Entanglement, Conformal Field Theory, and Interfaces

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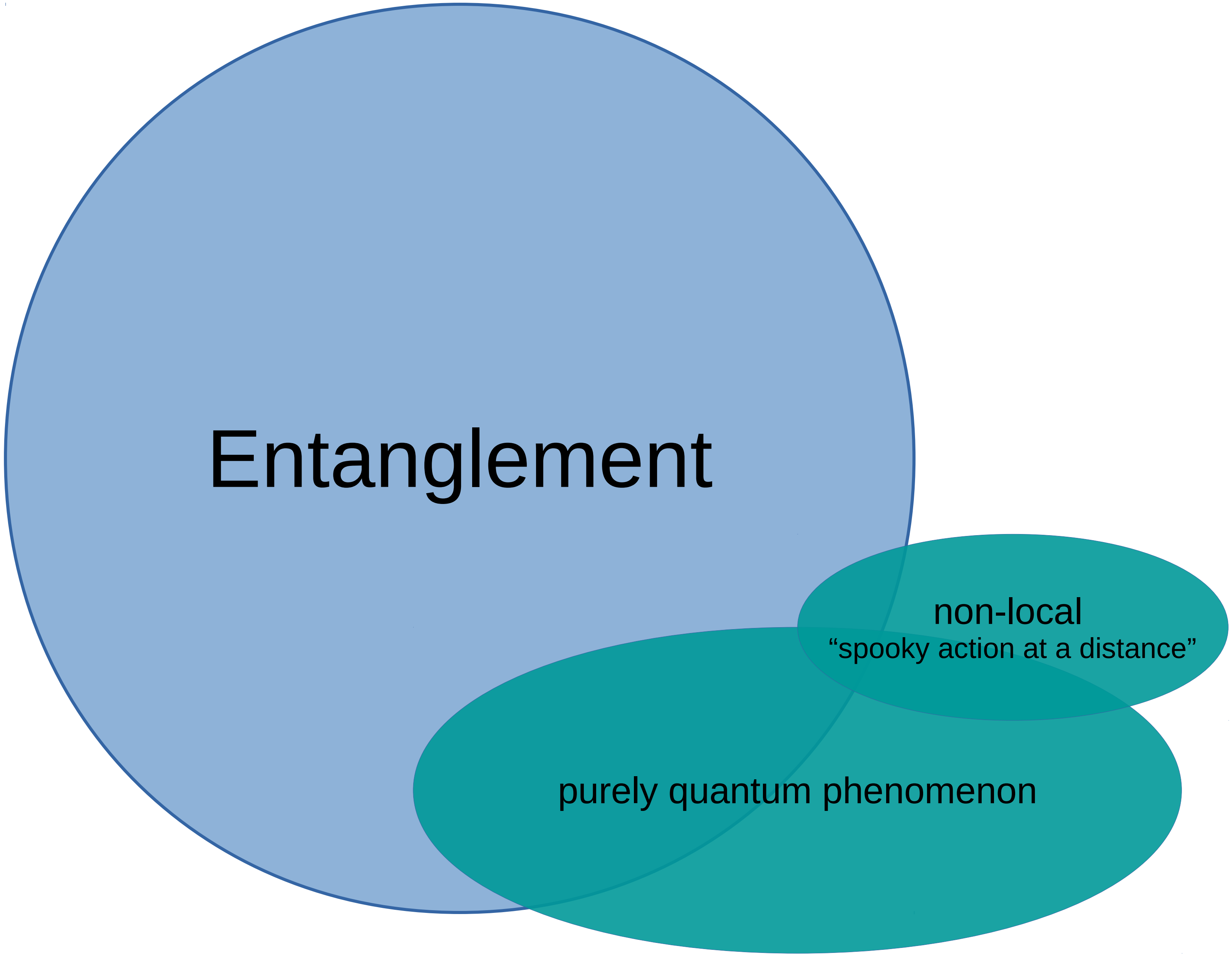
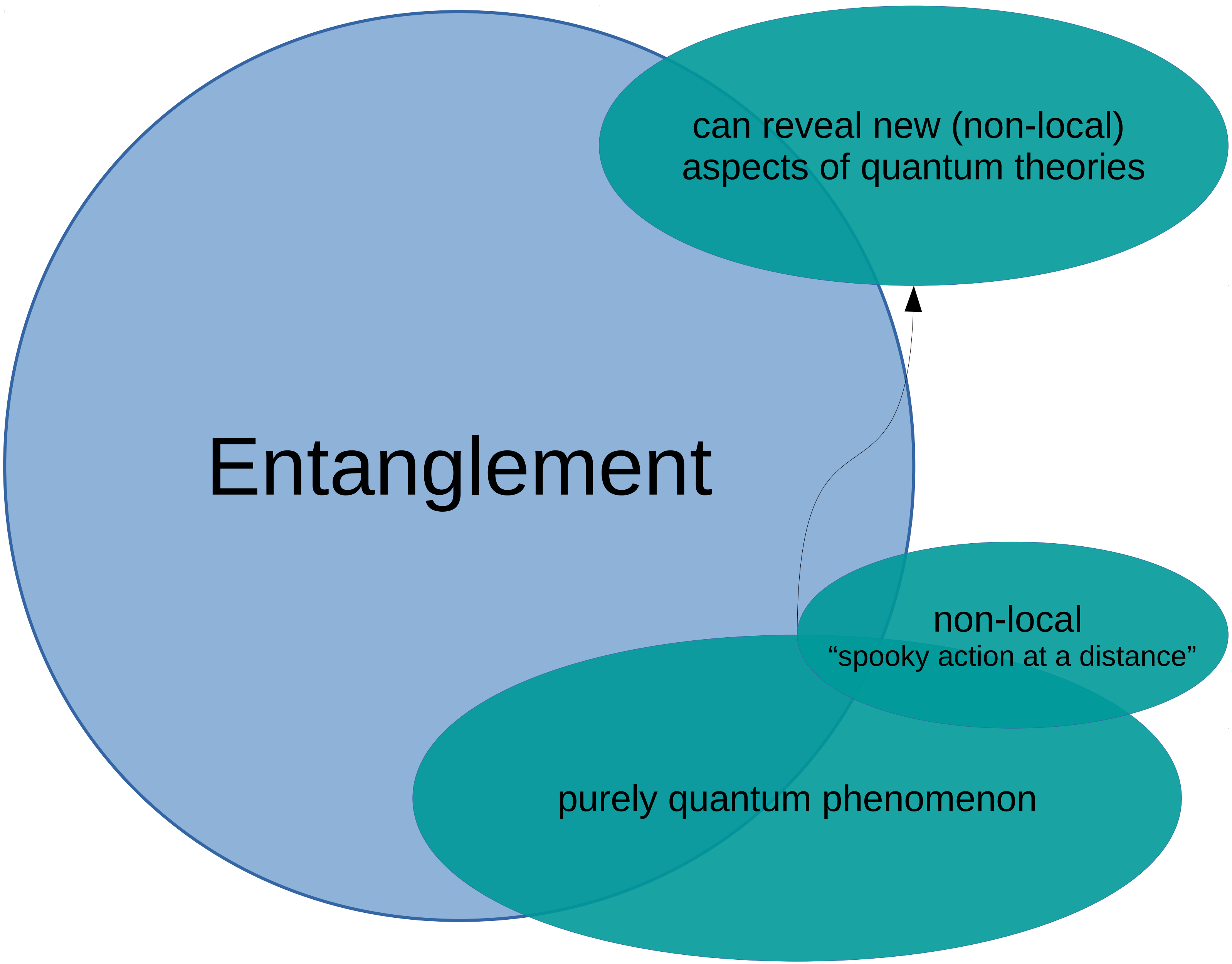
with Ilka Brunner,

Excellence Cluster Universe

Daniel Jaud,
Cornelius Schmidt Colinet

Entanglement


Entanglement
non-local
“spooky action at a distance”
purely quantum phenomenon



can reveal new (non-local)
aspects of quantum theories
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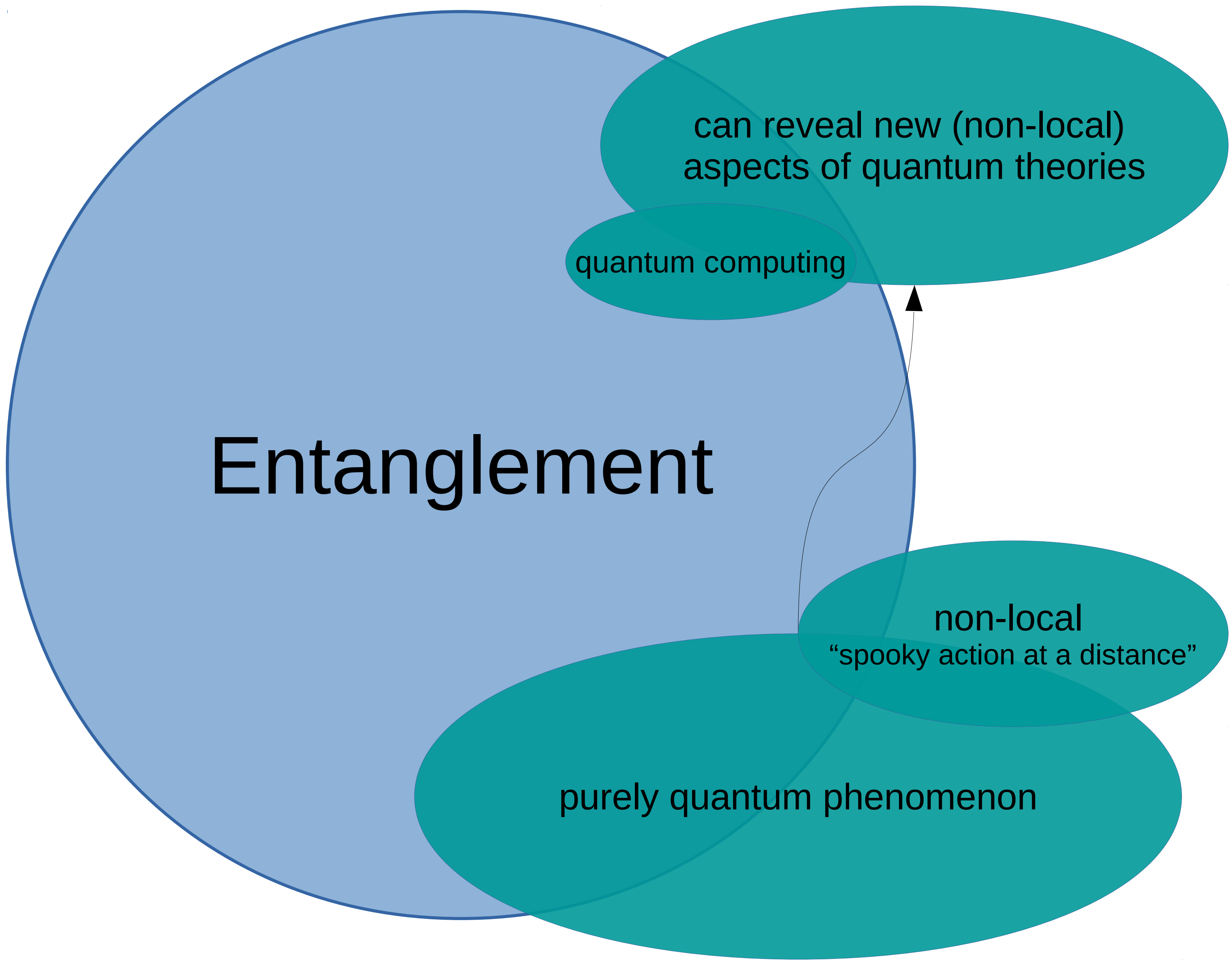
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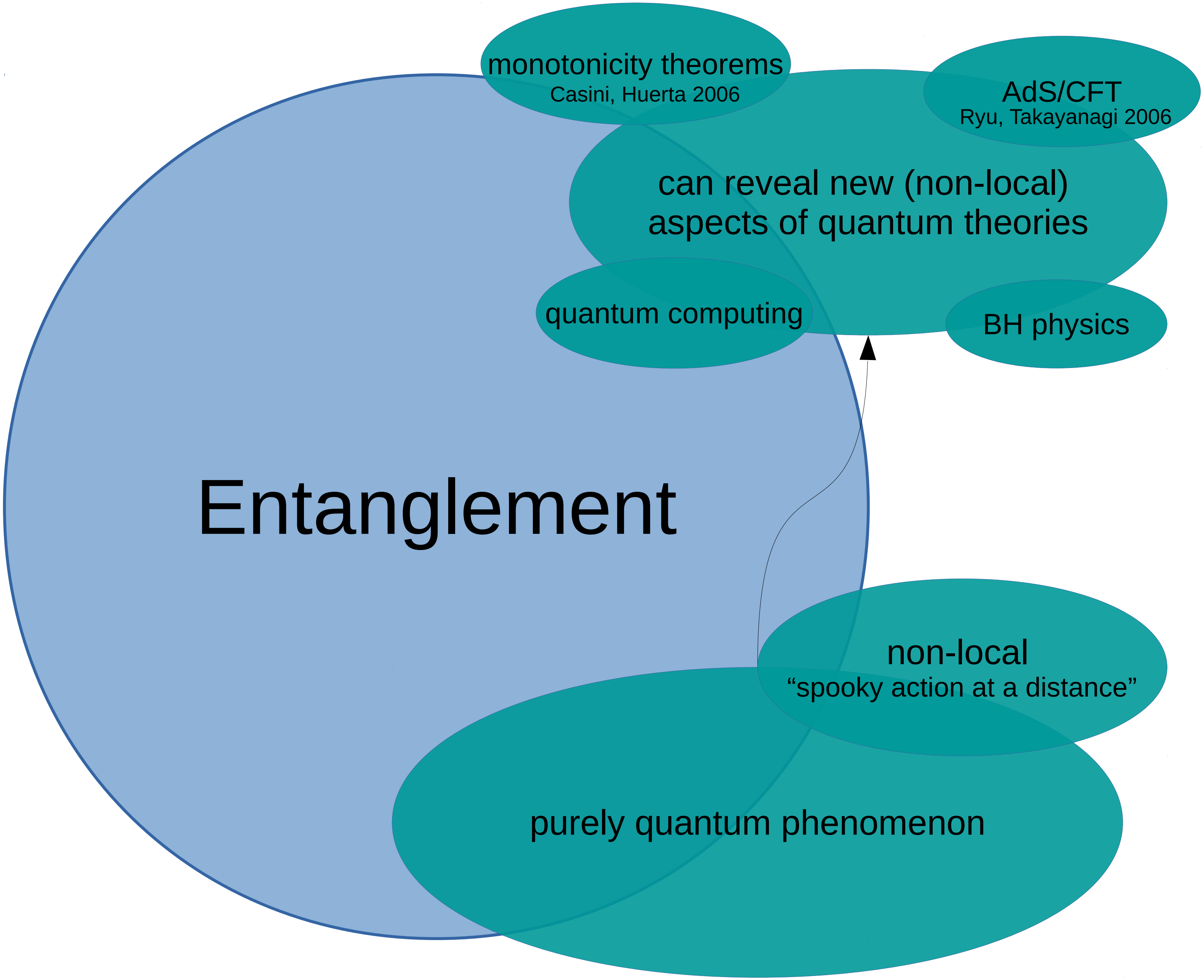
monotonicity theorems
Casini, Huerta 2006
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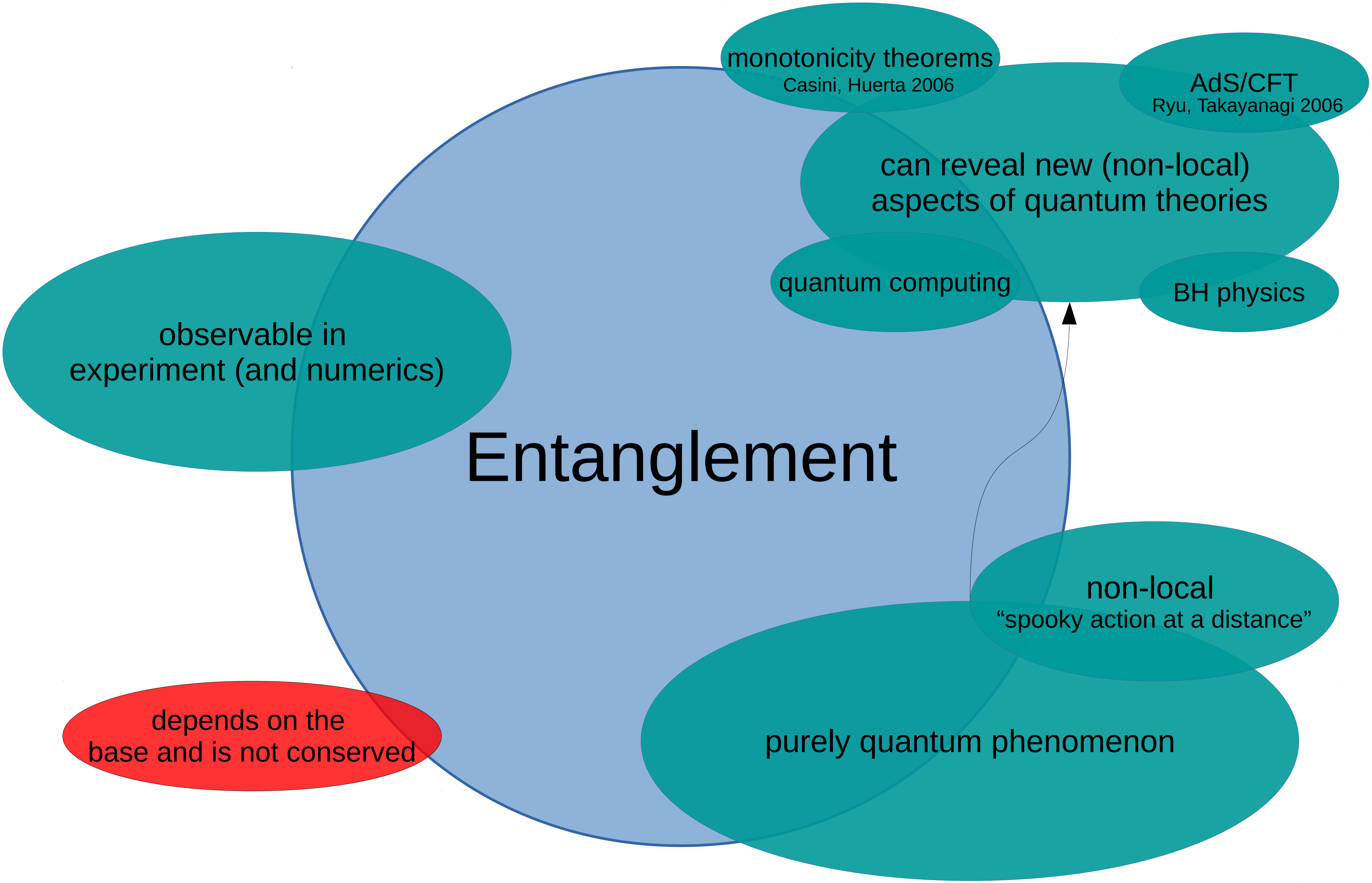
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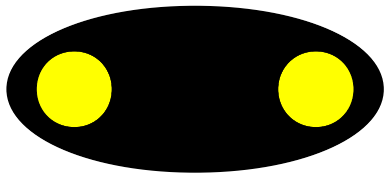
observable in
experiment (and numerics)
Entanglement
non-local
“spooky action at a distance”
depends on the
base and is not conserved
purely quantum phenomenon

Measure of Entanglement

separable state

fully entangled state,
e.g. Bell state



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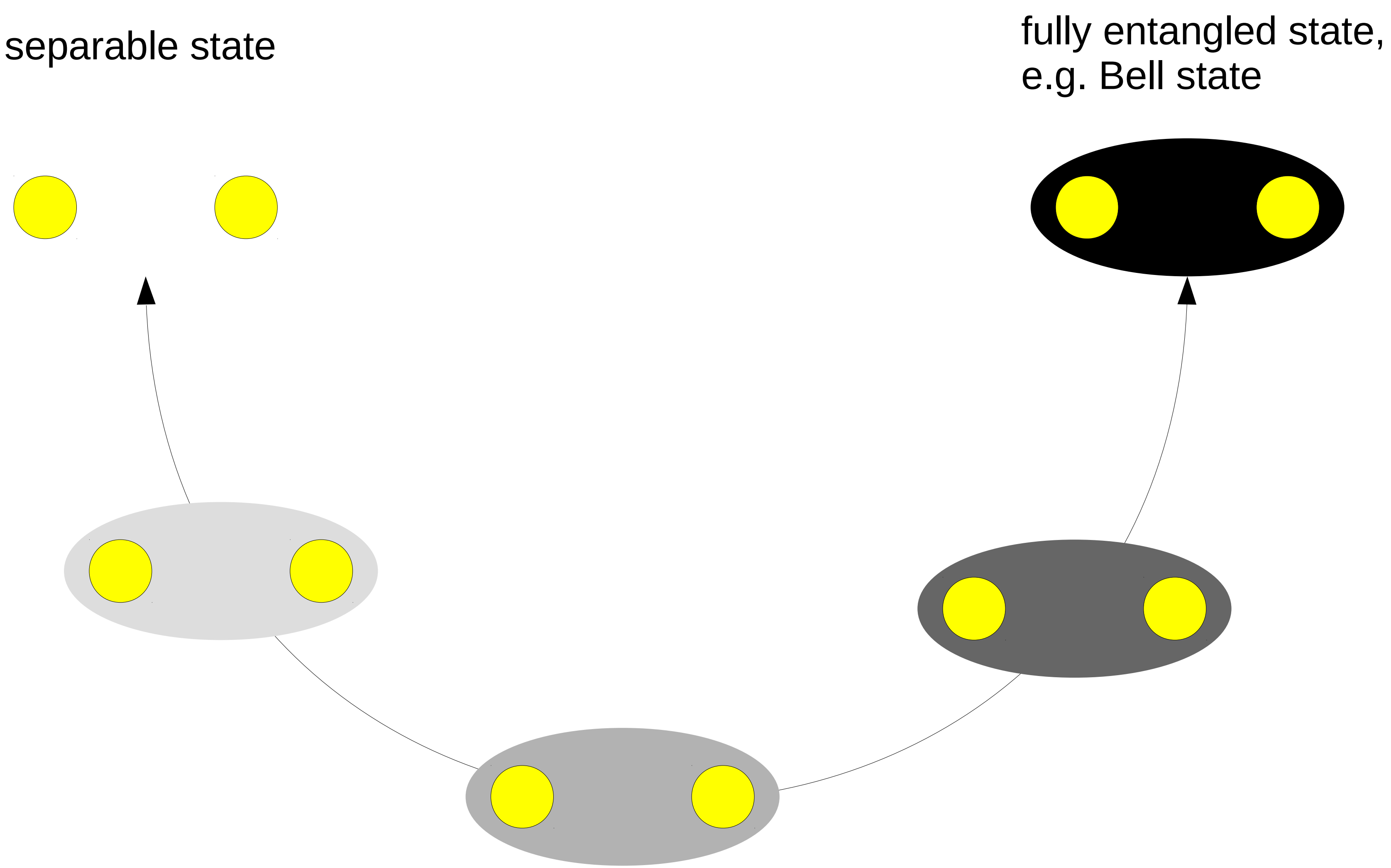

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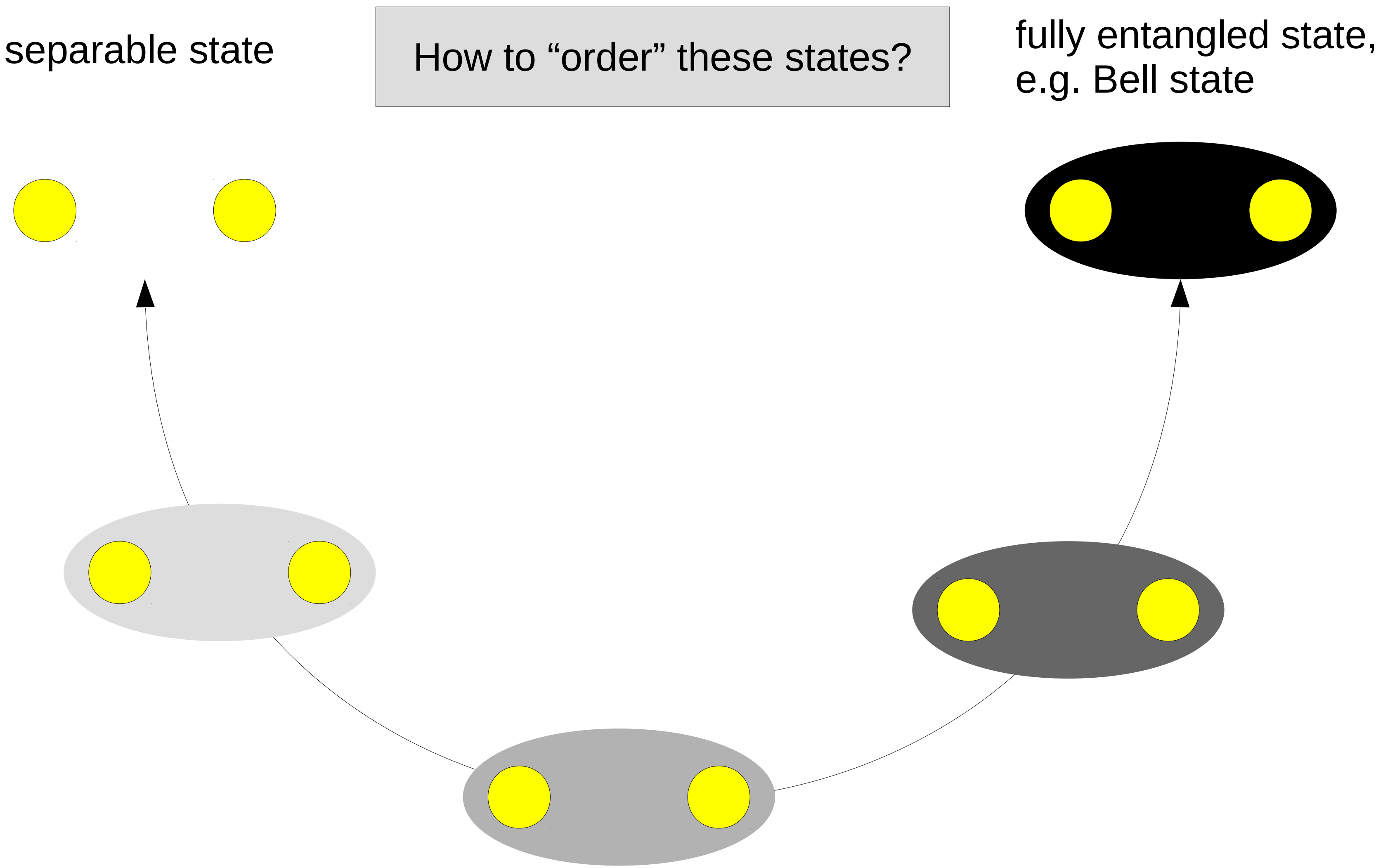
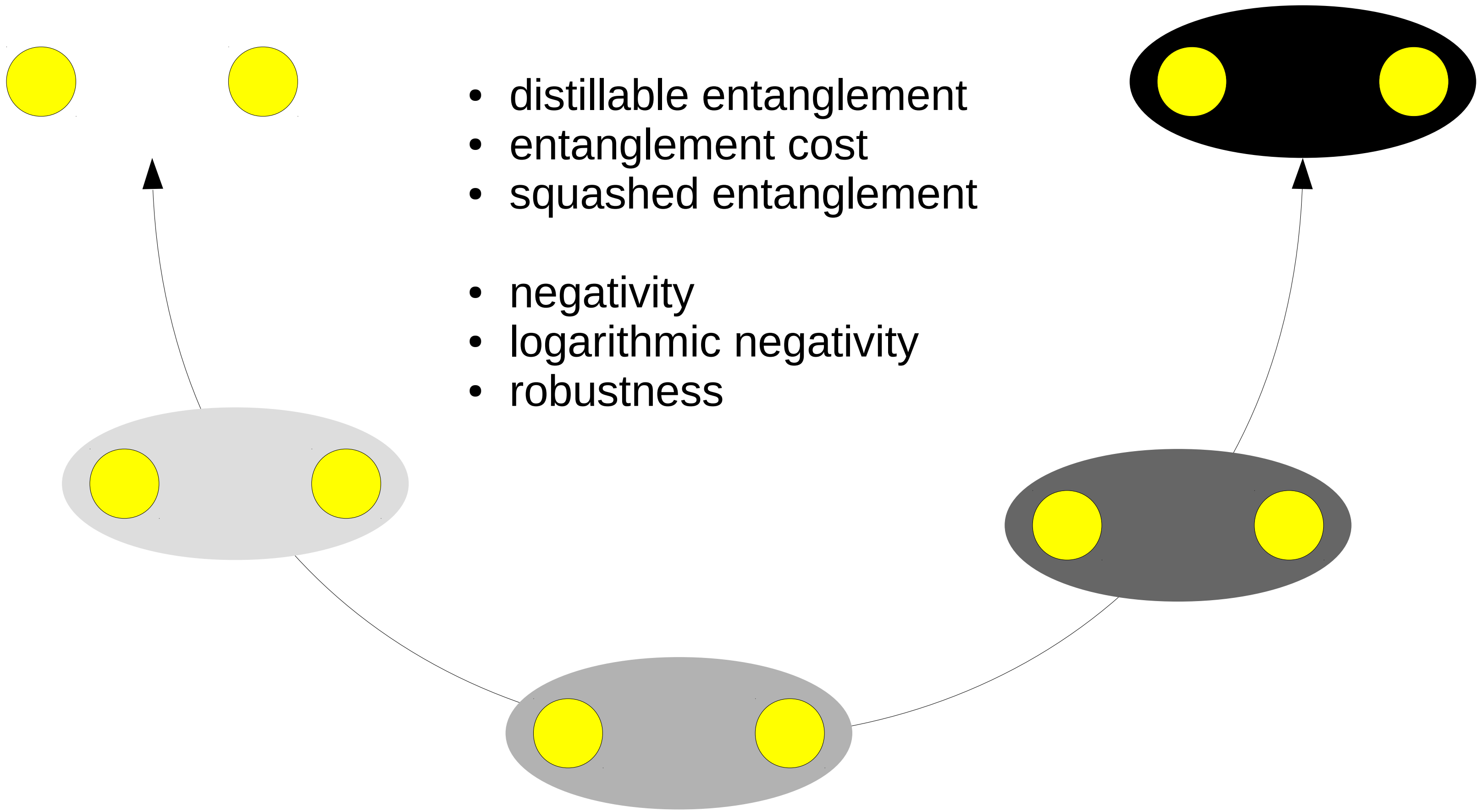


Measure of Entanglement

separable state

How to “order” these states?

fully entangled state,
e.g. Bell state



- distillable entanglement
- entanglement cost
- squashed entanglement

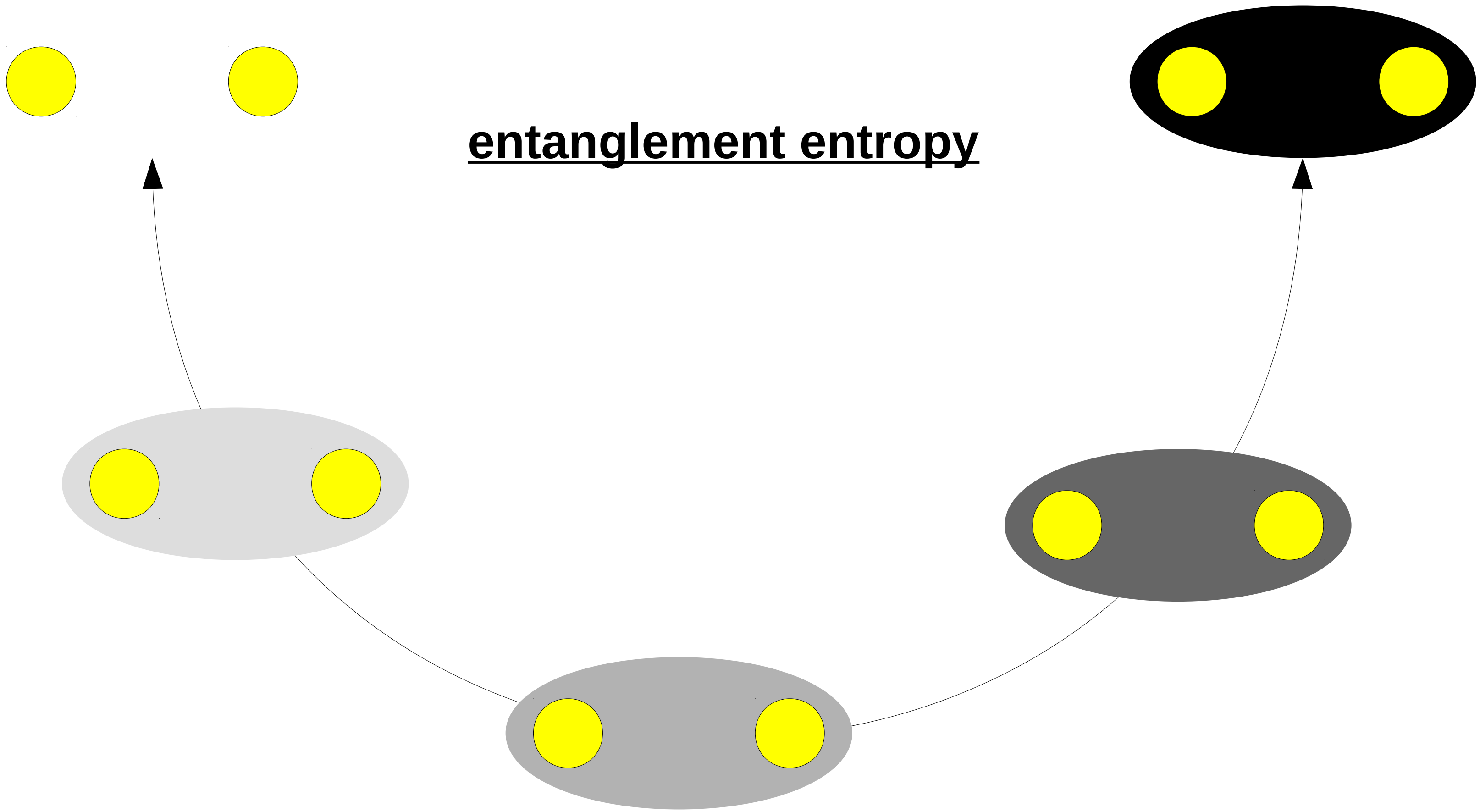
- negativity
- logarithmic negativity
- robustness

Measure of Entanglement

separable state

How to “order” these states?

fully entangled state,
e.g. Bell state

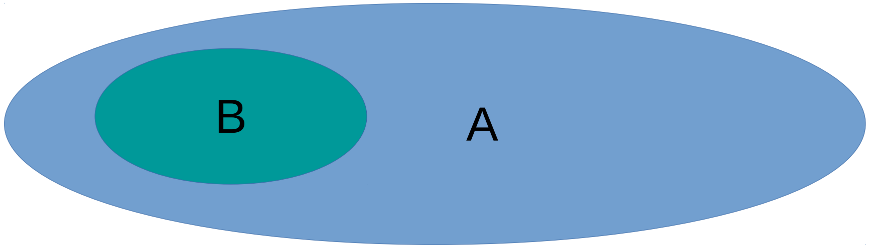


Entanglement Entropy

Definition: Let $\rho = |\psi\rangle\langle\psi|$ be the **density matrix** of a system in a pure quantum state $|\psi\rangle$. Let the Hilbert space be a direct product $\mathcal{H} = \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$. The reduced density matrix of A is $\rho_A = \mathrm{Tr}_\mathrm{B}\, \rho$. The **entanglement entropy** is the corresponding **von Neumann entropy**

$$S_A = -\mathrm{Tr}\, \rho_A \log \rho_A.$$

It measures the entanglement, i.e. quantum correlation, between the two sub-systems **A** and **B**.



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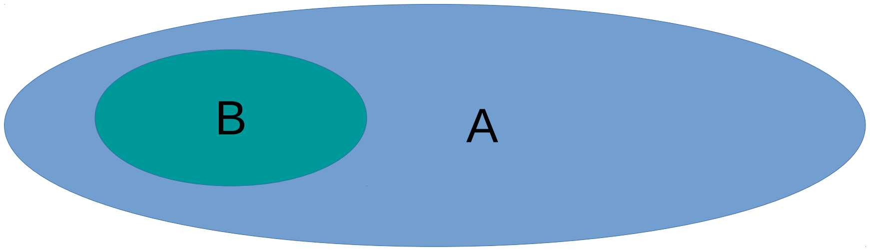
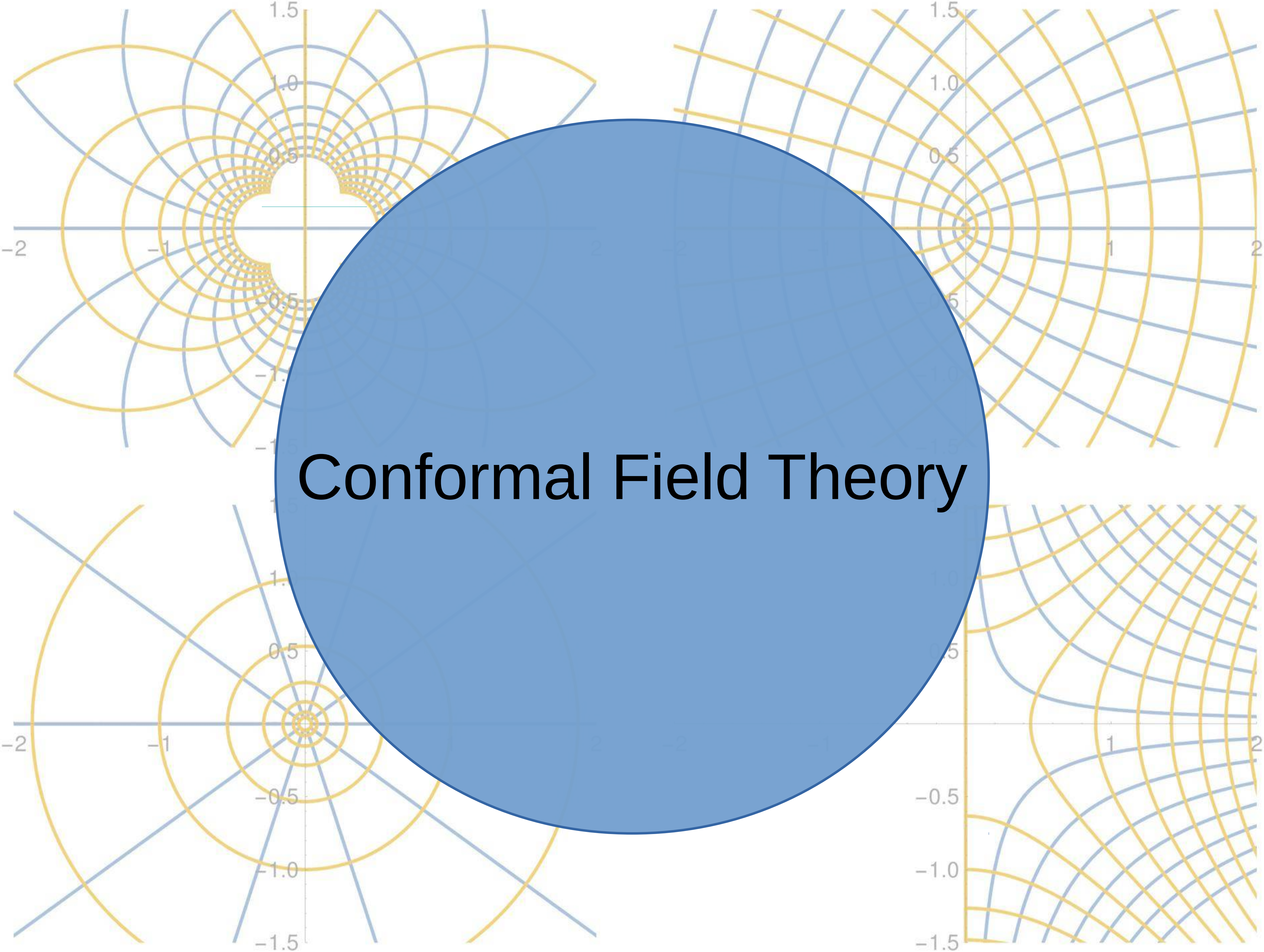
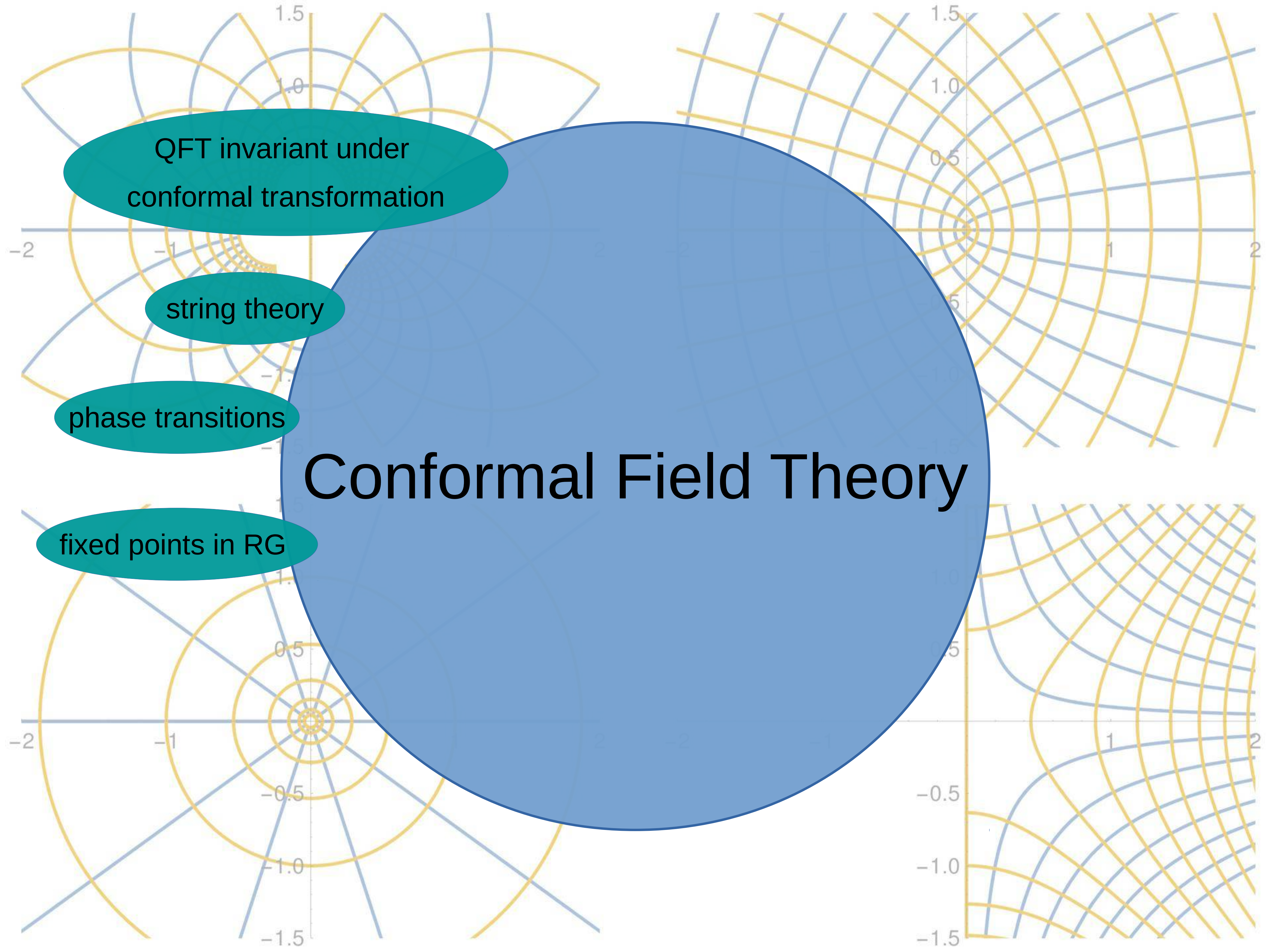


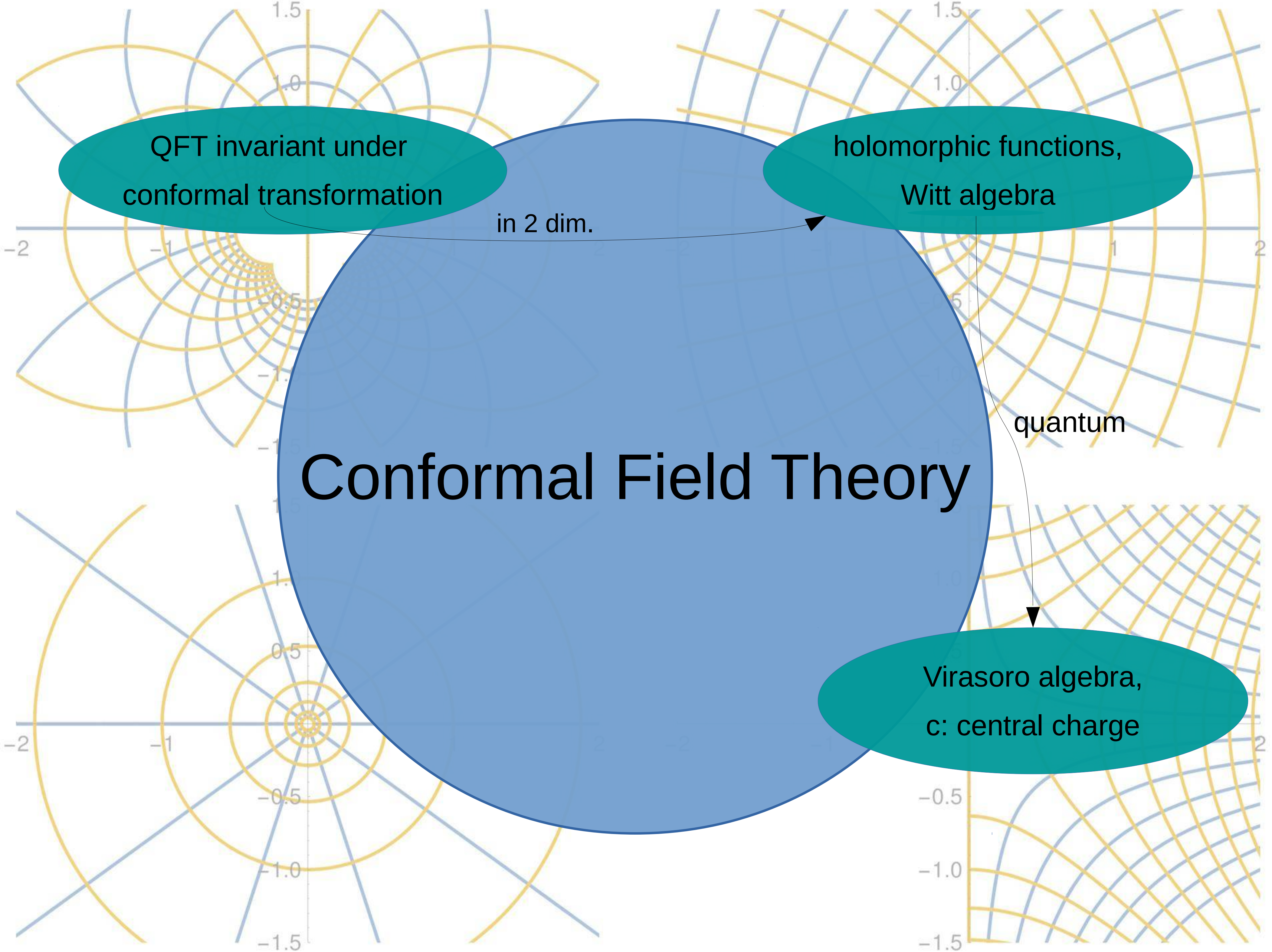
Replica trick...

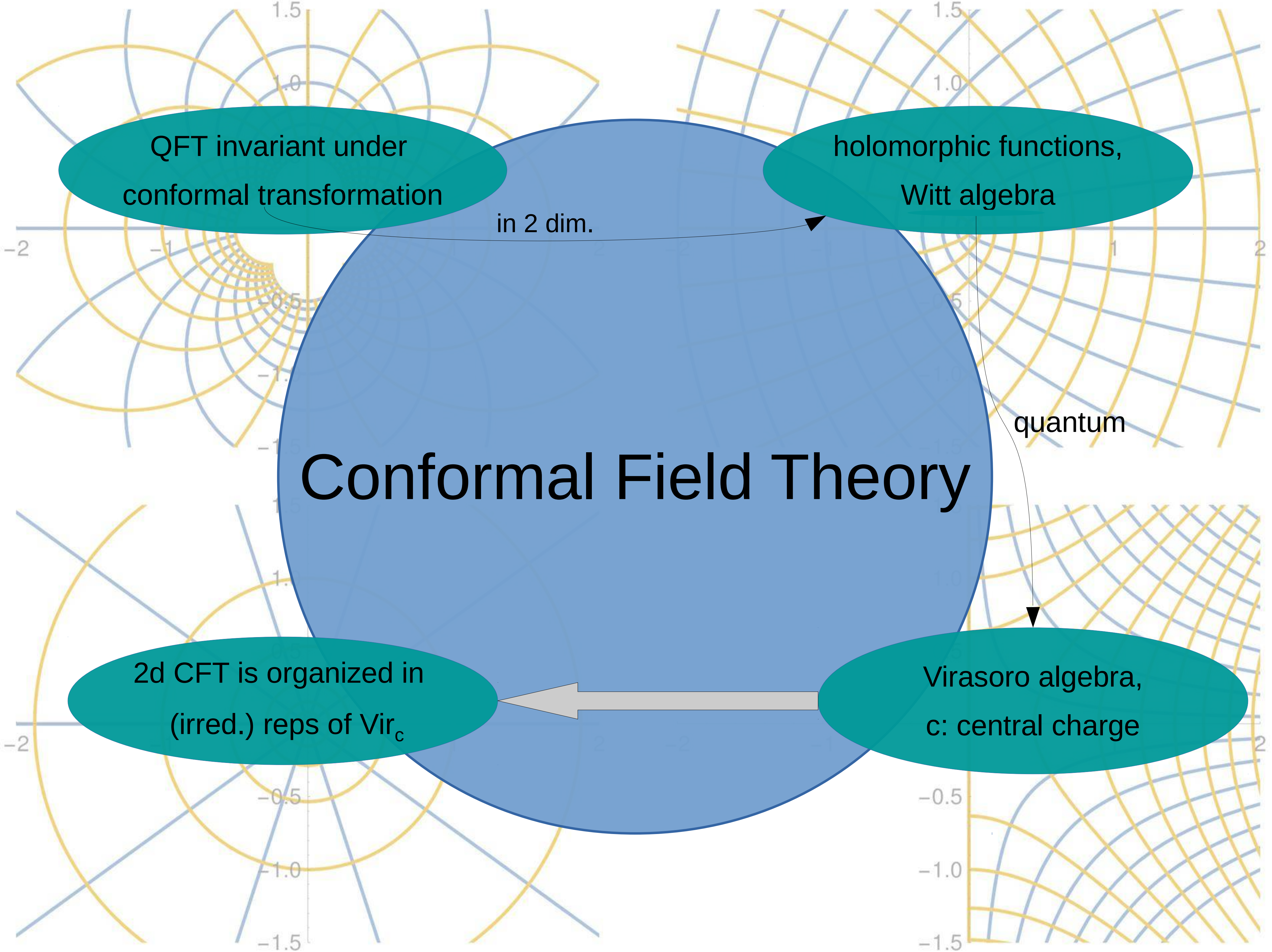
$$S_A = -\frac{\partial}{\partial n} \mathrm{Tr} \rho_A^n |_{n \to 1}$$

Conformal Field Theory



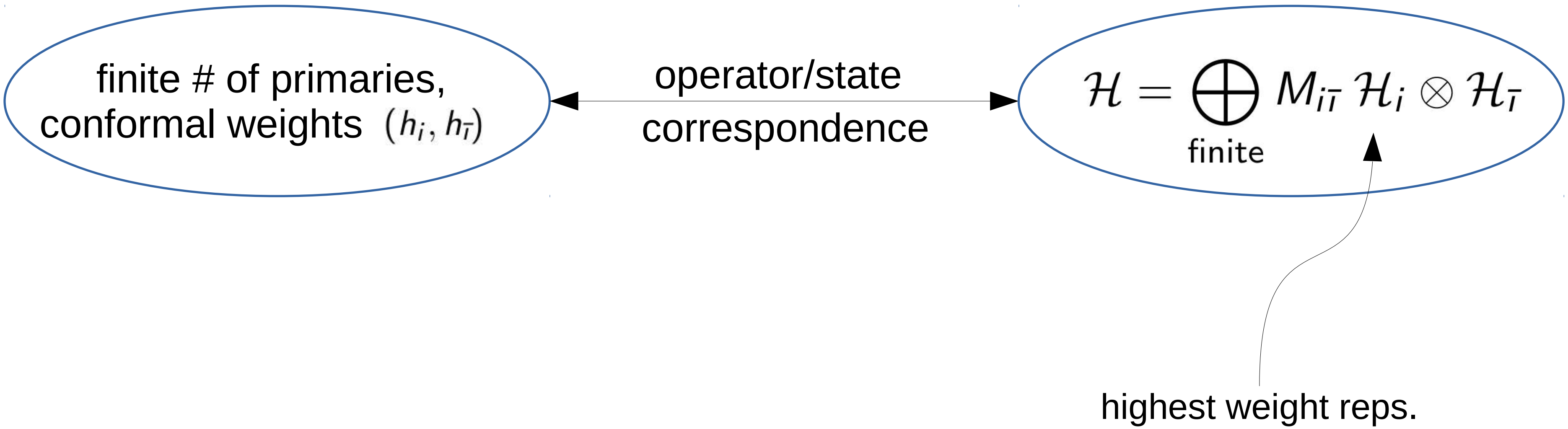

Conformal Field Theory
QFT invariant under conformal transformation
string theory
phase transitions
fixed points in RG


QFT invariant under
conformal transformation
in 2 dim.
holomorphic functions,
Witt algebra
quantum
Conformal Field Theory
Virasoro algebra,
c: central charge



Conformal Field Theory
QFT invariant under conformal transformation
in 2 dim.
holomorphic functions, Witt algebra
quantum
Virasoro algebra, c: central charge
2d CFT is organized in (irred.) reps of Vir_c

Rational Models



Rational Models

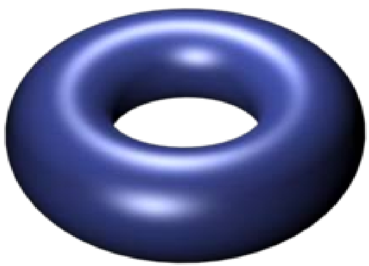
finite # of primaries,
conformal weights $(h_i, h_{\bar{\imath}})$

operator/state
correspondence

$$\mathcal{H} = \bigoplus_{\text{finite}} M_{i\bar{\imath}}\, \mathcal{H}_i \otimes \mathcal{H}_{\bar{\imath}}$$

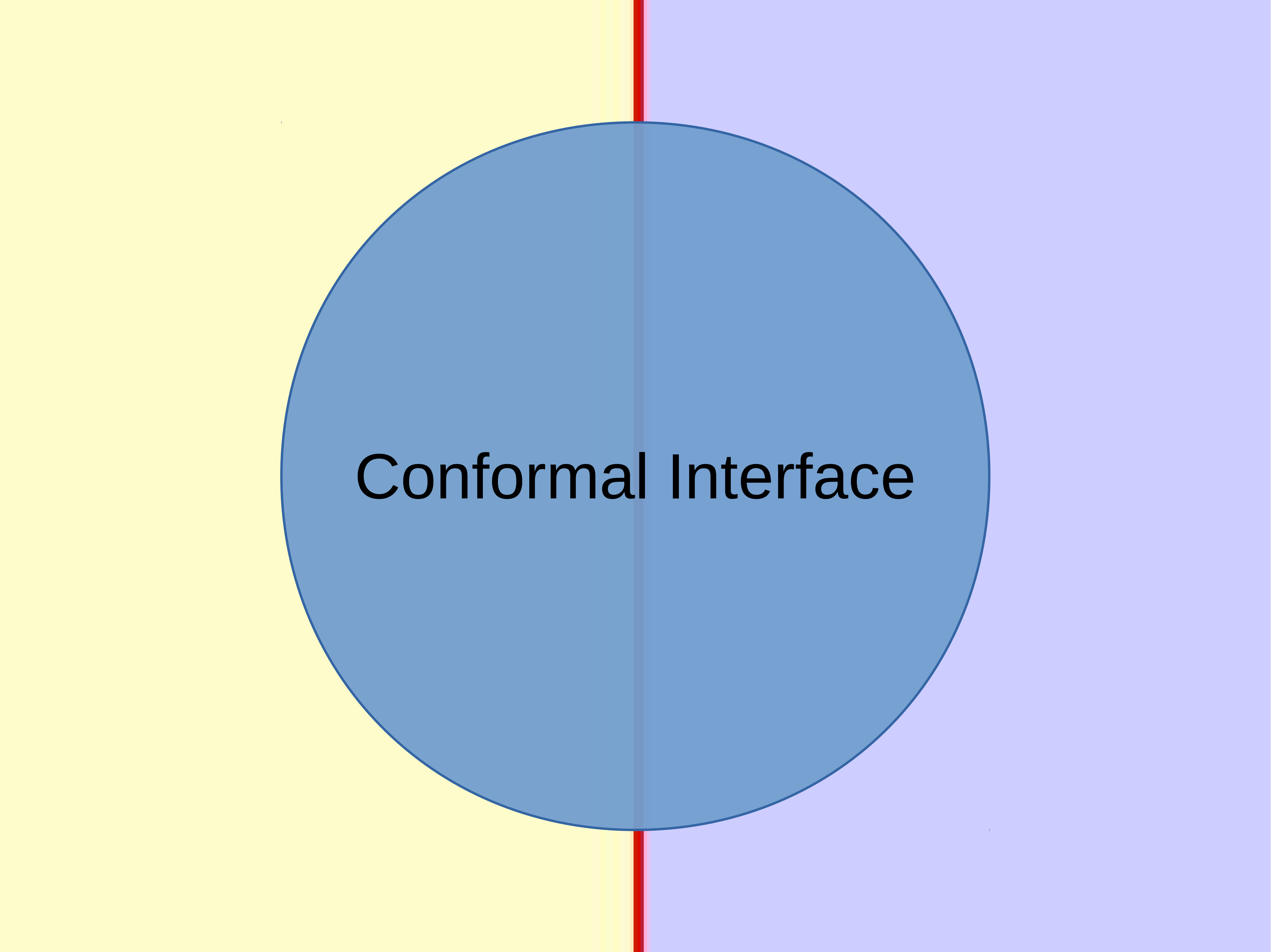
highest weight reps.

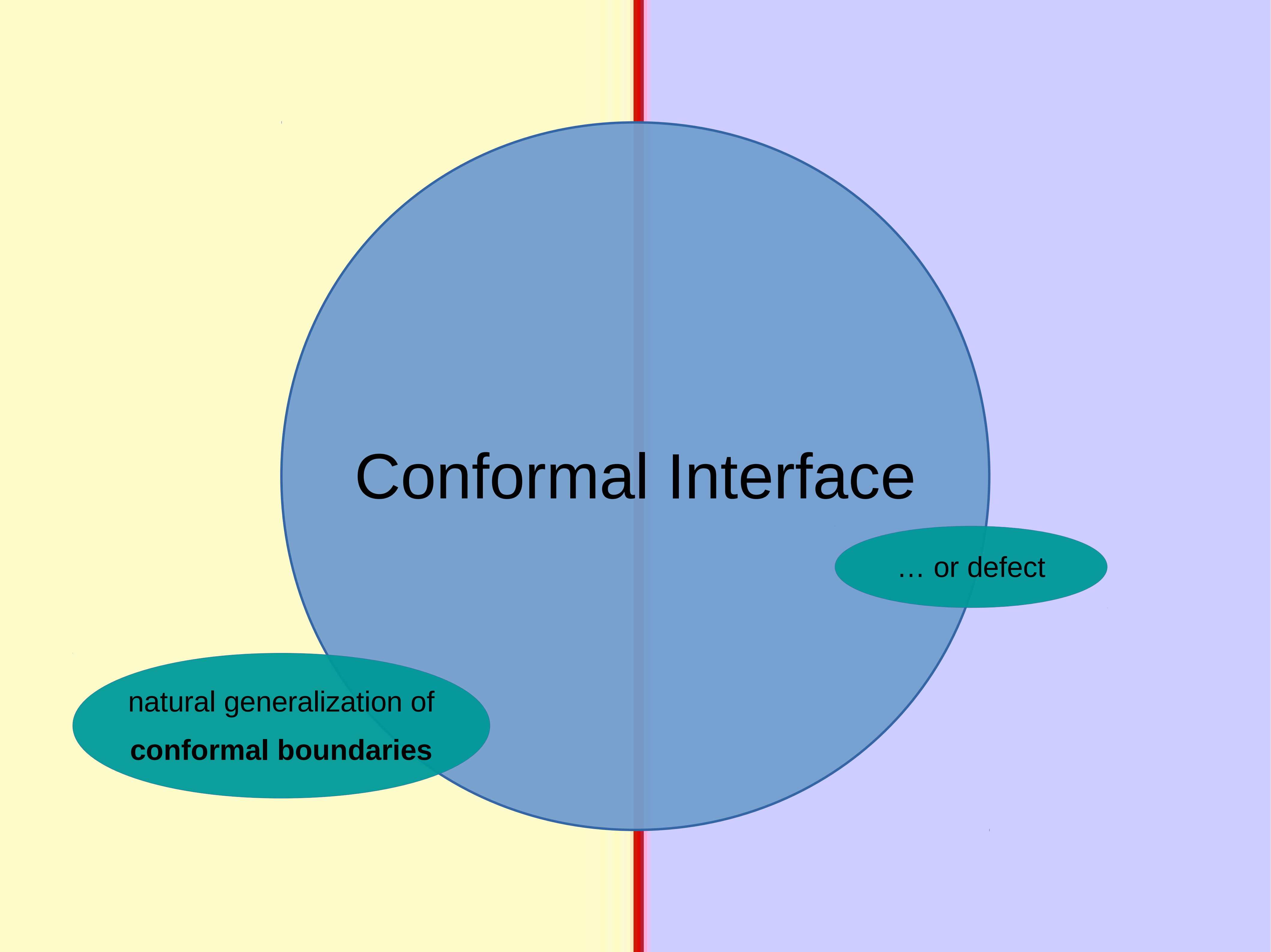
their characters

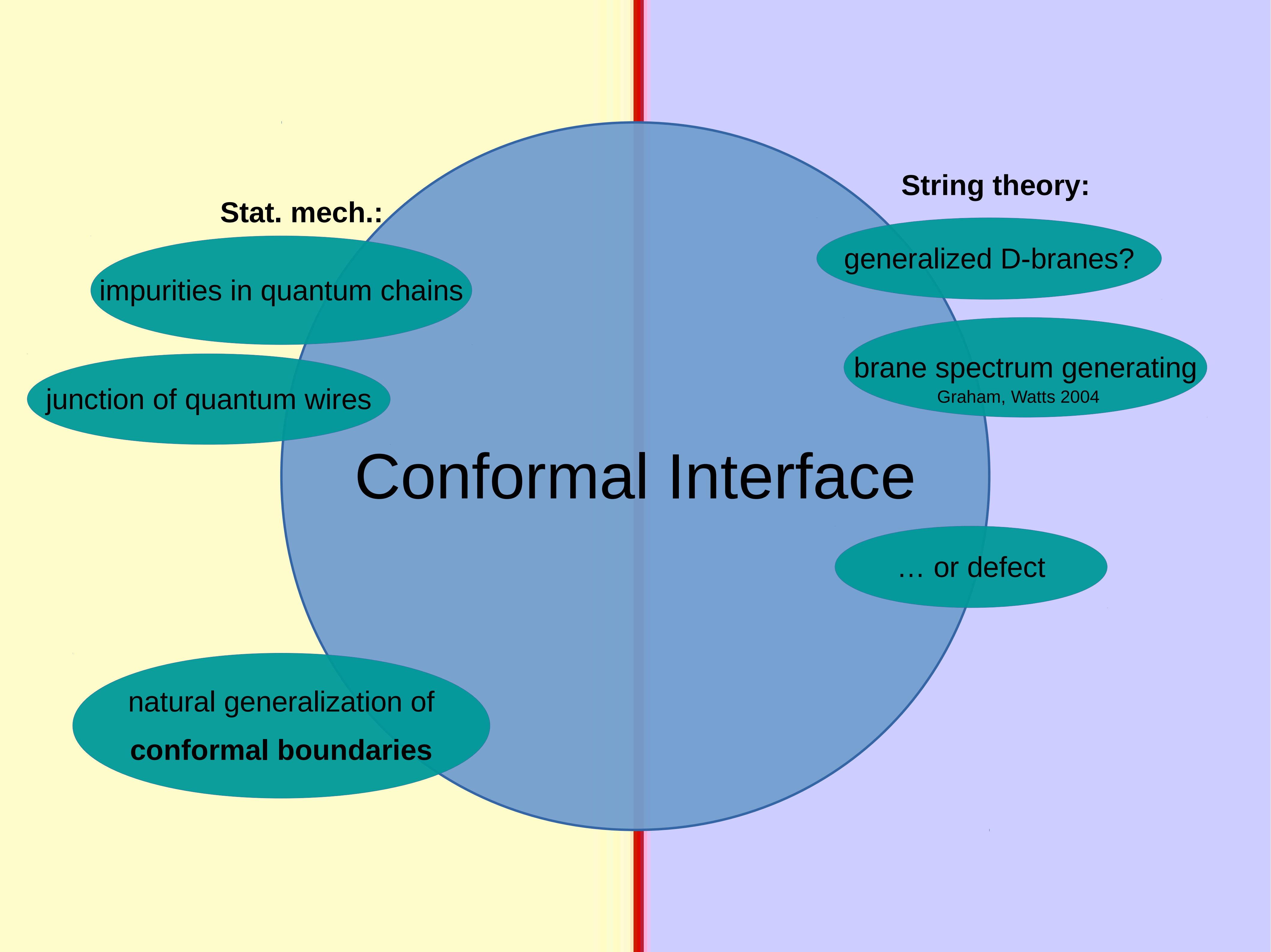


torus partition function

$$Z(\tau,\bar{\tau}) = \sum_{(i\bar{\imath})} M_{(i\bar{\imath})}\, \chi_i(q)\chi_{\bar{\imath}}(\bar{q}) = \sum_{(i\bar{\imath})} M_{(i\bar{\imath})}\, S_{ij}S_{\bar{\imath}\bar{\jmath}}\, \chi_j(\tilde{q})\chi_{\bar{\jmath}}(\bar{\tilde{q}})$$


Conformal Interface


Conformal Interface
… or defect
natural generalization of
conformal boundaries


Stat. mech.:
impurities in quantum chains
junction of quantum wires
natural generalization of
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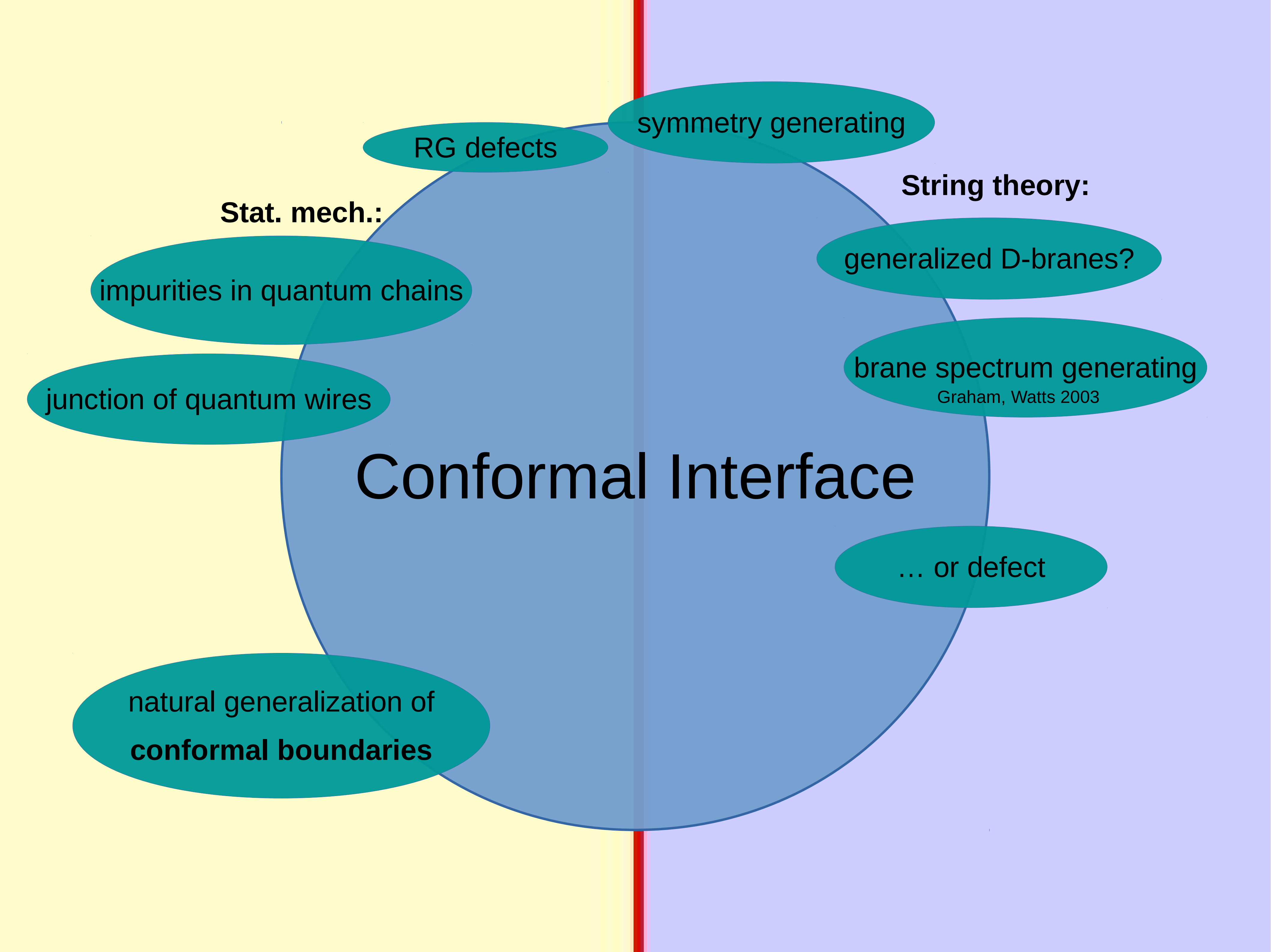
Conformal Interface
String theory:
generalized D-branes?
brane spectrum generating
Graham, Watts 2004
… or defect


symmetry generating
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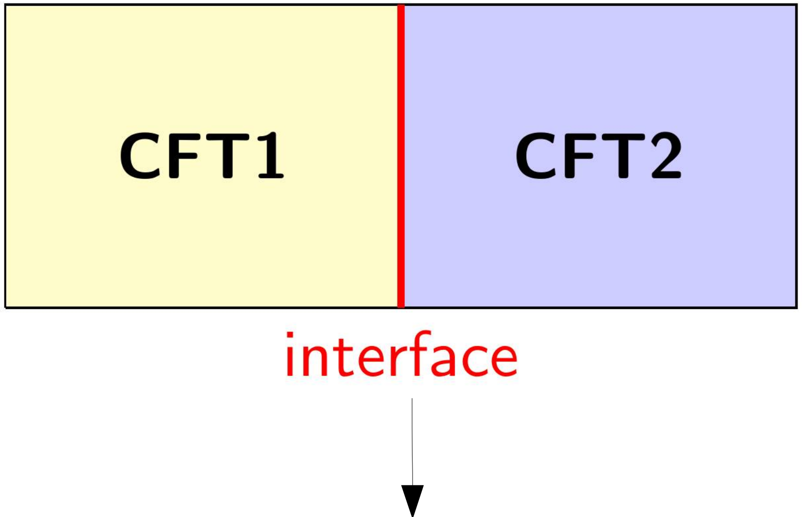
Conformal Interfaces

Bachas et al 2002

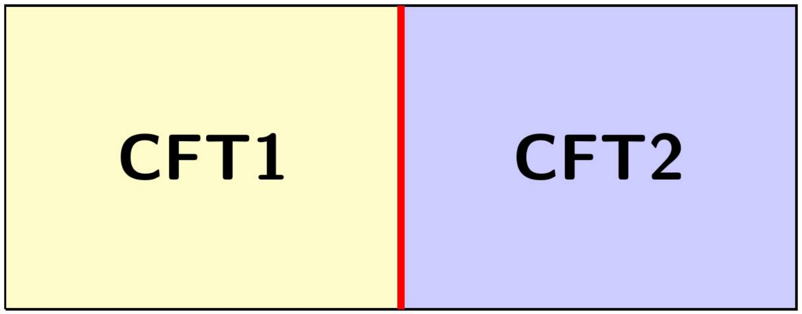


operator mapping states from on CFT to the other

$I_{1,2}$

Conformal Interfaces

Bachas et al 2002



interface

operator mapping states from on CFT to the other

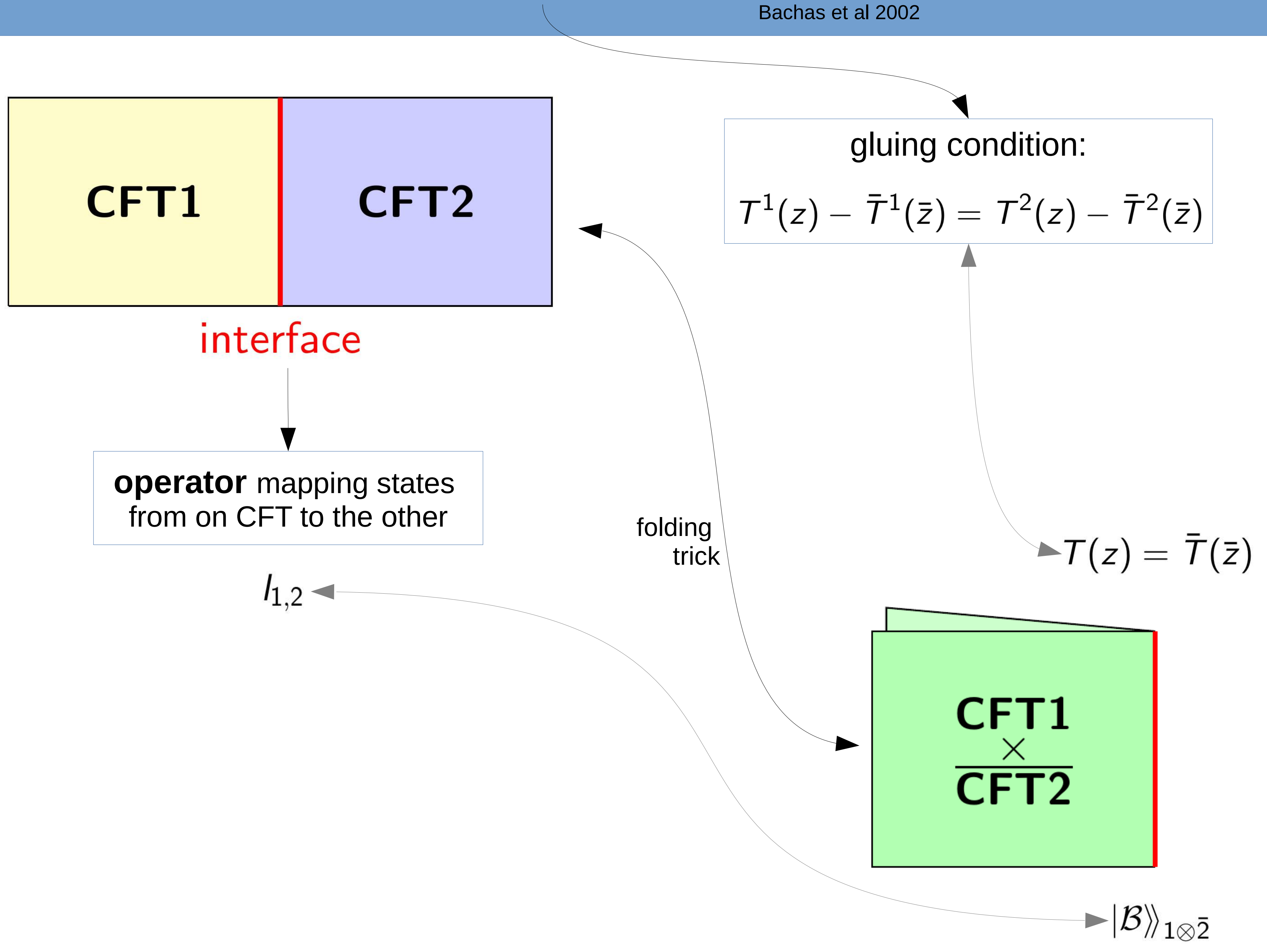
$I_{1,2}$

gluing condition:

$$T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z})$$

Conformal Interfaces

Bachas et al 2002



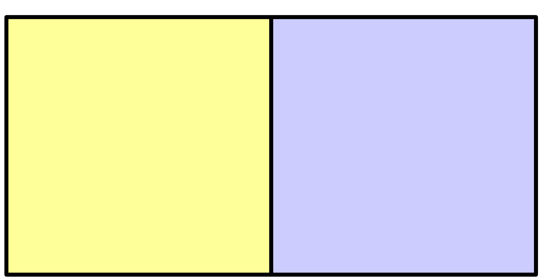
Special Gluing Conditions

$$T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z})$$

- Both sides vanish independently:

$$T^i(z) = \bar{T}^i(\bar{z})$$

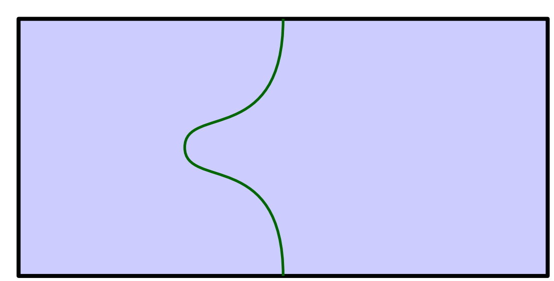
- **separate boundary conditions**
- In particular happens when one of the CFTs is **trivial**



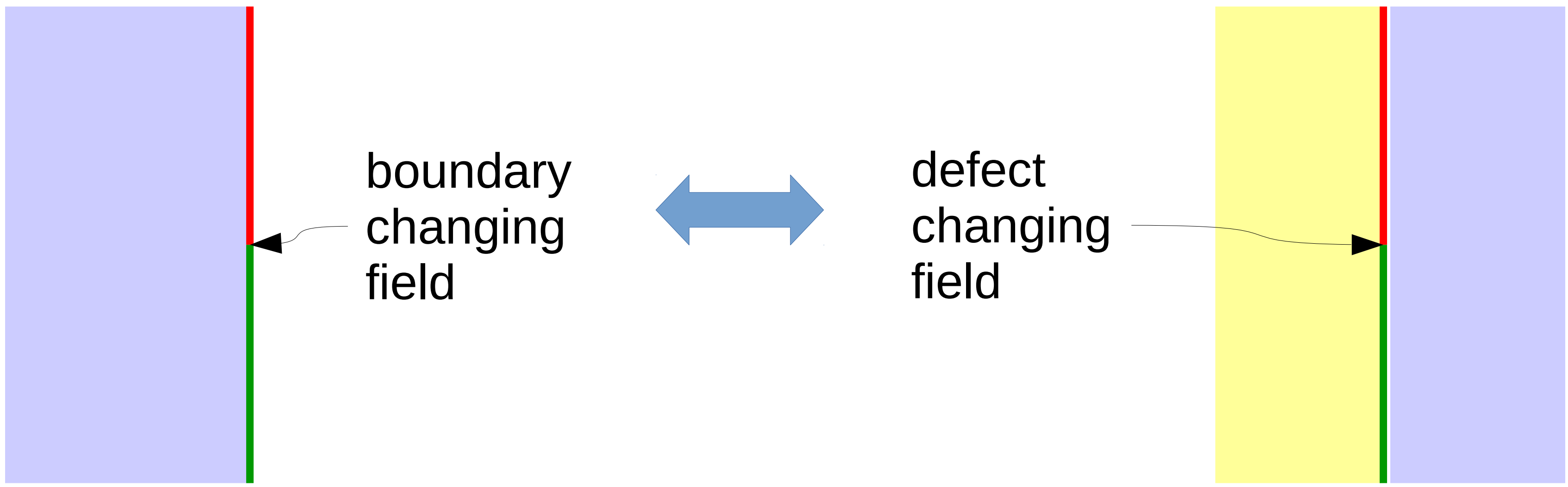
- The two components equal independently:

$$T^1(z) = T^2(z)\,, \quad \bar{T}^1(\bar{z}) = \bar{T}^2(\bar{z})$$

- $I_{1,2}$ also commutes with the Hamiltonian
- The interface can be moved around without cost of energy or momentum
- This is called a **topological interface**



What makes the difference?



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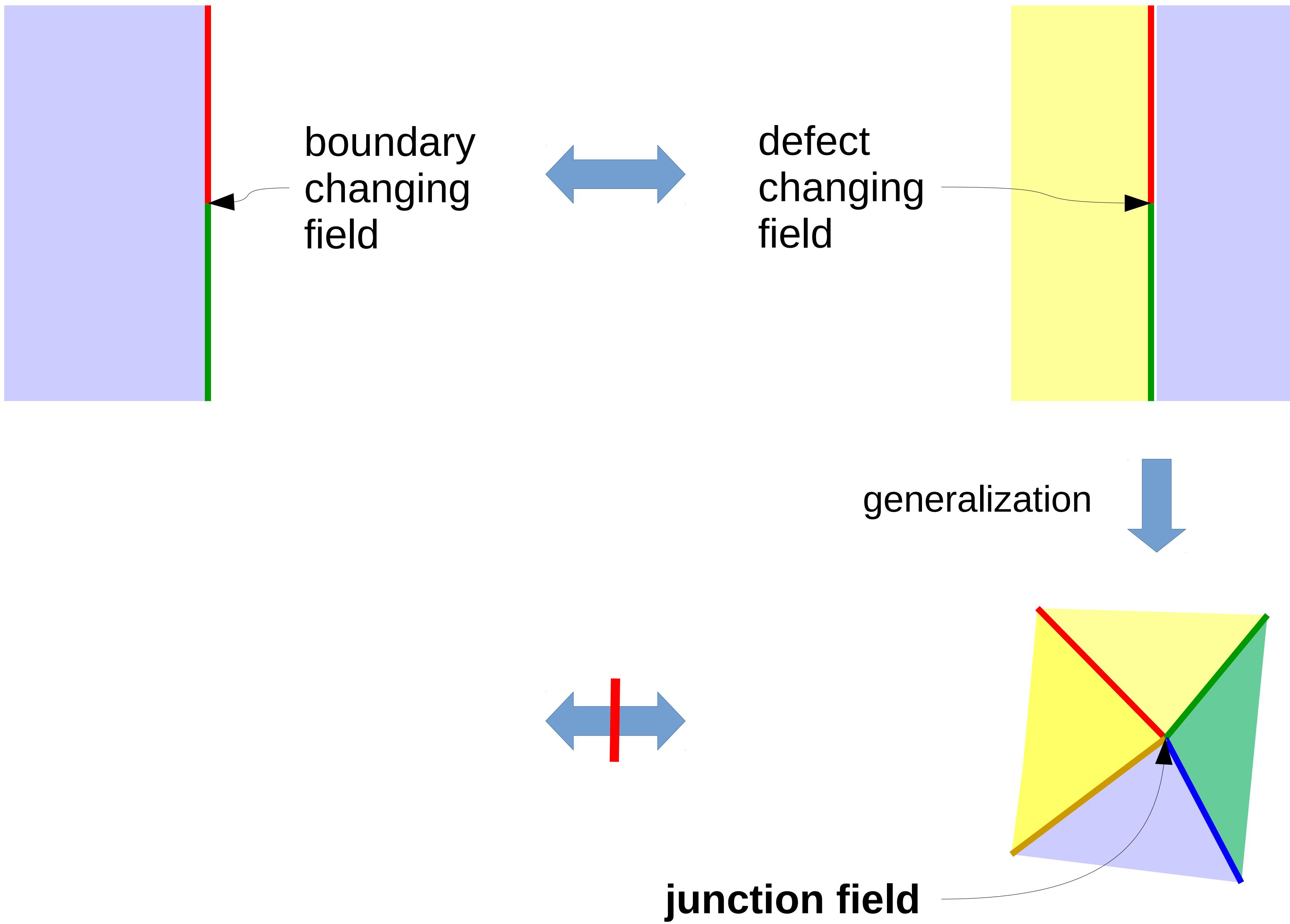


Topological Interfaces in a CFT

acts as a **constant map** between isomorphic **Virasoro representations**

Petkova, Zuber 2000

⇕

$$I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, ||\mathbf{i}||$$

Topological Interfaces in a CFT

acts as a **constant map** between isomorphic **Virasoro representations**

Petkova, Zuber 2000

$$I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, \|\mathbf{i}\|$$

invariance under S-trafo

$$\sum_{\mathbf{i}} S_{ij} S_{\bar{\imath}\bar{\jmath}} \, \mathrm{Tr}\, d_{A^*\mathbf{i}} d_{A\mathbf{i}} = \mathcal{N}_{j\bar{\jmath}} \in \mathbb{N}$$

example: diagonal rational theories

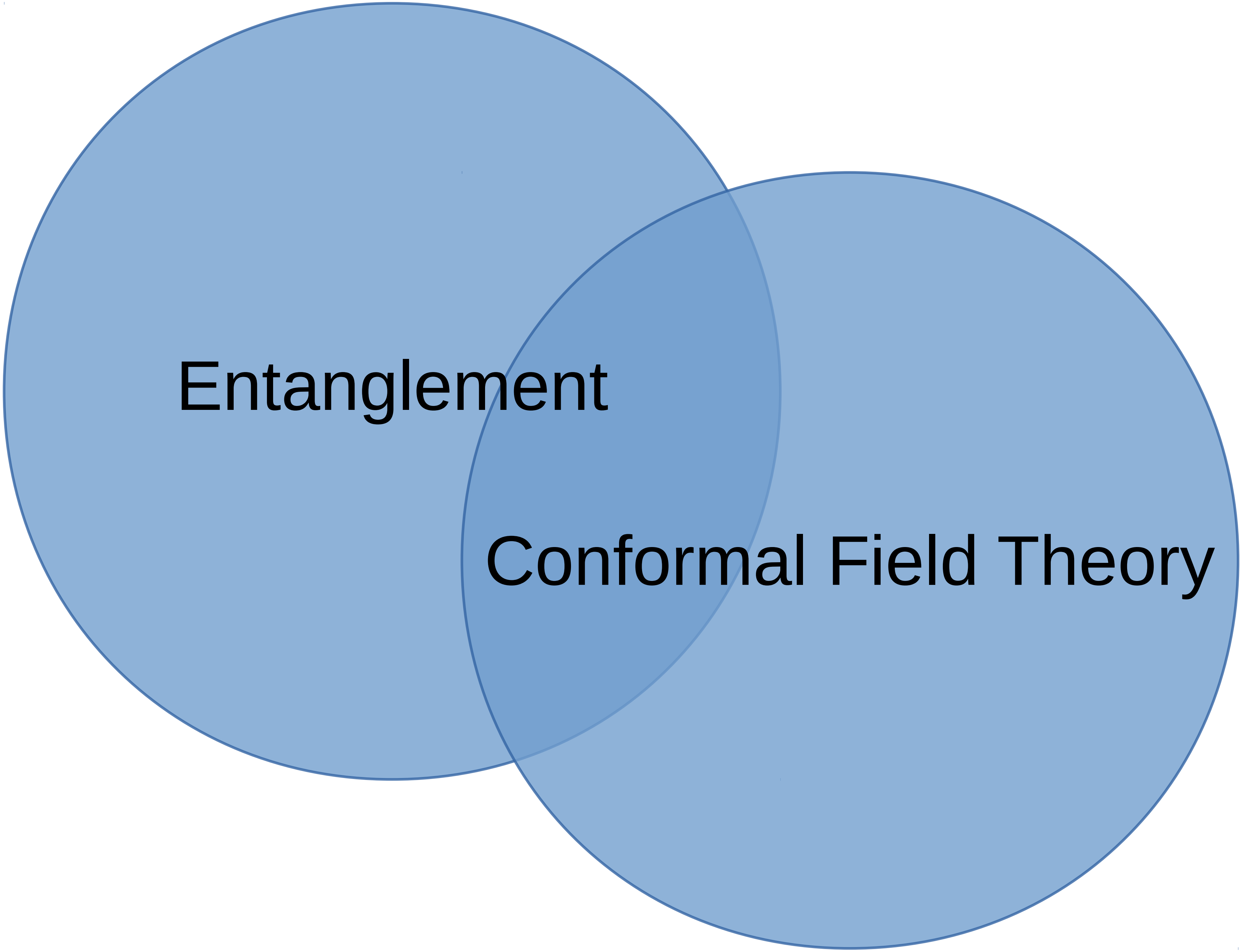
$$I_a = \sum_i \frac{S_{ai}}{S_{0i}} \, \|i\|$$

Example: Topological Interfaces of the Ising model

primary	conformal weight
id	(0,0)
ε	(1/2, 1/2)
σ	(1/16,1/16)

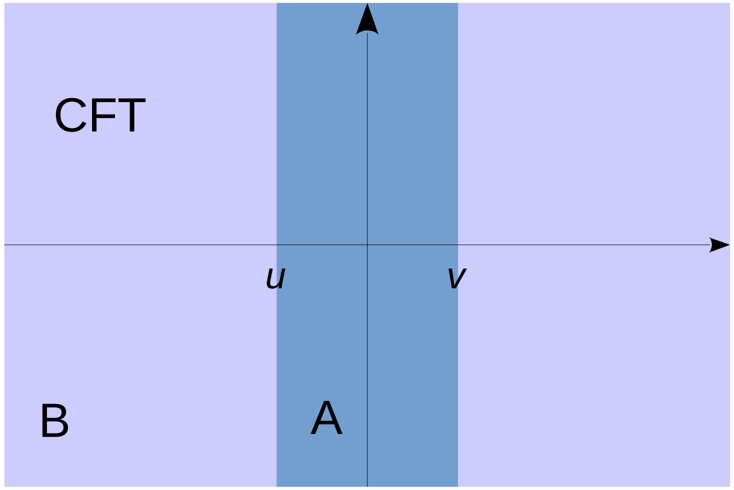
$$S_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$\begin{aligned} I_{id} &= \|id\| + \|\epsilon\| + \|\sigma\| \\ I_{\epsilon} &= \|id\| + \|\epsilon\| - \|\sigma\| \\ I_{\sigma} &= \sqrt{2}\|id\| - \sqrt{2}\|\epsilon\| \end{aligned}$$


Entanglement
Conformal Field Theory

Entanglement Entropy of a Finite Interval

Cardy, Calabrese 2009



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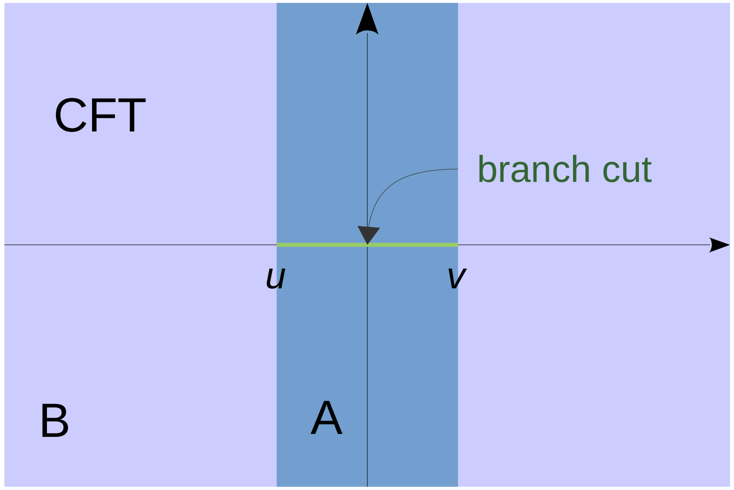
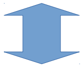


remember replica trick:

$$\mathrm{Tr}\rho_A^n$$



partition function *Z(n)* on a complicated Riemann surface

Entanglement Entropy of a Finite Interval

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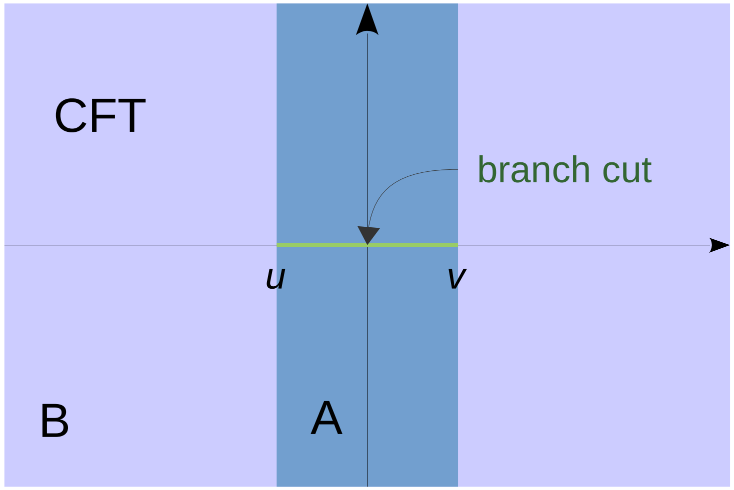

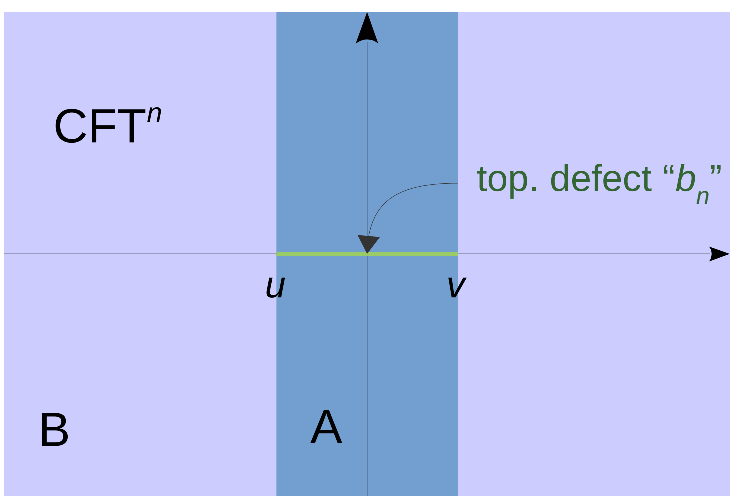

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$\Updownarrow$

partition function $Z(n)$ on a complicated Rieman surface

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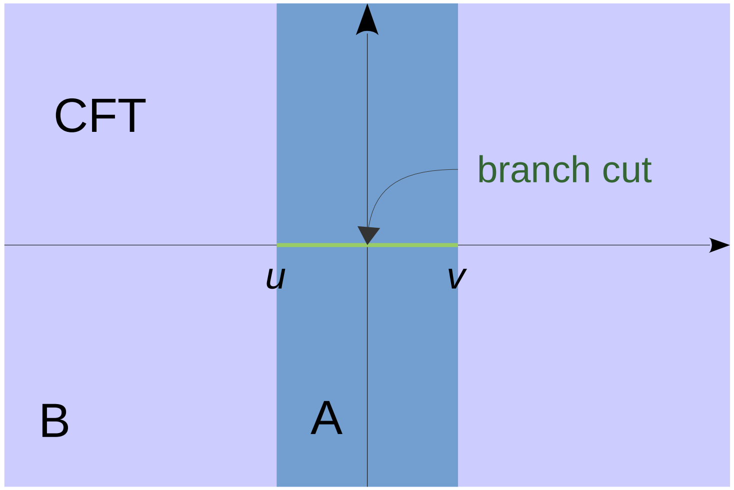

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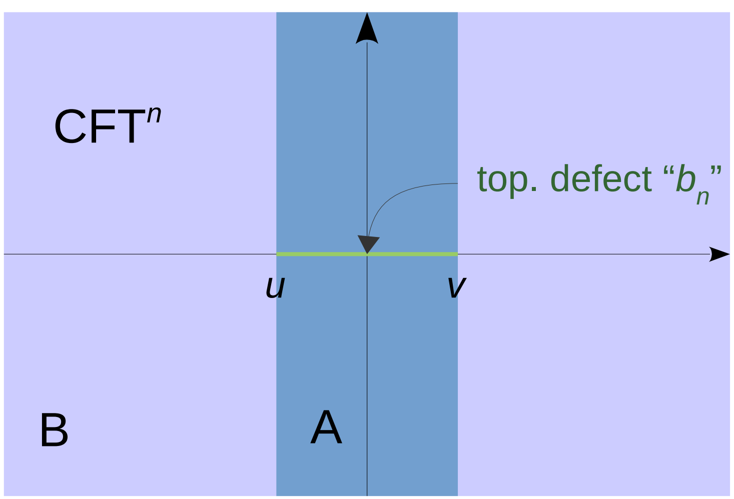
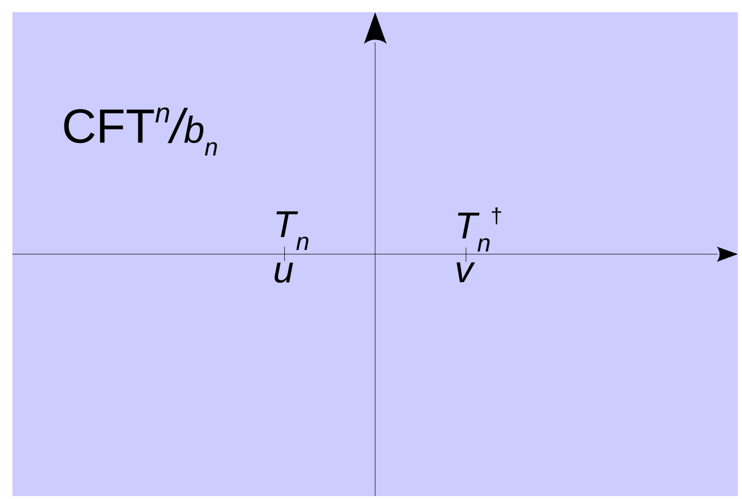
partition function *Z(n)* on a complicated Rieman surface

2-point function of twist fields

$$\langle T_n(u) T_n^\dagger(v) \rangle$$





EE of a Finite Interval

2-point function of **twist fields**

$$\langle T_n(u) T_n^\dagger(v) \rangle$$

“junction field” of lowest conformal weight

EE of a Finite Interval

2-point function of **twist fields**

$$\langle T_n(u) T_n^\dagger(v) \rangle$$

“junction field” of lowest conformal weight

state-operator correspondence

leading order for large L

$$q^{h_n - \frac{nc}{12}} = \langle T_n | q^{H^n_{b_n}} | T_n \rangle = Z_{\mathcal{H}^n_{b_n}}(\tau \gg 1)$$

Cardy condition

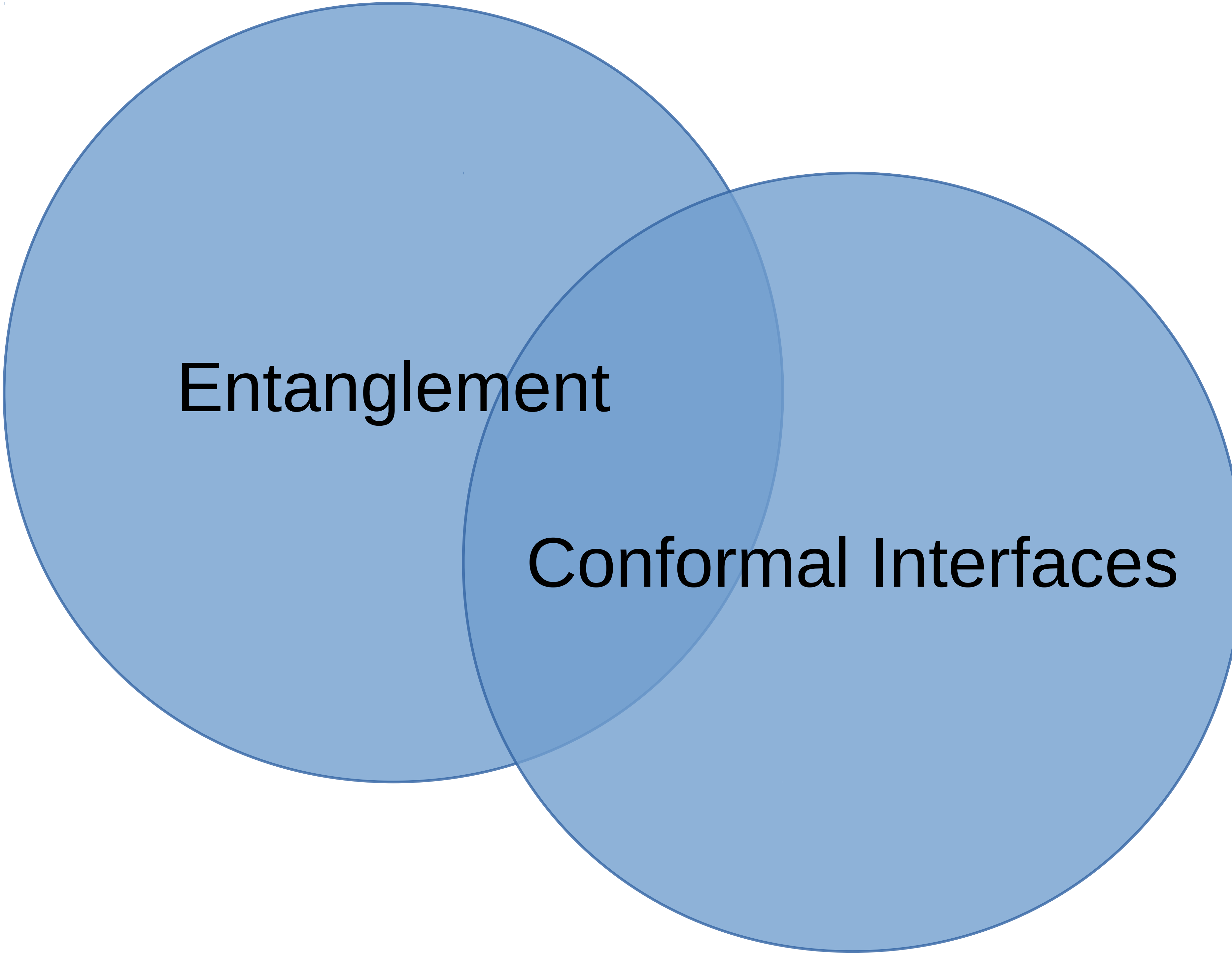
$$= \mathrm{Tr}(b_n \tilde{q}^{H^n}) = \mathrm{Tr}(\tilde{q}^{nH}) = \sum_{(i\bar{\imath})} \chi_i(\tilde{q}^n) \chi_{\bar{\imath}}(\tilde{q}^n)$$

S-trafo & leading order

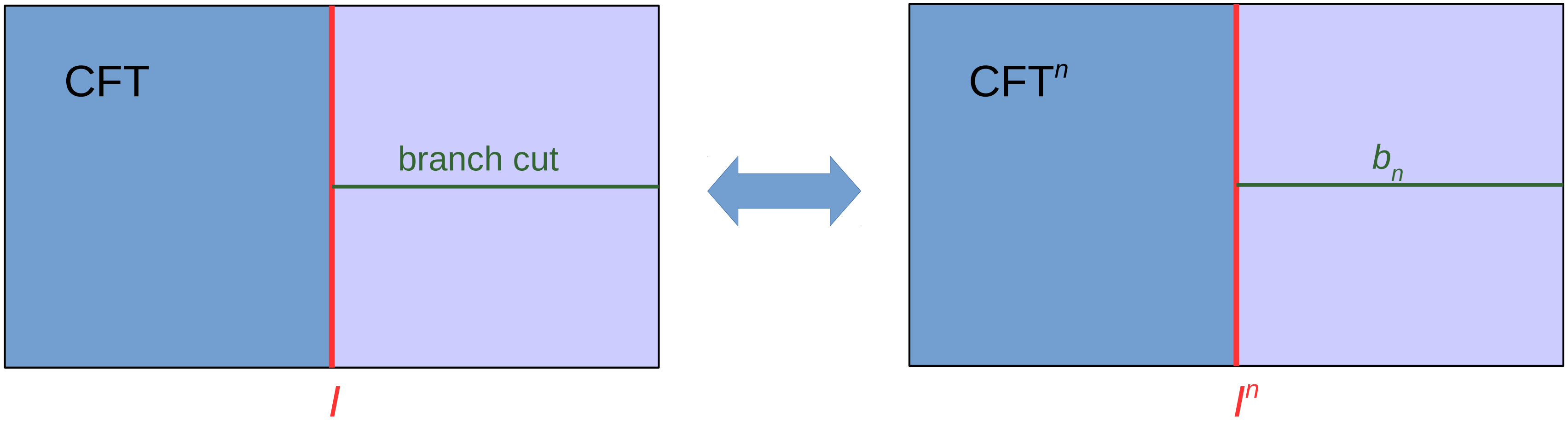
$$= q^{-\frac{c}{12n}}$$

$$h_n = \frac{c}{12}\left(n - \frac{1}{n}\right), \quad L = |v - u| \gg 1$$

$$S_A = \frac{c}{3} \log L + c_0$$


Entanglement
Conformal Interfaces

Entanglement through Conformal Interfaces



$$Z(n) = \mathrm{Tr}(b_n\, q^{H^n/4}\, I^n\, q^{H^n/2}\, (I^n)^\dagger\, q^{H^n/4})$$

$$= \mathrm{Tr}(I\, q^{H/2}\, I^\dagger\, q^{H/2})^n$$

Entanglement through Topological Interfaces

Remember: $I_A = \sum_{\mathbf{i}=(i\bar{\imath})} d_{A\mathbf{i}} \, ||\mathbf{i}||$ and $[I_A, H] = 0$

$$Z(n) = \mathrm{Tr}\left(\left(I_A I_A^\dagger\right)^n q^{nH}\right) = \sum_{(i\bar{\imath})} \mathrm{Tr}(d_{A\mathbf{i}} d_{A^*\mathbf{i}})^n \chi_i(q^n)\,\chi_{\bar{\imath}}(\bar{q}^n)$$

$$= \underbrace{\sum_{(i,\bar{\imath})} \mathrm{Tr}\,(d_{A^*\mathbf{i}}\, d_{A\mathbf{i}})^n\, S_{i0} S_{\bar{\imath}0}}_{\equiv A(n)}\;\; \tilde{q}^{-\frac{c}{12n}}$$

S-trafo &
leading order

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S-trafo & leading order

no change in the log term of the EE

$$\frac{c}{3}\log L$$

contributes to sub-leading term in the EE:

$$s(I_A) = -\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^A \, \log \frac{p_{\mathbf{i}}^A}{p_{\mathbf{i}}^{id}}$$

with

$$p_{\mathbf{i}}^A = \frac{d_{A^*\mathbf{i}}\, d_{A\mathbf{i}}\, S_{i0} S_{\bar{\imath}0}}{\mathcal{N}_{\mathbf{0}A}^{\;A}}$$

Entanglement through Topological Interfaces

relative entropy / Kullback–Leibler divergence:

$$s(I_A) = - \sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^{A} \log \frac{p_{\mathbf{i}}^{A}}{p_{\mathbf{i}}^{id}}$$

Entanglement through Topological Interfaces

relative entropy / Kullback–Leibler divergence:

$$s(I_A) = -\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^A \log \frac{p_{\mathbf{i}}^A}{p_{\mathbf{i}}^{id}}$$

diagonal
RCFTs

$p_i^a = |S_{ia}|^2$

$$s(I_a) = -\sum_i |S_{ia}|^2 \log \left| \frac{S_{ia}}{S_{i0}} \right|^2$$

Ising

$$s(I_a) = \begin{cases} -\log 2\,, & a = \sigma \\ 0\,, & a = id, \epsilon \end{cases}$$

$su(2)_{k \gg 1}$

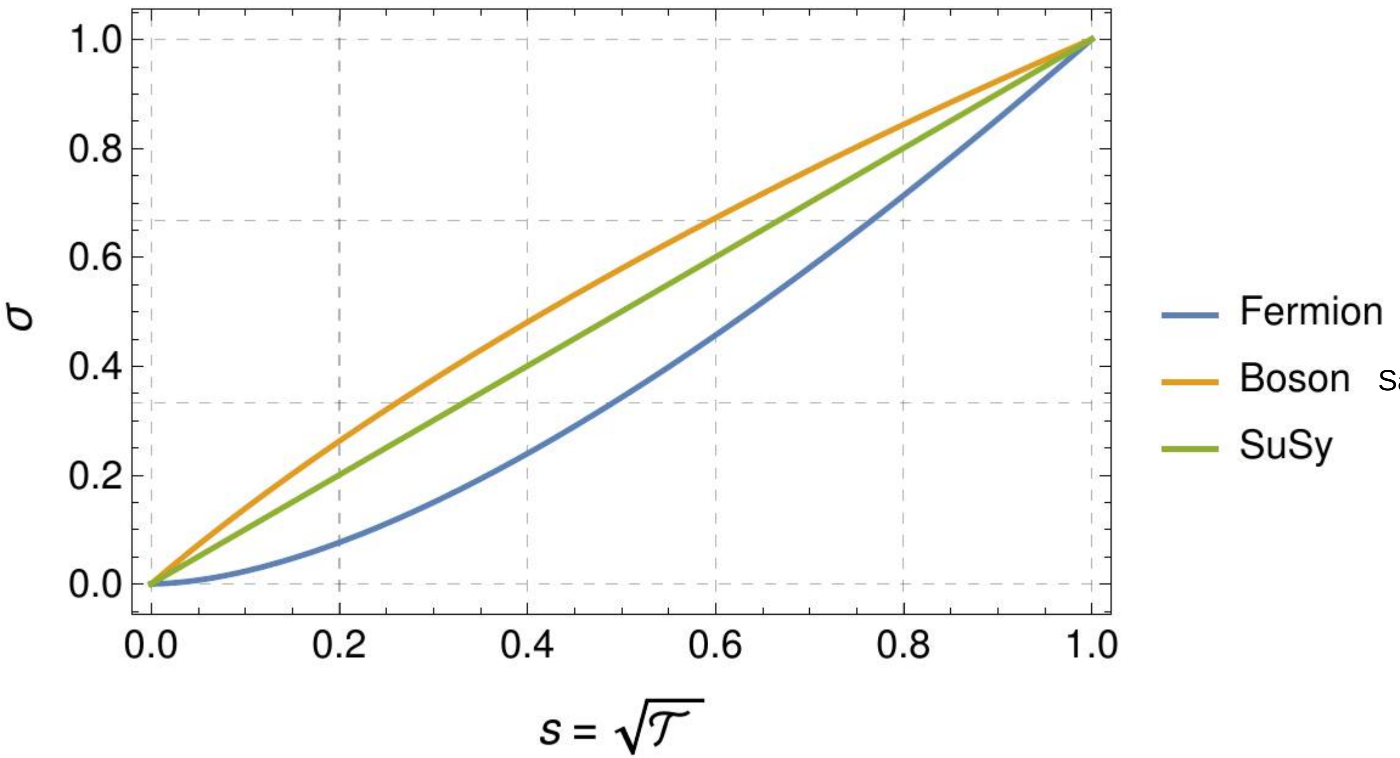
$$s(I_a) = -\frac{a}{a+1} \qquad (a \ll k)$$

Entanglement through Non-Topological Interfaces

they affect the leading order contribution ⇔ change the conformal weight of the twist field

Example: Interfaces of a single free boson or fermion:

$$S = \sigma(\mathcal{T})\,\frac{c}{3}\,\log L + c_1$$



Sakai, Satoh 2008

Entanglement through Non-Topological Interfaces

they affect the leading order contribution ⇔ change the conformal weight of the twist field

Some interesting questions:

- How does the EE behave for **general non-topological defects**?

- On which **features** of a general conformal defect does it depend? Keywords: **transmission coefficient**; **Casimir energy**; topological data**.**

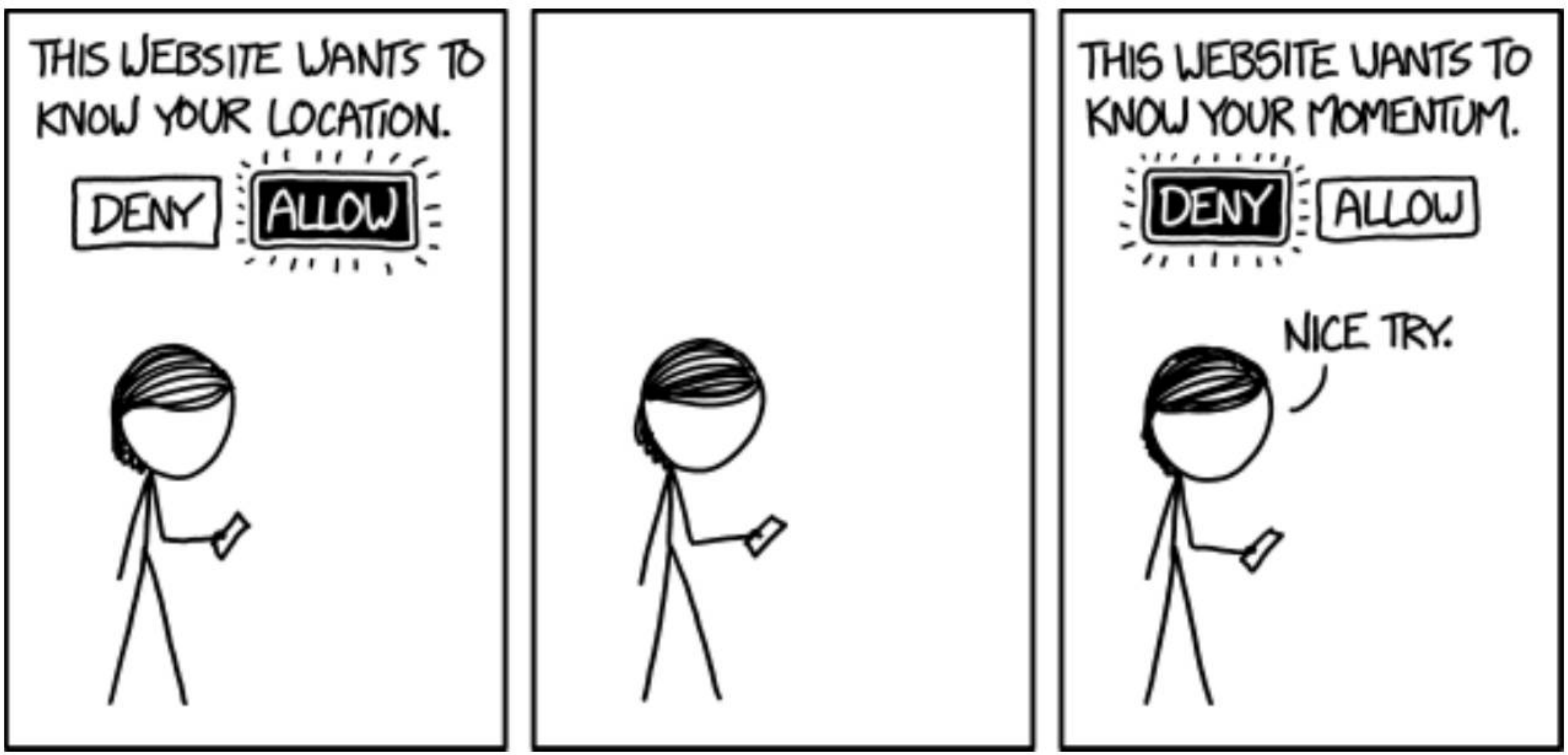
- Is the sub-leading term constant under **non-topological deformations** of a topological defect?

Final Words and Thoughts

- By **unfolding a boundary** one may always interpret it as a **top. defect in a chiral theory**
  - one can use the same techniques to derive the **left-right entanglement** at a boundary

- The entanglement through the defect is a **feature** of the defect itself.
- It might be possible to define more **structure** to the space of 2d CFTs
  - **define distances** between CFTs, by the help of conformal defects and the EE through them? (in the spirit of ideas of Bachas et al 2014)
  - the infinitesimal limit of the Kullback–Leibler divergence yields the **Fisher information metric**

Thank You!



https://xkcd.com/1473/

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More about relative entropy

Using the constrains for $d_{A\mathbf{i}}$:

$$\sum_{(i,\bar{\imath})} \mathrm{Tr}\, p_{\mathbf{i}}^{A} = 1$$

so they form a probability distribution.

$$s \leq \log \left(\sum_{(i,\bar{\imath})} T_{i\bar{\imath}} S_{i0} S_{\bar{\imath}0} \right)$$

$T_{i\bar{\imath}}$: $\min(M^1_{i\bar{\imath}}, M^2_{i\bar{\imath}})$

If the two CFTs are not the same: Their exists a defect s.t. the Kullback-Leibler divergence vanishes iff the **spectra are identical**.

Results for higher torus models

$$\mathcal{I}_{12}(\Lambda) = \sum_{\gamma \in \Gamma_{12}^{\Lambda}} d_{\Lambda\gamma} ||\gamma||$$

Bachas et al 2012

$$\Gamma_{12}^{\Lambda} = \{\gamma \in \Gamma_1 \,|\, \Lambda\gamma \in \Gamma_2\} = \Gamma_1 \cap \Lambda^{-1}\Gamma_2 \subset \Gamma_1$$

$$S = (1 - \partial_K)\log(Z(K))\big|_{K=1} = \frac{c}{3}\log(L) - \log|\Gamma_1/\Gamma_{12}^{\Lambda}|$$

is also the g-factor of the interface