Mass Ladder Operators from Spacetime Conformal Symmetry

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Introduction

In quantum mechanics, ladder operators are very powerful tools.
We can derive physical properties without a detailed knowledge of solutions.

Today, we show ladder operators for massive Klein-Gordon equations on curved spacetime.
I expect this will be also powerful tool.

Introduction

Purpose of this project:

- construct ladder operator for KG eq
- reproduce known results from different point of view
- find new applications

Introduction

My personal motivation:

A phenomena around an extremal black hole is effectively described by a massive KG eq in AdS2.

There exists a “conserved quantity” if the mass takes special values.

I guessed that there should be mathematically deeper understanding.

Contents

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mass ladder operator

In n Dim spacetime (or space), if there exists a closed conformal Killing vector ζ_ν

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = Q g_{\mu\nu} \quad (Q = n^{-1} \nabla_\mu \zeta^\mu)$$

$$\nabla_\mu \zeta_\nu - \nabla_\nu \zeta_\mu = 0$$

and ζ_ν is an eigen vector of Ricci tensor

$$R^\mu{}_\nu \zeta^\nu = \chi (n-1) \zeta^\mu \quad (\chi : \mathrm{const.})$$

then, $D_k := \mathcal{L}_{\zeta^\mu} - kQ$ satisfies

$$[\Box, D_k]\Phi = \chi(2k+n-2) D_k \Phi + 2Q(\Box + \chi k(k+n-1))\Phi$$

$$\begin{cases} m^2 := -\chi k(k+n-1) \\ m'^2 := -\chi(k-1)(k+n-2) \end{cases}$$

Eq. becomes

$$(\Box - m'^2)D_k\Phi = (D_k + 2Q)(\Box - m^2)\Phi$$

If Φ is a sol. of KG eq with m^2

$D_k\Phi$ becomes a sol. of KG eq with m'^2

D_k is mass ladder operator for KG eq

Both m^2, m'^2 are real $\iff$ k is real

$$m^2 = -\chi k(k+n-1)$$

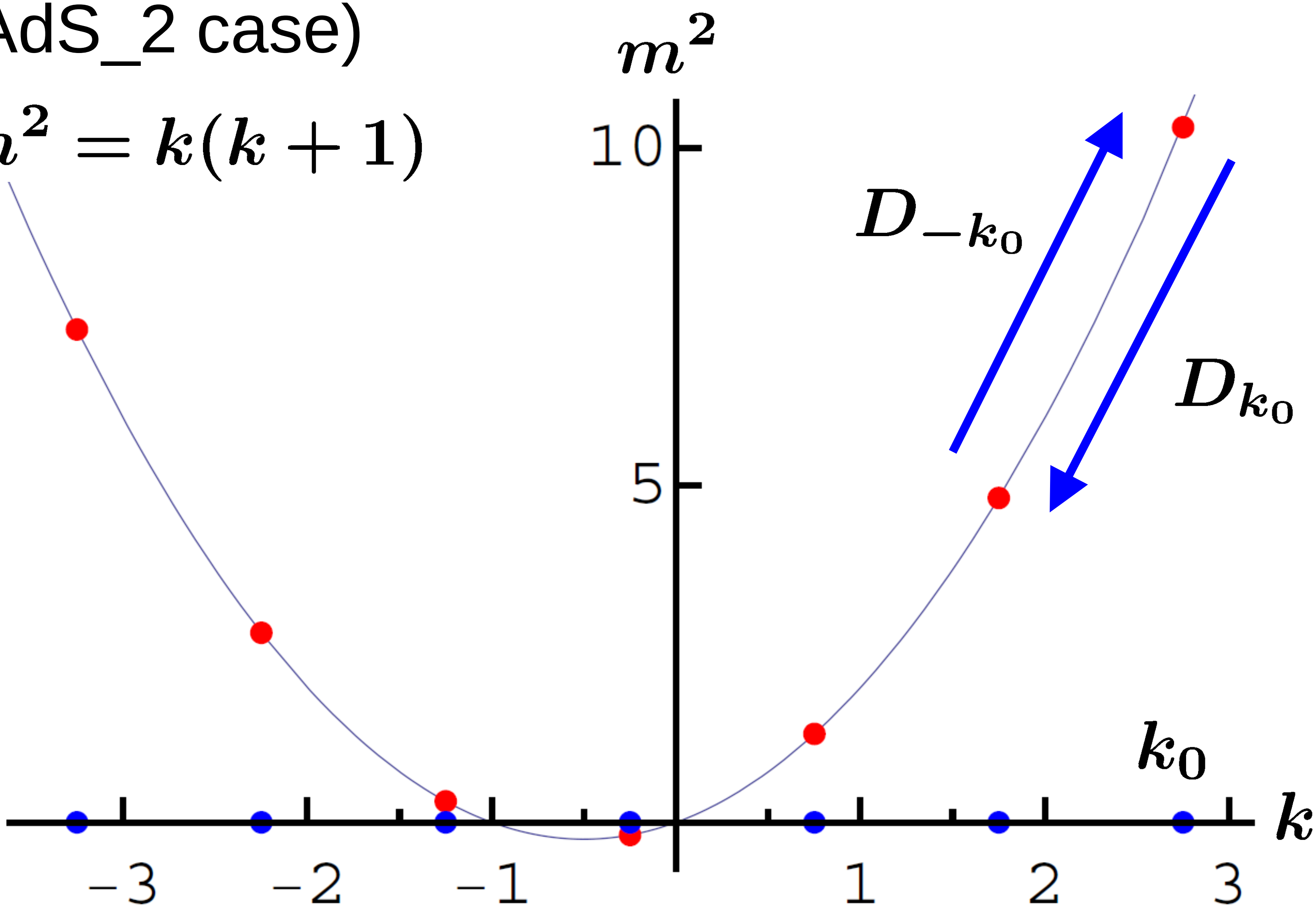
$$\implies k = k_\pm = \frac{1-n \pm \sqrt{(n-1)^2 - 4m^2/\chi}}{2}$$

$$\frac{\chi}{4}(n-1)^2 \leq m^2, \quad \chi < 0 \quad \text{(e.g. AdS)}$$

$$m^2 \leq \frac{\chi}{4}(n-1)^2, \quad \chi > 0 \quad \text{(e.g. dS)}$$

(AdS_2 case)

$m^2 = k(k+1)$



Comment

D_k is surjective (onto) map

We can construct all solutions for m'^2 from the solutions for m^2

(proof is straightforward, but need hard calculation)

In this sense, two different mass systems are “same”

S^2 and Spherical harmonics

$$(\triangle_{S^2}+\ell(\ell+1))Y_{\ell,m}=0$$

$$L_\pm Y_{\ell,m}=\sqrt{(\ell\mp m)(\ell\pm m+1)}Y_{\ell,m\pm 1}$$

$D_k=\sin\theta\partial_\theta-k\cos\theta$ can shift ℓ

$$D_\ell Y_{\ell,m}=-\sqrt{\frac{(2\ell+1)(\ell^2-m^2)}{2\ell-1}}Y_{\ell-1,m}$$

$$D_{-\ell}Y_{\ell-1,m}=\sqrt{\frac{(2\ell-1)(\ell^2-m^2)}{2\ell+1}}Y_{\ell,m}$$

S^2 and Spherical harmonics

$$(\triangle_{S^2})\frac{e^{i\phi}}{\tan\theta} = 0$$

$$D_{-1}\frac{e^{i\phi}}{\tan\theta} \propto Y_{11}$$

D_k can map singular sol. to regular sol.

AdS case

$$\mathbf{AdS}_n \quad ds^2 = \frac{dr^2}{r^2} + r^2\left(-dt^2 + \sum_{i=1}^{n-2}(dx^i)^2\right)$$

$$\zeta_{-1} = r^2\frac{\partial}{\partial r} \qquad Q_{-1} = r$$

$$\zeta_i = x^i r^2\frac{\partial}{\partial r} + \frac{1}{r}\eta^{ij}\frac{\partial}{\partial x^i} \qquad Q_i = x^i r \qquad (i = 0, 1, \cdots, n-2)$$

$$\zeta_{n-1} = (-1 + r^2\eta_{ij}x^i x^j)\frac{\partial}{\partial r} + \frac{2x^i}{r}\frac{\partial}{\partial x^i} \qquad Q_{n-1} = \frac{1}{r} + r\eta_{ij}x^i x^j$$

$$D_{\mu,k} = \mathcal{L}_{\zeta_\mu} - kQ_\mu \qquad D_{-1,k} = r^2\frac{\partial}{\partial r} - kr$$

$$D_{i,k} \sim x^i\left(r^2\frac{\partial}{\partial r} - kr\right)$$

$$D_{n-1,k} \sim (\eta_{ij}x^i x^j)\left(r^2\frac{\partial}{\partial r} - kr\right)$$

$$(\Box_{AdS_n} - m^2)\Phi = 0$$

$$\Phi \sim c_+(x^i) r^{\Delta_+} + c_-(x^i) r^{\Delta_-}$$

$$\Delta_\pm = \frac{-(n-1) \pm \sqrt{(n-1)^2 + 4m^2}}{2}$$

$D_{\mu,k} \sim f(x^i)\left(r^2 \frac{\partial}{\partial r} - k_+ r\right)$ maps (non)normalizable mode to (non)normalizable mode

If $m^2_{\mathrm{BF}} \leq m^2 \leq m^2_{\mathrm{BF}} + 1$ two modes are normalizable, so we need to be carefull

towards AdS/CFT

- D_k can map m^2 to m'^2

  This suggests some relations between CFT with different conformal dims

- D_k may be able to map a singular sol. into a regular sol.

  Singular sols. may have physical meaning in AdS/CFT context

- $D_{-k-n+2}D_k$ is a symmetry operator

  m^2 to m^2

KK mode in AdS5 x S5

$$\Box_{AdS_5 \times S^5} \Phi = 0 \qquad \Phi = Y_\ell \tilde{\Phi}$$

$$\Longrightarrow (\Box_{AdS_5} - \Lambda\ell(\ell+4))\tilde{\Phi} = 0 \quad (\ell = 0, 1, 2, \cdots)$$

mass spectrum corresponds to the masses which can be mapped from massless scalar fields in AdS5

there is a duality among the zero mode and Kaluza-Klein modes on massless scalar fields in AdS5 x S5

Comment on $\chi = 0$ case

$m^2 = -\chi k(k+n-1)$ $\quad R^\mu{}_\nu \zeta^\nu = \chi(n-1)\zeta^\mu$

D_k can be used for massless scalar if $\chi = 0$

We can construct another ladder operator if
$Q(= n^{-1}\nabla_\mu \zeta^\mu)$ is constant (homothetic case)

$\tilde{D}_\lambda := e^{\lambda \mathcal{L}_\zeta}$ can shift m^2 to $e^{2\lambda Q} m^2$

Minkowski case $\quad \tilde{D}_\lambda = e^{\lambda(x^\mu \partial_\mu + \xi^\mu \partial_\mu)}$

Supersymmetric quantum mechanics

- comformal tr of metric → CKV becomes Killing

$$\left[\Box - m^2\right] \Phi = 0$$

$$\Longrightarrow \left[\partial^2_{\bar{\lambda}} + \tilde{\Box} - V(\bar{\lambda}, m^2)\right] \bar{\Phi} = 0$$

V is at most 2nd order of $\frac{1}{\cos\bar{\lambda}}$ or $\frac{1}{\cosh\bar{\lambda}}$ or $\frac{1}{\bar{\lambda}}$

this is a potential for supersymmetric quantum mechanics which has shift shape invariance

D_k Corresponds to supercharge

Aretakis const.

Aretakis showed the “instability” of test scalar field on 4Dim extremal RN BH

[Aretakis 2011]

It is useful to use the Aretakis const.

$$\partial_r^{\ell+1}\Phi|_{\mathcal{H}} = \mathbf{const.}$$

Relation between Newman Penrose const.?

[Bizon, Friedrich, 2013]

We can derive Aretakis const from ladder operator D_k

Aretakis const in AdS_2

$$ds^2 = -r^2 dv^2 + 2dvdr$$

KG eq: $2\partial_v \partial_r \Phi + \partial_r (r^2 \partial_r \Phi) = m^2 \Phi$

If we assume $m^2 = \ell(\ell+1), (\ell = 0, 1, 2, \cdots)$

$$\partial_v \partial_r^{\ell+1} \Phi \Big|_{r=0} = 0$$

AdS2 is maximally sym, we can find a quantity which takes const. on every outgoing null hypersurface

$$\underbrace{\left(\partial_v + \frac{r^2}{2}\partial_r\right)}_{\text{outgoing null}} \underbrace{\left[\left(\frac{vr}{2}+1\right)^{2(\ell+1)} \partial_r^{\ell+1}\Phi\right]}_{A_k} = 0$$

Ladder operator D_k in AdS_2

$$ds^2 = -\frac{4|\Lambda|}{(x^+ - x^-)^2} dx^+ dx^-$$

closed conformal Killing vector :

$$\zeta_{-1} = \partial_+ - \partial_-$$

$$\zeta_0 = x^+ \partial_+ - x^- \partial_-$$

$$\zeta_1 = (x^+)^2 \partial_+ - (x^-)^2 \partial_-$$

$$D_{i,k} = \mathcal{L}_{\zeta_i} - kQ_i \quad (i = -1, 0, 1)$$

KG eq: $(\Box - \ell(\ell+1))\Phi = 0$

Acting D_k ℓ times, $D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi$ becomes massless

$\Box(D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi) = 0$

$$D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi = F(x^+) + G(x^-)$$

$$\frac{\partial}{\partial x^-} D_1 D_2 \cdots D_{\ell-1} D_\ell \Phi = G'(x^-)$$

This coincides with Aretakis const

$$\underbrace{\left(\partial_v + \frac{r^2}{2}\partial_r\right)}_{\text{outgoing null}} \underbrace{\left[\left(\frac{vr}{2}+1\right)^{2(\ell+1)} \partial_r^{\ell+1}\Phi\right]}_{A_k} = 0$$

$$\left(\frac{vr}{2}+1\right)^{2(\ell+1)} \partial_r^{\ell+1} =: L^{(\ell)}$$

$\frac{\partial}{\partial x^-} D_1 D_2 \cdots D_{\ell-1} D_\ell$ coincides with $L^{(\ell)}$

up to the function of x^- and $(\Box - m^2)$

$$L^{(2)} = \frac{1}{(x^-)^2}\left\{-\partial_- D_1 D_2 - \frac{(x^+)^2}{(x^+ - x^-)^2}(\Box_{AdS_2} - 2)\right\}$$

4D extremal RN black hole

$$ds^2 = -\left(1 - \frac{1}{\rho}\right)^2 dv^2 + 2dvd\rho + \rho^2 d\Omega_{S^2}$$

We can also derive Aretakis const in 4Dim extremal Reissner–Nordström black hole

$$\partial_\rho (D_1 D_2 \cdots D_\ell (e^{(\rho-1)/2}\Phi))\Big|_{\mathcal{H}} = \mathrm{const.}$$

Ladder operator is useful for less symmetric spacetimes which have approximate conformal symmetry

Summary, future works

- If there exists CCKV and it is an eigen vector of Ricci tensor, we can define mass ladder operator for KG eq.
- D_k can shift ℓ for $Y_{\ell,m}$
- We can derive Aretakis const.
- Higher derivative operator becomes non trivial
- (non)smoothness of extremal BHs
- AdS/CFT
- de Sitter case
- super symmetric quantum mechanics
- vector, tensor, spinor harmonics
- derivation from commutation relation