Matrix model of Chern-Simons matter theories beyond the spherical limit

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Chern-Simons matter theory

① (Condensed matter physics)

Fractional quantum hall effect, anyon [Zhan-Hansson-Kivelson '88] [Fradkin-Lopez '91]

② (Mathematics)

Knot theory, Jones polynomial [Witten '89]

③ (String theory)

Cubic string field theory, Open topological string theory [Witten '85] [Gopakumar-Vafa '98]

④ (M-theory)

Interaction between membranes. [BLG '07, ABJM '08]

⑤ (3d CFT)

Infinitely many interacting CFT (conformal zoo). [Moore_Seiberg '89] [Witten '89]

⑥ (AdS/CFT correspondence)

Dual CFT3 of (HS) gravity on AdS4 [Klebanov-Polyakov '02]

Pure (HS) gravity on AdS3 [Gaberdiel_Gopakumar '11]

Partition function of CSM

① ∃SUSY ⇒ Localization

Path integral ⇒ Matrix model [Pestun '07]

(i) S^3 Exact

[Kapustin-Willet-Yaakov '09] [Jafferis '10] [Hama-Hosomichi-Lee '10]

(ii) $S^2 \times S^1$ (superconformal index) Exact

[Bhattacharya-Minwalla '08] [Kim '09] [Imamura-Yokoyama '09]

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② Non-SUSY ⇒ Large N limit

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(ii) $S^2 \times S^1$ (near critical high temperature) Exact

[Jain-Minwalla-Sharma-Takimi-Wadia-SY '11]

Exact analysis is difficult for the S3 partition function...

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Exact analysis is difficult for the S3 partition function...

⇒ Study a class of matrix model!

Plan

✓ 1. Introduction

2. The class of matrix models

3. Loop equation

4. Solving the loop equation

    4.1 planar solution

    4.2 hole correction

    4.3 genus one correction

5. Application to pure CS theory

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Matrix model of $U(N)_k$ CSM

[SY '16]

① Consider partition function of CSM on S3

$$Z = \int \mathcal{D}A\mathcal{D}\Phi e^{-\frac{ik}{4\pi}\int_{\mathbf{S}^3}(A\wedge dA - \frac{2i}{3}A\wedge A\wedge A) - S[\Phi,A]}$$

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② Expand the gauge field by vector spherical harmonics on S3

$$A_\mu(x) = \sum_{s\in\frac{1}{2}\mathbf{N}}\left(\sum_{\substack{|l|\le s\\|r|\le s+1}} a_{l\,r}^{s\,s+1} Y_{l\,r}^{s\,s+1}{}_\mu(x) + \sum_{\substack{|l|\le s+1\\|r|\le s}} a_{l\,r}^{s+1\,s} Y_{l\,r}^{s+1\,s}{}_\mu(x) + \sum_{|l|,|r|\le s} a_{l,r}^{s,s} Y_{l\,r}^{s\,s}{}_\mu(x)\right)$$

Lorenz gauge $\quad a_{l,r}^{s,s} = 0 \qquad s > 0$

Residual gauge $\quad a_{0,0}^{0,0} = 0$ except its Cartan part $\Rightarrow \sigma$

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Expand the matter fields similarly.

③ Integrate out all the massive modes.

$$Z = \int d^N\sigma e^{-i\frac{k}{4\pi}\sum_{s=1}^N \sigma_s^2 + \cdots},$$

Matrix model of $U(N)_k$ CSM

[SY '16]

④ Pure Chern-Simons case ⇒ SUSY localization, Cohomological localization

$$Z \sim \int_{\mathbf{R}^N} d^N\sigma e^{-i\frac{k}{4\pi}\sum_{s=1}^N \sigma_s^2 + \sum_{t\neq s}^N \log 2\sinh(\frac{\sigma_s - \sigma_t}{2})}$$

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⑤ Inclusion of the matter fields will be $Z = \mathfrak{N}\int_{\mathbf{R}^N} d^N\sigma \prod_{t\neq s}^N 2\sinh(\frac{\sigma_s - \sigma_t}{2}) e^{-V[\sigma]}$

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⑥ Restrict the potential to consist of single trace operators:

$$V[\sigma] = N\sum_{s=1}^N W_\sigma(\sigma_s) \qquad W_\sigma(\sigma) = \frac{1}{2\tilde{\lambda}}\sigma^2 + \sum_{p=0}^\infty t_p e^{\sigma p} + \delta W_\sigma(\sigma)$$

Ex. (i) N=2 Chern-Simons theory with Nf fundamental chiral fields

$$\frac{1}{\tilde{\lambda}} = \frac{ik}{2\pi N} + \frac{iN_F}{4\pi N}, \quad t_0 = -\frac{N_F}{N}\frac{i\pi}{24}, \quad t_p = \frac{iN_F}{2N}\frac{(-)^{p-1}}{p^2}$$

$$\delta W_\sigma(\sigma) = \sum_{p=0}^\infty u_p \sigma e^{\sigma p}, \quad u_0 = i\frac{\zeta}{2N} - \frac{N_F}{4N}, \quad u_p = \frac{iN_F}{2\pi N}\frac{(-)^p}{p}$$

(ii) N=2 Chern-Simons theory with Nf pairs of anti/fundamental chiral fields

$$\tilde{\lambda} = -2\pi i\frac{N}{k}, \quad t_0 = \frac{N_F}{N}\log 2, \quad t_p = \frac{N_F}{N}\frac{(-)^p}{p} \quad (p\geq 1), \quad \delta W_\sigma = -\frac{i}{2N}\zeta + \frac{N_F}{2N}.$$

Matrix model of U(N)$_k$ CSM

[SY '16]

⑦ Change the integration variables such that $\sigma_s = e^{\phi_s}$

$$Z = \mathfrak{N} \int_{\mathbf{R}_+^N} d^N\phi \prod_{t \neq s}^{N} (\phi_s - \phi_t) e^{-N \sum_{s=1}^{N} W(\phi_s)}$$

$$W(\phi_s) = \frac{1}{2\tilde{\lambda}} (\log \phi_s)^2 + \log \phi_s + \sum_{p=0}^{\infty} t_p \phi_s^p + \delta W(\phi_s)$$

Logarithmic cut!

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Logarithmic cut!

This can be written by using a positive definite hermitian matrix Φ as

$$Z \propto \int \mathcal{D}\Phi e^{-N\mathrm{Tr}W(\Phi)}$$

$$W(\Phi) = \frac{1}{2\tilde{\lambda}} (\log\Phi)^2 + \log\Phi + \sum_{p=0}^{\infty} t_p \Phi^p + \delta W(\Phi)$$

The goal is to solve this class of matrix models in the 1/N expansion.

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Resolvent

Definition

$$\omega(z) := \frac{1}{N}\sum_{s=1}^{N}\left\langle \frac{1}{z-\phi_s}\right\rangle = \frac{1}{N}\mathrm{Tr}\left\langle \frac{1}{z-\Phi}\right\rangle = \frac{1}{N}\sum_{p\geq 0}\frac{\langle \mathrm{Tr}\Phi^p\rangle}{z^{p+1}}.$$

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Once the resolvent is determined, the free energy is determined as follows.

- $$\frac{d}{dT_z}F = N^2\left(\frac{1}{z} - \omega(z)\right) \qquad \frac{d}{dT_z} = \sum_{p\geq 1}\frac{-1}{z^{p+1}}\frac{\partial}{\partial t_p} \qquad \left(\because \frac{\partial F}{\partial t_p} = N\langle \mathrm{Tr}\Phi^p\rangle\right)$$

- $$\frac{\partial F}{\partial \tilde{\lambda}} = -\frac{\partial \log \mathfrak{N}}{\partial \tilde{\lambda}} - \frac{N}{2\tilde{\lambda}^2}\left\langle \sum_{s=1}^{N}(\log \phi_s)^2\right\rangle = -\frac{\partial \log \mathfrak{N}}{\partial \tilde{\lambda}} - \frac{N^2}{2\tilde{\lambda}^2}\int_{\mathbf{R}_+} dx\rho(x)(\log x)^2$$
$$= -\frac{\partial \log \mathfrak{N}}{\partial \tilde{\lambda}} - \frac{N^2}{2\tilde{\lambda}^2}\oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\omega(w)(\log w)^2$$

where $$\rho(x) := \frac{1}{N}\mathrm{Tr}\langle \delta(x-\Phi)\rangle = \frac{1}{N}\sum_{s=1}^{N}\langle \delta(x-\phi_s)\rangle$$

The density function and the resolvent are related by $\omega(x-i\epsilon) - \omega(x+i\epsilon) = 2\pi i\rho(x)$

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The correlator is also computed as follows.

$$\left(\prod_{k=1}^{n}\frac{d}{dT_{z_k}}\right)(-F) = N^n\left\langle \prod_{k=1}^{n}\mathrm{Tr}\left(\frac{1}{z_k-\Phi}\right)\right\rangle_{\mathrm{conn}}$$

Schwinger-Dyson equation

Expectation value of a generic operator O[ϕ]

$$\langle \mathcal{O}[\phi] \rangle = \frac{\mathfrak{N}}{Z} \int_{\mathbf{R}_+^N} d^N\phi \mathcal{O}[\phi] \left(\prod_{t \neq s}^{N} (\phi_s - \phi_t) e^{-N \sum_{s=1}^{N} W(\phi_s)} \right).$$

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Consider a one-to-one transformation on R+ denoted by ϕ s → ϕ s'

$$\frac{\mathfrak{N}}{Z} \int_{\mathbf{R}_+^N} d^N\phi \mathcal{O}[\phi] \left(\prod_{t \neq s}^{N} (\phi_s - \phi_t) e^{-N \sum_{s=1}^N W(\phi_s)} \right) = \frac{\mathfrak{N}}{Z} \int_{\mathbf{R}_+^N} d^N\phi' \mathcal{O}[\phi'] \left(\prod_{t \neq s}^{N} (\phi'_s - \phi'_t) e^{-N \sum_{s=1}^N W(\phi'_s)} \right)$$

Suppose the (infinitesimal) transformation $\phi'_s = \phi_s + a\delta\phi_s$

Expand the right-hand side in terms of a

$$\left\langle \sum_{s=1}^{N} (\mathcal{O}[\phi] \frac{\partial \delta\phi_s}{\partial \phi_s} + \frac{\partial \mathcal{O}[\phi]}{\partial \phi_s} \delta\phi_s) + 2 \sum_{s>t} \frac{\mathcal{O}[\phi]}{\phi_s - \phi_t} (\delta\phi_s - \delta\phi_t) - \sum_{s=1}^{N} N \mathcal{O}[\phi] \frac{\partial W(\phi_s)}{\partial \phi_s} \delta\phi_s \right\rangle = 0$$

Loop equation

Let us choose $\mathcal{O} = 1, \quad \delta\phi_s = \dfrac{\phi_s}{z - \phi_s}$

Then the transformation ϕs → ϕs' is one-to-one on R+.

$\because \quad \delta\phi_s|_{\phi_s=0} = 0 \quad a\dfrac{\partial\delta\phi_s}{\partial\phi_s} = \dfrac{az}{(z-\phi_s)^2} > 0$ for $az > 0$

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$$LHS = z\left\langle \sum_{s=1}^{N} \frac{1}{(z-\phi_s)^2} + 2\sum_{s>t}\frac{1}{(z-\phi_s)(z-\phi_t)} - \sum_{s=1}^{N} N\frac{\partial W(\phi_s)}{\partial\phi_s}\frac{1}{z-\phi_s}\right\rangle + \left\langle \sum_{s=1}^{N} N\frac{\partial W(\phi_s)}{\partial\phi_s}\right\rangle$$

$$1^{st} + 2^{nd} = \sum_{s,t=1}^{N}\left\langle \frac{1}{(z-\phi_s)(z-\phi_t)}\right\rangle = \frac{d}{dT_z}\omega(z) + N^2\omega(z)^2$$

$$3^{rd} = \left\langle -N\sum_{s=1}^{N}\frac{W'(\phi_s)}{z-\phi_s}\right\rangle = -N^2\int_{\mathbf{R}_+} dx\rho(x)\frac{W'(x)}{z-x} = -N^2\oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\frac{W'(w)}{z-w}\omega(w)$$

$4^{th} = 0$

$$\left(\begin{aligned} \left\langle \sum_n NW'(\phi_n)\right\rangle &= \frac{\mathfrak{N}}{Z}\int_{\mathbf{R}_+^N} d^N\phi \sum_n NW'(\phi_n)\prod_{s\neq t}(\phi_s-\phi_t)e^{-\sum_{s=1}^N NW(\phi_s)} = \frac{\mathfrak{N}}{Z}\sum_n\int_{\mathbf{R}_+^N} d^N\phi\prod_{s\neq t}(\phi_s-\phi_t)\left(-\frac{\partial}{\partial\phi_n}e^{-\sum_{s=1}^N NW(\phi_s)}\right) \\ &= \frac{\mathfrak{N}}{Z}\sum_n\int_{\mathbf{R}_+^N} d^N\phi\left(\frac{\partial}{\partial\phi_n}\prod_{s\neq t}(\phi_s-\phi_t)\right)e^{-\sum_{s=1}^N NW(\phi_s)} = \sum_n\left\langle \sum_{t\neq n}\frac{2}{\phi_n-\phi_t}\right\rangle = 0 \end{aligned} \right)$$

$$\Rightarrow \quad \boxed{\omega(z)^2 - \oint_{\mathcal{C}_{\mathbf{R}_+}}\frac{dw}{2\pi i}\frac{W'(w)}{z-w}\omega(w) + \frac{1}{N^2}\frac{d}{dT_z}\omega(z) = 0.}$$

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Solving the loop equation

[SY '16]

The matrix model potential is generically written as

$$W(\phi) = W_0(\phi) + \frac{1}{N} W_1(\phi),$$

Assume

$$W_0(\phi) = \frac{1}{2\widetilde{\lambda}} (\log \phi)^2 + \log \phi + \sum_{p=0}^{\infty} t_p \phi^p + \cdots .$$

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The consistent 1/N expansion of the resolvent will be

$$\omega(z) = \sum_{g=0}^{\infty} (N^{-2g} \omega_g(z) + N^{-2g-1} \omega_{g+\frac{1}{2}}(z)) = \sum_{\bar{g} \in \frac{1}{2}\mathbf{N}} N^{-2\bar{g}} \omega_{\bar{g}}(z)$$

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The loop equation boils down to

$$\omega_0^2(z) = \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_0'(w)\omega_0(w)}{z-w}, \qquad \hat{K}\omega_{\frac{1}{2}}(z) = -\oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_0(w)}{z-w},$$

$$\hat{K}\omega_g(z) = \sum_{g'=1}^{g-1} \omega_{g'}(z)\omega_{g-g'}(z) + \sum_{g'=0}^{g-1} \omega_{g',1}(z)\omega_{g-g'-\frac{1}{2}}(z) - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_{g-\frac{1}{2}}(w)}{z-w} + \frac{d}{dT_z}\omega_{g-1}(z),$$

$$\hat{K}\omega_{g+\frac{1}{2}}(z) = 2\omega_g(z)\omega_{\frac{1}{2}}(z) + \sum_{g'=1}^{g-1} 2\omega_{g'}(z)\omega_{g-g'+\frac{1}{2}}(z) - \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_g(w)}{z-w} + \frac{d}{dT_z}\omega_{g-\frac{1}{2}}(z),$$

where

$$\hat{K}f(z) := \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_0'(w)}{z-w} f(w) - 2\omega_0(z)f(z).$$

Solving the planar loop equation

Claim The planar loop equation contains the saddle point equation in the planar limit.

$\because$ Compute the discontinuity between x - i ε and x + i ε

$$LHS: \quad (\omega_0(x-i\epsilon)-\omega_0(x+i\epsilon))(\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon))$$
$$=2\pi i\rho_0(x)(\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon))$$

$$RHS: \quad \int_{\mathbf{R}_+} dy W_0'(y)\left(\frac{\rho_0(y)}{x-i\epsilon-y}-\frac{\rho_0(y)}{x+i\epsilon-y}\right)$$
$$=\int_{\mathbf{R}_+} dy W_0'(y)\rho_0(y)2\pi i\delta(x-y)=W_0'(x)2\pi i\rho_0(x).$$

➡ $$\omega_0(x-i\epsilon)+\omega_0(x+i\epsilon)=W_0'(x)$$

where $x\in \mathrm{supp}(\rho_0)$

Planar resolvent

The support of the density function consists of s distinct connected intervals

$$\mathrm{supp}(\rho_0) = \cup_{i=1}^{s} [a_{2i-1}, a_{2i}] \qquad 0 < a_1 < \cdots < a_{2s}$$

Each interval corresponds to a square root cut of the quadratic loop equation.

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Define a trial function such that $\omega_0(z) = h(z) H(z)$ $\quad h(z) = \sqrt{\prod_{i=1}^{2s} (z - a_i)}$

Assume (or construct a solution) that
there exists a limit approaching the infinity such that the resolvent behaves as

$$\omega_0(z) \overset{z\sim\infty}{\sim} \frac{1}{z} \quad \Rightarrow \quad H(z) \overset{z\sim\infty}{\sim} \frac{\pm 1}{z^{s+1}} \quad \Rightarrow \quad \oint_{\mathcal{C}_\infty} \frac{dw}{2\pi i} \frac{H(w)}{w - z} = 0$$

$$(z \notin \mathrm{supp}(\rho_0))$$

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$$(z \notin \mathrm{supp}(\rho_0))$$

Assume further that the trial function is analytic except supp(ρ_0)

$$\Rightarrow \quad H(z) = -\int_{\mathrm{supp}(\rho_0)} \frac{dy}{2\pi i} \frac{W_0'(y)}{(y - z)h(y)} = -\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{W_0'(w)}{(w - z)h(w)} \frac{1}{2}$$

$$\Rightarrow \quad \omega_0(z) = \frac{-h(z)}{2} \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{W_0'(w)}{(w - z)h(w)}$$

Planar resolvent

Determination of the 2s endpoints of the cut

① The asymptotic behavior of the resolvent $\omega_0(z) \overset{z\sim\infty}{\sim} \frac{1}{z}$

$$\frac{1}{2}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i}\frac{w^k W_0'(w)}{h(w)} = \pm\delta_{k,s} \qquad \forall k = 0,\cdots,s$$

② Stability against the tunneling of eigenvalues between different cuts

[David '90]

= "Equality of the Lagrange multiplier computed from the resolvent"

[Jurkiewicz '90]

$$\int_{a_{2i}-\epsilon}^{a_{2i+1}+\epsilon} dx\widetilde{\rho}_0(x) = 0 \quad \text{or} \quad \oint_{\beta_i} dw\omega_0(w) = 0 \qquad \forall i = 1,\cdots,s-1$$

Hole correction

[SY '16]

The genus half loop equation

$$\hat{K}\omega_{\frac{1}{2}}(z) = -\oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_1'(w)\omega_0(w)}{z-w},$$

where

$$\hat{K}f(z) := \oint_{\mathcal{C}_{\mathbf{R}_+}} \frac{dw}{2\pi i} \frac{W_0'(w)}{z-w} f(w) - 2\omega_0(z)f(z)$$

This contains the saddle point equation:

$$\omega_{\frac{1}{2}}(x-i\epsilon) + \omega_{\frac{1}{2}}(x+i\epsilon) = W_1'(x)$$

General solution is not known yet $\cdots$

Genus 1 correction

The genus 1 loop equation

$$\hat{K}\omega_1(z) = \frac{d}{dT_z}\omega_0(z).$$

$$\frac{d}{dT_z} = \sum_{p\geq 1} \frac{-1}{z^{p+1}}\frac{\partial}{\partial t_p}$$

A general solution [Ambjorn et.al. '92] [Akemann '96]

$$\omega_1(z) = \frac{1}{16}\sum_{i=1}^{2s}\chi_i^{(2)}(z) - \frac{1}{8}\sum_{i<j}\frac{\chi_i^{(1)}(z)-\chi_j^{(1)}(z)}{a_i-a_j} + \frac{1}{4}\sum_{i=1}^{2s}\sum_{l'=0}^{s-2}\chi_i^{(1)}(z)\alpha_{i,l'}a_i^{l'}$$

where

$$\chi_i^{(\mathfrak{n})}(z) = \frac{1}{M_i^{(1)}}\left(\frac{1}{h(z)(z-a_i)^{\mathfrak{n}}} - \sum_{k=1}^{\mathfrak{n}-1}M_i^{(\mathfrak{n}-k+1)}\chi_i^{(k)}(z)\right), \quad \hat{K}\chi_i^{(n)}(z) = \frac{1}{(z-a_i)^n}$$

$$M_i^{(k)} := \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{W_0'(w)}{h(w)(w-a_i)^k}.$$

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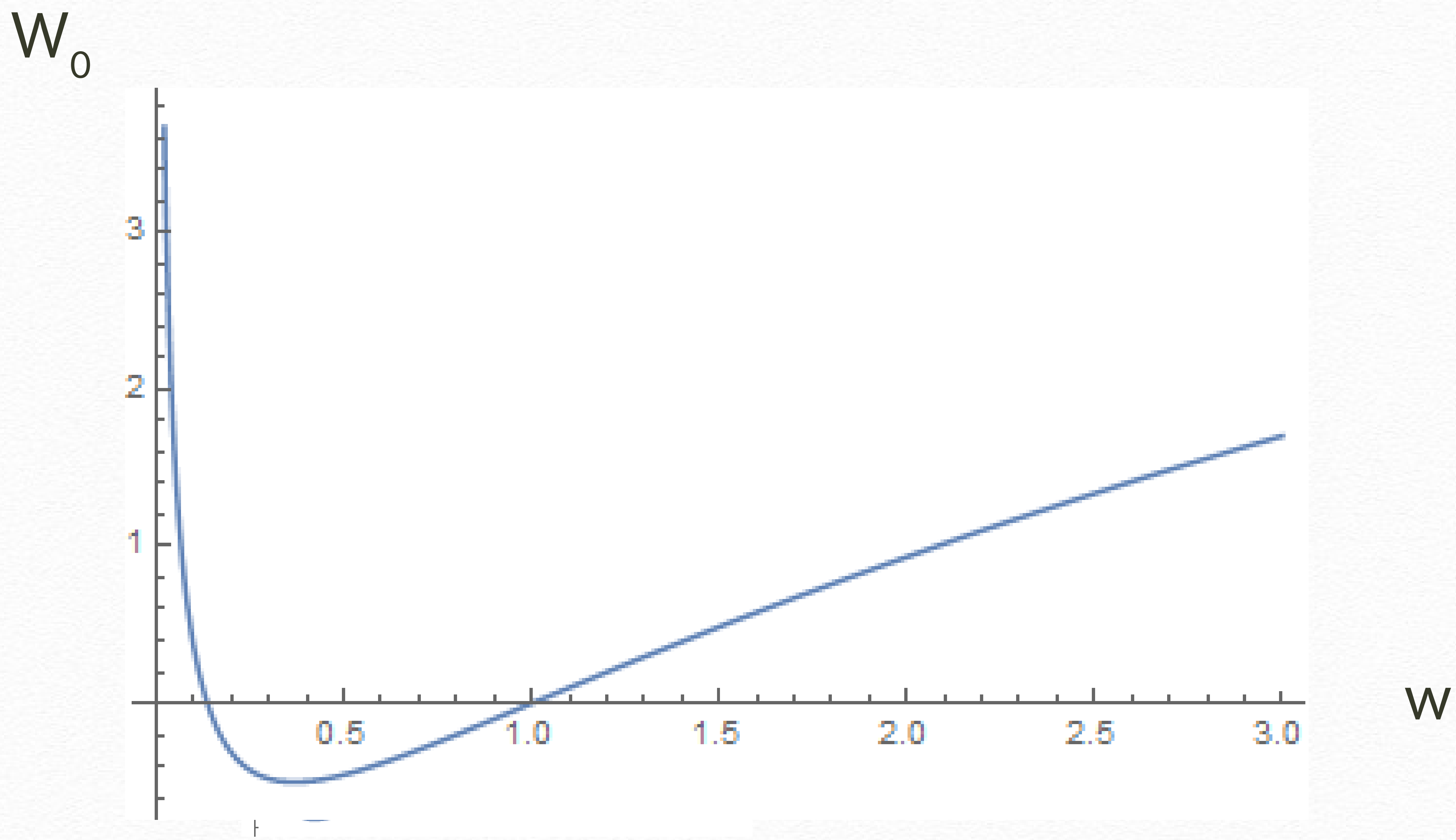
Application to pure CS theory

[SY '16]

Matrix model potential: $W_0(w) = \frac{(\log w)^2}{2\tilde{\lambda}} + \log w \qquad \tilde{\lambda} = -2\pi i \frac{N}{k}, \quad \mathfrak{N} = \frac{e^{\frac{-\pi(N-1)N(N+1)}{6ik}} i^{\frac{N^2}{2}}}{(2\pi)^N N!},$



$$\tilde{\lambda} = 1.0$$

➡ One cut solution!

Application to pure CS theory

[SY '16]

Support of the density function: $\mathrm{supp}(\rho_0) = [a_-, a_+]$

Planar resolvent:
$$\omega_0(z) = \frac{-h(z)}{2z}\oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}}\frac{dw}{2\pi i}\frac{wW_0'(w)}{(w-z)h(w)} = \frac{-h(z)}{2z}\oint_{\mathcal{C}_{[a_-,a_+]}}\frac{dw}{2\pi i}\frac{\frac{\log w}{\widetilde{\lambda}}+1}{(w-z)h(w)}$$
$$= \frac{\log\frac{(a_-+a_+)z-2a_-a_+-2\sqrt{a_-a_+}h(z)}{(-a_--a_+-2h(z)+2z)}}{2\widetilde{\lambda}z} + \frac{1}{2z}$$

$$\left(1^{\mathrm{st}} = \frac{h(z)}{2\widetilde{\lambda}z}\int_{-\infty}^{0}\frac{dw}{2\pi i}\frac{(\log|w|-\pi i)-(\log|w|+\pi i)}{(w-z)h(w)} = \frac{h(z)}{2\widetilde{\lambda}z}\frac{\log\frac{(a_-+a_+)z-2a_-a_+-2\sqrt{a_-a_+}h(z)}{z(-a_--a_+-2h(z)+2z)}}{h(z)} \right)$$

Application to pure CS theory

[SY '16]

Support of the density function: $\mathrm{supp}(\rho_0) = [a_-, a_+]$

Planar resolvent:
$$\omega_0(z) = \frac{-h(z)}{2z} \oint_{\mathcal{C}_{\mathrm{supp}(\rho_0)}} \frac{dw}{2\pi i} \frac{w W_0'(w)}{(w-z)h(w)} = \frac{-h(z)}{2z} \oint_{\mathcal{C}_{[a_-,a_+]}} \frac{dw}{2\pi i} \frac{\frac{\log w}{\tilde{\lambda}} + 1}{(w-z)h(w)}$$
$$= \frac{\log \frac{(a_-+a_+)z - 2a_-a_+ - 2\sqrt{a_-a_+}h(z)}{(-a_- - a_+ - 2h(z) + 2z)}}{2\tilde{\lambda} z} + \frac{1}{2z}$$

$$\left(1^{\mathrm{st}} = \frac{h(z)}{2\tilde{\lambda}z} \int_{-\infty}^{0} \frac{dw}{2\pi i} \frac{(\log|w| - \pi i) - (\log|w| + \pi i)}{(w-z)h(w)} = \frac{h(z)}{2\tilde{\lambda}z} \frac{\log \frac{(a_-+a_+)z - 2a_-a_+ - 2\sqrt{a_-a_+}h(z)}{z(-a_- - a_+ - 2h(z) + 2z)}}{h(z)} \right)$$

Edge of the cut:
$$\omega_0(z) \overset{z\sim-\infty}{\sim} \frac{1}{z} \quad \longleftrightarrow \quad \log \frac{a_- + a_+ + 2\sqrt{a_-a_+}}{2+2} = \tilde{\lambda}.$$
$$\omega_0(z) \overset{z\sim 0}{\sim} \text{finite} \quad \longleftrightarrow \quad \log \frac{-4a_-a_+}{-a_- - 2\sqrt{a_-a_+} - a_+} = -\tilde{\lambda}.$$
$$\Rightarrow \quad a_\pm = (e^{\frac{\tilde{\lambda}}{2}} \pm \sqrt{e^{\tilde{\lambda}} - 1})^2$$

$$\omega_0(z) = \frac{1}{\tilde{\lambda}z} \log \frac{z + 1 + h(z)}{2}. \quad \Rightarrow \quad \rho_0(x) = \frac{\tan^{-1} \frac{\sqrt{(x-a_-)(a_+-x)}}{1+x}}{\pi\tilde{\lambda}x}.$$

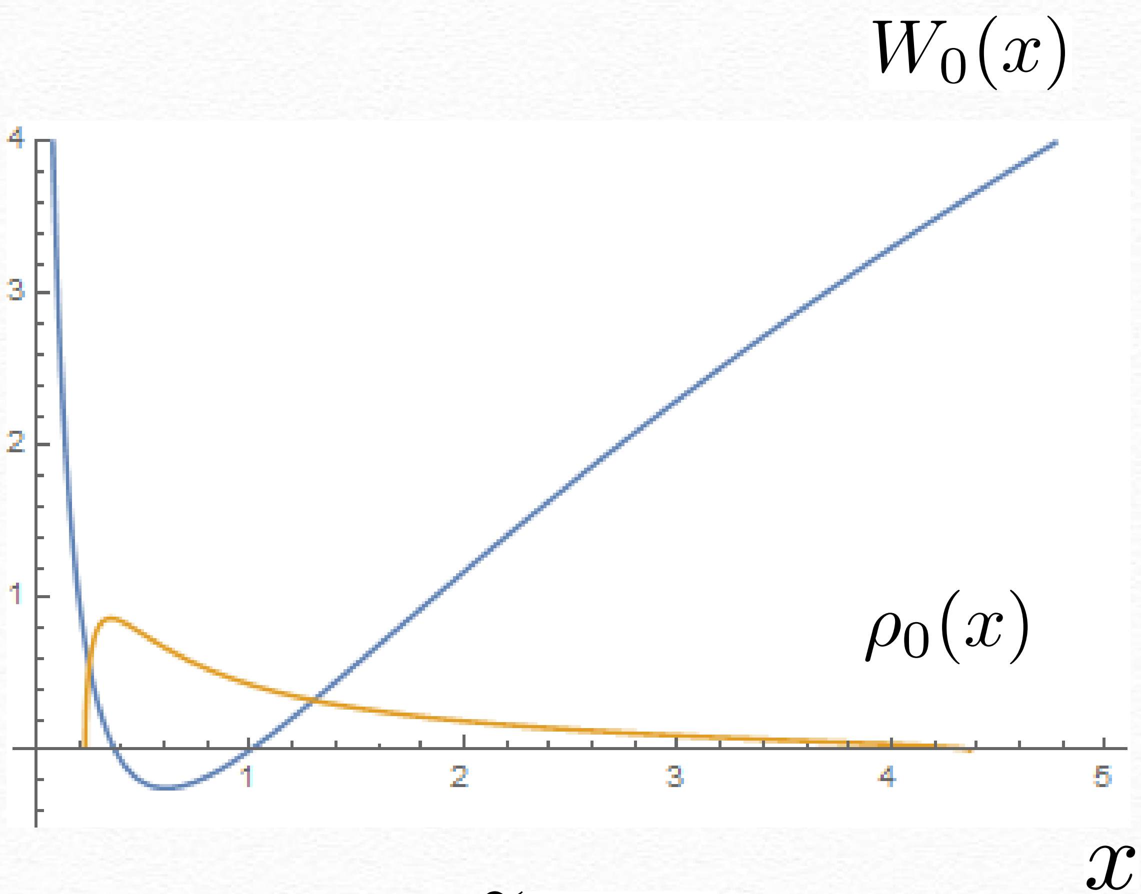
cf. [Marino '04]

Application to pure CS theory

[SY '16]

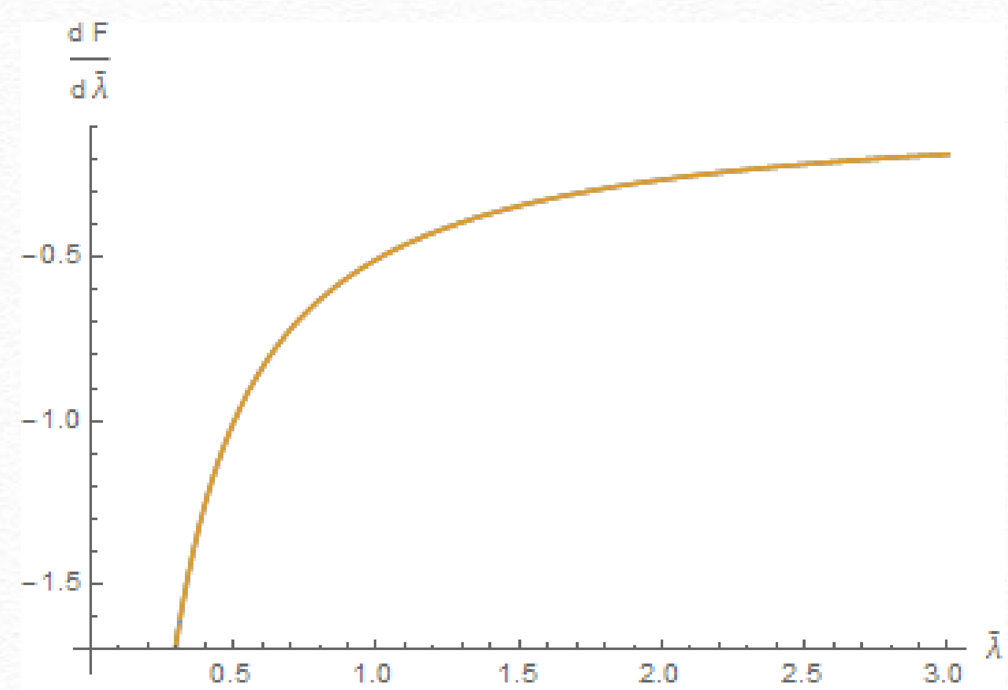
Planar result

Potential vs eigenvalue distribution



$\widetilde{\lambda} = 0.5$

Resolvent result vs Past result



$$\frac{\partial F_0}{\partial \widetilde{\lambda}} = \frac{1}{12} - \frac{1}{2\widetilde{\lambda}^2} \int_{a_-}^{a_+} dx \rho_0(x) (\log x)^2$$

$$\frac{\partial F_0}{\partial \widetilde{\lambda}} = -\frac{2\mathrm{Li}_3\left(e^{\widetilde{\lambda}}\right)}{\widetilde{\lambda}^3} + \frac{\mathrm{Li}_2\left(e^{\widetilde{\lambda}}\right)}{\widetilde{\lambda}^2} + \frac{2\zeta(3)}{\widetilde{\lambda}^3} + \frac{\pi^2}{6\widetilde{\lambda}^2} + \frac{1}{12}$$

Application to pure CS theory

[SY '16]

Genus 1 correction

Genus 1 correction: $$\omega_1(z) = \frac{1}{16}(\chi_-^{(2)}(z) + \chi_+^{(2)}(z)) - \frac{1}{8}\frac{1}{a_- - a_+}(\chi_-^{(1)}(z) - \chi_+^{(1)}(z)),$$

"Moments":

$$M_\pm^{(1)} = \frac{1}{a_\pm}\oint_{\mathcal{C}_{[a_-,a_+]}}\frac{dw}{2\pi i}\frac{wW_0'(w)}{h(w)(w-a_\pm)} = \frac{1}{\widetilde{\lambda}a_\pm}\oint_{\mathcal{C}_{[a_-,a_+]}}\frac{dw}{2\pi i}\frac{\log w}{h(w)(w-a_\pm)} + \frac{1}{a_\pm}\oint_{\mathcal{C}_{[a_-,a_+]}}\frac{dw}{2\pi i}\frac{1}{h(w)(w-a_\pm)}$$
$$= \frac{-1}{\widetilde{\lambda}a_\pm}\int_{-\infty}^{0}dy\frac{-1}{h(y)(y-a_\pm)} = \frac{-1}{\widetilde{\lambda}a_\pm}\left(\frac{2\sqrt{w-a_\mp}}{(a_\pm - a_\mp)\sqrt{w-a_\pm}}\right)\Bigg|_{-\infty}^{0} = \frac{-2}{\widetilde{\lambda}a_\pm\sqrt{a_\pm}(\sqrt{a}_\pm + \sqrt{a}_\mp)},$$

$$M_\pm^{(2)} = \cdots = \frac{2\left(5\sqrt{a_\pm} + 4\sqrt{a_\mp}\right)}{3a_\pm^{5/2}\widetilde{\lambda}\left(\sqrt{a_\pm} + \sqrt{a_\mp}\right)^2}.$$

Free energy: $$\frac{\partial F_1}{\partial\widetilde{\lambda}} = \frac{1}{12} - \frac{1}{2\widetilde{\lambda}^2}\int_{a_-}^{a_+}dx\rho_1(x)(\log x)^2 = \frac{1}{12} - \frac{1}{\widetilde{\lambda}^2}\int_{-\infty}^{0}dx\omega_1(x)\log(-x)$$
$$= \frac{e^{\widetilde{\lambda}}(\widetilde{\lambda}-2) + \widetilde{\lambda} + 2}{24(e^{\widetilde{\lambda}}-1)\widetilde{\lambda}} = \frac{\widetilde{\lambda}\coth\left(\frac{\widetilde{\lambda}}{2}\right) - 2}{24\widetilde{\lambda}}$$

This is in precise agreement with the past exact result!!

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Summary

- We have performed a general analysis on a class of matrix models describing CSM theory on three sphere incorporating the standard technique of 1/N expansion.

- We have derived the loop equation for all orders in the 1/N expansion including the hole correction.

- We presented explicit solution at the spherical order for a general potential and at the genus 1 order in the case where the potential does not have any 1/N correction.

- We have applied the formulation to pure CS theory and confirmed that the presented solution reproduces the exact result known in the past.

Future works

- Another iterative approach (topological expansion)?

  [Enyald '04]

- Generalization to 2-matrices? Application to ABJM?

  cf. [Marino-Putrov '10]

- Exact analysis on three sphere partition function and 3d non-SUSY duality?

- 2d CFT (or QFT?) description?

  cf. [Fukuma-Kawai-Nakayama '90] [Milnov-Morosov '90] [Dijkgraaf-Verlinde-Verlinde '91]

- Relation to 2d bosonization?

- AdS/CFT correspondence? Higher-spin (Vasiliev) theory?

⋮

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⋮

Thank you!!