Anomalies and SPT phases

Kazuya Yonekura, Kavli IPMU

- Review (mainly of [1508.04715] by Witten)
- [1607.01873] KY
- [1610.07010][1611.01601] collaboration with Yuji Tachikawa

Introduction

What is the most general ('t Hooft) anomaly of QFTs?

Perturbative anomalies (one-loop diagrams) are not enough!

- Global anomalies
- Anomalies of discrete symmetries (e.g. time-reversal)
- Anomalies of topological field theories
- …

Introduction

Anomalies appear in many areas:

- High energy physics
  (the standard model, string theory, etc.)
- Condensed matter physics
  (quantum Hall effect, topological insulator, etc.)
- Mathematics
  (index theorems, topological field theory, etc.)

Introduction

A toy question:

Suppose there is a "universe with **a** standard model":

- Space-time is 3-dimensional
- There is exact time-reversal symmetry
- "One generation" has just one Majorana fermion

Then it is possible to prove that the number of generations must be a multiple of 16 due to a global anomaly.

This has implications on the classification of topological superconductors.

Introduction

A question:

How many generations of **the** standard model is anomaly free at the non-perturbative level?

I don’t even have a direct answer to this question!

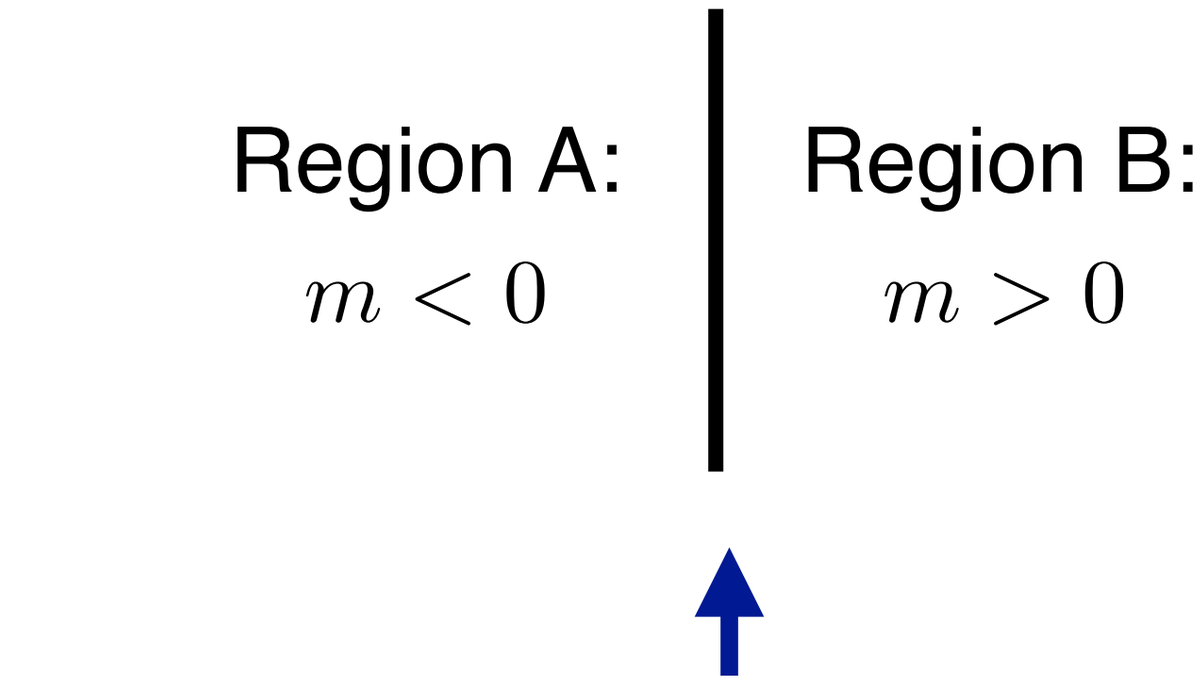
(Textbooks only discuss perturbative anomalies…)

Introduction

A chiral fermion in d-dimensions may be realized as a domain wall fermion in one higher dimensions.

Massive fermion in (d+1)-dimensions

$$\mathcal{L} = \bar{\Psi}(iD + m)\Psi \quad (\ D : \text{Dirac operator})$$

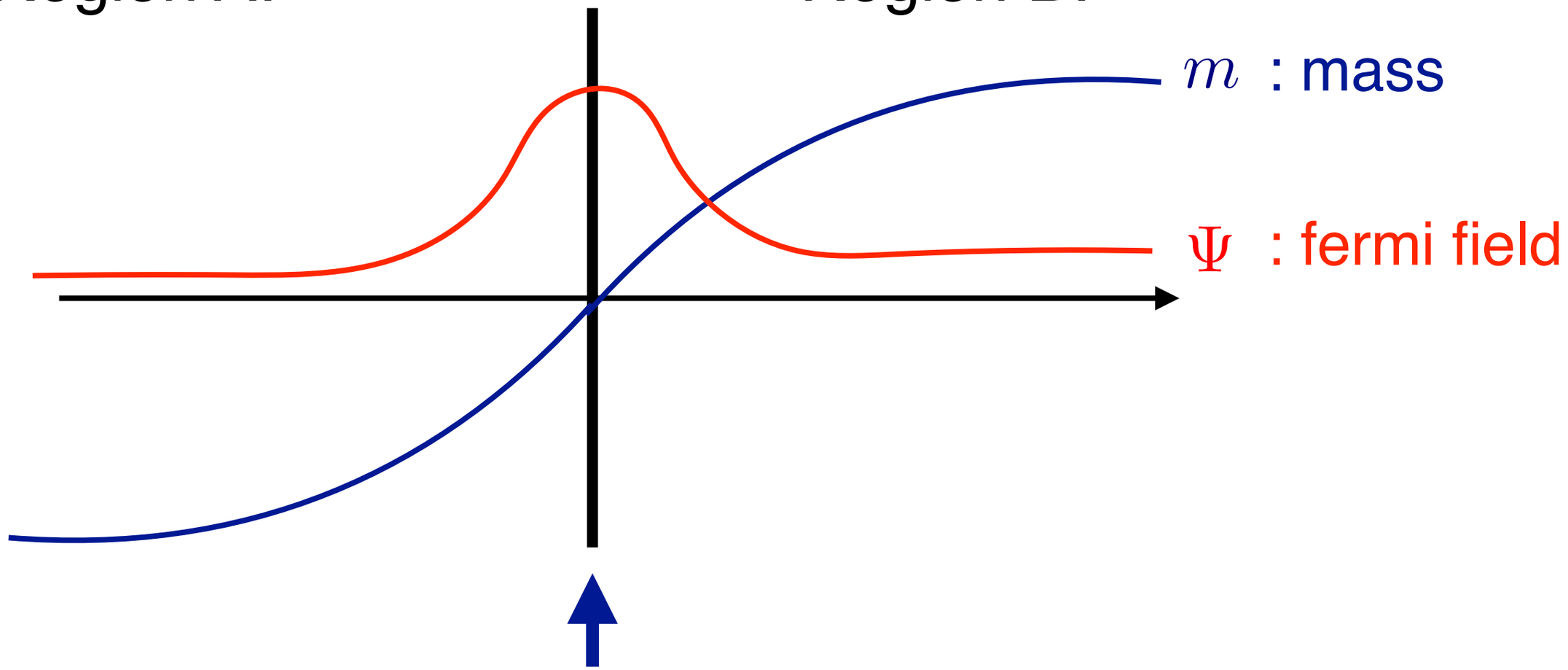


Chiral fermion between two regions.

Introduction

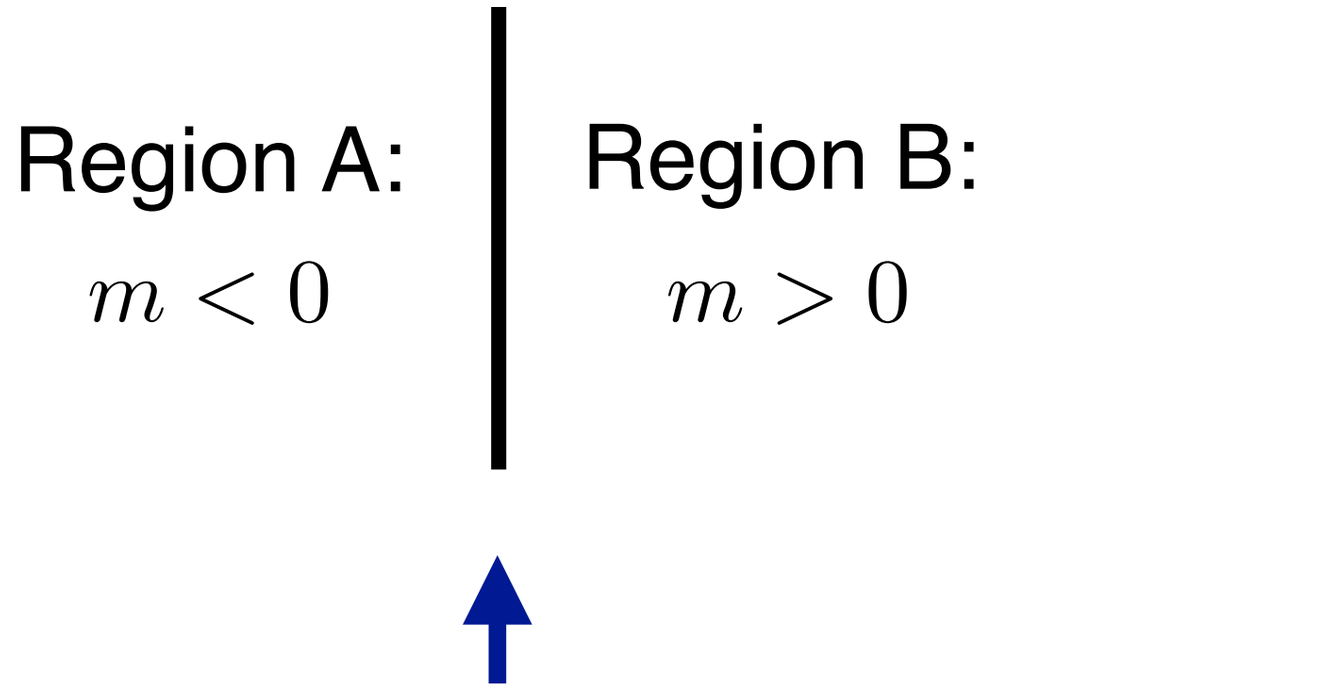
Region A: $m < 0$

Region B: $m > 0$



Localized zero mode = chiral fermion

Introduction



Massless chiral fermion between two regions.

Introduction

Modern understanding (to be explained in this talk):

Region A:	Region B:
$m < 0$	$m > 0$
→ SPT phase1	→ SPT phase2

↑

Massless chiral fermion between two regions.

→ Anomalous quantum field theory

Introduction

Modern understanding (to be explained in this talk):

Anomalies of d-dimensional theories are (believed to be) classified by (d+1)-dimensional SPT phases

Bulk SPT phase
d+1 dimensions

Boundary anomalous theory
d dimensions

Short summary of the talk

Message 1

- **Anomalies =**
  (generalized) Berry phase of SPT phase

Message 2

- **Topological QFTs can have (’t Hooft) anomalies**

Contents

1. Introduction

2.SPT phases

3.A class of SPT phases from massive fermions

4. Manifolds with boundary and fermion anomalies

5.Anomalies of topological quantum field theory

6.Summary

Remark

The basic tool I'm going to use is the partition function

$$Z[g_{\mu\nu}, A_\mu] = \int \mathfrak{D}\Psi e^{-S}$$: path integral

Background field (do not do path integral over them)

- Metric $g_{\mu\nu}$
- Gauge field for symmetries A_μ

SPT phases

G-protected Symmetry Protected Topological (SPT) phase

- It is a QFT with global symmetry G which can be coupled to background fields.
- The Hilbert spaces are always one-dimensional on any compact space (at least in low energy limit).
- Characterized by the partition function $Z[g_{\mu\nu}, A_\mu]$

Also called invertible field theory by mathematicians.

[Freed,,…]

SPT phases

Example: Integer quantum hall state

- 3=2+1 dimensions
- Global symmetry G=U(1) coupled to A_μ
- Partition function: Chern-Simons action of background

$$\log Z = \int \frac{ik}{4\pi} A dA$$

$A = A_\mu dx^\mu$: background field (connection) for G=U(1)

$k \in \mathbb{Z}$: integer

SPT phases

<u>Example</u>: Integer quantum hall state

SPT phase1

$$\log Z = \int \frac{ik}{4\pi} AdA$$

SPT phase2

trivial phase $\log Z = 0$

k massless 2-dim chiral fermions between two regions due to anomaly inflow.

$$\delta_{\text{gauge}} \log Z = \int_{\text{boundary}} \frac{ik}{4\pi} \alpha F$$

This is cancelled by the anomaly of chiral fermions.

Contents

1. Introduction

2.SPT phases

3.A class of SPT phases from massive fermions

4. Manifolds with boundary and fermion anomalies

5.Anomalies of topological quantum field theory

6.Summary

SPT from massive fermions

A class of SPT phases is obtained from massive fermions.

Massive fermion in (d+1)-dimensions

$$S = \int d^{d+1}x \ \bar{\Psi}(iD + m)\Psi$$

$$D = i\gamma^{\mu}(\partial_{\mu} - iA_{\mu} + \cdots)$$

: Dirac operator coupled to background G-field A_{μ} .

SPT from massive fermions

A class of SPT phases is obtained from massive fermions.

Massive fermion in (d+1)-dimensions

$$S = \int d^{d+1}x \ \bar{\Psi}(iD + m)\Psi$$

Low energy limit (RG flow)

SPT phase (invertible field theory)

SPT from massive fermions

Partition function:

Euclidean path integral on a compact manifold X

$$Z = \int \mathfrak{D}\Psi e^{-S}$$

$$\det(iD + m)$$

$$= \frac{}{\det(iD + \Lambda)}$$

$$= \prod_{\lambda \in \mathrm{Spec}(D)} \frac{(i\lambda + m)}{(i\lambda + \Lambda)}$$

Λ : Pauli-Villars regulator mass

λ : eigenvalues of the Dirac operator D

SPT from massive fermions

Low energy limit $\longleftrightarrow$ Taking the mass m very large

$$m \to \pm\Lambda$$

$$Z = \prod_{\lambda \in \mathrm{Spec}(D)} \frac{(i\lambda \pm \Lambda)}{(i\lambda + \Lambda)}$$

$$= \begin{cases} 1 & m = +\Lambda \\ \exp(-2\pi i \cdot \eta) & m = -\Lambda \end{cases} \quad (\Lambda \to \infty)$$

η : Atiyah-Patodi-Singer eta-invariant.

[Atiyah-Patodi-Singer, 1975]

SPT from massive fermions

SPT phase 1

$$m < 0$$

$$Z = \exp(-2\pi i \cdot \eta)$$

: nontrivial phase

SPT phase 2

$$m > 0$$

$$Z = 1$$

: trivial phase

The SPT phases realized by fermions are completely characterized by the Atiyah-Patodi-Singer eta-invariant.

SPT from massive fermions

<u>Example 1</u>: Integer quantum hall state

Suppose there are k copies of massive fermion coupled to

$A = A_\mu dx^\mu$: background field for G=U(1)

$$Z = \exp(-2\pi i \cdot \eta)$$
$$= \exp(\int \frac{ik}{4\pi} AdA)$$: Atiyah-Patodi-Singer index theorem

We recover $\log Z = \int \frac{ik}{4\pi} AdA$

SPT from massive fermions

<u>Example 2</u>: 3+1 dim. topological superconductors

- Low energy limit of ν copies of massive Majorana fermion in 3+1 dim.

- Symmetry G = time-reversal
  (or equivalently spatial reflection by CPT theorem)

- Background field for the time-reversal symmetry
  = Putting the theory on un-orientable manifold.

SPT from massive fermions

Example 2: 3+1 dim. topological superconductors

- Fact 1: Z is always a 16-th root of unity.
- Fact 2: $\mathbb{RP}^4$: real projective space (un-orientable)

$$Z(\mathbb{RP}^4) = \exp(-\pi i \cdot \eta(\mathbb{RP}^4)) \quad \text{(for Majorana)}$$
$$= \exp(\frac{2\pi i \nu}{16}) \qquad \text{[Witten,2015]}$$

3+1 dim. topological superconductors are classified by

$$\nu \in \mathbb{Z}_{16}$$

Berry phase of SPT phase

Berry phase of SPT phase

$$S = \int d^{d+1}x \ \bar{\Psi}(iD + m)\Psi$$

A unique ground state in the Hilbert space.

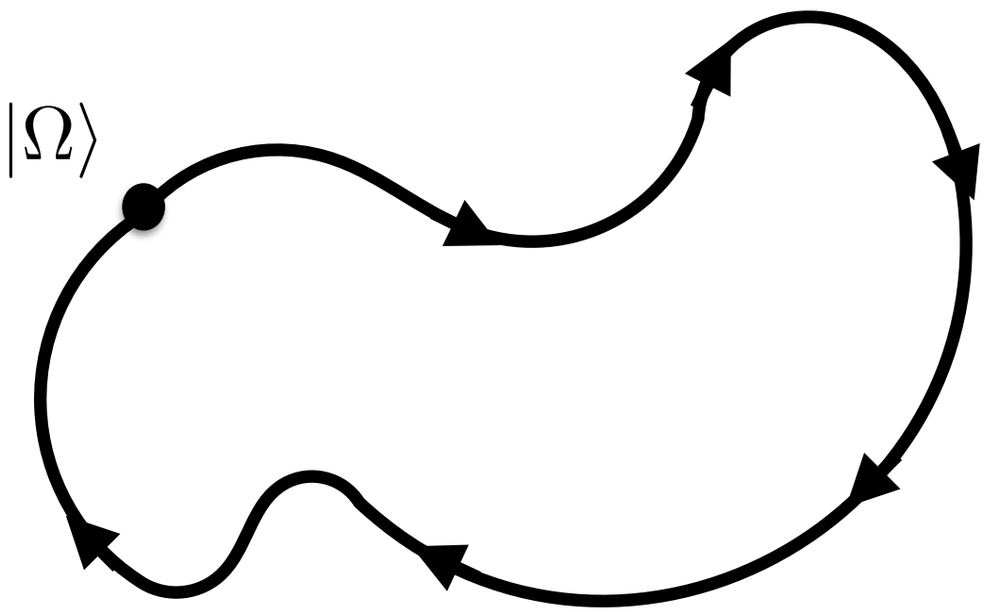
= The unique state of the SPT phase

$|\Omega\rangle$ on a spacial manifold Y
(d-dimensional)

Although the Hilbert space is one-dimensional,
it is nontrivial due to the Berry phase!

Berry phase of SPT phase

parameter space of background fields $\mathbf{g}_{\mu\nu}, \mathbf{A}_\mu$



$|\Omega\rangle$: the unique ground state

$|\Omega\rangle \to e^{i\mathcal{B}}|\Omega\rangle$: Berry phase

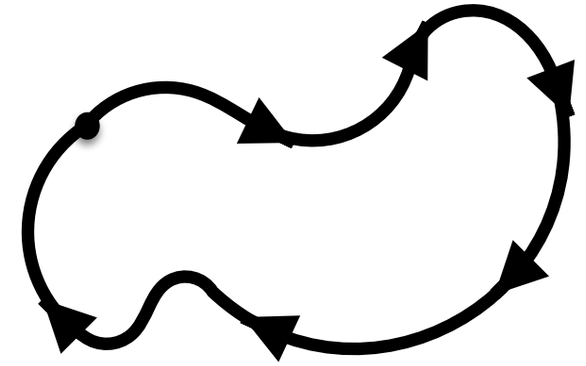
Berry phase of SPT phase

Change parameters as (Euclidean) time τ evolves.

$g_{ij}(\tau, \vec{x}),\ \vec{A}(\tau, \vec{x})$: slowly change as τ evolves

Returning to the same point:

$$\tau \in S^1$$

$S^1 =$ 

Berry phase = partition function on

$$S^1 \times Y$$

Euclidean time

d-dim. space

Berry phase of SPT phase

Berry phase formula:

$$e^{iB} = Z(S^1 \times Y)$$
$$= \exp(-2\pi i \cdot \eta(S^1 \times Y)) \qquad (m < 0)$$

For more general manifold X

$Z = \exp(-2\pi i \cdot \eta(X))$: generalized Berry phase (only in this talk)

Contents

1. Introduction

2.SPT phases

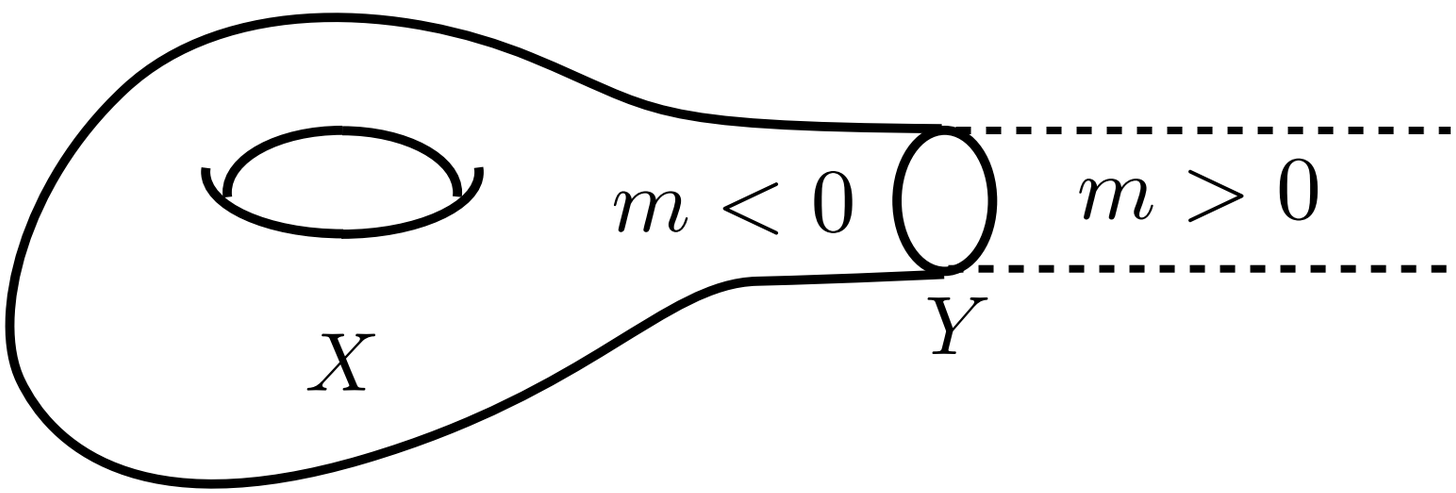
3.A class of SPT phases from massive fermions

4. Manifolds with boundary and fermion anomalies

5.Anomalies of topological quantum field theory

6.Summary

Manifolds with boundary



Suppose:

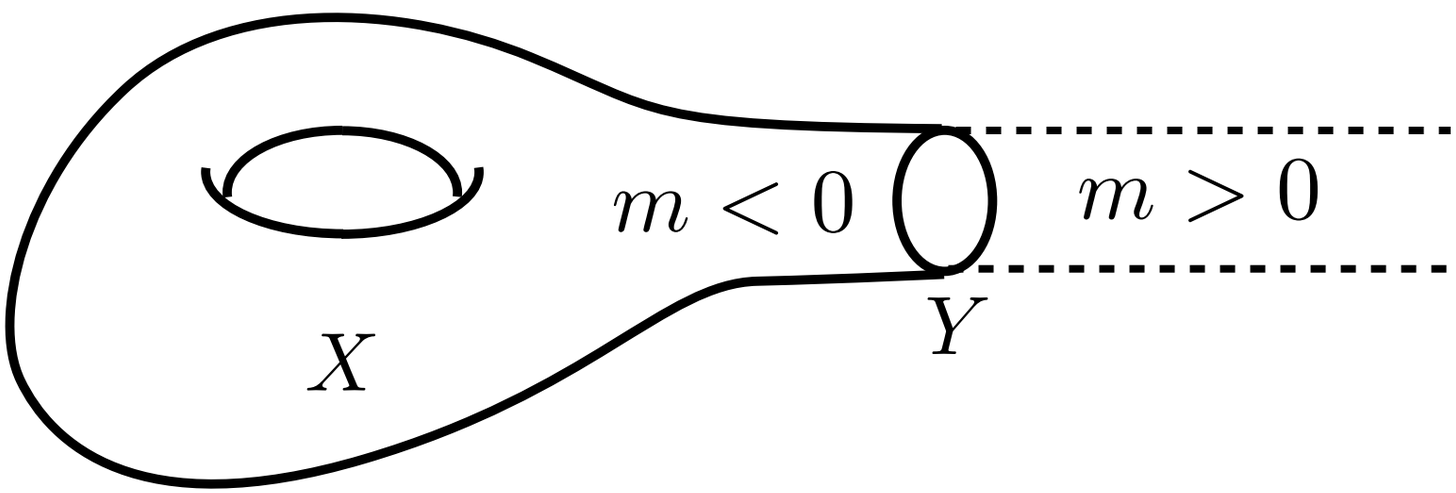
- Bulk X has the SPT phase with

$$Z_{\text{bulk}} = \exp(-2\pi i \cdot \eta)$$

- Boundary Y has chiral fermion with

$$Z_{\text{boundary}} = \det(\gamma^\mu D_\mu)_{\text{chiral}}$$

Manifolds with boundary



The point 1:

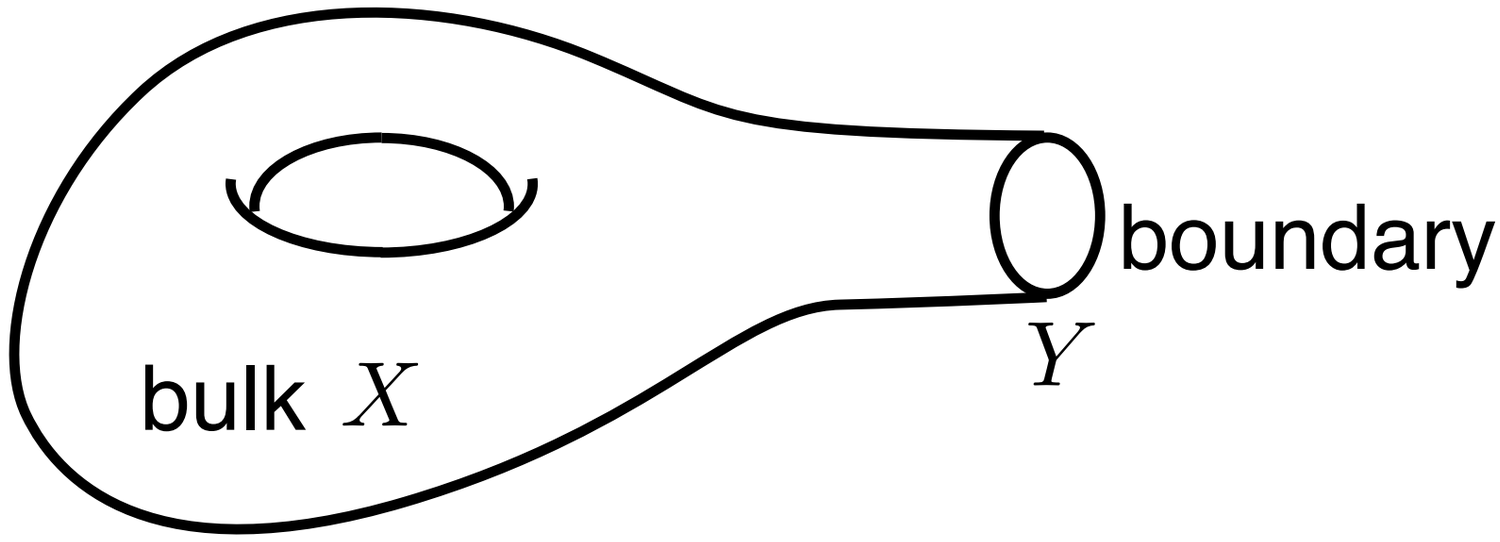
$$Z = Z_{\text{bulk}} \cdot Z_{\text{boundary}}$$

SPT　　chiral

: well defined and anomaly free because
d+1 dim massive fermion has lattice construction.

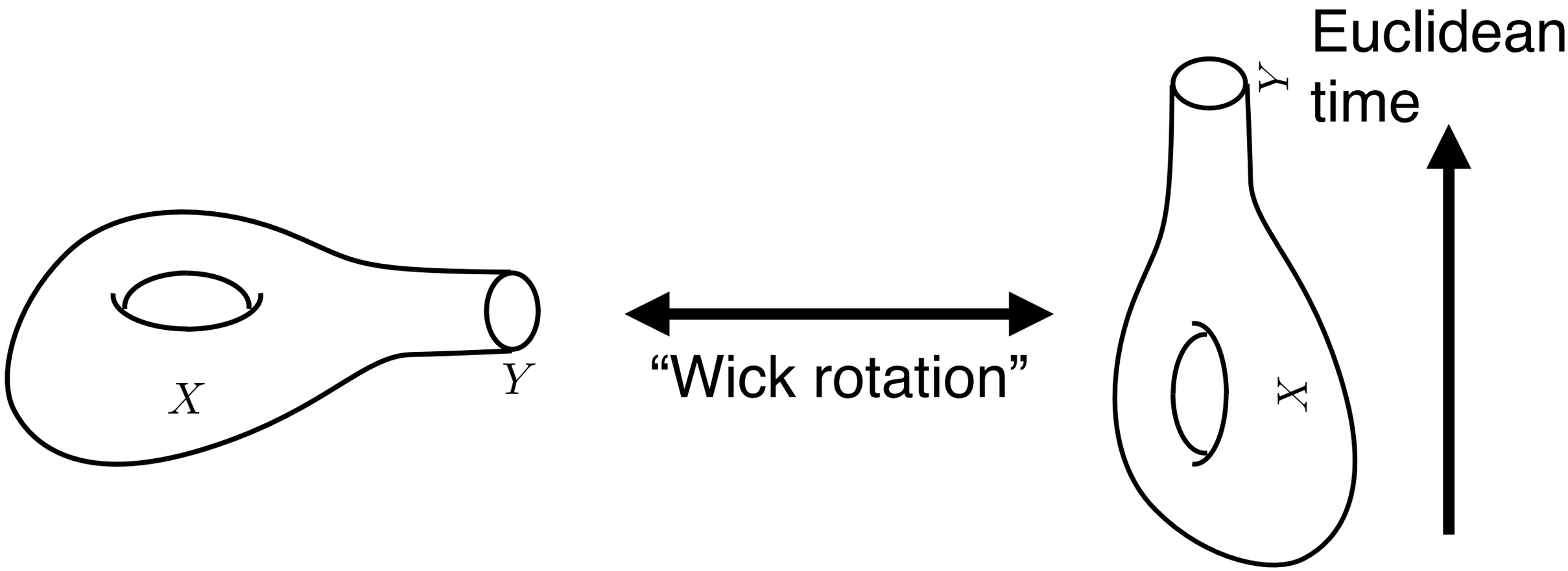
Manifolds with boundary

What is Z_{bulk} ?



The Z_{bulk} is now a partition function of bulk theory on a manifold X with boundary Y .

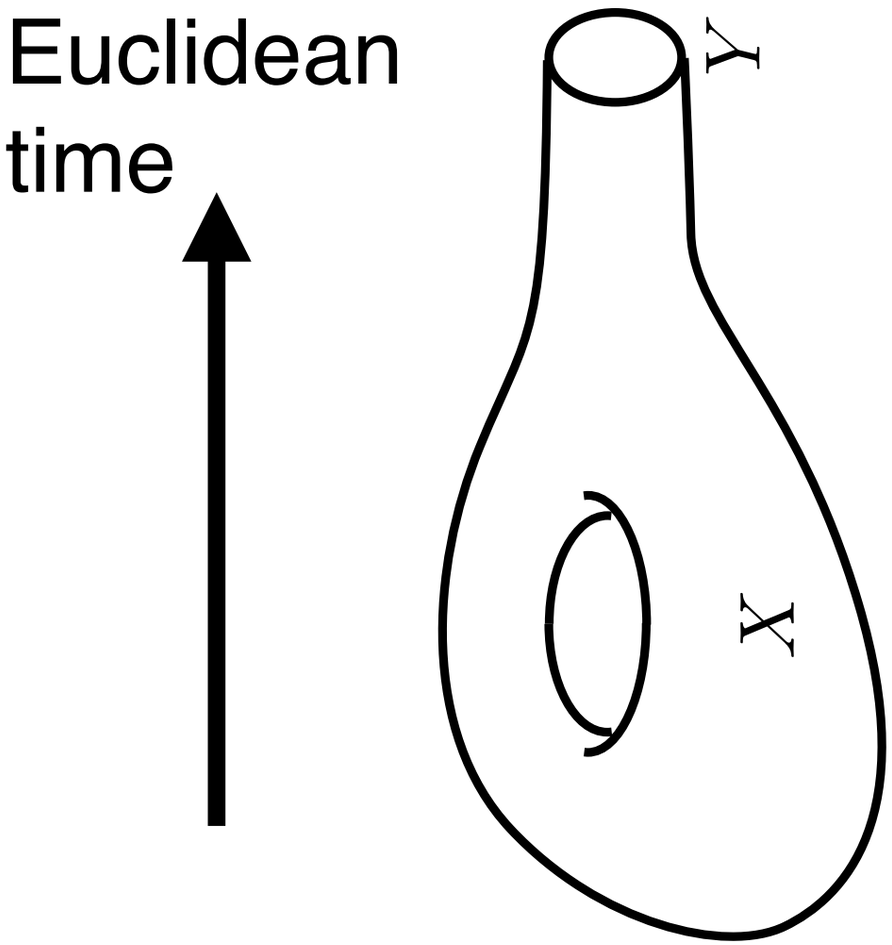
Manifolds with boundary



The boundary Y can be regarded as a time-slice on which we can consider physical states.

Manifolds with boundary

What is $Z_{\rm bulk}$?



$$Z_{\rm bulk} \sim |X\rangle$$

$|X\rangle$ is a state on Y created by the path integral on X

Surprisingly, it is not a number, but a state vector in the Hilbert space!

Ambiguity by Berry phase

- To get $Z_{\rm bulk}$ as a number, we might consider

$$Z_{\rm bulk} \overset{???}{:=} \langle \Omega | X \rangle$$

But this is ambiguous due to the Berry phase

$$|\Omega\rangle \to e^{i\mathcal{B}} |\Omega\rangle \ : \text{Berry phase}$$

The point 2:

$Z_{\rm bulk}$ is ambiguous by Berry phase
(or generalized Berry phase)

Ambiguity of $Z_{\rm bulk} = \exp(-2\pi i \cdot \eta)$ and its properties are known well in mathematics.

[Dai-Freed,1994]
[KY,2016]

Anomaly

The point 1:

$$Z = Z_{\text{bulk}} \cdot Z_{\text{boundary}}$$

: well defined and anomaly free

The point 2:

Z_{bulk} is ambiguous by Berry phase.

Conclusion:

Z_{boundary} is ambiguous: anomaly

(It is a generalization of $Z_{\text{boundary}}[A_\mu + \partial_\mu \alpha] \neq Z_{\text{boundary}}[A_\mu]$)

Anomaly descendant

Anomaly descendant

At the perturbative level, there is anomaly descendant equations.

$$I_d^{(1)} \text{ : anomaly (infinitesimal)}$$

$$dI_d^{(1)} = \delta_{\text{gauge}} I_{d+1}^{(0)}$$

$$dI_{d+1}^{(0)} = I_{d+2} \sim \text{tr} F^{d/2+1} \text{ (anomaly polynomial)}$$

By Atiyah-Patodi-Singer index theorem

$$\exp(2\pi i \int_Y I_d^{(1)}) \to \exp(2\pi i \int_X I_{d+1}^{(0)}) \sim \exp(-2\pi i \cdot \eta)$$

Anomaly descendant

$$\exp(2\pi i \int_X I^{(0)}_{d+1}) \sim \exp(-2\pi i \cdot \eta)$$: Berry phase

$$I_{d+2} = dI^{(0)}_{d+1}$$: Berry curvature

- Somewhat mysterious $I_{d+2},\ I^{(0)}_{d+1}$ have a concrete meaning in the SPT phase as Berry phase & curvature!

- At the non-perturbative level, the eta-invariant must be taken seriously;

  Sometimes $I_{d+2} = 0$ but $\eta \neq 0$

Contents

1. Introduction

2.SPT phases

3.A class of SPT phases from massive fermions

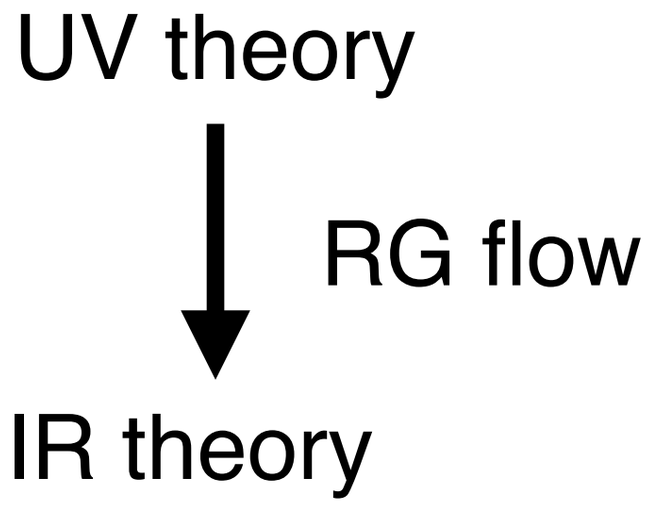
4. Manifolds with boundary and fermion anomalies

5.Anomalies of topological quantum field theory

6.Summary

't Hooft anomaly matching

UV anomaly = IR anomaly



't Hooft anomaly matching

Sometimes it is possible that

UV "chiral fermion"

RG flow

IR topological QFT (TQFT)

IR TQFT must have anomaly!

See e.g. [Seiberg-Witten,2016]
[Witten,2016]

Topological superconductor

I only discuss topological superconductor

- 3=2+1-dim. boundary (Chern-Simons theory, etc.)

- Symmetry G = time-reversal → unorientable manifold

- Classified in bulk by

$$\nu \in \mathbb{Z}_{16}$$

Time reversal anomaly

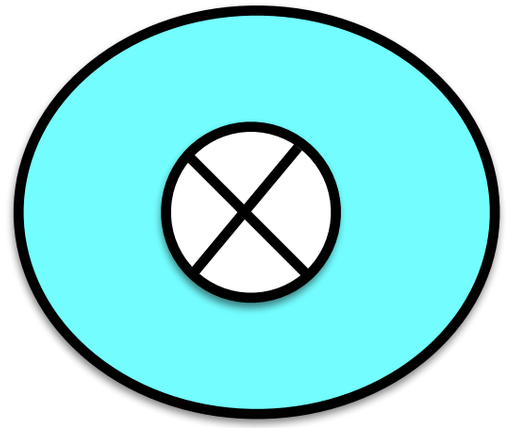
Setup to see the anomaly of TQFT:

$$Y' = CC^2 \times S^1 \text{ (3-dim.)}$$

$$\partial Y' = S^1 \times S^1 = T^2$$

$CC^2 =$ crosscap on a disk =

: unorientable

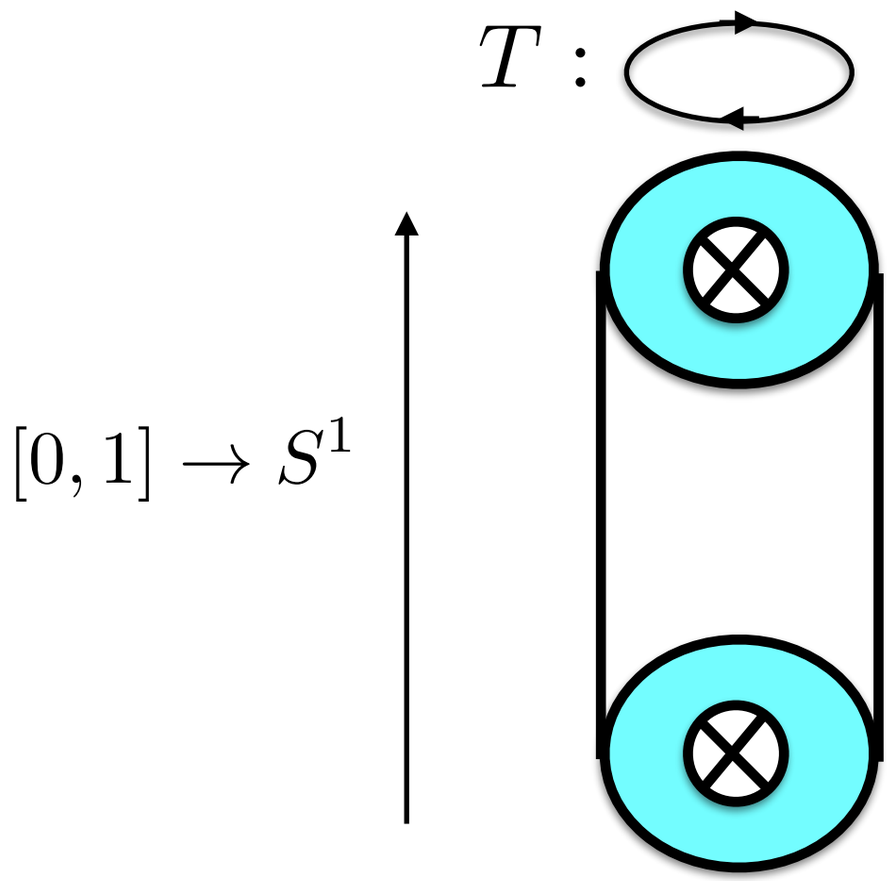


$|Y'\rangle$: state on the torus $\partial Y' = T^2$ obtained by the path integral on Y'

$T \in SL(2, Z)$: an element of modular transformation of the torus

Time reversal anomaly

Point : $T \in SL(2,Z)$ does not change the topology of the manifold $Y' = CC^2 \times S^1$



Action of T :

- Consider $CC^2 \times [0,1]$
- Twist it by 2π rotation
- Glue the top and bottom

Time reversal anomaly

Point : $T \in SL(2,Z)$ does not change the topology of the manifold $Y' = CC^2 \times S^1$

So naively $T|Y'\rangle = |Y'\rangle$, but actually it turns out

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle$$

$\exp(\frac{2\pi i\nu}{16})$: 't Hooft anomaly of time reversal

[Tachikawa-KY,2016]

In "surgery", this gives an ambiguity of partition function.

Remark

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle$$

Facts (which I don't explain):

- By anomaly matching with fermions, it is known that

$$\nu \in \mathbb{Z}_{16}$$

- For example, one can compute that

  $\nu = \pm 2$ for Chern-Simons theory $U(1)_2 \times U(1)_{-1}$

- General formula for time-reversal anomaly of TQFT is

$$\exp(\frac{2\pi i\nu}{16}) = \sum_p S_{0p} e^{2\pi i\theta_p} \eta_p$$

Contents

1. Introduction

2.SPT phases

3.A class of SPT phases from massive fermions

4. Manifolds with boundary and fermion anomalies

5.Anomalies of topological quantum field theory

6.Summary

Summary

- Ambiguity of partition functions on manifolds with boundary gives an interesting relation

Anomaly of boundary theory

= (Generalized) Berry phase of bulk SPT phase

- TQFTs can have ambiguities of partition functions

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle \qquad \begin{array}{c} Y' = CC^2 \times S^1 \\ T \in SL(2,Z) \end{array}$$

Future direction

- Systematically understand anomalies of TQFTs.

  Anomaly matching
  (non-trivial mathematical structure?)

- Deeper understanding of $Z = \exp(-2\pi i \cdot \eta)$