Recent developments in the complex Langevin method for solving the sign problem

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12th July 2016

Seminar at Osaka Univ.

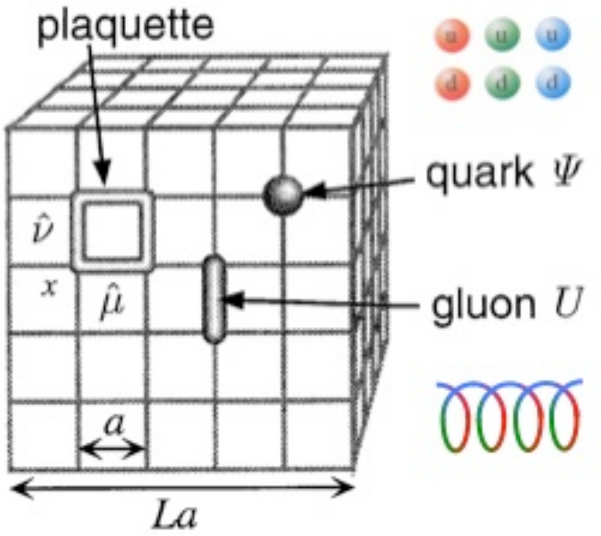
Ref.) Nagata-J.N.-Shimasaki 1606.07627 [hep-lat]
1604.07717 [hep-lat], to appear in JHEP

The sign problem in Monte Carlo methods

- Monte Carlo methods

  powerful tool to study quantum field theories and statistical systems nonperturbatively.

e.g.) lattice QCD



$$Z = \int dU\, d\Psi\, \mathrm{e}^{-S[U,\Psi]}$$

$$= \int dU\, \mathrm{e}^{-S_{\mathrm{g}}[U]} \det \mathcal{M}[U]$$

Generate configurations *U* with this probability and calculate <*O*> as expectation values.

The sign problem in Monte Carlo methods (cont'd)

- At finite baryon number density ($\mu \neq 0$),

$$
\begin{aligned}
Z &= \int dU\, d\Psi\, \mathrm{e}^{-S[U,\Psi]} \\
&= \int dU\, \mathrm{e}^{-S_{\mathrm{g}}[U]}\, \mathrm{det}\mathcal{M}[U]
\end{aligned}
$$

The fermion determinant becomes complex in general.

$$\mathrm{det}\mathcal{M}[U] = |\mathrm{det}\mathcal{M}[U]|\, \mathrm{e}^{i\Gamma[U]}$$

Generate configurations *U* with the probability $\mathrm{e}^{-S_{\mathrm{g}}[U]}\, |\mathrm{det}\mathcal{M}[U]|$

and calculate

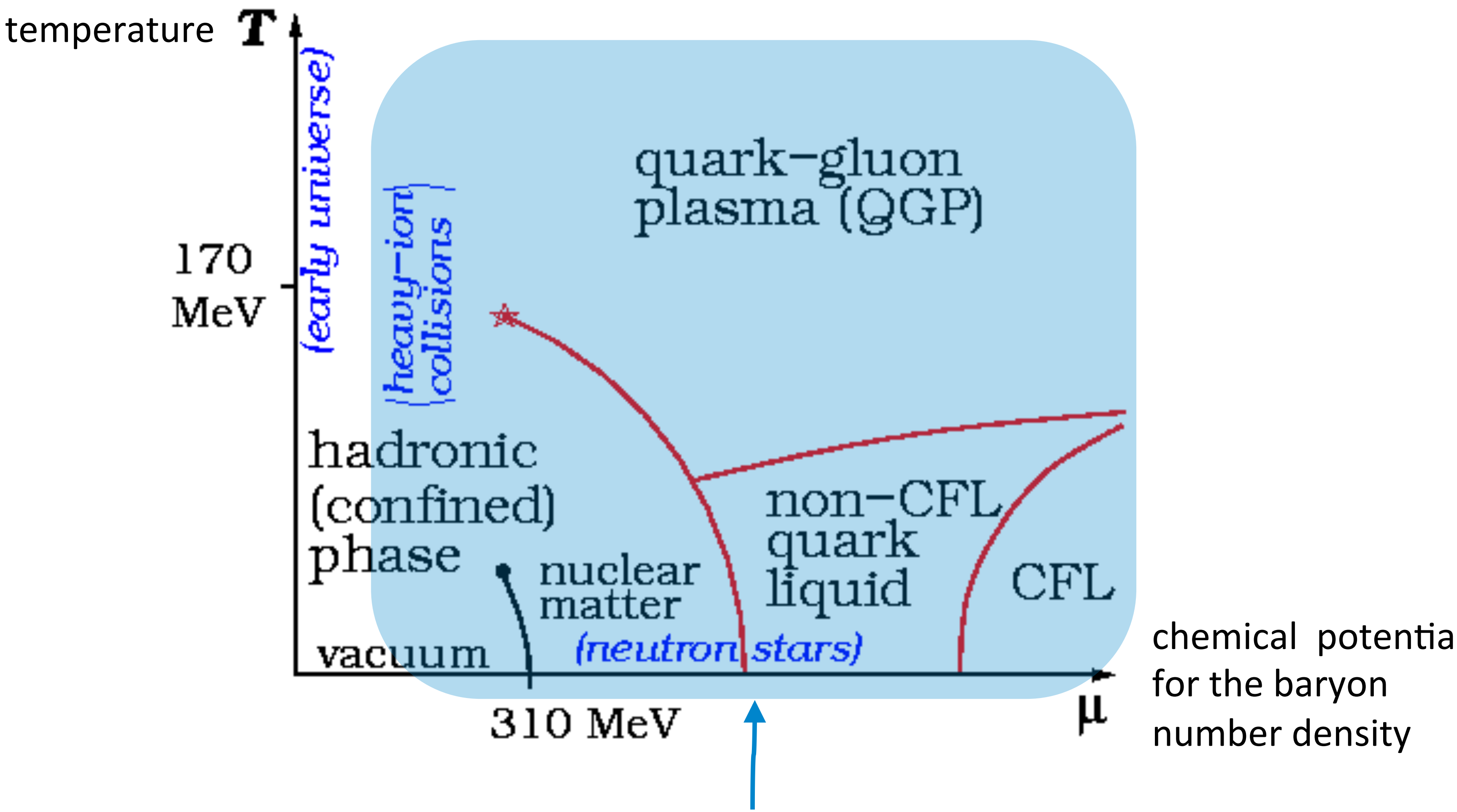
$$\langle \mathcal{O}[U] \rangle = \frac{\langle \mathcal{O}[U]\, \mathrm{e}^{i\Gamma[U]} \rangle_0}{\langle \mathrm{e}^{i\Gamma[U]} \rangle_0} \qquad \text{(reweighting)}$$

become exponentially small as the volume increases due to violent fluctuations of the phase Γ

Number of configurations needed to evaluate <O> increases exponentially.

"sign problem"

QCD phase diagram still in mystery



First principle calculations are difficult due to the sign problem

Sign problem occurs also in many other interesting cases

- a theta term
- Chern-Simons term
- real-time dynamics
- Yukawa interactions
- supersymmetric theories
- chiral fermions
- systems of strongly coupled electrons etc.

fermionic origin

Solving the sign problem will be a great breakthrough in theoretical physics !

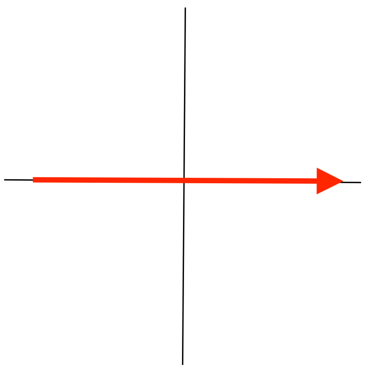
A new development toward solution to the sign problem

2011～

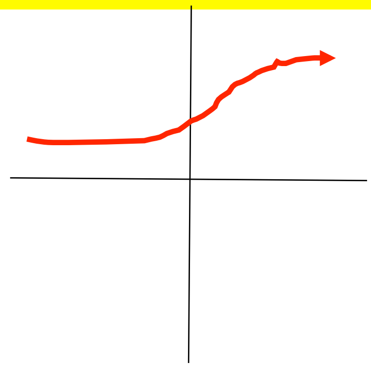
Key : complexification of dynamical variables

The original path integral

$$Z = \int dx\, w(x)$$



The phase of $w(x)$ oscillates violently (sign problem)

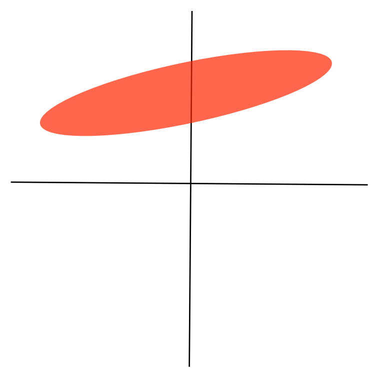


$$Z = \int dz\, w(z)$$

Minimize the sign problem by deforming the integration contour

“Lefschetz thimble approach”

This talk



An equivalent stochastic process of the complexified variables (no sign problem !)

“complex Langevin method”

The equivalence to the original path integral holds under certain conditions.

Plan of the talk

0. Introduction
1. Complex Langevin method
2. Argument for justification and
   the condition for correct convergence
3. Tests in two simple examples
4. Gauge cooling and its justification
5. Successful applications in semi-realistic cases
6. Summary and future prospects

1. Complex Langevin method

Stochastic quantization

Parisi-Wu ('81)
For review, see
Damgaard-Huffel ('87)

$$Z = \int dx\, w(x)$$

View this as the stationary distribution of a stochastic process.

Langevin eq. $$\frac{d}{dt}x^{(\eta)}(t) = v(x^{(\eta)}(t)) + \eta(t)$$ Gaussian noise

"drift term" $$v(x) \equiv \frac{1}{w(x)}\frac{\partial w(x)}{\partial x}$$

$$\langle \mathcal{O} \rangle = \lim_{t\to\infty} \langle \mathcal{O}(x^{(\eta)}(t)) \rangle_\eta \qquad \langle \cdots \rangle_\eta = \frac{\int \mathcal{D}\eta \cdots \mathrm{e}^{-\frac{1}{4}\int dt\, \eta^2(t)}}{\int \mathcal{D}\eta\, \mathrm{e}^{-\frac{1}{4}\int dt\, \eta^2(t)}}$$

Proof $$\langle \mathcal{O}(x^{(\eta)}(t)) \rangle_\eta = \int dx\, \mathcal{O}(x) P(x,t)$$

Probability distribution of $x^{(\eta)}(t)$ $$P(x,t) = \langle \delta(x - x^{(\eta)}(t)) \rangle_\eta$$

Fokker-Planck eq.

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - \frac{1}{w(x)}\frac{\partial w(x)}{\partial x}\right) P \qquad \lim_{t\to\infty} P(x,t) = \frac{1}{Z} w(x)$$

Extension to a complex-action system

Parisi ('83), Klauder ('83)

$$Z = \int dx\, w(x)$$

Let us naively apply the same method to the case with $w(x) \in \mathbb{C}$

Langevin eq. $$\frac{d}{dt} x^{(\eta)}(t) = v(x^{(\eta)}(t)) + \eta(t)$$

"drift term"

Gaussian noise

$$v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$$

The drift term becomes complex !

The solution becomes complex, even if $x^{(\eta)}(0) \in \mathbb{R}$ and $\eta(t) \in \mathbb{R}$

We need to complexify the dynamical variables !

$$z^{(\eta)}(t) = x^{(\eta)}(t) + i\, y^{(\eta)}(t)$$

complex Langevin method :

Gaussian noise

$$\frac{d}{dt} z^{(\eta)}(t) = \boxed{v(z^{(\eta)}(t))} + \eta(t)$$

$$\langle \mathcal{O} \rangle \stackrel{?}{=} \lim_{t\to\infty} \langle \boxed{\mathcal{O}(z^{(\eta)}(t))} \rangle_\eta$$

Note : The drift term $v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$ and the observables $\mathcal{O}(x)$ should be evaluated for complexified variables.

It is natural to define $v(z)$ and $\mathcal{O}(z)$ by analytic continuation.

$$\langle \mathcal{O}(z^{(\eta)}(t)) \rangle_\eta = \int dx dy\, \mathcal{O}(x+iy) P(x,y;t)$$

$$P(x,y;t) = \langle \delta(x - x^{(\eta)}(t)) \delta(y - y^{(\eta)}(t)) \rangle_\eta$$

The key identity :

$$\int dx dy\, \mathcal{O}(x+iy) P(x,y;t) \stackrel{?}{=} \int dx\, \mathcal{O}(x) \rho(x;t)$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} - \frac{1}{w(x)} \frac{\partial w(x)}{\partial x} \right) \rho$$

$$\Rightarrow \lim_{t\to\infty} \rho(x;t) = \frac{1}{Z} w(x)$$

History of the complex Langevin method (CLM)

1) CLM proposed by Parisi ('83), Klauder ('83)
2) Tested in many cases ('84-'86). It works beautifully in some highly nontrivial cases, but gives simply wrong results in the other cases.
   The reason for this was unclear, and the interest in this method declined.
3) Application to the real-time dynamics (Berges-Stamatescu '05)
4) Application to 4d scalar field theory with finite chemical potential (Aarts '09)
5) Justification of the CLM (Aarts-James-Seiler-Stamatescu '11)
6) The proposal of "gauge-cooling" technique (Seiler-Sexty-Stamatescu '13)
7) Application to finite density QCD in the heavy dense limit (Aarts-Seiler-Sexty-Stamatescu '14)
8) Application to finite density QCD in the deconfined phase (Sexty '14)
9) Application to RMT for finite density QCD (Mollgaard-Splittorff '15)
   "the singular-drift problem" at small quark mass (J.N.-Shimasaki '15)
10) Justification of the CLM including the gauge-cooling (Nagata-J.N.-Shimasaki '16)
11) "Gauge-cooling" for the singular-drift problem (Nagata-J.N.-Shimasaki '16)
12) Justification of the CLM revisited (Nagata-J.N.-Shimasaki '16)

The CLM is one of the hottest topics in the lattice QCD community !

2. Argument for justification and the condition for correct convergence

Ref.) Nagata-J.N.-Shimasaki 1606.07627 [hep-lat]

What was wrong in the previous argument by Aarts et al.

Aarts, James, Seiler, Stamatescu: Eur. Phys. J. C ('11) 71, 1756

- The previous argument started with the continuum Langevin time from the outset.
  However, starting from a discretized Langevin time with the step-size ϵ , it turns out that the $\epsilon \to 0$ limit is actually subtle.

- The time-evolved observable $\mathcal{O}(z;t)$, which plays an important role in the argument, can be ill-defined.

Fixing these loopholes, we establish the argument for justification.

⇒ a simple condition, which tells us clearly whether the results are reliable or not.

Discrete version of the complex Langevin eq.

$$\frac{d}{dt} z^{(\eta)}(t) = v(z^{(\eta)}(t)) + \eta(t) \ ,$$

discretize t

$$\boxed{z^{(\eta)}(t+\epsilon) = z^{(\eta)}(t) + \epsilon\, v(z^{(\eta)}(t)) + \sqrt{\epsilon}\,\eta(t)}$$

$$\eta(t) = \eta^{(\mathrm{R})}(t) + i\eta^{(\mathrm{I})}(t)$$

probability $\propto \mathrm{e}^{-\frac{1}{4}\sum_t \{\frac{1}{N_{\mathrm{R}}}\eta^{(\mathrm{R})}(t)^2 + \frac{1}{N_{\mathrm{I}}}\eta^{(\mathrm{I})}(t)^2\}}$

$$N_{\mathrm{R}} - N_{\mathrm{I}} = 1$$

t-evolution of the expectation value

$$\langle \mathcal{O}(z^{(\eta)}(t)) \rangle_\eta = \int dx dy\, \mathcal{O}(x+iy) P(x,y;t)$$

$$\langle \mathcal{O}(z^{(\eta)}(t+\epsilon)) \rangle_\eta = \int dx\, dy\, \mathcal{O}_\epsilon(x+iy)\, P(x,y;t)$$

$$\mathcal{O}_\epsilon(z) = \frac{1}{\mathcal{N}} \int d\eta\, \mathrm{e}^{-\frac{1}{4}\left\{\frac{1}{N_\mathrm{R}}\eta_k^{(\mathrm{R})2} + \frac{1}{N_\mathrm{I}}\eta_k^{(\mathrm{I})2}\right\}} O\left(z + \epsilon\, v(z) + \sqrt{\epsilon}\, \eta\right)$$

expanding this with respect to ϵ

$$O_\epsilon(z) = :\mathrm{e}^{\epsilon L}: O(z)$$

$$\left(\begin{aligned}
&\mathrm{e}^{\epsilon L} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \epsilon^n L^n \\
&L = \left\{\mathrm{Re} v(z) + N_\mathrm{R}\frac{\partial}{\partial x}\right\}\frac{\partial}{\partial x} + \left\{\mathrm{Im} v(z) + N_\mathrm{I}\frac{\partial}{\partial y}\right\}\frac{\partial}{\partial y} \\
&:(f(x)+\partial)^2: = f(x)^2 + 2f(x)\partial + \partial^2
\end{aligned}\right)$$

The role of holomorphy

Note that $\mathcal{O}(z)$ is holomorphic.

$$L = \left\{ \mathrm{Re}\, v(z) + N_{\mathrm{R}} \frac{\partial}{\partial x} \right\} \frac{\partial}{\partial x} + \left\{ \mathrm{Im}\, v(z) + N_{\mathrm{I}} \frac{\partial}{\partial y} \right\} \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial y} \mapsto i \frac{\partial}{\partial z}$$

$$\begin{aligned} \tilde{L} &= \left\{ v(z) + (N_{\mathrm{R}} - N_{\mathrm{I}}) \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z} \\ &= \left\{ v(z) + \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$N_{\mathrm{R}} - N_{\mathrm{I}} = 1$$

$$\begin{aligned} O_\epsilon(z) &= :\mathrm{e}^{\epsilon L}: O(z) \\ &= :\mathrm{e}^{\epsilon \tilde{L}}: O(z) \end{aligned}$$

$\mathcal{O}_\epsilon(z)$ is also a holomorphic function of z.

Subtlety in the ϵ expansion

$$\langle \mathcal{O}(z^{(\eta)}(t+\epsilon))\rangle_\eta = \int dx\, dy\, \mathcal{O}_\epsilon(x+iy)\, P(x,y;t)$$

$$O_\epsilon(z) = :\mathrm{e}^{\epsilon \tilde{L}}: O(z)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \epsilon^n \boxed{\int dx\, dy\, \{:\tilde{L}^n: \mathcal{O}(x+iy)\}\, P(x,y;t)}$$

Convergence of this integral is not obvious !!!

$$\tilde{L} = \left\{ v(z) + \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z}$$

The ϵ expansion is justified if and only if the probability of the drift term is suppressed more strongly than any power law at large magnitude.

$$\langle \mathcal{O}(z^{(\eta)}(t+\epsilon))\rangle_\eta = \int dx\, dy\, \{(1+\epsilon\tilde{L})\, \mathcal{O}(x+iy)\}\, P(x,y;t) + O(\epsilon^2)$$

taking the $\epsilon \to 0$ limit

$$\frac{d}{dt}\langle \mathcal{O}(z^{(\eta)}(t))\rangle_\eta = \int dx\, dy\, \{\tilde{L}\, \mathcal{O}(x+iy)\}\, P(x,y;t)$$

finite t evolution of the observable

$$\frac{d}{dt}\langle \mathcal{O}(z^{(\eta)}(t))\rangle_\eta = \int dx\, dy \boxed{\{\tilde{L}\, \mathcal{O}(x+iy)\}} P(x,y;t)$$

Repeating the argument

holomorphic c.f.) $\tilde{L} = \left\{ v(z) + \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z}$

$$\langle \mathcal{O}(z^{(\eta)}(t+\tau))\rangle_\eta = \sum_{n=0}^{\infty} \frac{1}{n!} \tau^n \boxed{\int dx\, dy\, \{\tilde{L}^n\, \mathcal{O}(x+iy)\}\, P(x,y;t)}$$

In order for this expression to be valid for finite τ ,
the infinite series should have a finite convergence radius.

$$\tilde{L} = \left\{ v(z) + \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z}$$

This requires that the probability of the drift term should be suppressed exponentially at large magnitude.

Conditions on the probability distribution of the drift term

- The probability distribution of the magnitude of the drift term

$$p(u;t) = \int dx dy \, \delta(u - |v(z)|) \, P(x,y;t)$$

- In order for $\int_0^\infty du \, u^n \, p(u;t)$ to be finite, $p(u;t)$ need to fall off at large *u* faster than any power law.

- E.g., if $p(u;t) \sim \mathrm{e}^{-\kappa u}$, $\int_0^\infty du \, u^n \, p(u;t) \sim \int_0^\infty du \, u^n \, \mathrm{e}^{-\kappa u} \sim \frac{n!}{\kappa^{n+1}}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tau^n \int_0^\infty du \, u^n \, p(u;t) \sim \frac{1}{\kappa} \sum_{n=0}^{\infty} \left(\frac{\tau}{\kappa} \right)^n$$

Convergence radius can be estimated as $\tau_{\mathrm{conv}} \sim \kappa$

The validity of ϵ expansion → fall off faster than any power

Finite *t*-evolution of the observable → fall off faster than exponential

the key identity

$$\boxed{\begin{gathered}\int dxdy\, \mathcal{O}(x+iy)P(x,y;t) = \int dx\, \mathcal{O}(x)\rho(x;t) \\ P(x,y;0) = \rho(x)\,\delta(y)\ , \qquad \rho(x;0) = \rho(x)\end{gathered}}$$

Here $\rho(x;t)$ satisfies

$$\boxed{\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - v(x)\right)\rho}$$

Fokker-Planck equation

$$v(x) \equiv \frac{1}{w(x)}\frac{\partial w(x)}{\partial x}$$

$$\lim_{t\to\infty} \rho(x;t) = \frac{1}{Z} w(x)$$

This is true if the key identify holds and if P(x,y;t) converges to a unique function.
(J.N-Shimasaki '15)

$$\lim_{t\to\infty} \int dxdy\, \mathcal{O}(x+iy)P(x,y;t) = \frac{1}{Z}\int dx\, \mathcal{O}(x)w(x)$$

Proof of the key identity

We prove the following identity for any *k* by induction w.r.t. *t*

$$\int dxdy \left\{ \tilde{L}^k \, \mathcal{O}(x+iy) \right\} P(x,y;t) = \int dx \left\{ (L_0)^k \, \mathcal{O}(x) \right\} \rho(x;t)$$

$$\tilde{L} = \left\{ v(z) + \frac{\partial}{\partial z} \right\} \frac{\partial}{\partial z} \ , \qquad L_0 = \left(v(x) + \frac{\partial}{\partial x} \right) \frac{\partial}{\partial x}$$

At *t=0*, trivially satisfied because:

$$P(x,y;0) = \rho(x)\, \delta(y) \ , \qquad \rho(x;0) = \rho(x)$$

Assume that it is satisfied at some *t* . Then for $\tau < \tau_{\text{conv}}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tau^n \int dx\, dy \, \{ \tilde{L}^{n+k} \, \mathcal{O}(x+iy) \} \, P(x,y;t) = \sum_{n=0}^{\infty} \frac{1}{n!} \tau^n \int dx \, \{ (L_0)^{n+k} \, \mathcal{O}(x) \} \, \rho(x;t)$$

$$\begin{aligned} \int dxdy \left\{ \tilde{L}^k \, \mathcal{O}(x+iy) \right\} P(x,y;t+\tau) &= \int dx \left\{ \mathrm{e}^{\tau L_0} (L_0)^k \, \mathcal{O}(x) \right\} \rho(x;t) \\ &= \int dx \left\{ (L_0)^k \, \mathcal{O}(x) \right\} \mathrm{e}^{\tau (L_0)^\top} \rho(x;t) \\ &= \int dx \left\{ (L_0)^k \, \mathcal{O}(x) \right\} \rho(x;t+\tau) \end{aligned}$$

3. Tests in two simple examples

Ref.) Nagata-J.N.-Shimasaki 1606.07627 [hep-lat]

3.1 A model with a singular drift

A model with a singular drift

$$Z = \int dx\, w(x)\ , \qquad w(x) = (x + i\alpha)^p \, \mathrm{e}^{-x^2/2}$$

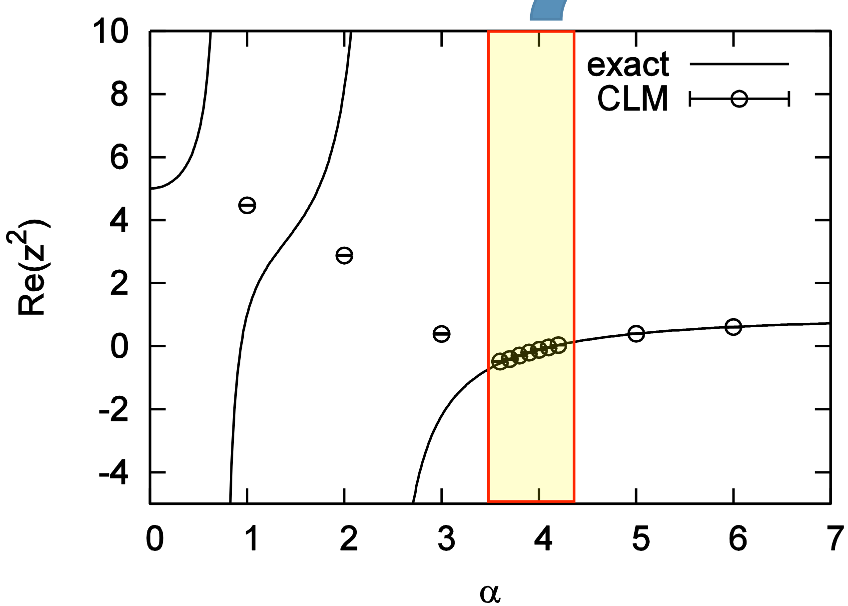
J.N.-Shimasaki, PRD 92 (2015) 1, 011501 arXiv:1504.08359 [hep-lat]

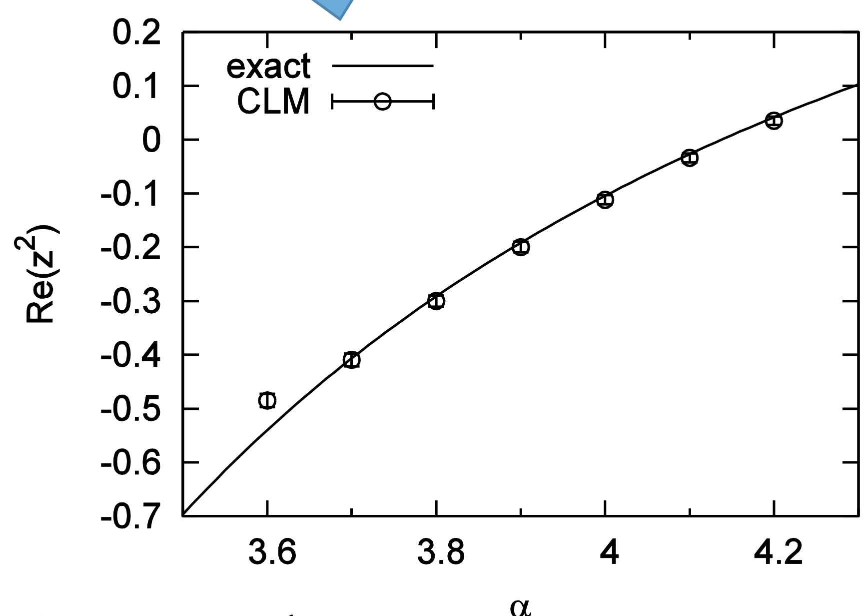
Drift term : $v(x) = \dfrac{1}{w(x)} \dfrac{dw(x)}{dx} = \dfrac{p}{x + i\alpha} - x$ $\xrightarrow{\text{anal. cont.}}$ $v(z) = \dfrac{p}{z + i\alpha} - z$

singular at $z = -i\alpha$

$p = 4$

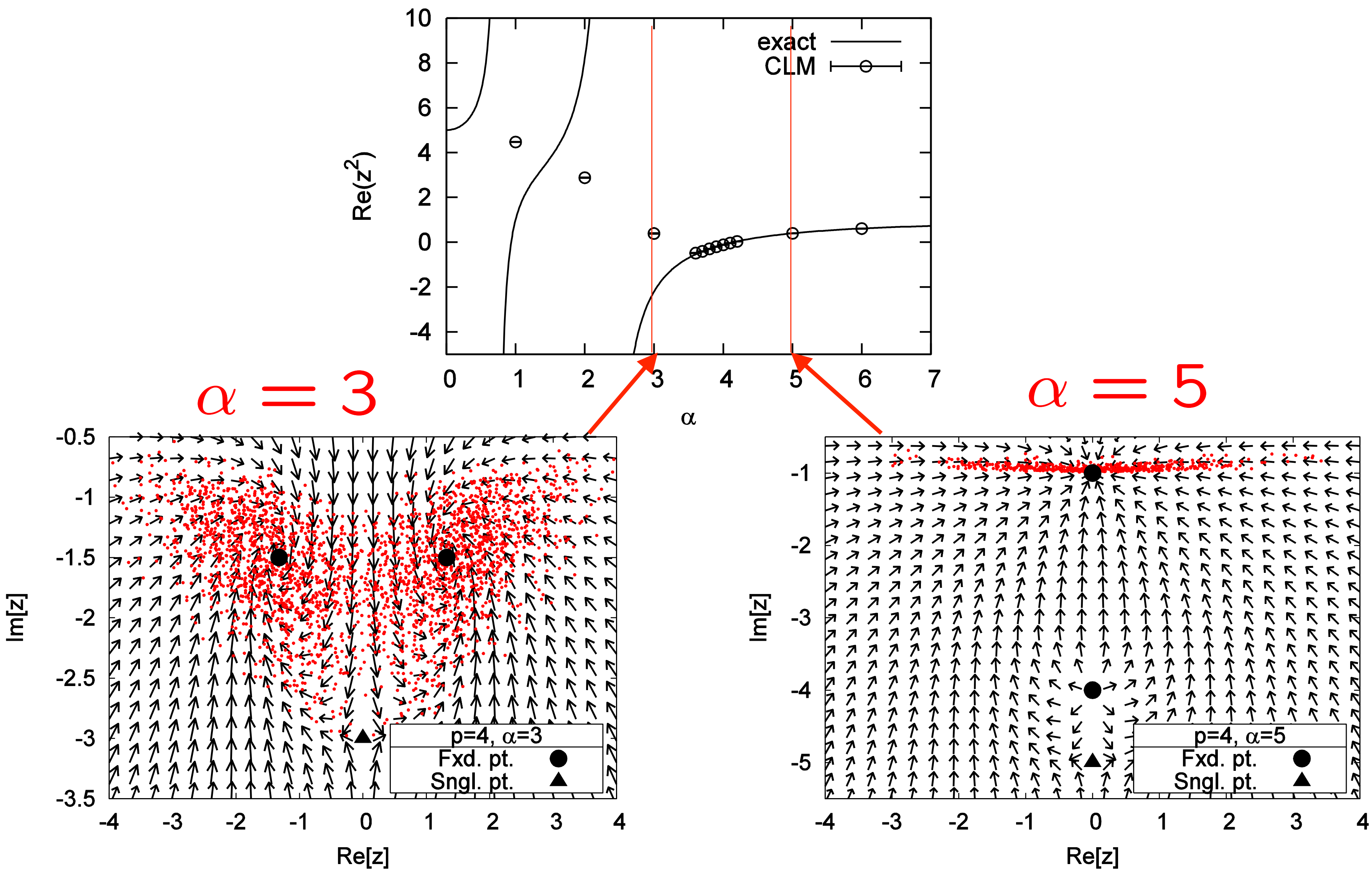
zoom up of this region





CLM starts to fail at $\alpha \lesssim 3.6$

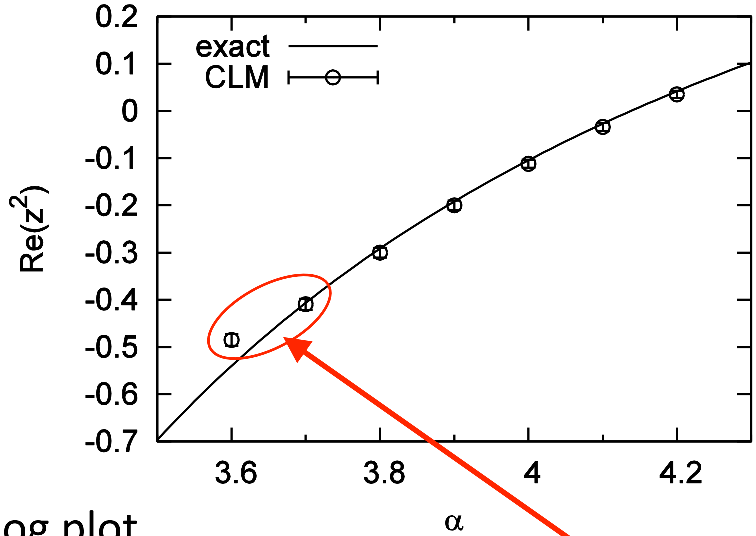
The reason of the failure at small α



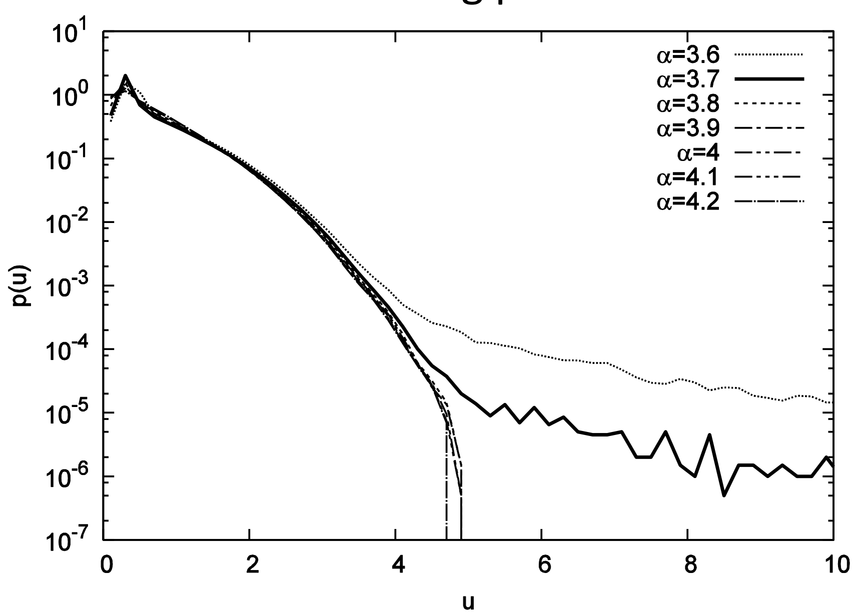
Many configurations near singularity

No configurations near singularity

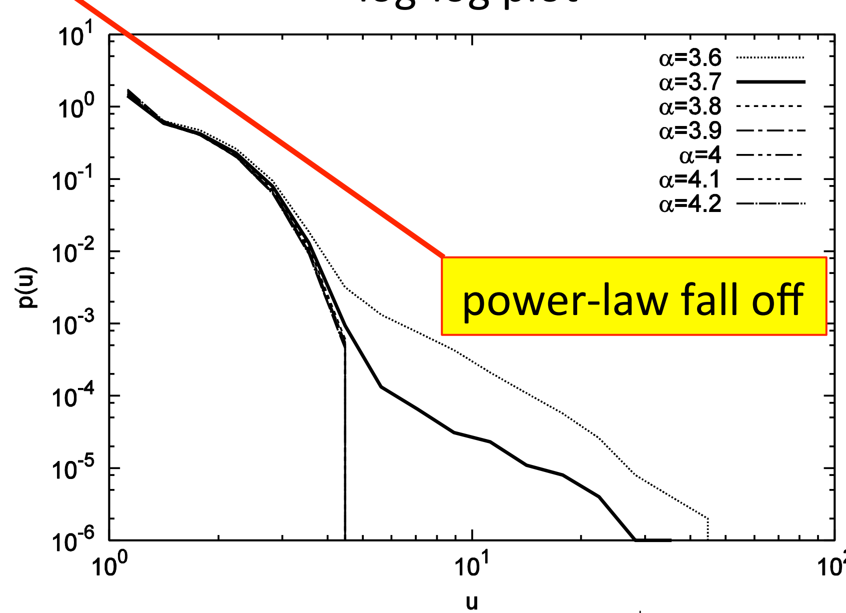
Demonstration of our condition



semi-log plot



log-log plot



The probability distribution of the magnitude of the drift term $u \equiv |v(z)| = \left| \frac{p}{z + i\alpha} - z \right|$

3.2 A model with a possibility of excursions

A model with a possibility of excursions

$$Z = \int dx\, w(x) \ , \qquad w(x) = \mathrm{e}^{-\frac{1}{2}(A+iB)x^2 - \frac{1}{4}x^4}$$

Aarts-Giudice-Seiler ('13)

$$v(x) = \frac{1}{w(x)}\frac{dw(x)}{dx} = -(A+iB)x - x^3$$

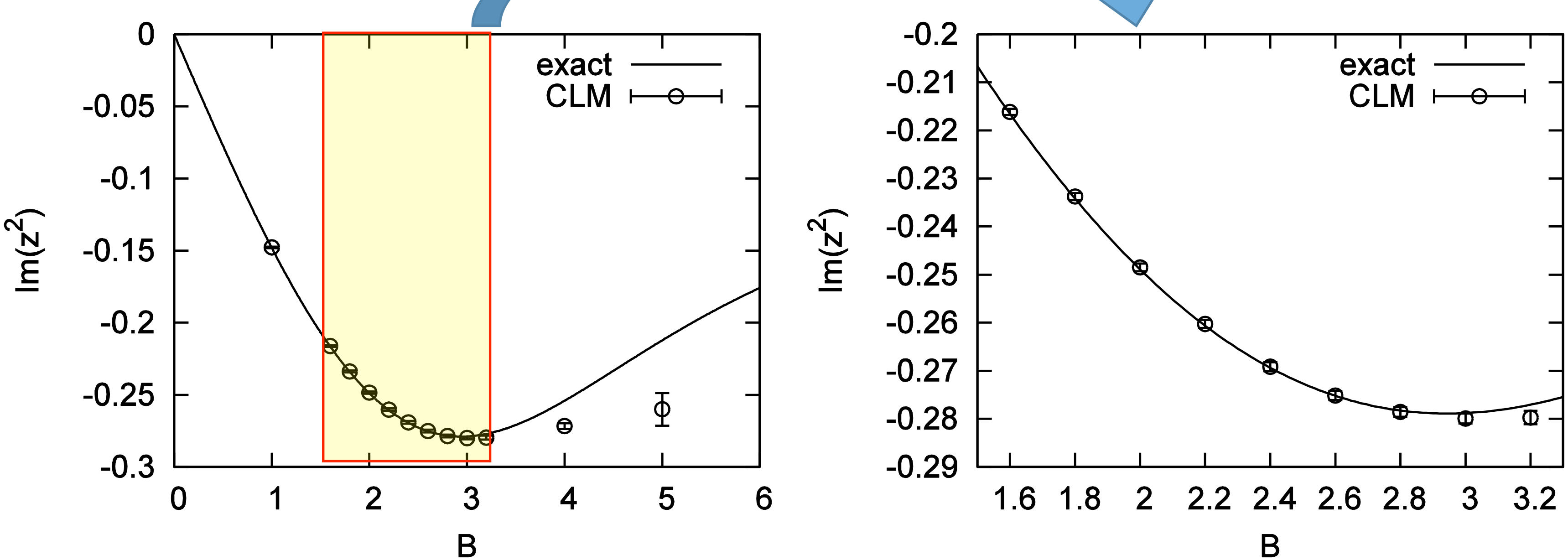
anal. cont.

$$v(z) = -(A+iB)z - z^3$$

potential danger of excursions into large |z| regime

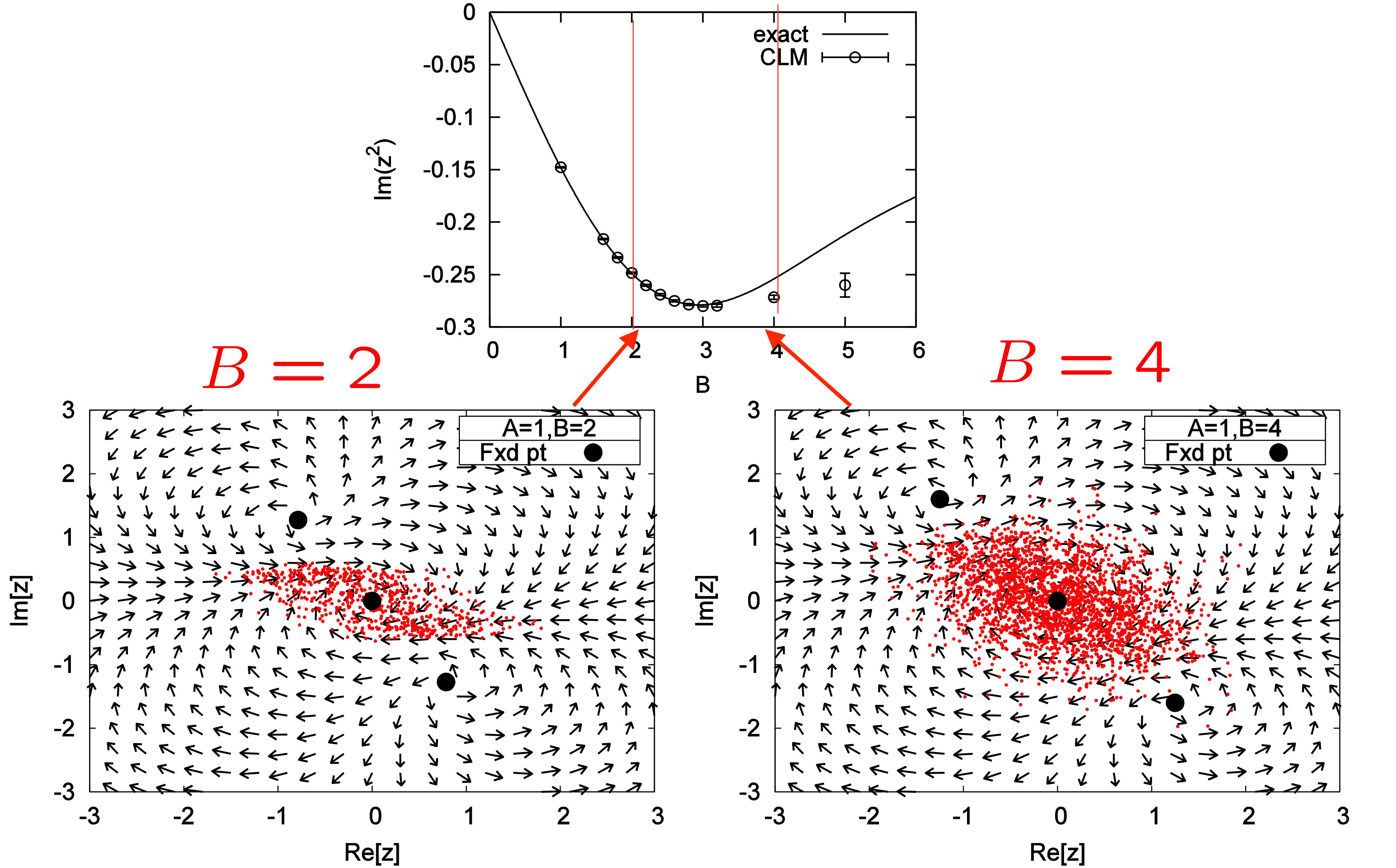
$A = 1$

zoom up of this region



CLM starts to fail at $B \gtrsim 3.0$

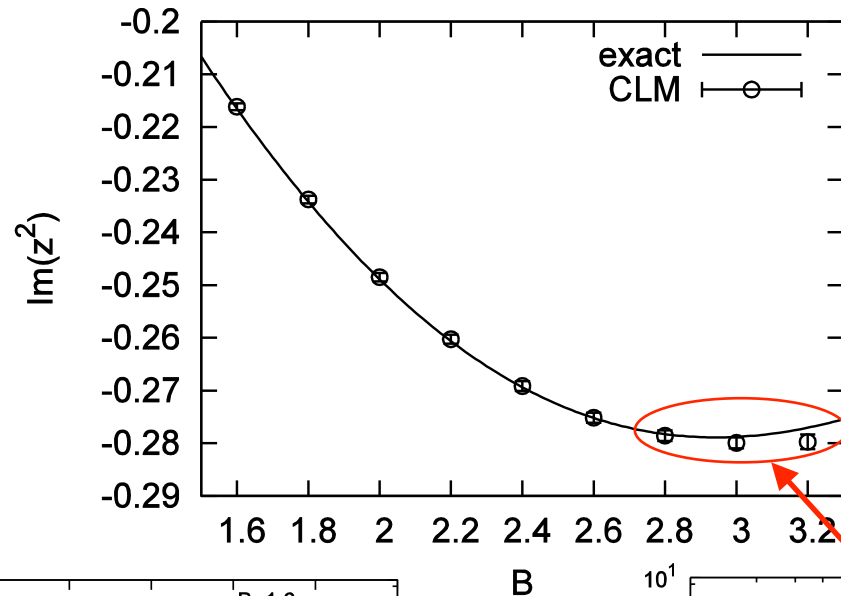
The reason of the failure at large B

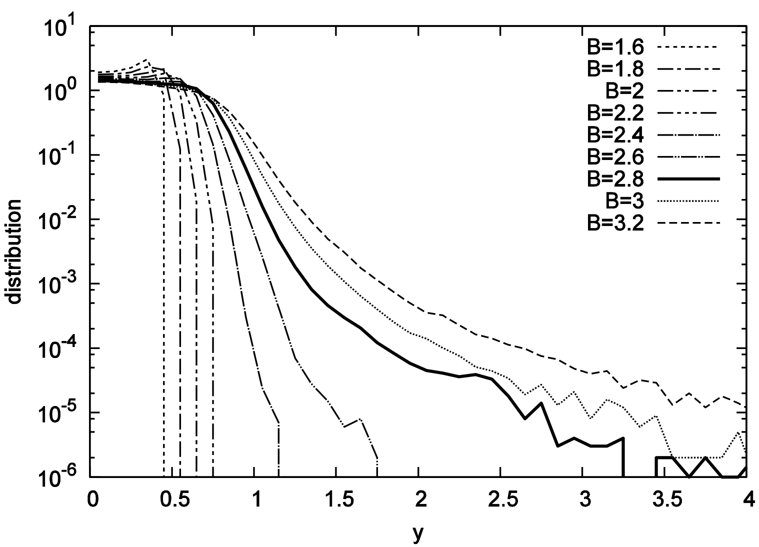


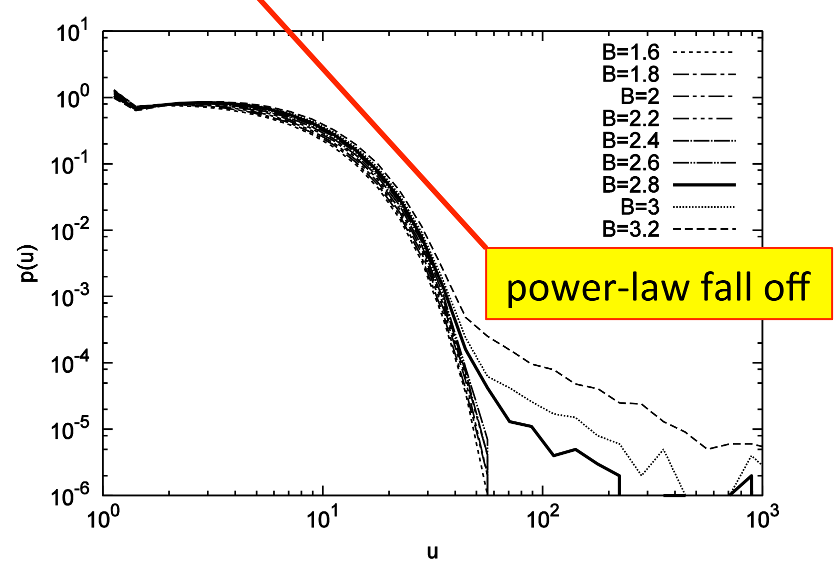
Configurations are restricted within |y|<C

Configurations spreads out into large |y| regime

Demonstration of our condition







The probability distribution of the magnitude of the drift term $u \equiv |v(z)| = \left|-(A+iB)z - z^3\right|$

These two problems are relevant to lattice QCD at finite density

$$w(U) = \mathrm{e}^{-S_{\mathrm{plaq}}[U]} \det M[U]$$

$$S_{\mathrm{plaq}}(U) = -\beta \sum_n \sum_{\mu\neq\nu} \mathrm{tr}\,(U_{n\mu} U_{n+\hat{\mu},\nu} U^{-1}_{n+\hat{\nu},\mu} U^{-1}_{n\nu})$$

generators of SU(3)

$$v_{an\mu}(U) = \frac{1}{w(U)} D_{an\mu} w(U) \qquad D_{an\mu} f(U) = \frac{\partial}{\partial x} f(\mathrm{e}^{ixt_a} U_{n\mu}) \Big|_{x=0}$$

complexification of dynamical variables : $U_{n\mu} \mapsto \mathcal{U}_{n\mu} \in \mathrm{SL}(3,\mathbb{C})$

discretized complex Langevin eq.

$$\mathcal{U}^{(\eta)}_{n\mu}(t+\epsilon) = \exp\left\{ i \sum_a \left(\epsilon\, v_{an\mu}(\mathcal{U}) + \sqrt{\epsilon}\, \eta_{an\mu}(t) \right) t_a \right\} \mathcal{U}^{(\eta)}_{n\mu}(t)$$

The drift term can become large when

1) the link variables $\mathcal{U}_{n\mu}$ becomes very non-unitary
2) eigenvalues of $M[\mathcal{U}]$ come close to zero.

c.f.) the drift term involves $\mathrm{tr}\,(M[\mathcal{U}]^{-1} \mathcal{D}_{an\mu} M[\mathcal{U}])$

"Gauge cooling" can be used to cure these problems.

4. Gauge cooling and its justification

Ref.) Nagata-J.N.-Shimasaki 1606.07627 [hep-lat]

Nagata-J.N.-Shimasaki, PTEP 2016 (2016) no.1, 013B01 [arXiv:1508.02377 [hep-lat]]

“gauge cooling”

Seiler-Sexty-Stamatescu, PLB 723 (2013) 213
arXiv:1211.3709 [hep-lat]]

E.g.) a system of N real variables x_k

$$Z = \int dx\, w(x) = \int \prod_k dx_k\, w(x)$$

$$v_k(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x_k}$$

Symmetry properties of the drift term $v_k(z)$ and the observables $\mathcal{O}(z)$

$$x'_j = g_{jk} x_k$$

enhances upon complexification of variables

$$z'_j = g_{jk} z_k$$

$g \in$ complexified Lie group

One can modify the Langevin process as :

$$\boxed{\tilde{z}_k^{(\eta)}(t) = g_{kl}\, z_l^{(\eta)}(t)}$$

“gauge cooling”

$$z_k^{(\eta)}(t+\epsilon) = \tilde{z}_k^{(\eta)}(t) + \epsilon\, v_k(\tilde{z}^{(\eta)}(t)) + \sqrt{\epsilon}\, \eta_k(t)$$

Justification of the gauge cooling

Nagata-J.N.-Shimasaki 1606.07627 [hep-lat]

$$\langle \mathcal{O}(z^{(\eta)}(t+\epsilon))\rangle_\eta = \sum_{n=0}^{\infty} \frac{1}{n!}\epsilon^n \int dx\, dy \left(:\tilde{L}^n : \mathcal{O}(z) \right)\Big|_{z^{(g)}} P(x,y;t)$$

$$z_k^{(g)} = g_{kl}(x,y)\, z_l$$

$$\left(:\tilde{L}^n : \mathcal{O}(z) \right)\Big|_{z^{(g)}} = :\tilde{L}^n : \mathcal{O}(z)$$

The only effect of gauge cooling disappears from this expression !

$\mathcal{O}(z)$ and $\tilde{L} = \left(v_k(z) + \frac{\partial}{\partial z_k} \right) \frac{\partial}{\partial z_k}$ are invariant under complexified symmetry transformations.

Note, however, that P(x,y;t) changes non-trivially because the noise term does not transform covariantly under the complexified symmetry.

probability $\propto e^{-\frac{1}{4}\sum_t \{\frac{1}{N_R}\eta^{(R)}(t)^2 + \frac{1}{N_I}\eta^{(I)}(t)^2\}}$

One can use this freedom to satisfy the condition for correct convergence !

5. Successful applications in semi-realistic cases

5.1 Random Matrix Theory for finite density QCD

Ref.) Nagata-J.N.-Shimasaki 1604.07717 [hep-lat]
to appear in JHEP

Random Matrix Theory for finite density QCD

$$Z = \int d\Phi d\Psi \, [\mathrm{det}(D+m)]^{N_\mathrm{f}} \mathrm{e}^{-S_\mathrm{b}}$$

Bloch-Bruckmann-Kieburg-Splittorff-Verbaarschot, PRD 87 (2013) no. 3, 034510 [arXiv:1211.3990 [hep-lat]].

$$S_b = 2N\mathrm{Tr}(\Phi^\dagger\Phi + \Psi^\dagger\Psi)$$

$$D = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

$$\begin{aligned} X &= e^{\mu}\Phi + e^{-\mu}\Psi, \\ Y &= -e^{-\mu}\Phi^\dagger - e^{\mu}\Psi^\dagger \end{aligned}$$

exact result for all values of N_f and N

$$Z_N^{N_\mathrm{f}}(m) = \frac{1}{(2m)^{1/2N_\mathrm{f}(N_\mathrm{f}-1)}} \mathrm{det}\left[\left(\frac{d}{dm}\right)^a L_{N+b}^{(\nu)}(-nm^2)\right]_{a=0,\ldots,N_\mathrm{f}-1;B=0,\ldots,N_\mathrm{f}-1}$$

independent of μ

The partition function is dominated by pions, which have zero quark charge.

CLM for Random Matrix Theory

$$S_b = 2N\mathrm{Tr}(\Phi^\dagger\Phi + \Psi^\dagger\Psi)$$

$$D = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

$$\begin{aligned} X &= e^{\mu}\Phi + e^{-\mu}\Psi, \\ Y &= -e^{-\mu}\Phi^\dagger - e^{\mu}\Psi^\dagger \end{aligned}$$

$$\begin{aligned} \Phi &= \Phi_1 + i\Phi_2 \\ \Psi &= \Psi_1 + i\Psi_2 \end{aligned}$$

$\Phi_1,\ \Phi_2,\ \Psi_1,\ \Psi_2$: Hermitian

Complexification of dynamical variables :

$\Phi_1,\ \Phi_2,\ \Psi_1,\ \Psi_2$: general complex matrices

$$\begin{aligned} \Phi_\pm &= \Phi_1 \pm i\Phi_2 \\ \Psi_\pm &= \Psi_1 \pm i\Psi_2 \end{aligned}$$

$$S_b = 2N\mathrm{Tr}[\Phi_-\Phi_+ + \Psi_-\Psi_+]$$

$$D = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

$$\begin{aligned} X &= e^{\mu}\Phi_+ + e^{-\mu}\Psi_+ \\ Y &= -e^{-\mu}\Phi_- - e^{\mu}\Psi_- \end{aligned}$$

"gauge cooling" in random matrix theory

Nagata-J.N.-Shimasaki 1604.07717 [hep-lat]
to appear in JHEP

$$D = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

$$\begin{aligned} X &= e^{\mu}\Phi_{+} + e^{-\mu}\Psi_{+} \\ Y &= -e^{-\mu}\Phi_{-} - e^{\mu}\Psi_{-} \end{aligned}$$

Symmetry of the system :

$$\begin{aligned} \Phi_{+} &\mapsto \Phi'_{+} = g\Phi_{+}h^{-1} \\ \Phi_{-} &\mapsto \Phi'_{-} = h\Phi_{-}g^{-1} \end{aligned} \qquad \begin{aligned} \Psi_{+} &\mapsto \Psi'_{+} = g\Psi_{+}h^{-1} \\ \Psi_{-} &\mapsto \Psi'_{-} = h\Psi_{-}g^{-1} \end{aligned}$$

complexfication of variables

$$g, h \in \mathrm{SU}(N)$$

$$\Downarrow$$

$$g, h \in \mathrm{SL}(N, \mathbf{C})$$

Using this complexified symmetry, we apply "gauge cooling" after each Langevin step so that problems are avoided.

- ✓ non-Hermiticity of the configurations
- ✓ zeroes of fermion determinant

Choices of the norm in the "gauge cooling"

- To avoid excursions into non-Hermitian regime

$$\mathcal{N}_{\mathrm{h}} = -\frac{1}{N}\mathrm{tr}\sum_{i=1,2}[(\Phi_i - \Phi_i^\dagger)^2 + (\Psi_i - \Psi_i^\dagger)^2]$$

- To avoid zero eigenvalues of D+m

$$D = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$$

$$\mathcal{N}_1 = \frac{1}{N}\mathrm{Tr}(X + Y^\dagger)(X^\dagger + Y)$$

$$\begin{aligned} X &= e^{\mu}\Phi_+ + e^{-\mu}\Psi_+ \\ Y &= -e^{-\mu}\Phi_- - e^{\mu}\Psi_- \end{aligned}$$

tries to make D closer to anti-Hermitian $D^\dagger \simeq -D$

$$\mathcal{N}_2 = \mathrm{Tr}e^{-\xi\, M^\dagger M}$$

$$M = D + m$$

- We take linear combinations like :

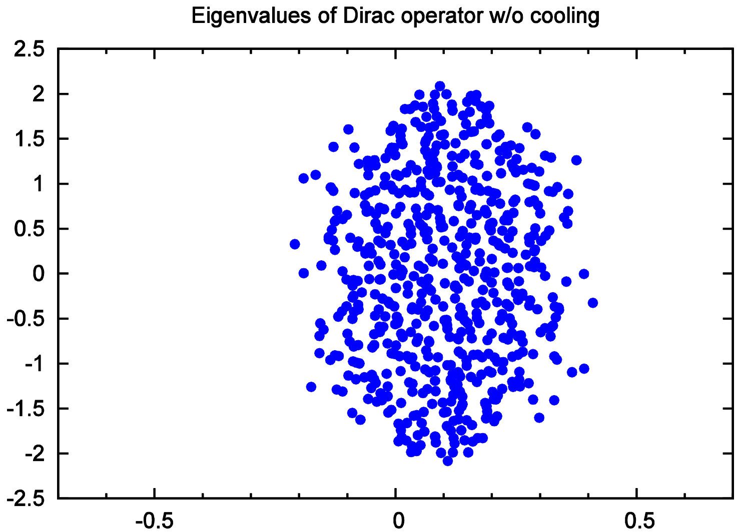
$$\mathcal{N}_{i,r} = r\mathcal{N}_{\mathrm{h}} + (1-r)\mathcal{N}_i,\ (0 \le r \le 1\ )$$

Results of eigenvalue distribution of D

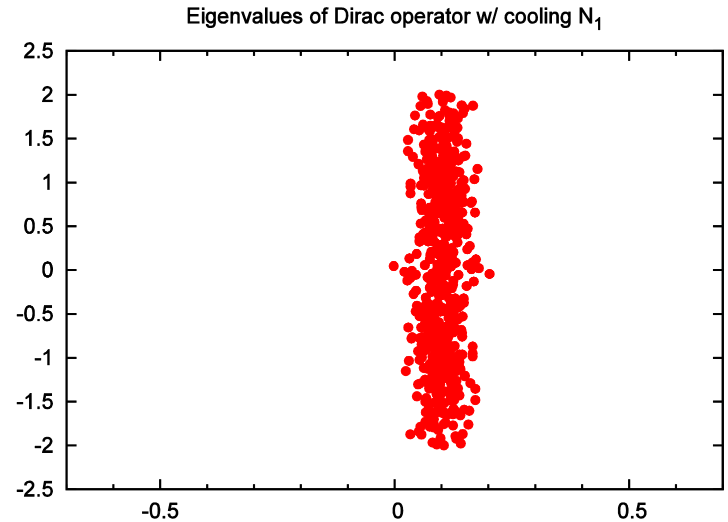
Nagata-J.N.-Shimasaki 1604.07717 [hep-lat] to appear in JHEP

$$\tilde{m} = mN = 2$$

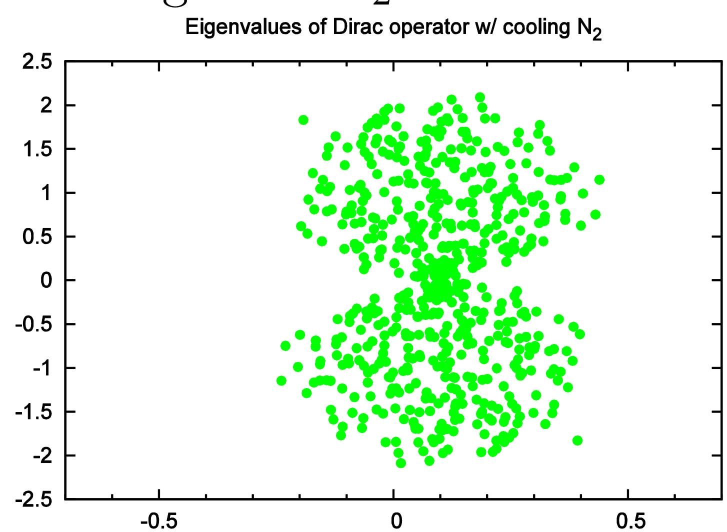
no cooling



cooling with $\mathcal{N}_1$



cooling with $\mathcal{N}_2$



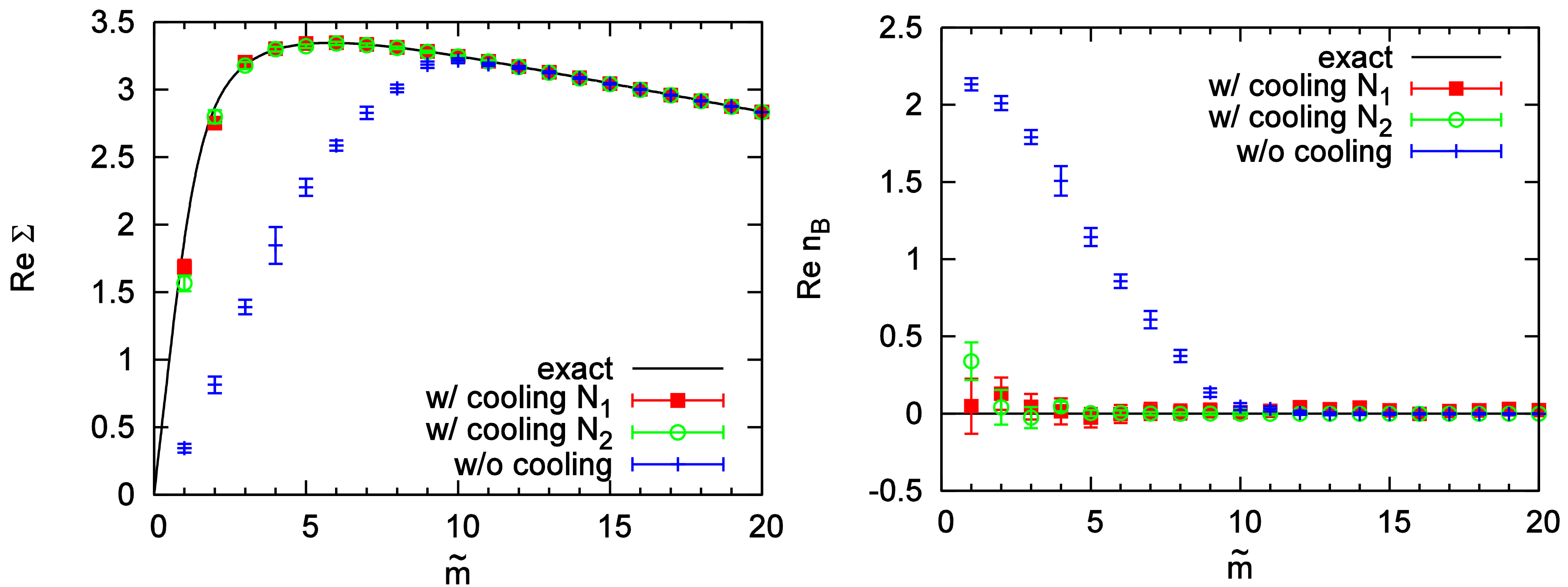
"Gauge cooling" can be used to avoid the problem of the singular drift by choosing an appropriate norm.

Results of the CLM w/ and w/o "gauge-cooling"

$$N_f = 2,\ N = 30,\ \sqrt{N}\mu = 2$$

$r = 0$ for $\mathcal{N}_1$
$r = 0.01$ for $\mathcal{N}_2$

Nagata-J.N.-Shimasaki
1604.07717 [hep-lat]
to appear in JHEP



Correct results for chiral condensate and baryon number density are reproduced by CLM with "gauge cooling"
even in the small mass region !!!

5.1 SSB of rotational symmetry in a simplified IKKT-type matrix model

Ito-J.N., in preparation

a simplified IKKT-type matrix model

J.N. PRD 65, 105012 (2002), hep-th/0108070

$$Z = \int \mathrm{d}A \, \mathrm{d}\psi \, \mathrm{d}\bar{\psi} \, \mathrm{e}^{-(S_\mathrm{b}+S_\mathrm{f})}$$

$$S_\mathrm{b} = \frac{1}{2} N \, \mathrm{tr} \, (A_\mu)^2$$

$$\mu = 1, 2, 3, 4$$

$$S_\mathrm{f} = \bar{\psi}^f_\alpha \, (\Gamma_\mu)_{\alpha\beta} \, A_\mu \, \psi^f_\beta$$

$$\alpha, \beta = 1, 2$$

$$f = 1, \cdots, N_\mathrm{f}$$

$$\Gamma_1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \Gamma_2 = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \Gamma_3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \Gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

SSB of SO(4) rotational symmetry
in the $N \to \infty$ limit with fixed $r = \frac{N_\mathrm{f}}{N}$
due to the complex fermion determinant

order parameters of the SSB of SO(4) symmetry

$$T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$$ 4×4 real symmetrix matrix

eigenvalues : $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$

If $\langle \lambda_i \rangle$ are different in the $N \to \infty$ limit

➡ SSB of SO(4) symmetry

Note : $$\sum_{i=1}^{4} \langle \lambda_i \rangle = \sum_{\mu=1}^{4} \left\langle \frac{1}{N}\mathrm{tr}\,(A_\mu)^2 \right\rangle = 4\left(1 - \frac{1}{N^2}\right) + 2\,r$$

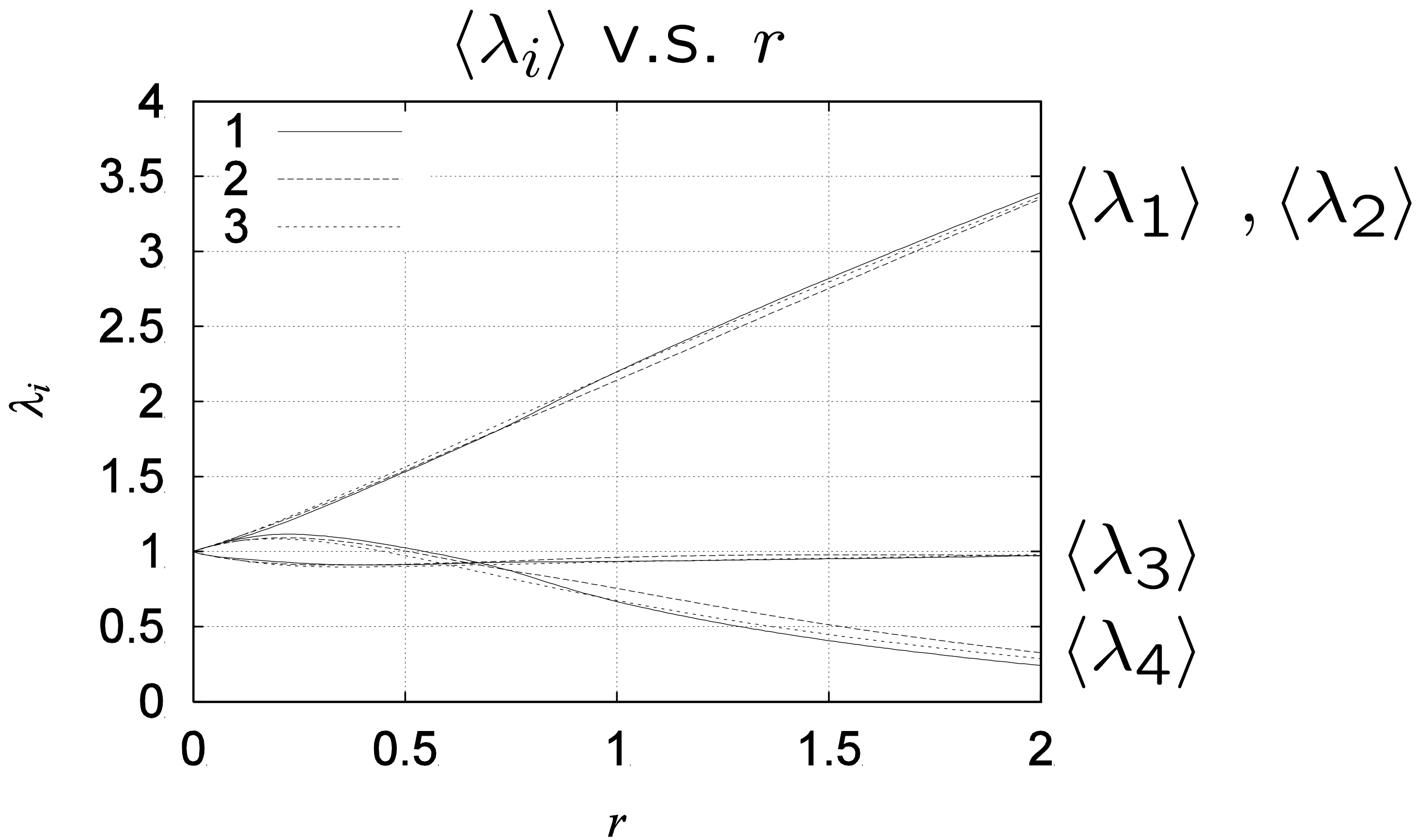
$$\langle \lambda_1 \rangle = \langle \lambda_2 \rangle = \langle \lambda_3 \rangle = \langle \lambda_4 \rangle$$

for the phase-quenched model $\quad |\mathrm{det}D|$ instead of $\mathrm{det}D$

The phase of $\mathrm{det}D$ induces SSB.

Results of the Gaussian expansion method

J.N., Okubo, Sugino: JHEP 0210 (2002) 043 [hep-th/0205253]



SSB $SO(4) \mapsto SO(2)$ occurs !!!

E.g.) $\langle \lambda_1 \rangle = \langle \lambda_2 \rangle = 2.1$, $\langle \lambda_3 \rangle = 1.0$, $\langle \lambda_4 \rangle = 0.8$ at $r = 1$

Application of the complex Langevin method

Ito-J.N., in preparation

$$A_\mu \ : \ \text{Hermitian} \ \mapsto \mathcal{A}_\mu \ : \ \text{general complex}$$

$$S_{\text{eff}} = \frac{1}{2} N \, \text{tr} \, (\mathcal{A}_\mu)^2 - \log \det \, (\Gamma_\mu \mathcal{A}_\mu)$$

In order to investigate the SSB, we introduce
an infinitesimal SO(4) breaking terms :

$$S_{\text{breaking}} = \frac{1}{2} \epsilon N \sum_{i=1}^{4} m_i \, \text{tr} \, (\mathcal{A}_i)^2$$

$$m_1 < m_2 < m_3 < m_4$$

and calculate :

$$\langle \lambda_i \rangle = \lim_{\epsilon \to 0} \lim_{N \to 0} \left\langle \frac{1}{N} \text{tr} \, (\mathcal{A}_i)^2 \right\rangle_{\text{CL}}$$

$$\text{no sum over } i = 1, 2, 3, 4$$

Results of the CLM

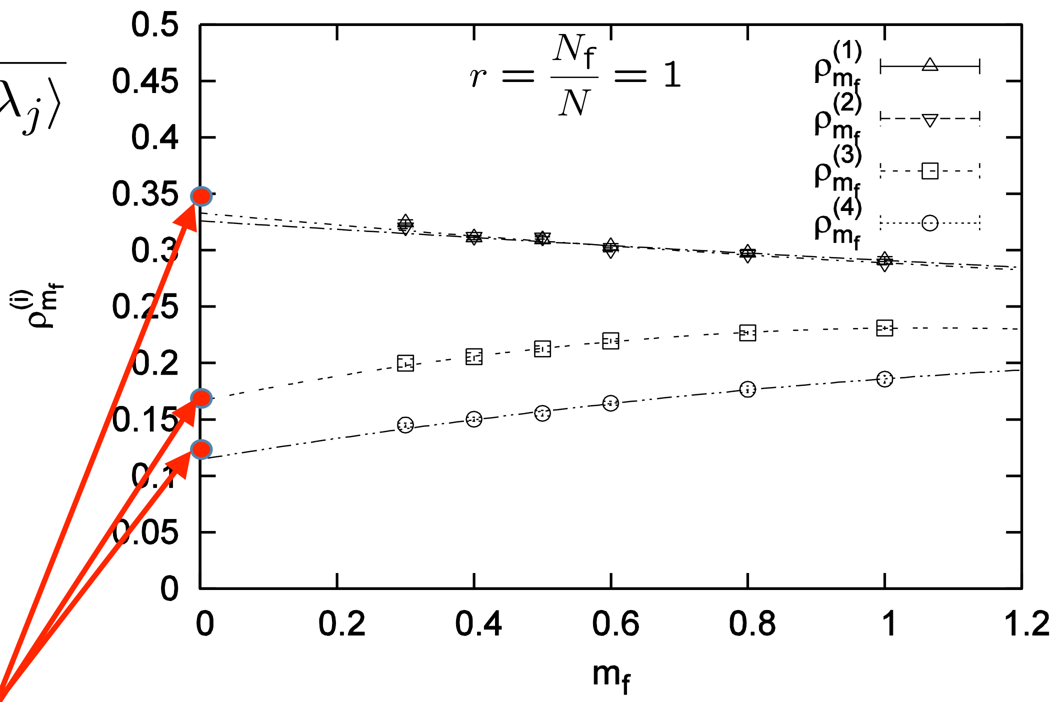
Ito-J.N., in preparation

In order to cure the singular-drift problem, we deform the fermion action as:

$$S_{\mathsf{f}} = \bar{\psi}^f_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi^f_\beta \boxed{+ m_4 \bar{\psi}^f_\alpha (\Gamma_4)_{\alpha\beta} \psi^f_\beta}$$

Explicitly breaks SO(4) $\mapsto$ SO(3)

$$\rho_i \equiv \frac{\langle \lambda_i \rangle}{\sum_{j=1}^{4} \langle \lambda_j \rangle}$$



GEM result : $\rho_1 = \rho_2 = 0.35$, $\rho_3 = 0.17$, $\rho_4 = 0.13$

$\langle \lambda_1 \rangle = \langle \lambda_2 \rangle = 2.1$, $\langle \lambda_3 \rangle = 1.0$, $\langle \lambda_4 \rangle = 0.8$ at $r = 1$

CLM reproduces the SSB of SO(4) induced by complex fermion determinant !

6. Summary and future prospects

Summary

- Complex Langevin method
  a promising approach to the sign problem

We have established the argument for the justification of the method.

The condition for correct convergence :
The probability distribution of the drift term should be suppressed exponentially at large magnitude.

If this condition is satisfied, the $\epsilon \to 0$ limit can be taken,
and the finite t evolution of the observable can be used in the argument.

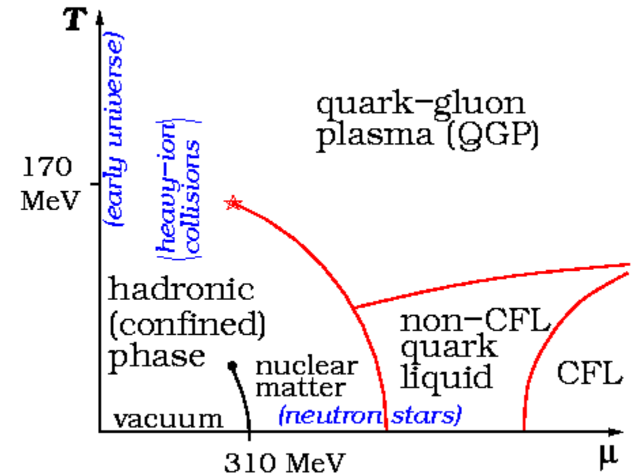
- "Gauge cooling" can be implemented in the argument.

Singular drift problem and the excursion problem can be cured.

- Successful applications to RMT for finite density QCD and SSB of rotational symmetry in a simplified IKKT-type matrix model.

Future prospects

- Exploration of the QCD phase diagram
  - ➢ Reliable determination of the critical end point.
    (important for heavy ion collision experiments.)
  - ➢ EoS of the nuclear matter and quark liquid
    at large μ (a big impact on astrophysics)



- Does SSB $SO(10) \mapsto SO(4)$ occur
  in the IKKT model for superstring theory ?
  Ishibashi-Kawai-Kitazawa-Tsuchiya ('97)

- Some of the following listed below may become feasible.

Some work already exists.

- a theta term
- Chern-Simons term
- real-time dynamics
- Yukawa interactions
- supersymmetric theories
- chiral fermions
- systems of strongly coupled electrons etc.

A whole new world of theoretical physics may be awaiting for us !