Entanglement, Conformal Field Theory, and Interfaces

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Entanglement
Entanglement

- purely quantum phenomenon
- non-local
- "spooky action at a distance"
Entanglement is a purely quantum phenomenon that can reveal new (non-local) aspects of quantum theories. Non-local phenomena were once termed "spooky action at a distance."
Entanglement

- purely quantum phenomenon
- can reveal new (non-local) aspects of quantum theories
- non-local "spooky action at a distance"
- quantum computing

Entanglement

purely quantum phenomenon

non-local "spooky action at a distance"

quantum computing
Entanglement

- purely quantum phenomenon
- can reveal new (non-local) aspects of quantum theories
- “spooky action at a distance”
- quantum computing
- AdS/CFT
  - Casini, Huerta 2006
- BH physics
  - Ryu, Takayanagi 2006
- monotonicity theorems
Entanglement

- purely quantum phenomenon
- non-local
- “spooky action at a distance”
- observable in experiment (and numerics)
- can reveal new (non-local) aspects of quantum theories
- quantum computing
- non-local
- “spooky action at a distance”
- depends on the base and is not conserved
- AdS/CFT
  - Casini, Huerta 2006
- Ryu, Takayanagi 2006
- BH physics
- monotonicity theorems
- Casini, Huerta 2006
- Ryu, Takayanagi 2006
- purely quantum phenomenon
Measure of Entanglement

separable state

fully entangled state, e.g. Bell state
Measure of Entanglement

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Measure of Entanglement

separable state

How to “order” these states?

fully entangled state, e.g. Bell state
Measure of Entanglement

separable state

fully entangled state, e.g. Bell state

How to “order” these states?

- distillable entanglement
- entanglement cost
- squashed entanglement

- negativity
- logarithmic negativity
- robustness
Measure of Entanglement

- separable state
- How to “order” these states?
- fully entangled state, e.g. Bell state
- entanglement entropy
**Entanglement Entropy**

**Definition:** Let $\rho = |\psi\rangle \langle \psi|$ be the density matrix of a system in a pure quantum state $|\psi\rangle$. Let the Hilbert space be a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The reduced density matrix of A is $\rho_A = \text{Tr}_B \rho$. The entanglement entropy is the corresponding von Neumann entropy

$$S_A = -\text{Tr} \rho_A \log \rho_A.$$ 

It measures the entanglement, i.e. quantum correlation, between the two sub-systems A and B.
**Entanglement Entropy**

**Definition:** Let $\rho = |\psi \rangle \langle \psi|$ be the **density matrix** of a system in a pure quantum state $|\psi\rangle$. Let the Hilbert space be a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The reduced density matrix of $A$ is $\rho_A = \text{Tr}_B \rho$. The **entanglement entropy** is the corresponding **von Neumann entropy**

$$S_A = -\text{Tr} \rho_A \log \rho_A.$$

It measures the entanglement, i.e. quantum correlation, between the two sub-systems $A$ and $B$.

**Replica trick...**

$$S_A = -\frac{\partial}{\partial n} \text{Tr} \rho_A^n |_{n \rightarrow 1}$$
Conformal Field Theory
Conformal Field Theory

- QFT invariant under conformal transformation
- String theory
- Phase transitions
- Fixed points in RG
Conformal Field Theory

- QFT invariant under conformal transformation
- holomorphic functions, Witt algebra
- Virasoro algebra, c: central charge
- in 2 dim.

Quantum
Conformal Field Theory

QFT invariant under conformal transformation

holomorphic functions, Witt algebra

in 2 dim.

2d CFT is organized in (irred.) reps of $\text{Vir}_c$

Virasoro algebra, $c$: central charge

quantum
Rational Models

finite # of primaries, conformal weights \((h_i, h_i)\)

operator/state correspondence

\[ \mathcal{H} = \bigoplus M_{i\bar{\imath}} \mathcal{H}_i \otimes \mathcal{H}_{i\bar{\imath}} \]

finite

highest weight reps.
Rational Models

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operator/state correspondence

\(\mathcal{H} = \bigoplus M_{i\bar{i}} \mathcal{H}_i \otimes \mathcal{H}_{\bar{i}}\)

finite

highest weight reps.
their characters

torus partition function

\[ Z(\tau, \bar{\tau}) = \sum_{(i\bar{i})} M_{i\bar{i}} \chi_i(q)\chi_i(\bar{q}) = \sum_{(i\bar{i})} M_{i\bar{i}} S_{ij} S_{\bar{j}\bar{\bar{i}}} \chi_j(\bar{q})\chi_{\bar{j}}(\bar{\bar{q}}) \]
Conformal Interface
Conformal Interface

... or defect

natural generalization of conformal boundaries
Conformal Interface

Stat. mech.:
- impurities in quantum chains
- junction of quantum wires
- natural generalization of conformal boundaries

String theory:
- generalized D-branes?
- brane spectrum generating
  Graham, Watts 2004

... or defect
Conformal Interface

Stat. mech.: natural generalization of conformal boundaries
impurities in quantum chains
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String theory:
symmetry generating
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brane spectrum generating
Graham, Watts 2003

… or defect
Conformal Interfaces

Bachas et al 2002

A diagram showing a conformal interface between CFT1 and CFT2, with an operator mapping states from one CFT to the other.

$\mathcal{I}_{1,2}$
Conformal Interfaces

operator mapping states from on CFT to the other

\[ I_{1,2} \]

gluing condition:

\[ T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z}) \]

Bachas et al 2002
Conformal Interfaces

operator mapping states from one CFT to the other

interface

$T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z})$

folding trick

$T(z) = \bar{T}(\bar{z})$

$|B\rangle_{1 \otimes \bar{2}}$
Special Gluing Conditions

\[ T^1(z) - \bar{T}^1(\bar{z}) = T^2(z) - \bar{T}^2(\bar{z}) \]

- Both sides vanish independently:
  \[ T^i(z) = \bar{T}^i(\bar{z}) \]
- separate boundary conditions
- In particular happens when one of the CFTs is trivial

- The two components equal independently:
  \[ T^1(z) = T^2(z), \quad \bar{T}^1(\bar{z}) = \bar{T}^2(\bar{z}) \]
- \( I_{1,2} \) also commutes with the Hamiltonian
- The interface can be moved around without cost of energy or momentum
- This is called a topological interface
What makes the difference?

boundary changing field

\[\leftrightarrow\]

defect changing field
What makes the difference?

boundary changing field

defect changing field

generalization

junction field
acts as a constant map between isomorphic Virasoro representations

\[ l_A = \sum_{i=(i\tilde{r})} d_{Ai} \|i\| \]
acts as a constant map between isomorphic Virasoro representations

\[ I_A = \sum_{i=(i\bar{r})} d_{Ai} \|i\| \]

invariance under S-trafo

example: diagonal rational theories

\[ \sum_i S_{ij} S_{\bar{ij}} \text{Tr} \, d_{A^i} d_{A^i} = N_{j\bar{j}} \in \mathbb{N} \]

\[ I_a = \sum_i \frac{S_{ai}}{S_{0i}} \|i\| \]
**Example: Topological Interfaces of the Ising model**

<table>
<thead>
<tr>
<th>primary</th>
<th>conformal weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>id</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>(1/2, 1/2)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>(1/16, 1/16)</td>
</tr>
</tbody>
</table>

$$S_{ij} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$l_{id} = \| id \| + \| \varepsilon \| + \| \sigma \|$$

$$l_{\varepsilon} = \| id \| + \| \varepsilon \| - \| \sigma \|$$

$$l_{\sigma} = \sqrt{2} \| id \| - \sqrt{2} \| \varepsilon \|$$
Entanglement

Conformal Field Theory
Entanglement Entropy of a Finite Interval

Cardy, Calabrese 2009
remember replica trick:
\[ \text{Tr} \rho_A^n \]

partition function \( Z(n) \) on a complicated Riemann surface
Entanglement Entropy of a Finite Interval

Cardy, Calabrese 2009

Remember replica trick:

\[ \text{Tr} \rho^n_A \]

Partition function \( Z(n) \) on a complicated Riemann surface
Entanglement Entropy of a Finite Interval

Cardy, Calabrese 2009

Remember replica trick:

\[ \text{Tr} \rho_A^n \]

Partition function \( Z(n) \) on a complicated Riemann surface

2-point function of twist fields

\[ \langle T_n(u) T_n^+(v) \rangle \]
2-point function of **twist fields**

\[ \langle T_n(u) T_n^\dagger(\nu) \rangle \]

“junction field” of lowest conformal weight

\[ T_n \quad b_n \]
EE of a Finite Interval

2-point function of twist fields

\[ \langle T_n(u) T_n^\dagger(v) \rangle \]

"junction field" of lowest conformal weight

\[ q^{h_n - \frac{nc}{12}} = \langle T_n | q^{H_{bn}} | T_n \rangle = Z_{H_{bn}} (\tau \gg 1) \]

\[ = \text{Tr} (b_n \tilde{q}^{H_n}) = \text{Tr} (\tilde{q}^{nH}) = \sum_{(i\bar{i})} \chi_i (\tilde{q}^n) \chi_{\bar{i}} (\tilde{q}^n) \]

Cardy condition

\[ = q^{-\frac{c}{12n}} \]

S-trafo & leading order

\[ h_n = \frac{c}{12} (n - \frac{1}{n}), \quad L = |v - u| \gg 1 \]

\[ S_A = \frac{c}{3} \log L + c_0 \]
Entanglement through Conformal Interfaces

\[ Z(n) = \text{Tr}(b_n q^{H_n/4} I^n q^{H_n/2} (I^n)\dagger q^{H_n/4}) \]

\[ = \text{Tr}(I q^{H/2} I\dagger q^{H/2})^n \]
Entanglement through Topological Interfaces

Remember: \( I_A = \sum_{i=(i\bar{i})} d_{Ai} \|i\| \) and \([I_A, H] = 0\)

\[
Z(n) = \text{Tr} \left( \left( I_A I_A^\dagger \right)^n q^{nH} \right) = \sum_{(i\bar{i})} \text{Tr}(d_{Ai} d_{A^*i})^n \chi_i(q^n) \chi_{\bar{i}}(\tilde{q}^n)
\]

\[
= \sum_{(i,\bar{i})} \text{Tr} \left( d_{A^*i} d_{Ai} \right)^n S_{i0} S_{r0} \tilde{q}^{-\frac{c}{12n}} \equiv A(n)
\]
Remember: \( I_A = \sum_{i=(i\bar{i})} d_A i || i \| \) and \([I_A, H] = 0\)

\[
Z(n) = \text{Tr} \left( \left( I_A I_A^{\dagger} \right)^n q^{nH} \right) = \sum_{(i\bar{i})} \text{Tr} (d_A i d_A^{*} i)^n \chi_i (q^n) \chi_{\bar{i}} (\bar{q}^n)
\]

\[
= \sum_{(i\bar{i})} \text{Tr} (d_A^{*} i d_A i)^n S_{i0} S_{\bar{i}0} \bar{q}^{-\frac{c}{12n}}
\]

\( \equiv A(n) \)

no change in the log term of the EE

\[ \frac{c}{3} \log L \]

contributes to sub-leading term in the EE:

\[
s(I_A) = - \sum_{(i\bar{i})} \text{Tr} p_i^A \log \frac{p_i^A}{p_i^{id}}
\]

with

\[
p_i^A = \frac{d_A^{*} i d_A i S_{i0} S_{\bar{i}0}}{N_{0A}^A}
\]
Entanglement through Topological Interfaces

relative entropy / Kullback–Leibler divergence:

\[ s(l_A) = - \sum_{(i,j)} \text{Tr} \ p_i^A \log \frac{p_i^A}{p_i^A} \]
Entanglement through Topological Interfaces

relative entropy / Kullback–Leibler divergence:

\[ s(l_A) = - \sum (i, \bar{i}) \text{Tr} \ p_i^A \log \frac{p_i^A}{p_i^{id}} \]

\[ s(l_a) = \begin{cases} -\log 2 & a = \sigma \\ 0 & a = id, \epsilon \end{cases} \]

\[ s(l_a) = -\frac{a}{a+1} \quad (a \ll k) \]
they affect the leading order contribution change the conformal weight of the twist field

Example: Interfaces of a single free boson or fermion:

\[ S = \sigma(T) \frac{c}{3} \log L + c_1 \]
they affect the leading order contribution

change the conformal weight of the twist field

Some interesting questions:

➢ How does the EE behave for general non-topological defects?

➢ On which features of a general conformal defect does it depend?
  Keywords: transmission coefficient; Casimir energy; topological data.

➢ Is the sub-leading term constant under non-topological deformations of a topological defect?
Final Words and Thoughts

➢ By unfolding a boundary one may always interpret it as a top. defect in a chiral theory
  ➢ one can use the same techniques to derive the left-right entanglement at a boundary

➢ The entanglement through the defect is a feature of the defect itself.
➢ It might be possible to define more structure to the space of 2d CFTs
  ➢ define distances between CFTs, by the help of conformal defects and the EE through them? (in the spirit of ideas of Bachas et al 2014)
  ➢ the infinitesimal limit of the Kullback–Leibler divergence yields the Fisher information metric
Thank You!

https://xkcd.com/1473/


More about relative entropy

Using the constrains for $d_{\alpha_i}$:

$$\sum_{(i, \bar{i})} \text{Tr} \, p_i^A = 1$$

so they form a probability distribution.

$$s \leq \log \left( \sum_{(i, \bar{i})} T_{i \bar{i}} \, S_{i0} \, S_{\bar{i}0} \right) - \min(M_{i \bar{i}}^1, M_{i \bar{i}}^2)$$

If the two CFTs are not the same: Their exists a defect s.t. the Kullback-Leibler divergence vanishes iff the spectra are identical.
Results for higher torus models

\[ I_{12}(\Lambda) = \sum_{\gamma \in \Gamma_{12}^\Lambda} d_{\Lambda \gamma} ||\gamma|| \]  

\[ \Gamma_{12}^\Lambda = \{\gamma \in \Gamma_1 | \Lambda \gamma \in \Gamma_2\} = \Gamma_1 \cap \Lambda^{-1} \Gamma_2 \subset \Gamma_1 \]

\[ S = (1 - \partial_K) \log(Z(K)) \big|_{K=1} = \frac{c}{3} \log(L) - \log |\Gamma_1 / \Gamma_{12}^\Lambda| \]

is also the g-factor of the interface