $SU(N_\mathrm{F})$

$$\Omega = \mathrm{diag.}\left[1, e^{2\pi i/N_\mathrm{F}}, e^{4\pi i/N_\mathrm{F}}, \cdots, e^{2(N_\mathrm{F}-1)\pi i/N_\mathrm{F}}\right] \in SU(N_\mathrm{F})$$

$\mathbb{Z}_{N_\mathrm{F}}$ twisted bounday condition

$$H(x_1, x_2 + L) = H(x_1, x_2)\Omega$$

BPS **Fractional instanton** with $S = 2\pi/(g^2 N_\mathrm{F})$, $Q = 1/N_\mathrm{F}$

$$H_0(z) = \left(0, \cdots, 0, 1, \lambda e^{i\theta} e^{+2\pi z/(N_\mathrm{F} L)}, 0, \cdots, 0\right)$$

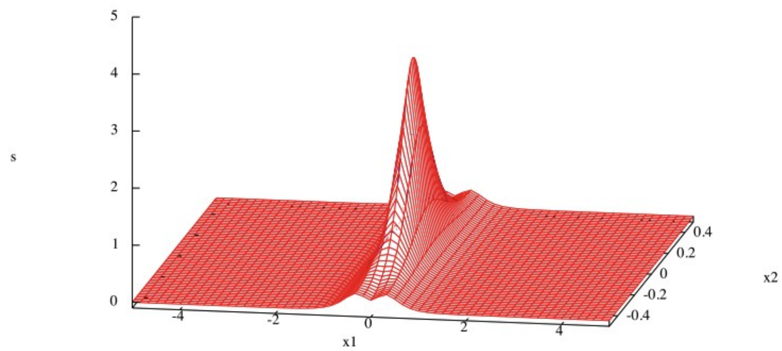
Twisted bound. cond. +Holomorphy $\longrightarrow$ Dependence on x_1 $\longrightarrow$ Kinks

Fractionalized instantons at $x_1 = \frac{NL}{2\pi}\log\frac{1}{\lambda_1}, \cdots, \frac{NL}{2\pi}\log\frac{\lambda_{N-1}}{\lambda_N}$

D-brane configurations are useful to understand (fractional) instantons

Fractional instanton (vortex) become Kinks on cylinder (**T-duality**)

Bion = composite of Fractional instanton and anti-instanton



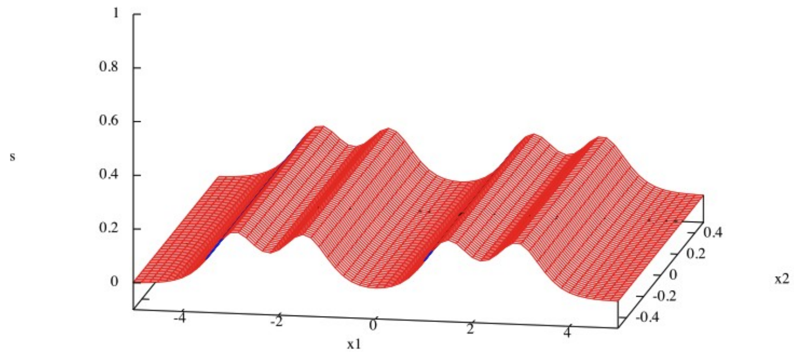


Figure 1: Energy density $\boldsymbol{s}(\mathbf{x})$ for the BPS instanton configuration of the Grassman sigma model $\boldsymbol{G_{4,2}}$.

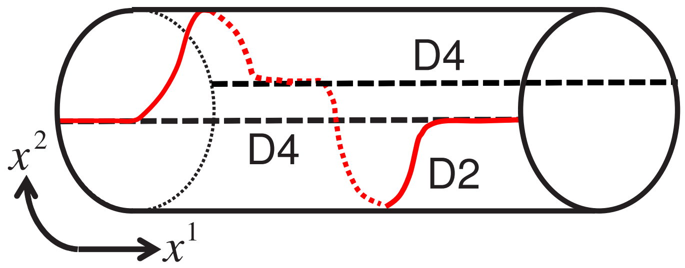


Figure 2: T-dual picture for $N_{\rm F} = 2, N_{\rm C} = 1, k = 1$.

1-loop analysis of holonomy for $R \times S^1$

Phases (Holonomy) of components of ω are dynamical variable:

$\mathbf{1}$-loop effective potential is reliable for small radius of S^1

$\longrightarrow$ Holonomy is stabilized at $Z_{N_{\rm F}}$ twisted boundary condition

G.Dunne, M.Ünsal, JHEP **1211**, 170 (2012); Phys.Rev.D **87**, 025015 (2013); $\cdots$

4 Bions in Grassmann sigma models

Fractional instantons are labeled by vacua at left and right infinities

Increment of flavor labels from left to right vacuum : $(k_1, k_2, \cdots, k_{N_{\rm C}})$

Instanton charge Q is given by

$$Q = \frac{1}{N_F} \sum_{a=1}^{N_C} k_a$$

Left vacuum label : $\langle f_1, f_2, \cdots, f_{N_C} \rangle$ (right vac label f')

(k_a) and $< f_a >$: **complete characteriz.** of fractional instantons

[Def] **Bion** : vanishing total instanton number $\sum_{a=1}^{N_C} f_a = \sum_{a=1}^{N_C} f'_a$

1. Neutral bions: the left and right vacuum label vectors are identical

$$f_a = f'_a, \quad a = 1, \cdots, N_C$$

2. Charged bions: the vacuum label vectors in the left and right vacua are different

$$\langle f_a, f_2, \cdots, f_{N_C} \rangle \neq \left\langle f'_1, f'_2, \cdots, f'_{N_C} \right\rangle$$

$\longrightarrow$ No charged bions in $\mathbb{C}P^{N_F-1}$ models

Pair creation of fractional instanton and antiinstanton gives generally bions

Reducible bions

can be decomposed by changing moduli

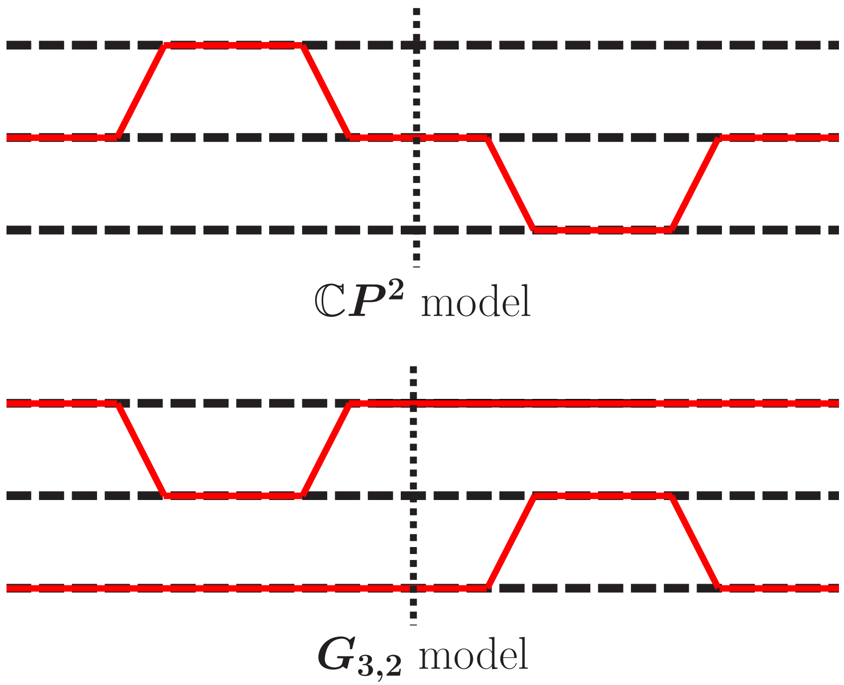


Figure 3: Reducible neutral bions in the dual pair $\mathbb{C}\boldsymbol{P}^{\mathbf{2}} = \boldsymbol{G}_{\mathbf{3,1}}$ and $\boldsymbol{G}_{\mathbf{3,2}}$ models.

We wish to list only **irreducible** bions

Irreducible neutral bions in $\boldsymbol{G_{4,2}}$ model : Adjacent flavor case

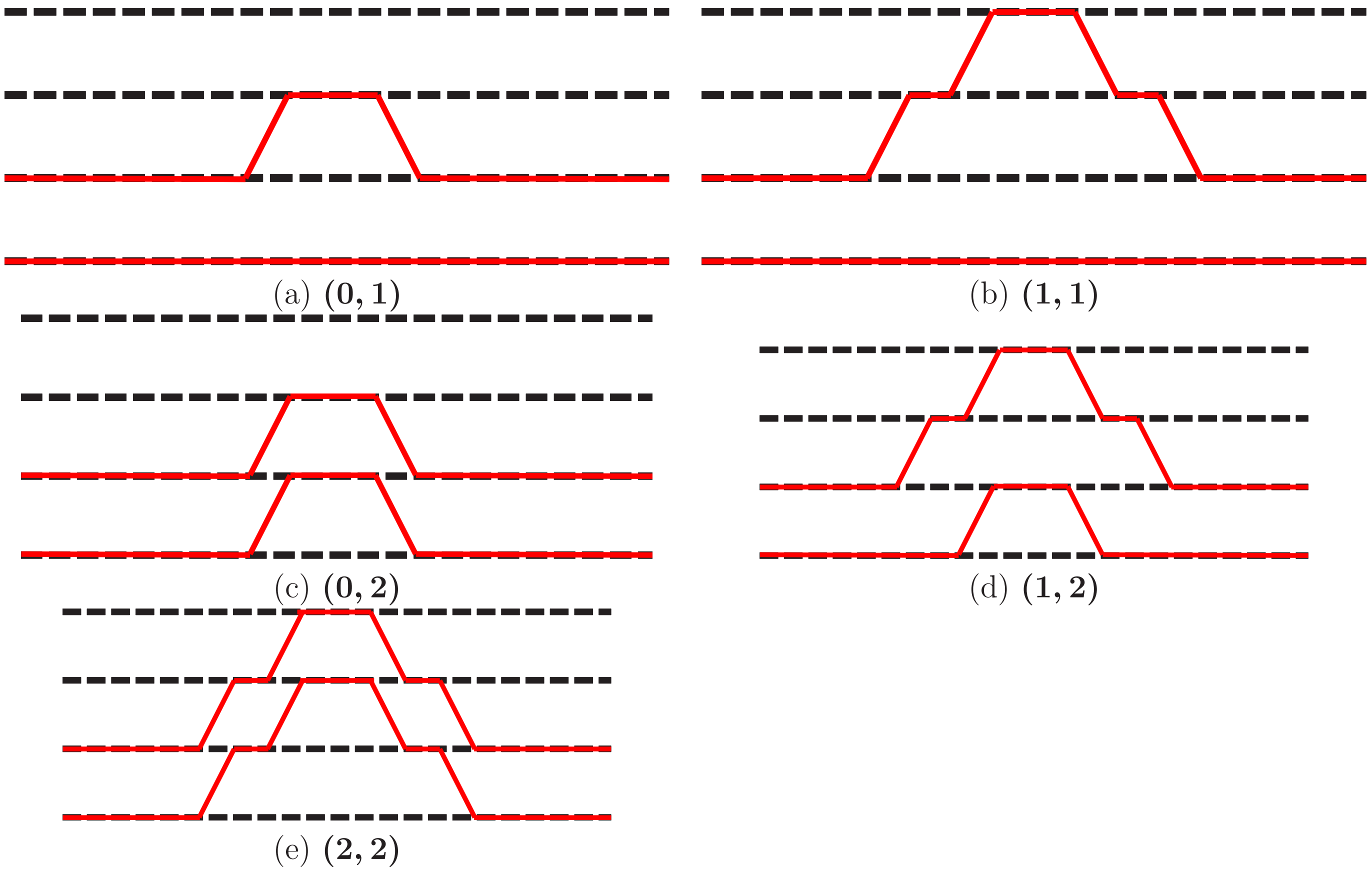
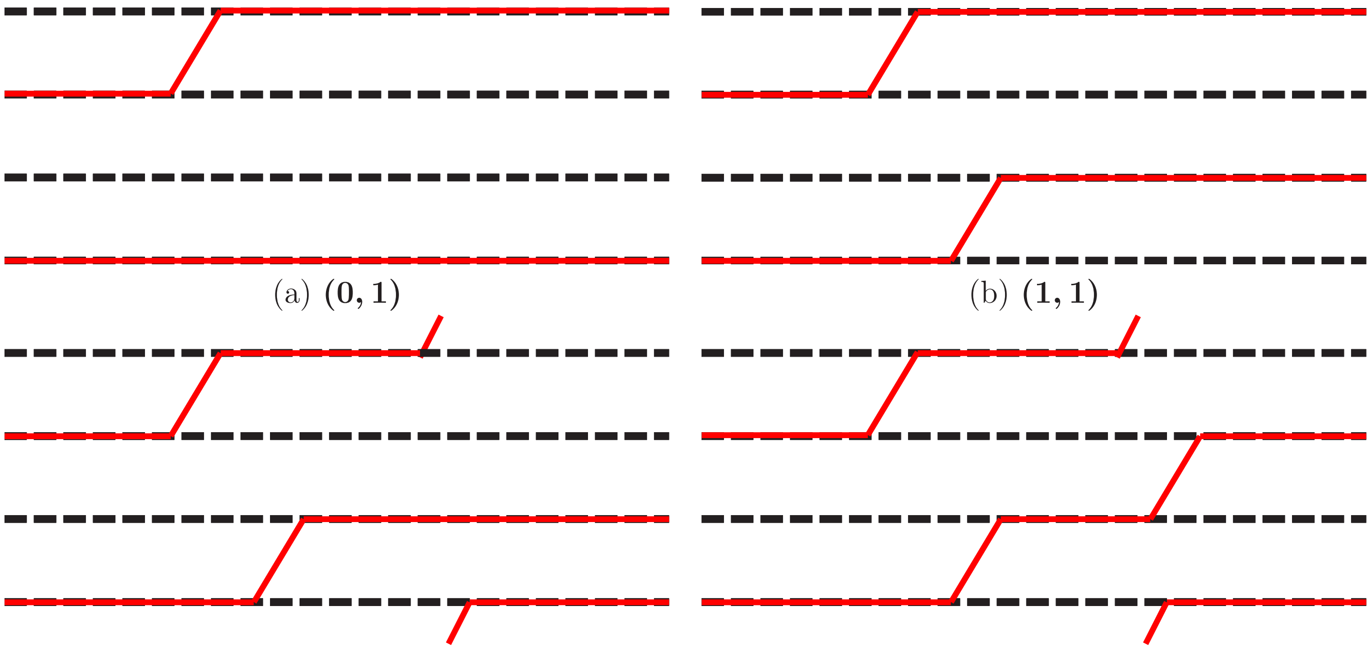


Figure 4: Neutral bions in $\boldsymbol{G_{4,2}}$, labeled by indices (a) $(\mathbf{0,1})$, (b) $(\mathbf{0,2})$, (c) $(\mathbf{1,1})$, (d) $(\mathbf{1,2})$, (e) $(\mathbf{2,2})$. (a)–(e) are elementary neutral bions in $\boldsymbol{G_{4,2}}$.

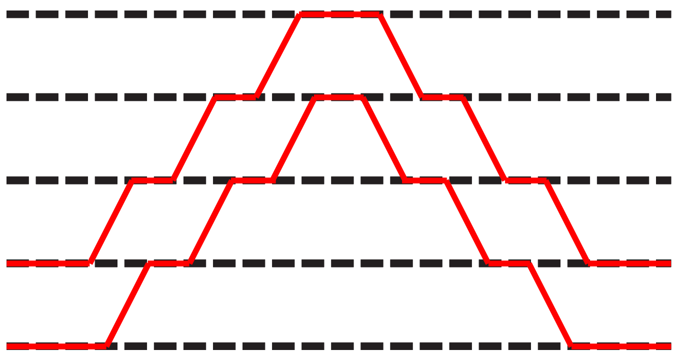
Irreducible neutral bions in $\boldsymbol{G_{4,2}}$ model : Non-adjacent flavor case



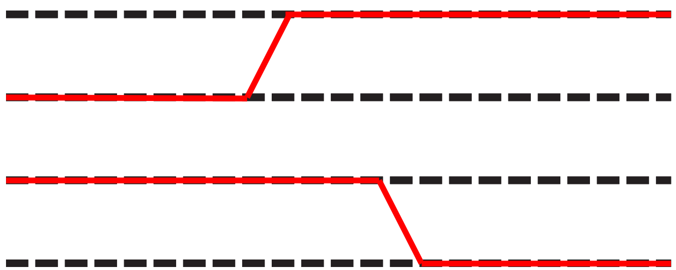
(a) $\mathbf{(0,1)}$ (b) $\mathbf{(1,1)}$ (c) $\mathbf{(1,2)}$ (d) $\mathbf{(2,2)}$

Figure 5: Fractional instantons for two non-adjacent flavors occupied by colors at the left vacuum in $\boldsymbol{G_{4,2}}$, labeled by indices (a) $\mathbf{(0,1)}$, (b) $\mathbf{(1,1)}$, (c) $\mathbf{(1,2)}$, and (d) $\mathbf{(2,2)}$. Diagram (a) is an elementary fractional instanton, (b) and (c) are composite fractional instantons, (d) is a BPS instanton.

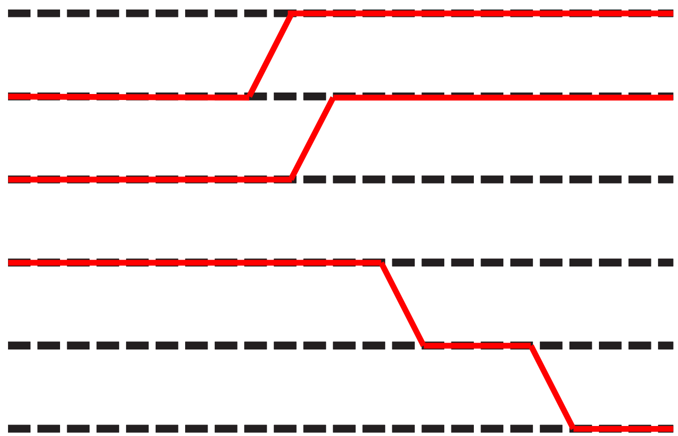
There are irreducible neutral bions with instanton charge of order $\boldsymbol{N_{\mathrm{C}}}$



Non-BPS exact solutions of charged bions



charged bion in $\boldsymbol{G_{4,2}}$



in $\boldsymbol{G_{6,3}}$

5 Conclusions

1. Perturbation series is factorially divergent, but can be made sense by means of **Borel summation**. Asymptotic free theories give **renomalon** singularities in the physical region of the Borel plane.

2. Singularities in the physical region of the Borel plane cause (path-dependent) imaginary ambiguities, which are cancelled by nonperturbative contributions called (neutral) **bions** (molecules of fractional instantons). This is called **resurgence**, and indicates that divergent perturbation series contains all informations including nonperturbative effects.

3. **Bions** in Grassmann models (including $\mathbb{C}P^{N_F-1}$ models) with $\mathbb{Z}_{N_F}$ **twisted boundary condition** are classified completely.

4. We work out explicitly an **Ansatz** for bions that becomes the exact solution in large separation limit.

5. Attractive interactions of constituent fractional instantons at short distances agree with the results of far-separated instanton calculus. This result supports the bion contribution for **short** ($R \geq 1$) as well as **long separations** ($R \gg 1$), leading to the **resurgence** phenomenon .

6. **D-brane configuration** offers a powerful tool to understand fractional instantons and bions.

7. **Charged bions** do not exist in $\mathbb{C}P^{N_F-1}$ models, but exist in Grassmann models ($N_C \geq 2$).

8. Neutral bions consisting of fractional instantons with instanton charge of order N_C are found.

9. **Non-BPS exact solutions** of charged bions are found.