Lattice energy-momentum tensor from the Yang-Mills gradient flow

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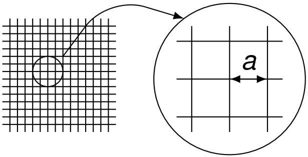
- H.S., Prog. Theor. Exp. Phys. (2013) 083B03 [arXiv:1304.0533 [hep-lat]]
- M. Asakawa, T. Hatsuda, E. Itou, M. Kitazawa, H.S. (FlowQCD Collaboration), Phys. Rev. D90 (2014) 011501 [arXiv:1312.7492 [hep-lat]]
- H. Makino and H.S., Prog. Theor. Exp. Phys. (2014) 063B02 [arXiv:1403.4772 [hep-lat]], arXiv:1410.7538 [hep-lat]
- . . . and recent unpublished numerical simulations

Lattice gauge theory and the energy–momentum tensor (EMT)

- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime. . .



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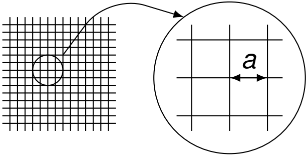

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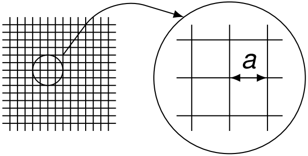

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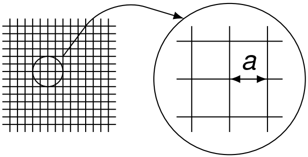

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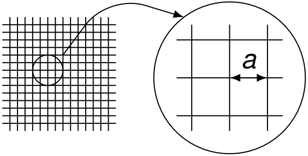

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- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $\{T_{\mu\nu}\}_R(x)$.
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \stackrel{a\to 0}{\to} 1.$$

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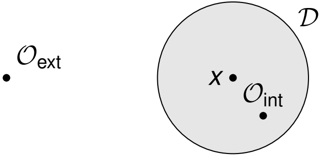

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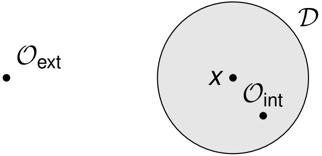

- This contains the correct normalization and the conservation law.

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- If such a construction is possible, we expect wide application to physics related to spacetime symmetries: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, . . .
- The present work is also an attempt to understand EMT in quantum field theory in the non-perturbative level.

EMT on the lattice (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$\{T_{\mu\nu}\}_R(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)\big|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x)\psi(x),$$

and, Lorentz non-covariant ones:

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

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- Seven non-universal coefficients Z_i must be determined by lattice perturbation theory or by a non-perturbative method

Yang–Mills gradient flow (Lüscher, (2009–))

- Yang–Mills gradient flow is an evolution of the gauge field $A_\mu(x)$ along a fictitious time $t \in [0, \infty)$, according to

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\text{YM}}}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

where

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

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- RHS is the Yang–Mills equation of motion, the gradient in function space if S_{YM} is regarded as a potential height. So the name of the gradient flow.

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- The key is the UV finiteness of the gradient flow

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

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- This equation can be formally solved as

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

by using the heat kernel,

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right].$$

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- R is the non-linear terms

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]].$$

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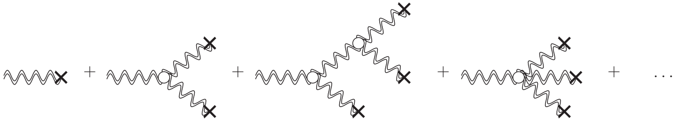
is represented pictorially as (double lines: K, crosses: A_μ, white circles: R),



Backup: Justification of the "gauge fixing term"

- Under the infinitesimal gauge transformation

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu\partial_\nu B_\nu(t,x),$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu\partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu\partial_\nu)\omega(t,x).$$

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$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = -\delta\alpha_0 \partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0,$$

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  α_0 can be changed accordingly

$$\alpha_0 \to \alpha_0 + \delta\alpha_0.$$

- Thus, a gauge invariant quantity (in usual 4D sense) is independent of α_0, as far as it does not contain the flow time derivative ∂_t.

Quantum correlation functions

- Quantum correlation function of the flowed gauge field is obtained by the functional integral over the initial value $A_\mu(x)$:

$$
\begin{aligned}
&\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle \\
&= \frac{1}{\mathcal{Z}} \int \mathcal{D}A_\mu \, B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \, e^{-S_{\mathrm{YM}} - S_{\mathrm{gf}} - S_{c\bar{c}}}.
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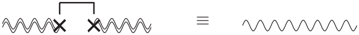
- For example, the contraction of two A_μ's

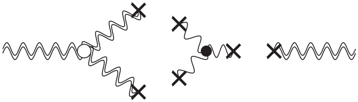


produces the free propagator of the flowed field

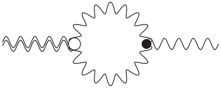
$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0$$
$$=\delta^{ab}g_0^2\int_p \frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2-p_\mu p_\nu)e^{-(t+s)p^2}+\frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].$$

Quantum correlation functions

- Similarly, for (black circle: Yang–Mills vertex)



we have the loop flow-line Feynman diagram



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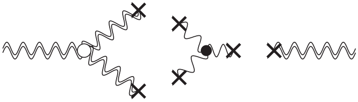

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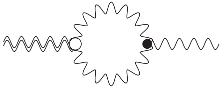
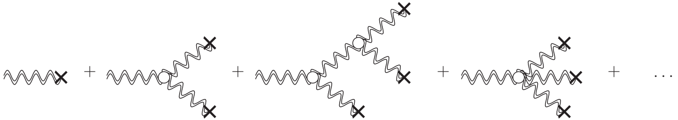


- Recall that the flowed gauge field is represented as



Renormalizability of the gradient flow I (Lüscher–Weisz (2011))

- Correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.

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- Two-point function in the tree level (in the Feynman gauge $\lambda_0 = \alpha_0 = 1$)

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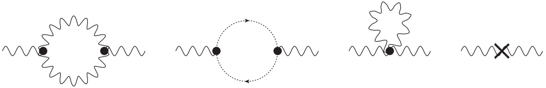
- One-loop corrections (consisting only from Yang–Mills vertices)



  where the last counter term arises from the parameter renormalization

$$g_0^2 = \mu^{2\epsilon} g^2 Z, \qquad \lambda_0 = \lambda Z_3^{-1}.$$

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.

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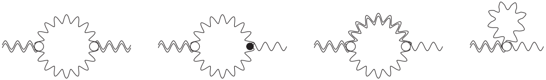

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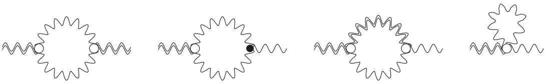

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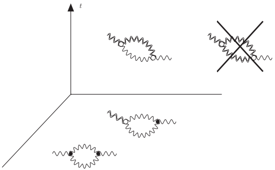
- All order proof of this fact, using a local $D + 1$-dimensional field theory



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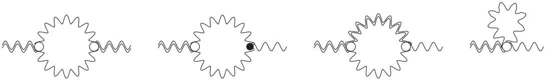

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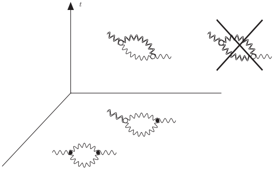

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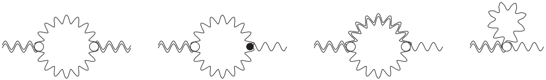
Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)



It turns out that these provide the same effect as the wave function renormalization!

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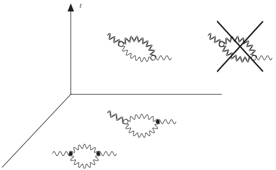


- No bulk ($t > 0$) counterterm: because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator.
- No boundary ($t = 0$) counterterm besides Yang–Mills ones: because of a BRS symmetry.

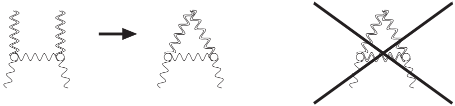
Renormalizability of the gradient flow II

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

remains finite even for the equal-point product

$$t_1 \to t_2, \qquad x_1 \to x_2.$$



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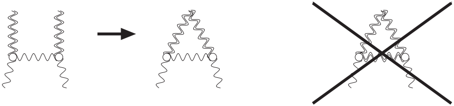

- The new loop always contains the gaussian damping factor $\sim e^{-tp^2}$ which makes integral finite; no new UV divergences arise.

Renormalizability of the gradient flow II

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- $\Rightarrow$ Construction of the energy–momentum tensor in lattice gauge theory.

Our strategy for lattice EMT (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool.

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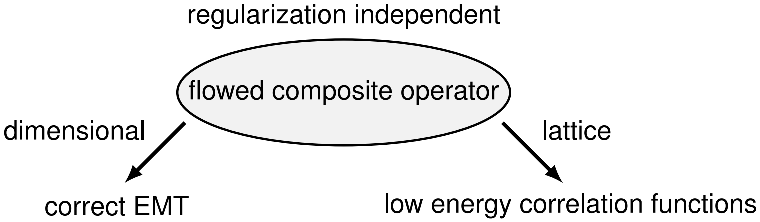
- Schematically,



EMT in the dimensional regularization

- The action

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\,[F_{\mu\nu}(x)F_{\mu\nu}(x)] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

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- Under the localized translation (plus the gauge transformation),

$$\delta A_\mu(x)=\xi_\nu(x)F_{\nu\mu}(x),$$
$$\delta\psi(x)=\xi(x)_\mu D_\mu\psi(x),\qquad \delta\bar{\psi}(x)=\xi(x)_\mu\bar{\psi}(x)\overleftarrow{D}_\mu,$$

we have

$$\delta S=-\int d^Dx\,\xi_\nu(x)\partial_\mu\left[T_{\mu\nu}(x)+A_{\mu\nu}(x)\right],$$

where

$$A_{\mu\nu}(x)=\frac{1}{4}\bar{\psi}(x)\left(\gamma_\mu\overleftrightarrow{D}_\nu-\gamma_\nu\overleftrightarrow{D}_\mu\right)\psi(x)$$

is the generator of the local Lorenz transformation and is neglected here, and. . .

EMT in dimensional regularization

- … and $T_{\mu\nu}(x)$ is the symmetric EMT:

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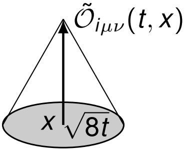
- Small flow-time expansion (Lüscher–Weisz (2011)):



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

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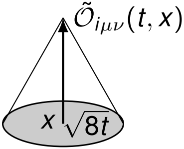

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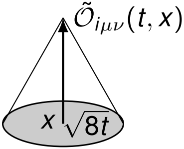

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- Inverting this relation,

$$\mathcal{O}_{i\mu\nu}(x) - \mathrm{VEV} = \lim_{t\to 0} \left\{ \sum_j \left(\zeta^{-1}\right)_{ij}(t) \left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x) \right\rangle \mathbb{1} \right] \right\}.$$

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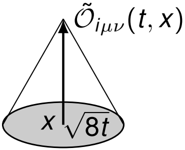

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- So, if we know the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$, the 4D operator in the LHS can be extracted as the $t \to 0$ limit.

A renormalization group argument

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- If all the composite operators in this relation are made out from bare quantities,

$$\left(\mu \frac{\partial}{\partial \mu} \right)_0 \zeta_{ij}(t) = 0,$$

and $\zeta_{ij}(t)$ are indep. of the renormalization scale μ, when expressed in terms of running parameters. We may take, for example, $\mu = 1/\sqrt{8t}$, and

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- For $t \to 0$, $\bar{g}(1/\sqrt{8t}) \to 0$ because of the asymptotic freedom; use of perturbation theory is thus justified!

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t \chi(t,x) = [\Delta - \alpha_0 \partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x) = \psi(x),$$
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EMT from the gradient flow

- We take following composite operators of flowed fields:

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$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle\tilde{\mathcal{O}}_{i\mu\nu}(t,x)\right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t)\left[\mathcal{O}_{j\mu\nu}(x) - \langle\mathcal{O}_{j\mu\nu}(x)\rangle \mathbb{1}\right] + O(t).$$

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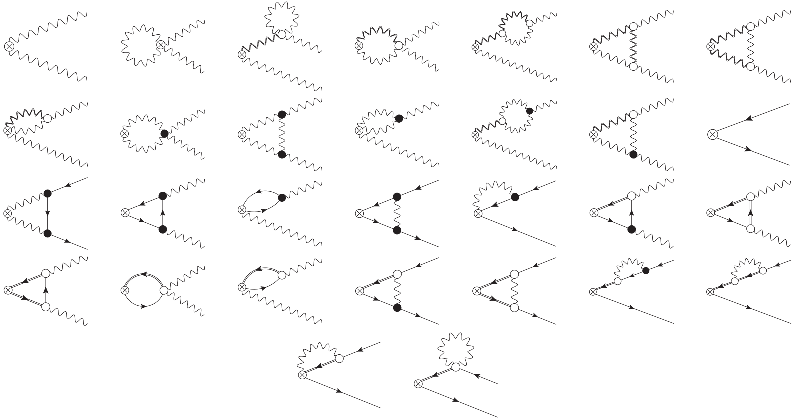
- We compute $\zeta_{ij}(t)$ to the one-loop order and substitute

$$\mathcal{O}_{i\mu\nu}(x) - \langle\mathcal{O}_{i\mu\nu}(x)\rangle\,\mathbb{1} = \lim_{t\to 0}\left\{\sum_j \left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle\tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle\mathbb{1}\right]\right\},$$

in the expression of the EMT in the dimensional regularization

Computation of expansion coefficients $\zeta_{ij}(t)$

- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



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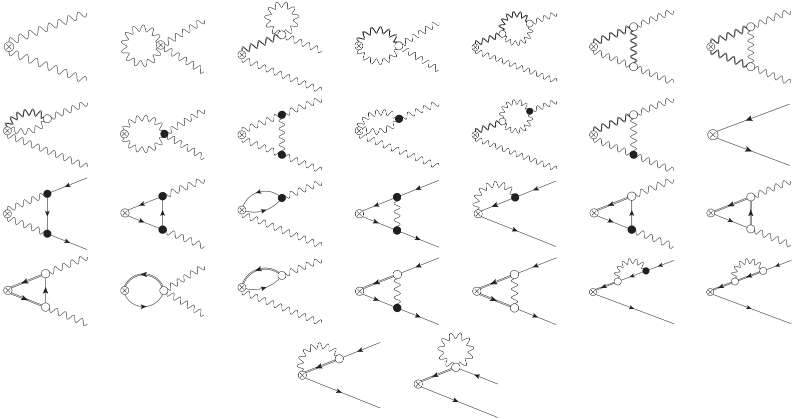
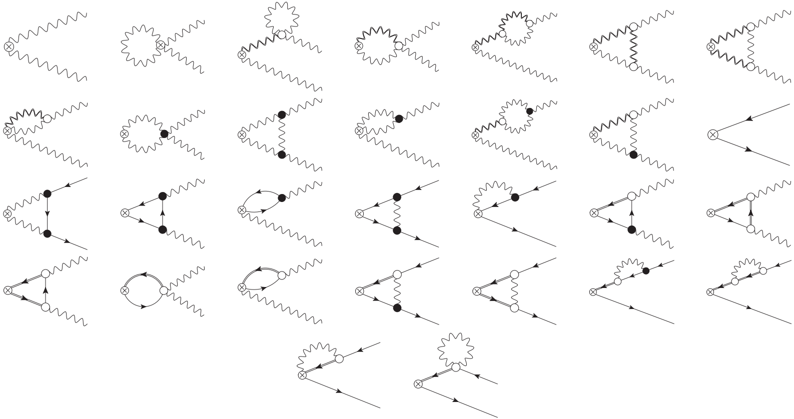


- Even to write down correct set of diagrams is tedious. . .

Computation of expansion coefficients $\zeta_{ij}(t)$

- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



- Even to write down correct set of diagrams is tedious. . .
- . . . and it is very easy to make mistakes in the loop calculation, as I actually did!

Universal formula for EMT

- For the system containing fermions (with Makino, arXiv:1403.4772),

$$\{T_{\mu\nu}\}_R(x)$$

$$= \lim_{t\to 0}\Big\{c_1(t)G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) + \left[c_2(t) - \frac{1}{4}c_1(t)\right]\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x)$$

$$+ c_3(t)\mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x)$$

$$+ [c_4(t) - 2c_3(t)]\,\delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x) + c_5'(t)\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x) - \mathrm{VEV}\Big\},$$

where (for the MS scheme; for $\overline{\mathrm{MS}}$ scheme, set $\ln\pi \to \gamma_E - 2\ln 2$)

$$c_1(t) = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - b_0\ln\pi - \frac{1}{(4\pi)^2}\left[\frac{7}{3}C_2(G) - \frac{3}{2}T(R)N_f\right],$$

$$c_2(t) = \frac{1}{8}\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G) + \frac{11}{3}T(R)N_f\right],$$

$$c_3(t) = \frac{1}{4}\left\{1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[\frac{3}{2} + \ln(432)\right]\right\},$$

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$$c_5'(t) = -\bar{m}(1/\sqrt{8t})\left\{1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[3\ln\pi + \frac{7}{2} + \ln(432)\right]\right\}.$$

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- “Universality” holds however only when one removes the regulator.
- Thus, we have to first take the continuum limit $a \to 0$ and then take the small flow time limit $t \to 0$.
- Practically, we cannot simply take $a \to 0$ and may take t as small as possible in the fiducial window,

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

Thus the usefulness with presently-accessible lattice parameters is not obvious a priori. . .

Application to thermodynamics of $SU(3)$ pure Yang–Mills theory (arXiv:1312.7492)

- Asakawa–Hatsuda–Itou–Kitazawa–H.S. (FlowQCD Collaboration).

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- Thermal average of diagonal elements of EMT: the trace part (the trace anomaly),

$$\langle \varepsilon - 3p \rangle_T = - \left\langle \{T_{\mu\mu}\}_R(x) \right\rangle_T ,$$

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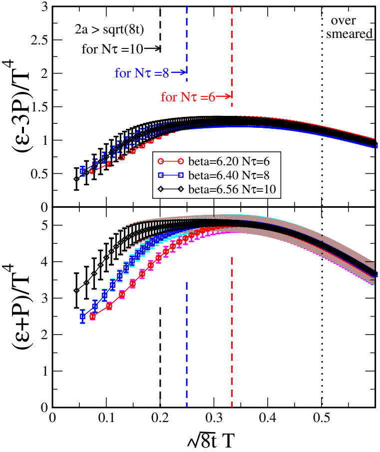
- Experiment setting:
  - Wilson plaquette action.
  - $N_S^3 \times N_T = 32^3 \times (6, 8, 10, 32)$, $\beta = 5.89$–6.56, $\sim$ 300 configurations.
  - Wilson flow: 2th order Runge–Kutta with $\epsilon / a^2 = 0.025$.
  - Scale setting: $\beta \leftrightarrow a\Lambda_{\overline{\mathrm{MS}}}$ from ALPHA Collaboration, aT_c at $\beta = 6.20$ from Boyd et al.
  - 4-loop running coupling in the $\overline{\mathrm{MS}}$ scheme.
  - Clover field strength $G^a_{\mu\nu}(x)$.

Application to thermodynamics of $SU(3)$ pure Yang–Mills theory

- Thermal expectation values versus the flow time $\sqrt{8t}$ at $T = 1.65\,T_c$:



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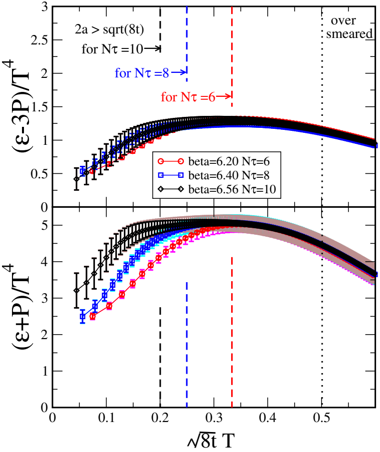
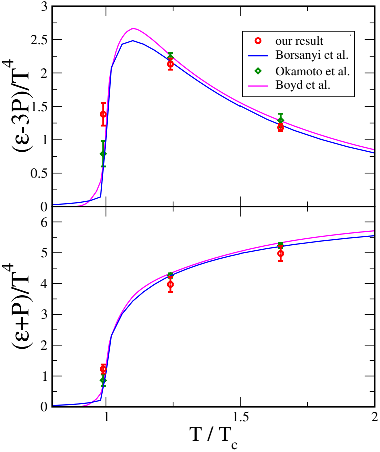


- We observe stable behavior for $2a < \sqrt{8t} < 1/(2T)$ which indicates (!!!) the $t \to 0$ limit.

Application to thermodynamics of $SU(3)$ pure Yang–Mills theory

- Continuum limit (from values at $\sqrt{8t}T = 0.40$):



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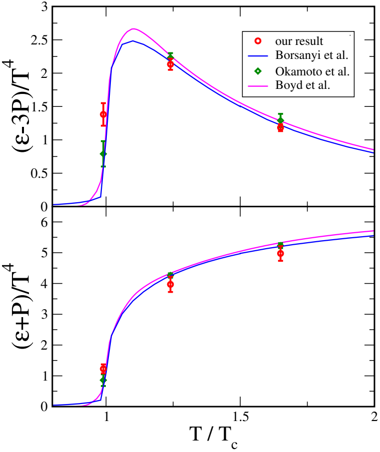


- That our simple method produces results being consistent with past comprehensive studies indicates that our reasoning is correct.

Recent status in quenched QCD

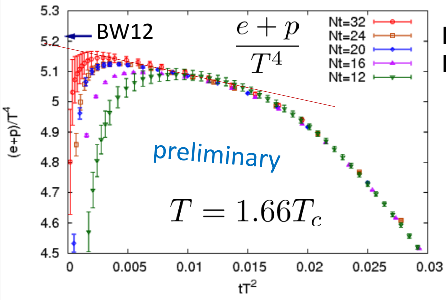
- Asakawa–Hatsuda–Iritani–Itou–Kitazawa–H.S. (FlowQCD Collaboration); Kitazawa's slide at Lattice 2015

New Results: Thermodynamics (e+p)

$$\tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}$$

FlowQCD, in prep.

$$T^R_{\mu\nu} = \tilde{T}_{\mu\nu}(t) + O(t)$$



BW12:Budapest-Wuppertal, 2012

- ☐ Existence of O(t) effect
- ☐ Linear behavior for
  $tT^2 < 0.015$
  $(\sqrt{8t} < 0.35T^{-1})$
  - t→0 limit is necessary

Summary and prospects

- We developed a formula that relates a correctly-normalized conserved EMT and composite operators defined through the gradient flow:

$$\{T_{\mu\nu}\}_R(x) = \lim_{t\to 0}\Big\{c_1(t)G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x) + \left[c_2(t) - \frac{1}{4}c_1(t)\right]\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho
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- One-point functions at the finite temperature show encouraging results; the method appears promising even practically!

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- Further theoretical applications of the gradient flow...