Osaka University, April 2016

Moonshine phenomena in string theory

About five years ago together with collaborators I have found some curious phenomenon in string theory, i.e. appearance of exotic new symmetry of the theory. This is now called as moonshine phenomenon and is now under intensive study. Prof.Nambu was curious to hear about the story, however, there was no a chance to tell him the details. Thus, today I would like to give you a brief introduction to moonshine phenomena which may possibly play an interesting role in string theory in the future.

Monstrous moonshine

Modular J function

$$J(q)=\frac{1}{q}+744+196884q+21493760q^2+864299970q^3$$
$$+20245856256q^4+333202640600q^5+\cdots$$

$$q=e^{2\pi i\tau},\ Im(\tau)>0,\ J(\tau)=J(\frac{a\tau+b}{c\tau+d}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,Z)$$

It turns out q-expansion coefficients of J-function are decomposed into a sum of dimensions of some irreducible representations of the monster group M

$$196884 = 1 + 196883, \ \ 21493760 = 1 + 196883 + 21296876,$$
$$864299970 = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326,$$
$$20245856256 = 1 \times 1 + 3 \times 196883 + 2 \times 21296876$$
$$+842609326 + 19360062527, \cdots$$

Dimensions of some irreducible representations of monster :

$$\{1, \ 196883, \ 21296876, \ 842609326,$$

$$18538750076,\ 19360062527\cdots\}$$

Monster group: largest sporadic discrete group, of order $\approx 10^{53}$.

McKay observation (1978) : strange relationship between modular form and an isolated discrete group

To be precise

$$J_1(\tau) = J(q) - 744 = \sum_{n=-1} c(n)q^n, \qquad c(0) = 0$$

$$= \sum_{n=-1} Tr_{V(n)} 1 \times q^n, \qquad dim V(n) = c(n)$$

McKay-Thompson series is given by

$$J_g(\tau) = \sum_{n=-1} Tr_{V(n)}\, g \times q^n, \quad g \in M$$

where $Tr_{V(n)}\, g$ denotes the character of a group element g in the representation $V(n)$. This depends on the conjugacy class g of M. If McKay-Thompson series is known for all conjugacy classes, decomposition of $V(n)$ into irreducible representations become uniquely determined. Series J_g are modular forms with respect to subgroups of $SL(2, Z)$ and possess similar properties like the modular J-function such as the genus=0 (Hauptmodul) property.

Phenomenon of monstrous moonshine has been solved mathematically in early 1990's using the technology of vertex operator algebra. However, we still do not have a 'simple' explanation of this phenomenon.

Mathieu moonshine

K_3 surface :

We consider string theory compactified on K_3 surface. K_3 surface is a complex 2-dimensional hyperKähler manifold. It possesses $SU(2)$ holonomy and a holomorphic 2-form and ubiquitous in string theory.

It is well-known that string theory on K_3 has an N=4 superconformal symmetry with the central charge $c = 6$ which contains $SU(2)_{k=1}$ affine symmetry.

Now instead of modular J-function we consider the elliptic genus of K_3 surface. Elliptic genus describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Using world-sheet variables it is written as

$$Z_{elliptic}(z;\tau) = Tr_{\mathcal{H}_L \times \mathcal{H}_R}(-1)^{F_L+F_R} e^{4\pi i z J^3_{L,0}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{c}{24}}$$

Here L_0 denotes the zero mode of the Virasoro operators and F_L and F_R are left and right moving fermion numbers. J^3_0 denotes the Cartan generator of affine $SU(2)_1$. In elliptic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the left-moving Hilbert space $\mathcal{H}_L$ contribute.

In general it is difficult to compute elliptic genera, however, we were able to evaluate it making use of Gepner models

EOTY

$$Z_{K3}(\tau,z)=8\left[\left(\frac{\theta_2(\tau,z)}{\theta_2(\tau,0)}\right)^2+\left(\frac{\theta_3(\tau,z)}{\theta_3(\tau,0)}\right)^2+\left(\frac{\theta_4(\tau,z)}{\theta_4(\tau,0)}\right)^2\right]$$

Here $\theta_i(\tau,z)$ are Jacobi theta functions.

We want to see how the Hilbert space $\mathcal{H}_L$ in elliptic genus decompose into irreducible representations of N=4 superconformal algebra (SCA).

Highest weight states of N=4 SCA are parametrized by the eigenvalues of L_0 and J_0^3.

$$L_0|h,\ell\rangle = h|h,\ell\rangle, \qquad J_0^3|h,\ell\rangle = \ell|h,\ell\rangle$$

There are two different types of representations in $c=6$ SCA.

In R sector

BPS (massless) rep. $h=\frac{1}{4}; \quad \ell=0,\ \frac{1}{2}$

non-BPS (massive) rep. $h>\frac{1}{4}; \quad \ell=\frac{1}{2}$

Character of a representation is given by

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

where $\mathcal{R}$ denotes the representation space.

Index is given by the value of the character at $z=0$,

$$\text{Index}(\mathcal{R})=Tr_{\mathcal{R}}(-1)^F q^{L_0}$$

BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell = 0) = 1$$
$$\text{index (BPS, } \ell = \frac{1}{2}) = -2$$

while non-BPS reps. have vanishing indices

$$\text{index (non-BPS, } \ell = \frac{1}{2}) = 0$$

Characters are given explicitly as ET

$$ch^{BPS}_{h=\frac{1}{4},\ell=0}(\tau, z) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau)=\frac{-ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n(-1)^n\frac{q^{\frac{1}{2}n(n+1)}e^{2\pi inz}}{1-q^ne^{2\pi iz}}$$

non-BPS characters are given by

$$ch^{non-BPS}_{h,\ell=\frac{1}{2}}=q^{h-\frac{3}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3},\quad h>\frac{1}{4}$$

Function $\mu(\tau,z)$ is a typical example of Mock theta function (Lerch sum or Appel function). Moch theta functions look like theta functions but they have anomalous modular transformation laws and difficult to handle. Recently there were developments in understanding the nature of Moch theta functions due to Zwegers. He has introduced a method of regularization which is similar to the ones used in physics. It improves the modular property of Moch theta functions so that they transform as analytic Jacobi forms. Jacobi form with weight k and index m transforms as

$$\varphi(\tau, z + a\tau + b) = e^{-2\pi i m(a^2\tau + 2az)}\varphi(\tau, z)$$
$$\varphi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}}\varphi(\tau, z)$$

Now let us make a decomposition of elliptic genus into a sum of characters of N=4 representaions

$$Z_{K3}(\tau,z)=24ch^{BPS}_{h=\frac{1}{4},\ell=0}(\tau,z)+2\sum_{n\geq 0}A(n)ch^{non-BPS}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(\tau,z)$$

At smaller values of n, expansion coefficients $A(n)$

may be obtained by direct series expansion of Z_{K3}.

We find, $A(0)=-1$

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	45	231	770	2277	5796	13915	30843	65550	$\ldots$

Surprize: Dimensions of some irreducible reps. of

Mathieu group M_{24} appear

dimensions : $\{ \; 45 \quad 231 \quad 770 \quad 990 \quad 1771 \quad 2024 \quad 2277 \quad 3312 \quad 3520 \quad 5313 \quad 5544 \quad 5796 \quad 10395 \quad \cdots \}$

$$A(6) = 13915 = 3520 + 10395,$$
$$A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$$

T.E.-Ooguri-Tachikawa, 2010

M_{24} is a subgroup of S_{24} (permutation group of 24 objects). Its order is given by $\approx 10^9$.
M_{24} is known for its many interesting arithmetic properties and in particular intimately tied to the Golay code of efficient error corrections.

Monster $\supset$ Conway $\supset$ Mathieu $\supset \cdots$

Mathieu moonshine conjecture:

Expansion coefficients of K_3 elliptic genus into N=4 characters are given by the sum of dimensions of representations of Mathieu group M_{24}

We were able to derive analogues of McKay-Thompson series Gaberdiel et al, T.E. and Hikami

And then the multiplicities $C_R(n)$ of the decomposition of $A(n)$ into representations R

$$A(n) = \sum_R C_R(n)\, dim R$$

were unambiguously determined. It tuned out that $C_R(n)$ are all positive integers up to $n \approx 1000$ and this gives a very strong evidence of Mathieu moonshine conjecture.

n	1	23	252	253	1771	3520	$\frac{45}{45}$	$\frac{990}{990}$	$\frac{1035}{1035}$	$1035'$	$\frac{231}{231}$	$\frac{770}{770}$	483
1	0	0	0	0	0	0	1	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	2	0	0	0	0	0	0	0
7	0	0	0	0	2	0	0	0	0	0	0	0	0
8	0	0	0	0	0	2	0	1	1	0	0	0	0
9	0	0	0	0	2	4	0	0	2	2	0	2	2
10	0	0	0	2	4	8	0	2	2	2	2	0	2
11	0	0	0	0	8	12	0	4	4	6	0	4	0
12	0	2	2	4	12	30	0	8	8	4	2	6	4
13	0	0	4	2	26	44	2	14	14	18	2	10	6
14	0	0	4	6	38	86	0	24	24	22	8	16	14
15	0	0	12	8	78	144	2	40	44	46	8	38	18
16	0	2	18	22	122	252	2	72	72	68	18	50	36
17	0	2	30	26	212	410	8	116	124	130	25	94	54
18	0	6	50	58	342	704	6	194	202	192	50	148	100
19	0	4	80	72	582	1116	18	318	332	346	68	252	150
20	0	14	128	138	904	1836	20	516	536	520	126	390	254
21	2	20	214	200	1476	2902	40	814	860	872	182	652	396
22	2	32	328	346	2302	4616	55	1298	1348	1336	314	988	640
23	2	40	512	496	3638	7166	98	2020	2118	2144	460	1590	972
24	0	80	798	824	5584	11192	132	3140	3278	3236	744	2426	1544
25	8	108	1232	1208	8654	17084	234	4814	5038	5084	1106	3764	2336
26	6	174	1860	1904	13090	26148	322	7348	7670	7626	1742	5677	3602
27	12	252	2836	2802	19914	39436	514	11092	11618	11666	2560	8688	5394
28	16	398	4238	4310	29772	59330	742	16686	17418	17356	3922	12912	8160
29	26	560	6328	6286	44512	88280	1154	24840	25994	26078	5758	19380	12090
30	34	876	9368	9486	65776	131020	1642	36824	38480	38368	8642	28580	18008
31	58	1236	13802	13764	97060	192538	2500	54178	56660	56800	12582	42218	26384
32	76	1866	20166	20356	141714	282074	3564	79320	82884	82730	18576	61574	38738
33	122	2664	29396	29374	206524	410062	5286	115334	120644	120798	26830	89868	56226
34	166	3900	42474	42810	298508	593800	7542	166990	174510	174330	39066	129694	81546
35	248	5536	61184	61234	430134	854284	10988	240304	251292	251544	55956	187094	117138

The conjecture is now proved mathematically using the method of mathematical induction. Gannon

Unfortunately the proof so far did not provide much insight into the nature of Mathieu moonshine. The situation looks a bit like the case of monstrous moonshine. 24 of M_{24} will certainly be the Euler number of K_3 and M_{24} permutes homology classes. There are, however, various indications that string theory on K_3 can not have such a high symmetry as M_{24}.

Instead of the total Hilbert space the BRS subsector of the theory may possibly possess an enhanced symmetry. It will be interesting to look into the algebraic structures of BPS states to explain Mathieu moonshine.

More Moonshine Phenomena

Mathieu moonshine exists at the intersection of string theory, K_3 surface (geometry), (moch) modular forms and sporadic discrete symmetry and appears to possess interesting mixture of physics and mathematics. Recently there have been intense interests in exploring new types of moonshine phenomena other than Mathieu moonshine. Already several types of new moonshine phenomena have been discovered.

- Umbral moonshine   Cheng,Duncan and Harvey
- fermions on 24 dim. lattice

- spin 7 manifold
- N=2 extremal Jacobi form ,,,,

Due to short of time we discuss only about Umbral moonshine. Umbral moonshine has a mysterious relationship to Niemeier lattice. It is known there are 23 types of self-dual lattices in 24 dimensions. It is given by the combination of root lattices of A-D-E type together with appropriate weight vectors so that the lattice becomes self-dual. The simplest ones are

$$A_1^{24} \qquad (k=1)$$
$$A_2^{12} \qquad (k=2)$$
$$A_3^{8} \qquad (k=3)$$
$$\cdots$$

etc. Automorphism group of Niemeier lattices are

$$M_{24} \times 2^{24}$$
$$M_{12} \times 3!^{12}$$
$$G \times 4!^{8}$$
$$\cdots$$

Corresponding to each of these there exists a moonshine phenomenon whose discrete symmetry given by

$$\frac{[\text{automorphism group of lattice}]}{[\text{Weyl group of root lattice}]}$$

The first one A_1^{24} corresponds to Mathieu moonshine and the rest are generalizations. Second one, for instance, A_2^{12} is assumed to be related to 4-dimensional hyperKähler manifold with $c = 12(k = 2)$ and $\mathcal{N} = 4$ superconformal symmetry.

Analogue of K_3 elliptic genus is given by

$$Z(k=2) = 4\left[\left(\frac{\theta_2(z)\theta_3(z)}{\theta_2(0)\theta_3(0)}\right)^2 + \left(\frac{\theta_2(z)\theta_4(z)}{\theta_2(0)\theta_4(0)}\right)^2 + \left(\frac{\theta_3(z)\theta_4(z)}{\theta_3(0)\theta_4(0)}\right)^2\right]$$

By expanding $Z(k = 2)$ in terms of characters of representations of $c = 12, \mathcal{N} = 4$ algebra one finds the expansion coefficients decompose in the symmetry group M_{12}.

Here, however, there is something awkward: $Z(k = 2)$ does not contain the contribution of vacuum operator ($h = 0$ in NS sector) thus the theory appears to describe the geometry of a (singular) non-compact four-fold. The rest of Umbral moonshine series has the same property (absence of identity operator) and their geometrical interpretation is somewhat obscure.

Recently we noted that if one uses the duality of $\mathcal{N}$=4 Liouville theory we may possibly improve our geometrical understanding of Umbral moonshine. T.E. and Y.Sugawara, See also M.Cheng and S.Harrison It is known that a large N=4 SCFT contains two independent $SU(2)$ algebras and when one adjusts its coupling (dilaton coupling strength) to a particular value one of the $SU(2)$'s, say, $SU(2)_{k_-}$ decouples from the theory and one obtains a small N=4 theory with $SU(2)_{k_+=1}$. At another value of coupling constant $SU(2)_{k_+}$ decouples and one obtains a small N=4 theory with $SU(2)_{k_-}$. These two theories are dual to each other and Mathieu moonshine sits at the self-dual point. Seiberg and Witten

Under this duality, one can relate $c = 6$ and $c = 6k_-$ theories while keeping the moonshine symmetry left invariant.

$$Z^{case,1}(\tau,z) = \frac{N-1}{12}\phi_{0,1}(\tau,z) + \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}H^{(N)}(\tau)$$

where

$$Z^{case,1}(\tau,z)$$
$$= -\frac{\theta_1(\tau,z)}{i\eta(\tau)^3}\sum_{r=1}^{N-1}\sum_{a\in Z_N} ch^{(\tilde{R})}_{r-1,r+2a}(\tau,z)\widehat{F}^{(N)}(r,a;\tau,-z)$$

$$F^{(N)}(v,a;\tau,z) = \sum_{n\in a+NZ}\frac{(yq^n)^{\frac{v}{N}}y^{\frac{2n}{N}}q^{\frac{n^2}{N}}}{1-yq^n}$$

$$H^{(N)}(\tau) = \frac{\eta(\tau)^3}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\widehat{f}^{(N)}(\tau, w)}{i\theta_1(\tau, 2w)} e^{(N-2)G_2(\tau)w^2}$$

$$Z^{case,2}(\tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r=1}^{N-1} \sum_{a \in Z_N} ch^{(\tilde{R})}_{r-1, r+2a}(\tau, z) \widehat{F}^{(N)}(r, a; \tau, (N-1)z)$$

$$Z^{case,1} \ (c = 6) \overset{dual}{\Longleftrightarrow} Z^{case,2} \ (c = 6(N-1))$$

$N = 2$ in the above is a self-dual point corresponding to Mathieu moonshine.

It seems likely $Z^{case,1}$ describes the elliptic genera of ALE spaces and umbral moonshine may be mapped into these geometries under duaity.

Moonshine symmetries recently discovered in string theory are still very mysterious and we may encounter many more surprises in the near future.