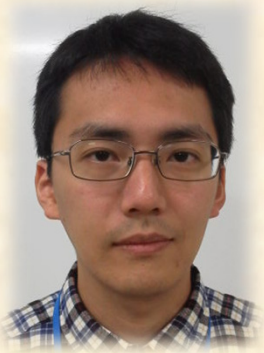
RMT and supersymmetric SYK



RIKEN

Part I

Universität Regensburg

JHEP 09 (2017) 050
arXiv:1706.03044

T. Kanazawa **(speaker)**
T. Wettig

Osaka University, Oct 31, 2017

Sachdev-Ye-Kitaev (SYK) model

$$H = \sum_{1 \le i < j < k < l \le N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l$$

Majorana fermions $\{\chi_i, \chi_j\} = \delta_{ij}$

Quenched disorder

$$P(J_{ijkl}) \sim \exp\left(-12 N^3 J_{ijkl}^2 / J^2\right)$$

3 properties

- Solvable.

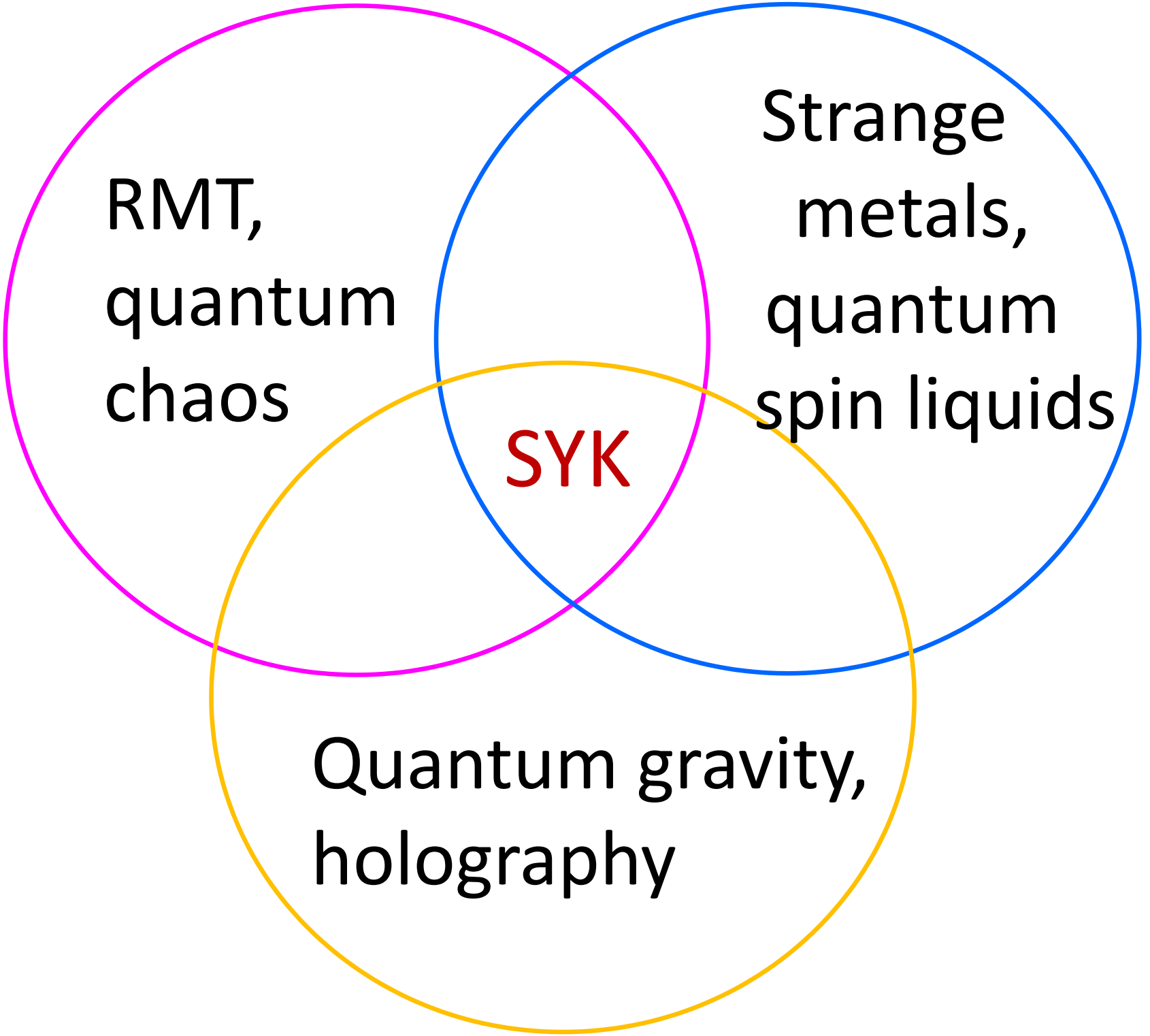
  Can compute correlation functions at large N

- Emergent conformal invariance in IR
- Maximally chaotic

  At strong coupling, has same Lyapunov exponent as a black hole

Holographic?

Sachdev '10, Kitaev '15


RMT,
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Quantum gravity,
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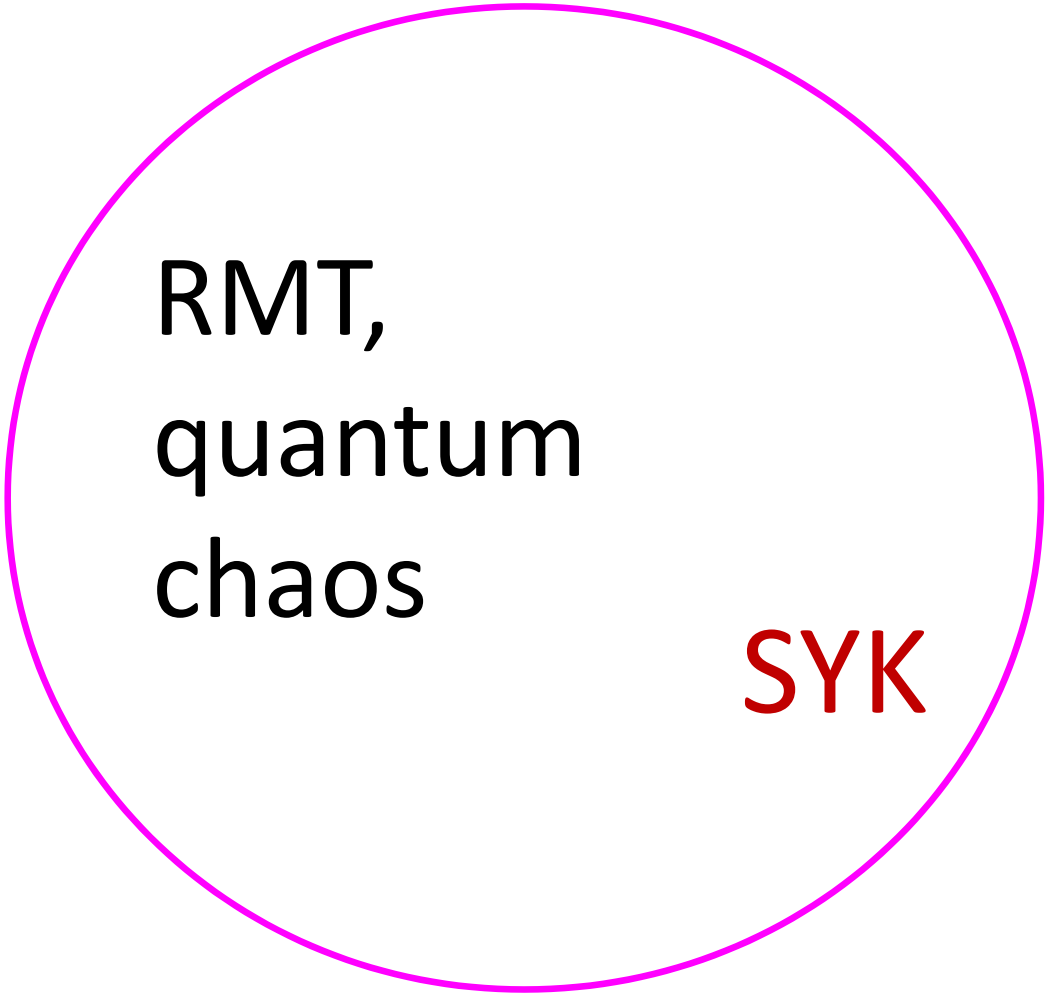
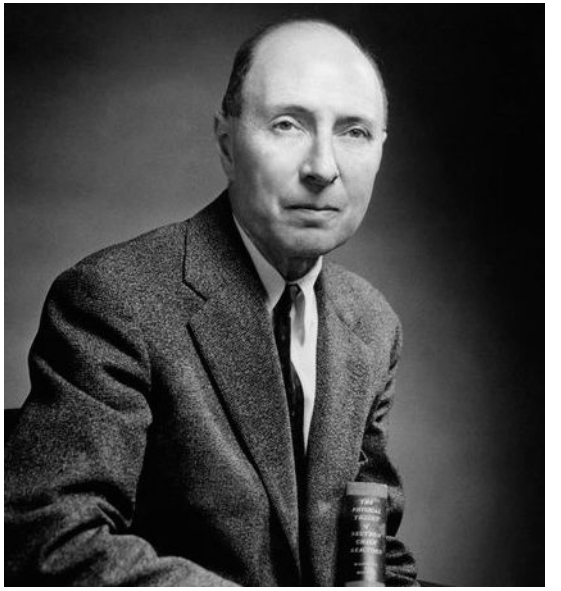
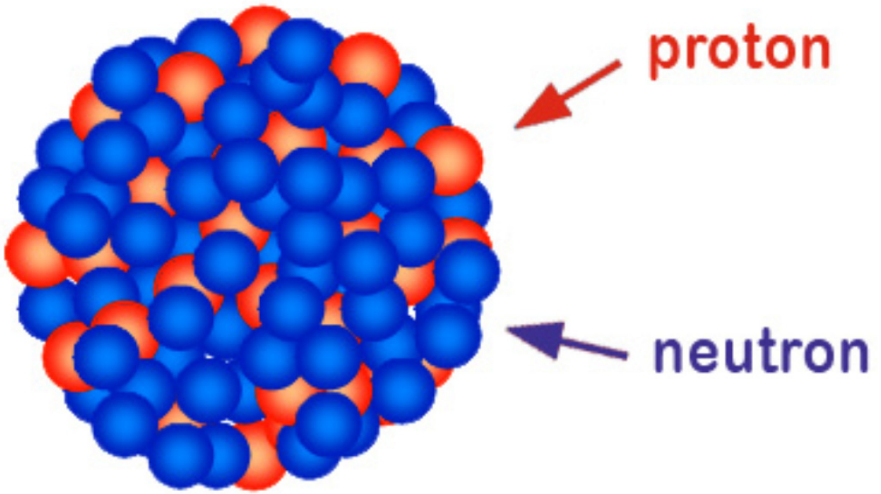

RMT,
quantum
chaos
SYK

- In 1955, Wigner proposed to use RMT as a simplistic quantum description of complicated atomic nuclei.



- Major mathematical progress was made by Dyson in 1960s.

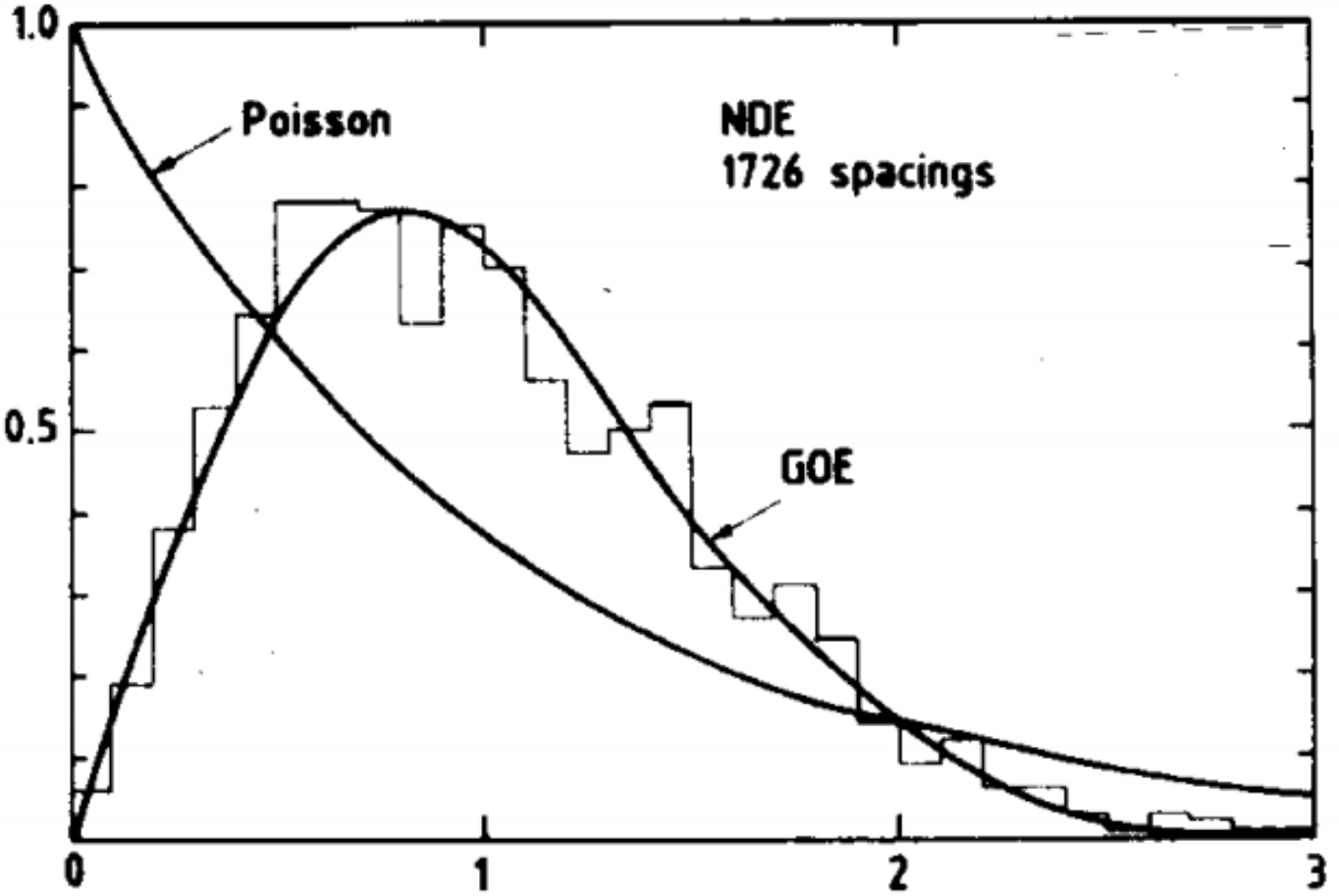


- In standard statistical mechanics, one considers an ensemble of physical systems governed by the same Hamiltonian but differing in initial conditions etc., and calculates thermodynamic functions by averaging over this ensemble.

- In RMT, one considers ensembles of systems governed by different Hamiltonians with some common symmetry property. It focusses attention on the generic properties which are common to (almost) all members of the ensemble and which are determined by the underlying fundamental symmetries.

$$P(H) \propto \exp(-\mathrm{Tr}\, H^2)\,, \quad H = H^\dagger$$

Nearest neighbor spacing distribution



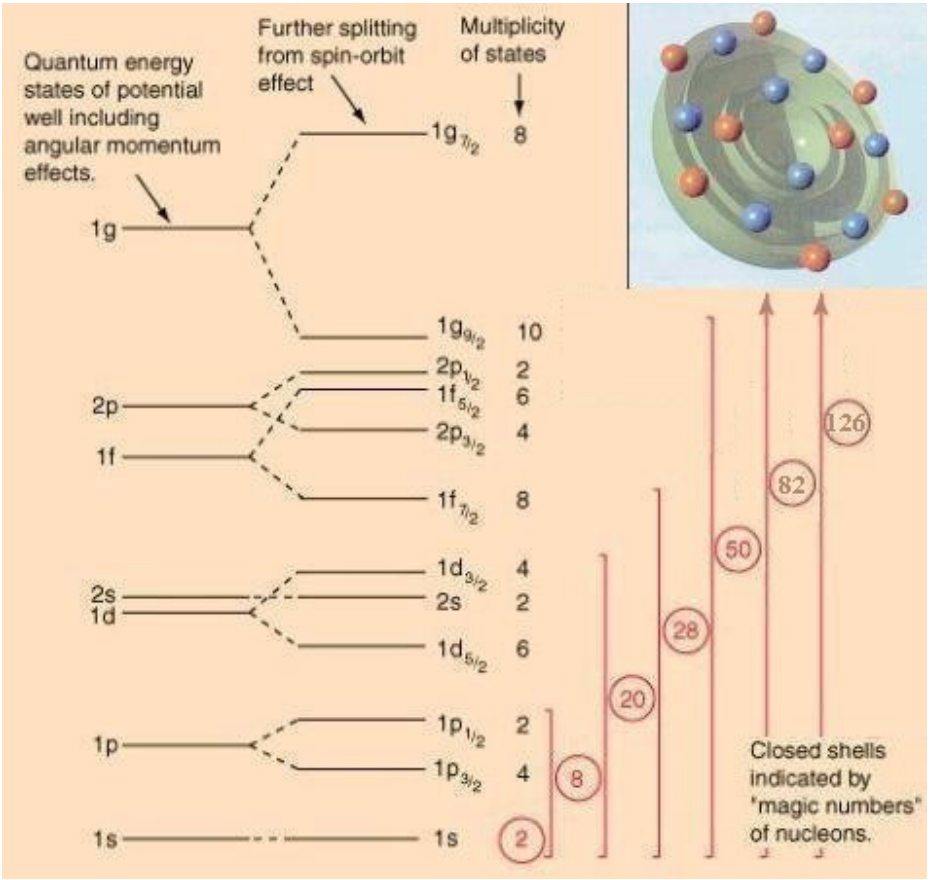
Taken from: O. Bohigas, R.U. Haq, and A. Pandey, in
Nuclear Data for Science and Technology, K.H. Bochhoff (ed.), Reidel, Dordrecht (1983), p.809.

- Gaussian Unitary Ensemble (GUE),
  Gaussian Orthogonal Ensemble (GOE),
  Gaussian Symplectic Ensemble (GSE)
  … classical 3 Wigner-Dyson classes

- Unitary invariance of RMT: $H \to UHU^{\dagger}$; there is no preferred basis of the Hilbert space
- In RMT, every state in Hilbert space is connected to every other state by a matrix element which does not vanish and is assumed to be independent

Success of the nuclear shell model

- Nucleons create a “mean-field potential” $U(x)$ and interacts with each other via a weak residual 2-body interaction



$$H = \sum_{i}^{A} (T_i + U_i) + \frac{1}{2} \sum_{i \neq j}^{A} V_{ij}$$

Criticism:

RMT is too simplistic. It is an ensemble of multi-body, not two-body interactions.

"k-body embedded ensembles" = EGE(k)

Mon and French 1975

Consider m Fermions in ℓ degenerate single-particle states interacting via a random k-body interaction.

ex) $m = 2, \ell = 5$

Hilbert space dimension

$$= \binom{\ell}{m}$$

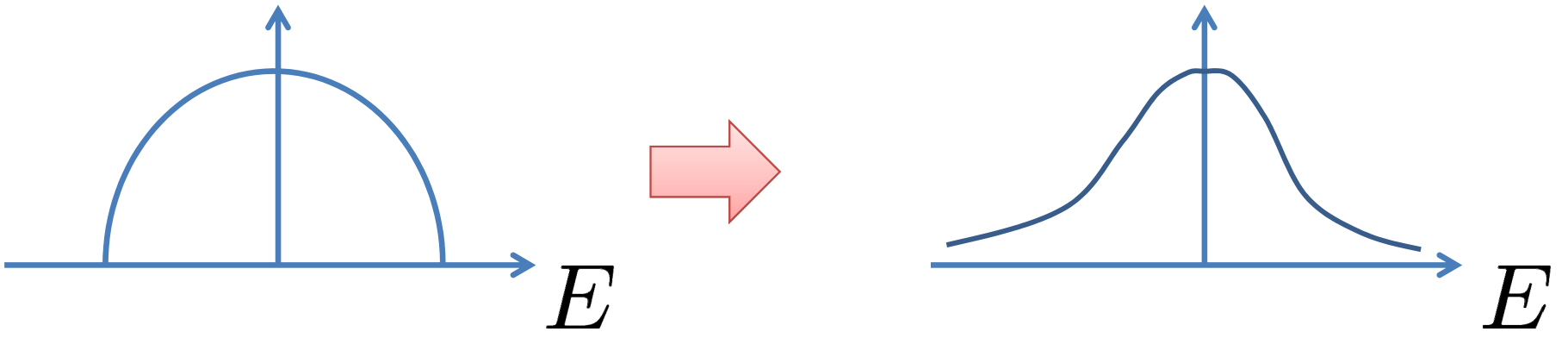
$$\psi^\dagger_{k,\alpha} = \psi^\dagger_{j_1,j_2,\ldots,j_k} = \prod_{s=1}^{k} a^\dagger_{j_s}, \qquad j_1 < j_2 < \ldots < j_k$$

$$H = \sum_{\alpha\gamma} v_{k;\alpha\gamma} \psi^\dagger_{k,\alpha} \psi_{k,\gamma} \ .$$

- The coefficients are random Gaussian variables with mean 0 and variance

$$\overline{v_{k;\alpha\gamma} v^*_{k;\alpha'\gamma'}} = v^2 \ \delta_{\alpha\alpha'} \delta_{\gamma\gamma'} \ .$$

- H becomes a sparse matrix
- The spectral density changes to a Gaussian



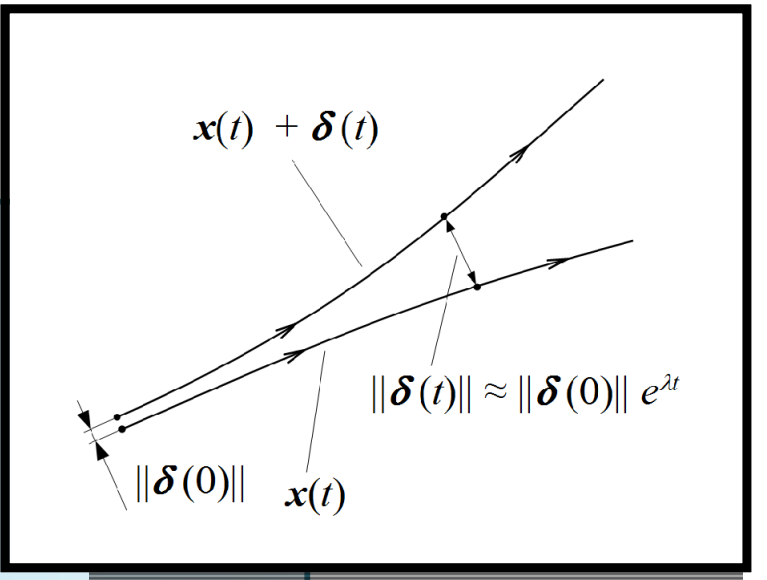
For $k = 2$, $H = \sum v_{ij;k\ell} a_i^\dagger a_j^\dagger a_k a_\ell$.

- This model (+ chemical potential) is exactly same as the "complex SYK model" considered by Sachdev (2015)

- Replacing the complex fermions by Majorana fermions leads to the SYK model proposed by Kitaev (2015)

EGE is now resurrected in a totally different context!

Classical Chaos

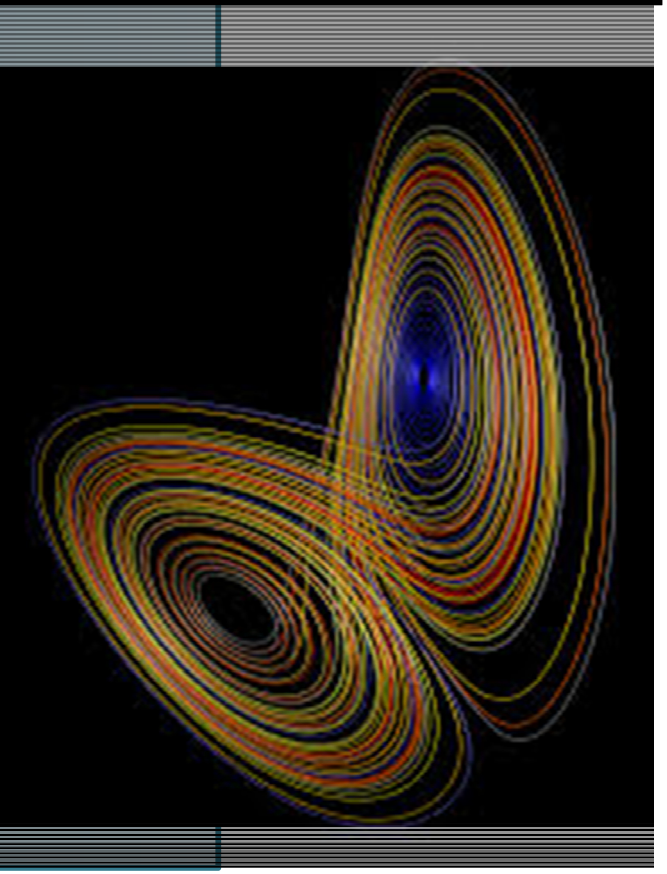


- Deterministic and bounded
- Lyapunov exponent > 0

  (= *hyperbolicity*)
- The orbit fills out all the phase space

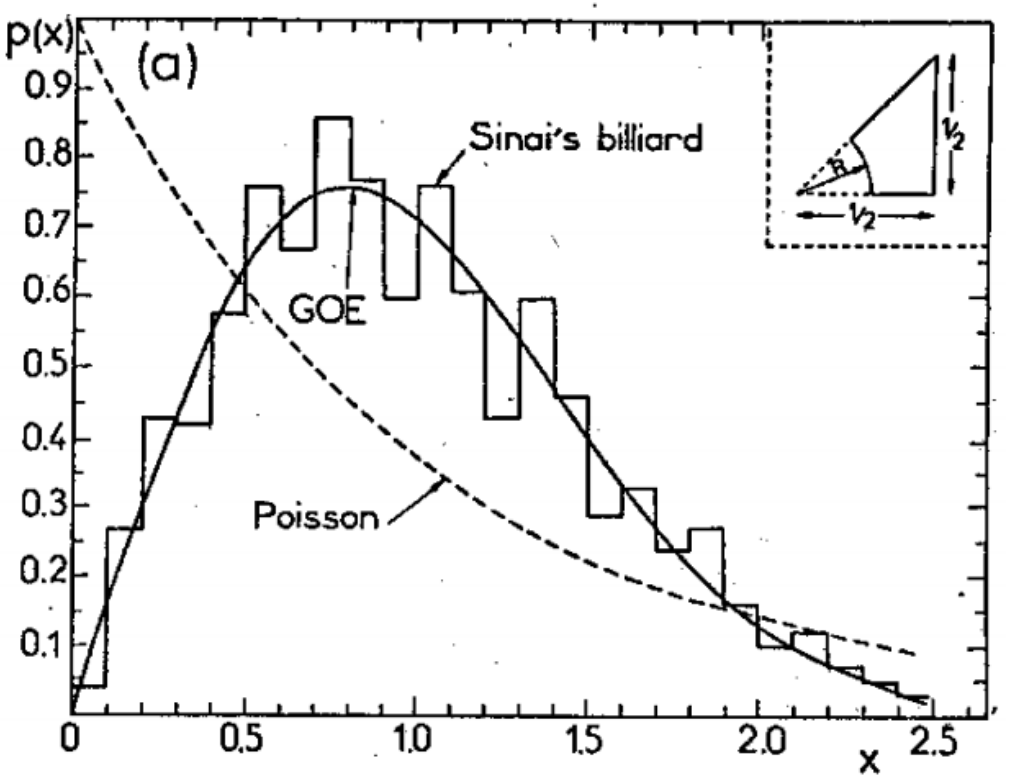
  (= *ergodicity*)
- Presence of attractors →

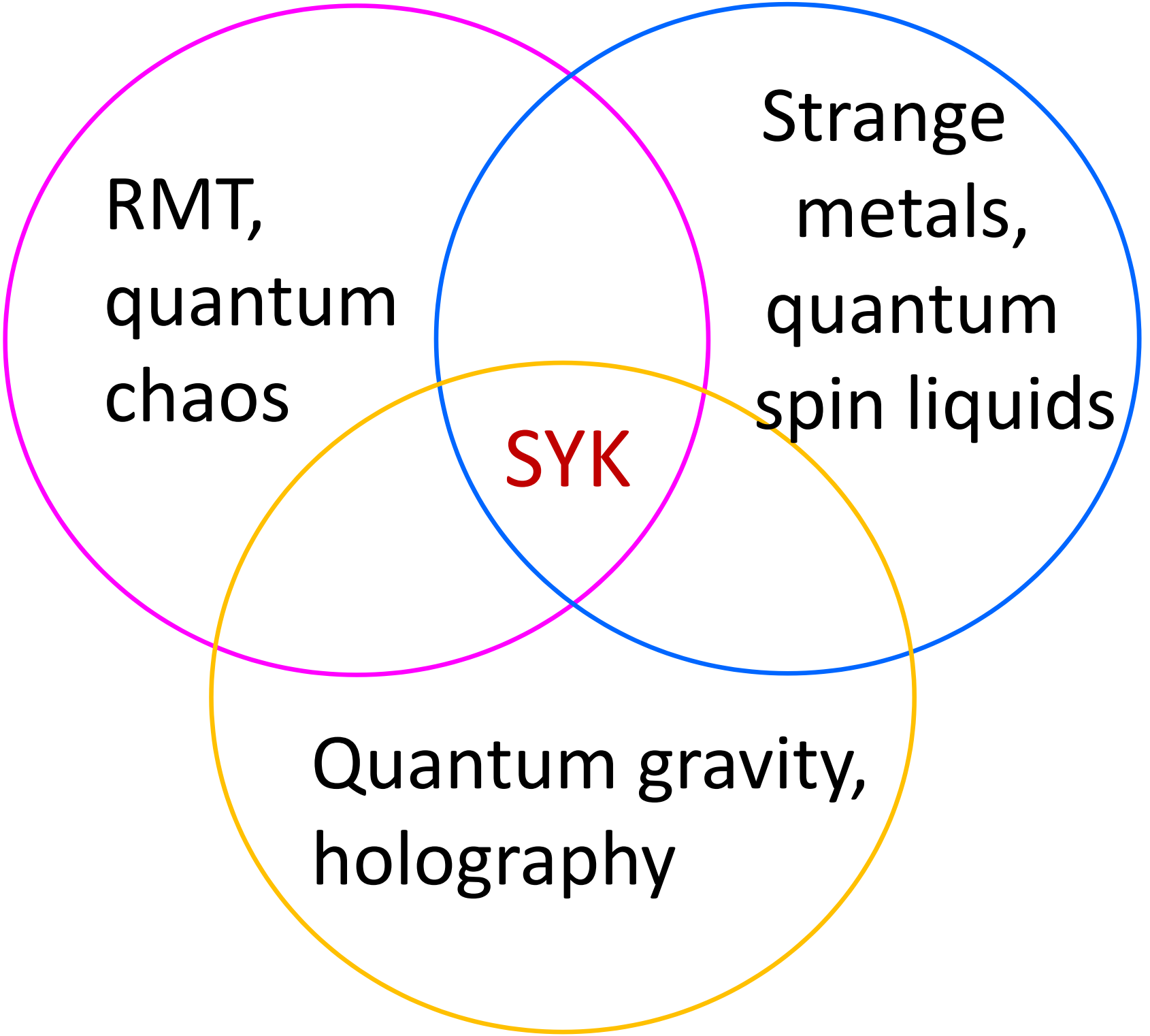
  (absent in Hamiltonian systems)



Quantum Chaos

- Level repulsion → Spectral correlations coincide with RMT (*BGS conjecture*, 1984)
- Delocalized energy eigenstates

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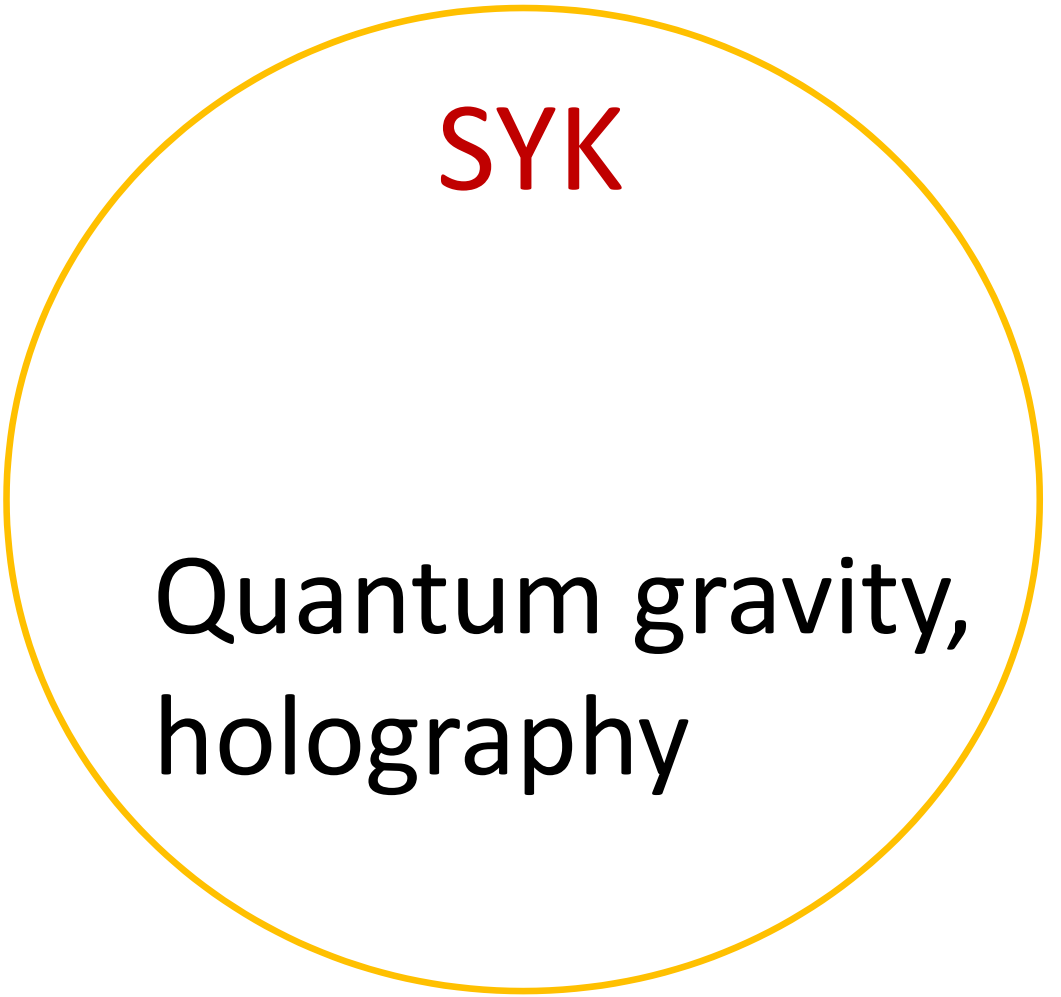
SYK
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Grand challenge

The "BH thermodynamics"

Bekenstein 1973, Hawking 1975

$$T_{\rm H} \sim \frac{1}{GM}$$

$$S_{\rm BH} \sim \frac{A}{G}$$

$(A \sim G^2 M^2$: area of the event horizon$)$

Grand challenge

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!

classical GR

How to reconcile them?

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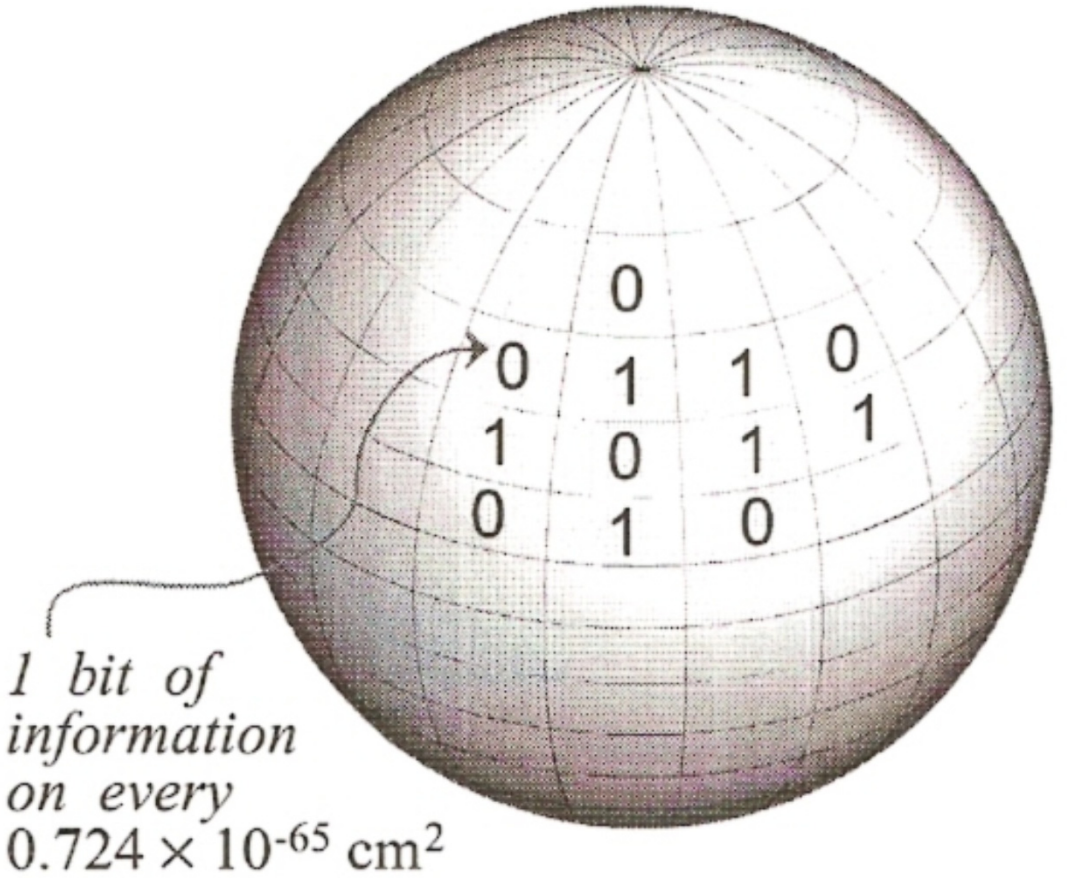
!

classical GR

How to reconcile them?

Holographic principle

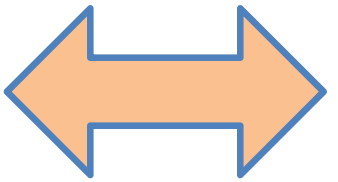
't Hooft 1993, Susskind 1995



AdS/CFT correspondence

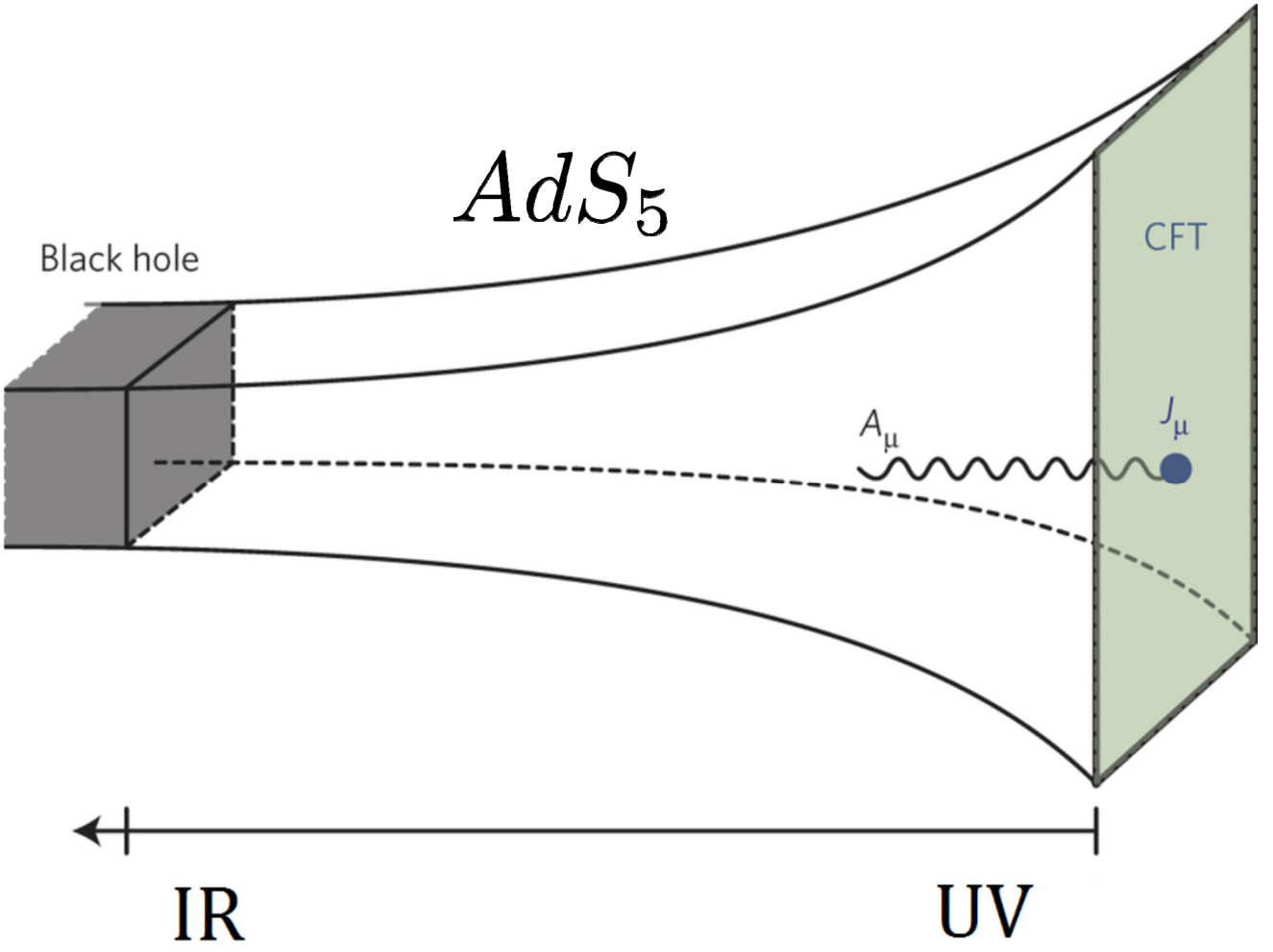
Maldacena '97, Gubser et al. '98, Witten '98

- It stipulates the duality

Quantum gravity in (4+1)-dim.  Gauge theory *without* gravity in (3+1)-dim.

- Similar to the duality in the Ising model and electromagnetism ($\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$), but now *the two theories live in different dimensions!*

- In modern interpretation, the 5-th dimension is not a "hidden extra dimension" of our universe, but rather the energy scale of QFT living on the 4-dim. boundary.



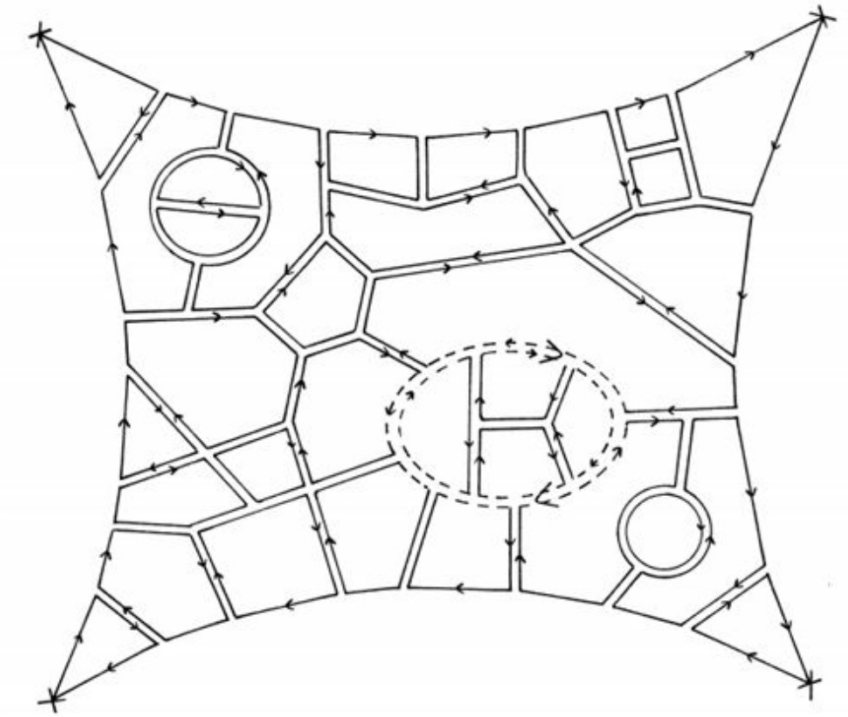
Requirements of (standard) AdS/CFT:

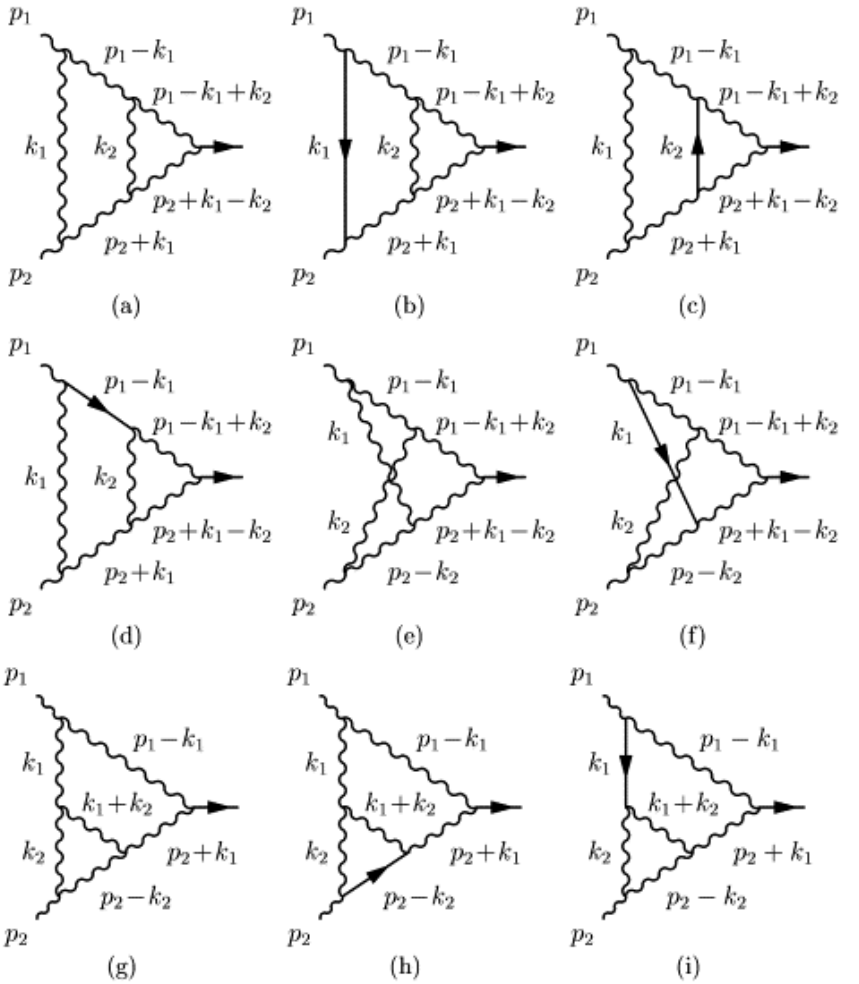
1. The boundary SU(N) gauge theory must be a conformal field theory. ($\mathcal{N}=4$ SUSY)

2. You must take both the large-N limit and the strong-coupling limit $1 \ll g_{\mathrm{YM}}^2 N \ll N$ to recover Einstein's GR. (➔ Use GR to learn QFT)

3. Quantum gravity corresponds to SU(N) gauge theory with $N \sim g_{\mathrm{YM}} \sim \mathrm{O}(1)$, but it is not solvable.

Summing over all *planar* diagrams (leading order in $1/N$ expansion) is already too hard.





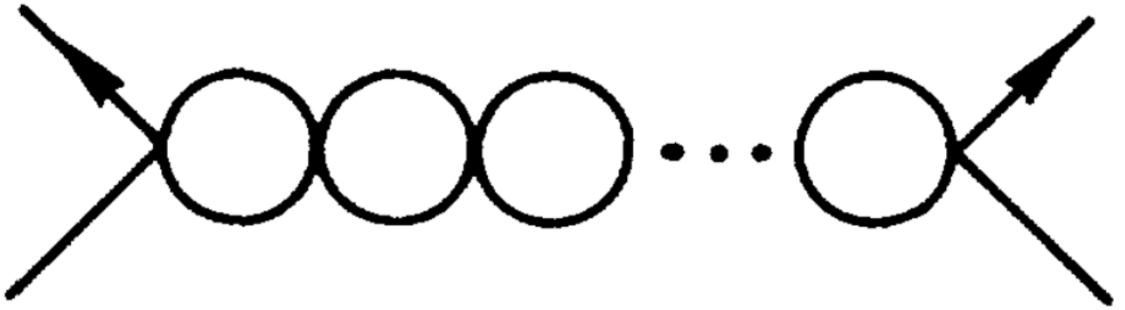
Look for a simpler duality?

Duality for O(N) vector model

WF fixed point

Klebanov & Polyakov 2002

Easily solved at large N,



but the gravity side becomes a theory with an infinite number of massless higher-spin gauge fields, known as the Vasiliev theory.

Duality in *lower* dimensions

- AdS_3/CFT_2 ← very successful!
  Strominger (hep-th/9712251) reproduced the Bekenstein-Hawking entropy of BH by counting states of CFT on the boundary.
  (This is exceptional, because the conformal group is infinite-dimensional only in 2d.)

- AdS_2/CFT_1 ← *puzzling* for a long time…

$\rho(E) \propto \delta(E)$ or $\frac{1}{E}$? IR singular, or topological…

AdS_2 is sick. (unstable to fragmentation, no finite-energy excitation, etc.)

Maldacena et al., JHEP 02 (1999) 011

- Fundamental aspects of holography remain mysterious

- A pedagogical toy model (like a hydrogen atom in QM) which is analytically solvable and nontrivial enough, is needed

Conditions for the existence of a gravity dual ??

◆ List of qualifications (tentative)

1. Conformally invariant (gapless).
2. Large-N limit can be taken.
3. Strongly coupled.
4. Extensive entropy at $T = 0$. $\cdots\cdots$(★)
5. Fastest information scrambler. (New)

(★)$\cdots\cdots$The Reissner-Nordstrøm-AdS BH, which corresponds to a finite charge-density system, retains its event horizon even as $T \to 0$, so the entropy S_{BH} is nonzero.

◆ List of qualifications (tentative)

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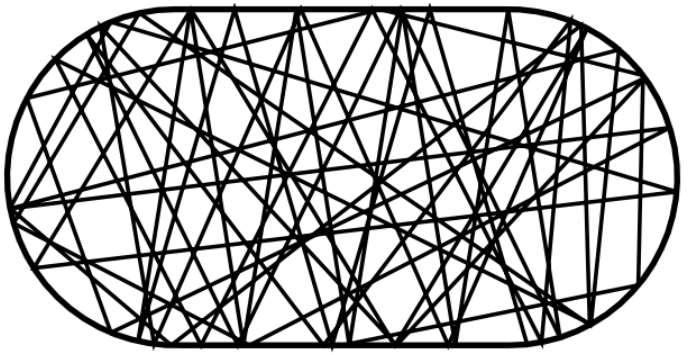
(6. Analytically solvable.)

Very very restrictive !

Quantum Chaos

- Level repulsion
- Delocalized energy eigenstates



Is BH quantum chaotic?

- No Hamiltonian for BH is available…
- Use *Quantum Information theory* and holography to infer the interior of BH

Preskill, Hayden,Susskind, Shenker, Stanford, Roberts, …

1. "Dissipation time" (phase coherence time)

Bounded as $\tau > \frac{\hbar}{k_B T}$. Temporal correlators decay over this time scale. ($\gtrsim$ "width of the quasi-particle pole" $\sim$ "thermalization time"; For insulators with a gap Δ, $\tau \sim \exp\left(\frac{\Delta}{kT}\right)$; for a Fermi liquid, $\tau \sim 1/T^2$)

S. Sachdev, *Quantum Phase Transitions*, Cambridge (1999)

2. "Information scrambling time"

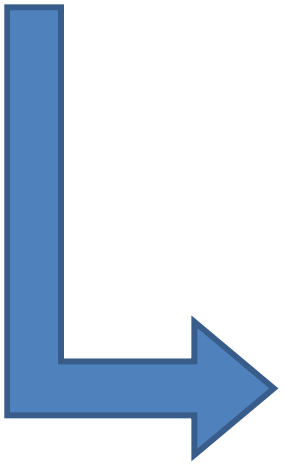
The time needed for localized quantum information to spread over the entire system ;

i.e., the time beyond which information of the initial state *cannot* be fully recovered by any *local* measurement.

Unitary time evolution

⇓

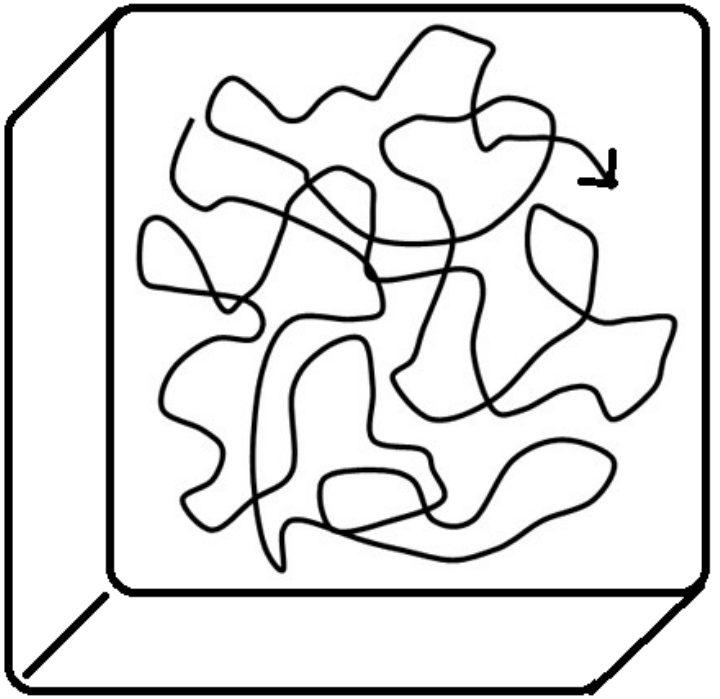
Information is not lost, just scrambled.



Intuitive picture of BH

In disordered metals, this time scale is related to the Thouless energy directly.

For $\frac{1}{t} \sim E \ll \frac{D}{L^2}$, the particle wander around the entire sample of size L . The wave function is delocalized and 0-dimensional RMT applies.



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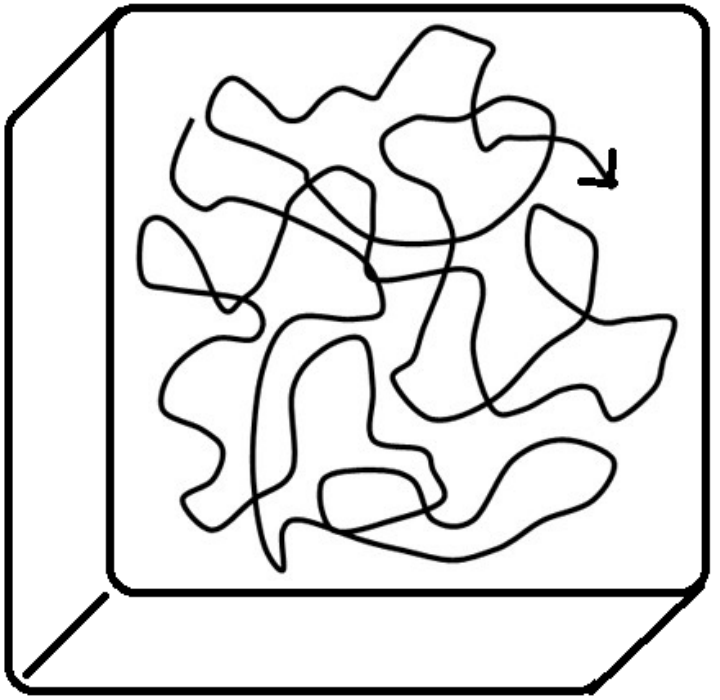


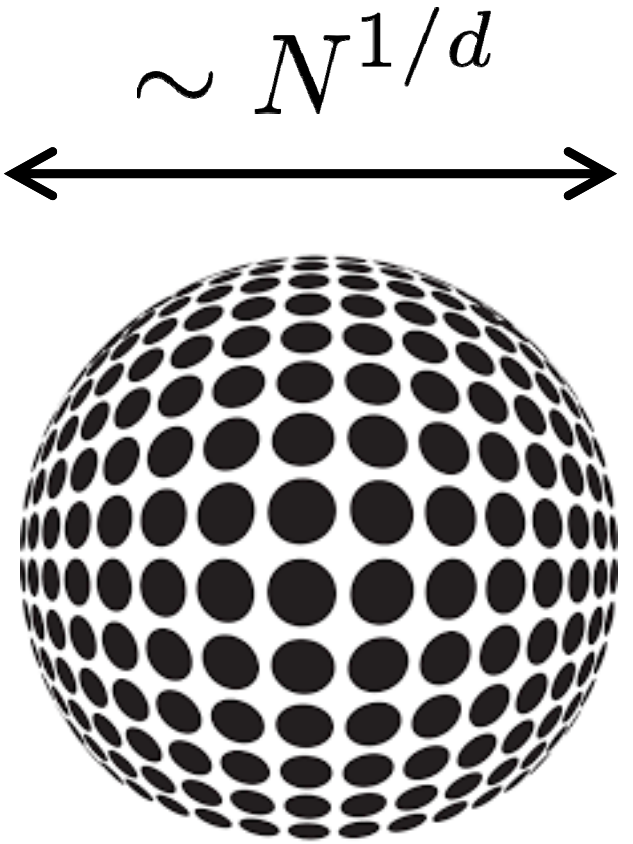
$$\#\ \text{Total d.o.f.} \equiv N \propto L^d$$

$$\downarrow$$

t_* (scrambling time) $\propto L^2 \propto N^{2/d}$.

◆ If the communication between d.o.f. is not diffusive but *ballistic*,

t_* will be pushed down to $N^{1/d}$.



◆ This looks like the lower bound for any real materials with short-range interactions:

$$t_* \gtrsim \beta \cdot N^{1/d}. \qquad (\beta = 1/k_B T.)$$

◆ Caution: t_* may not be well-defined for integrable systems (→ recurrences) and few-body systems.

Fast scrambling conjecture

Sekino & Susskind 2008

cf. Susking 2011
Shenker & Stanford 2014

(1) BH's scrambling time is given by

$$t_* \sim \frac{\beta}{2\pi} \log S = \frac{\beta}{2\pi} \log N \,.$$

(2) BH is the *fastest information scrambler* in the Universe.

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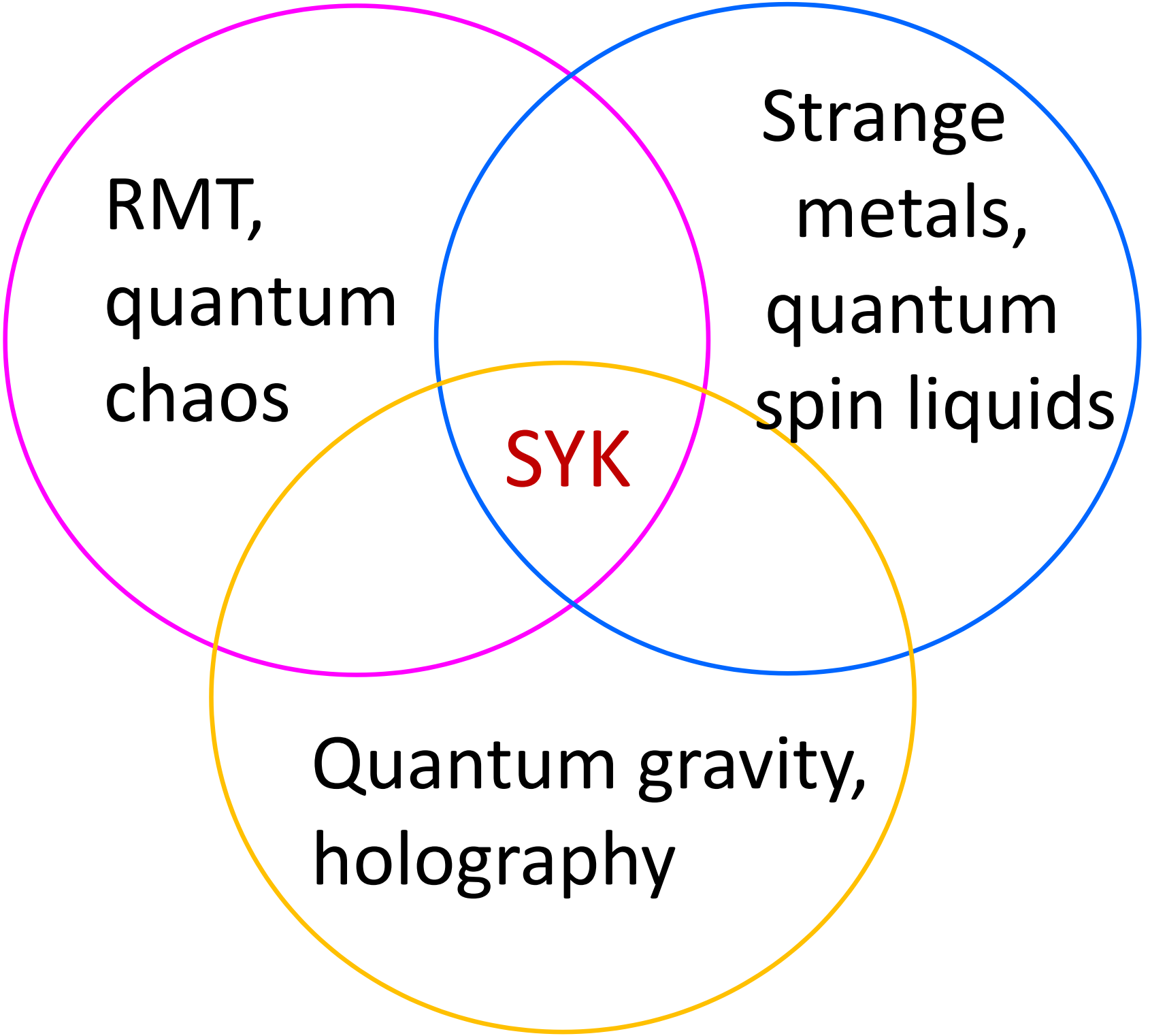
Remarks ✓ No well-defined quasiparticle

✓ Under physically plausible conditions on the Hamiltonian, this is likely to be the true lower bound. Lashkari et al. 2013

✓ If this bound is violated, then one of the fundamental principles of QM ("no-cloning theorem") is also violated.

Hayden & Preskill 2007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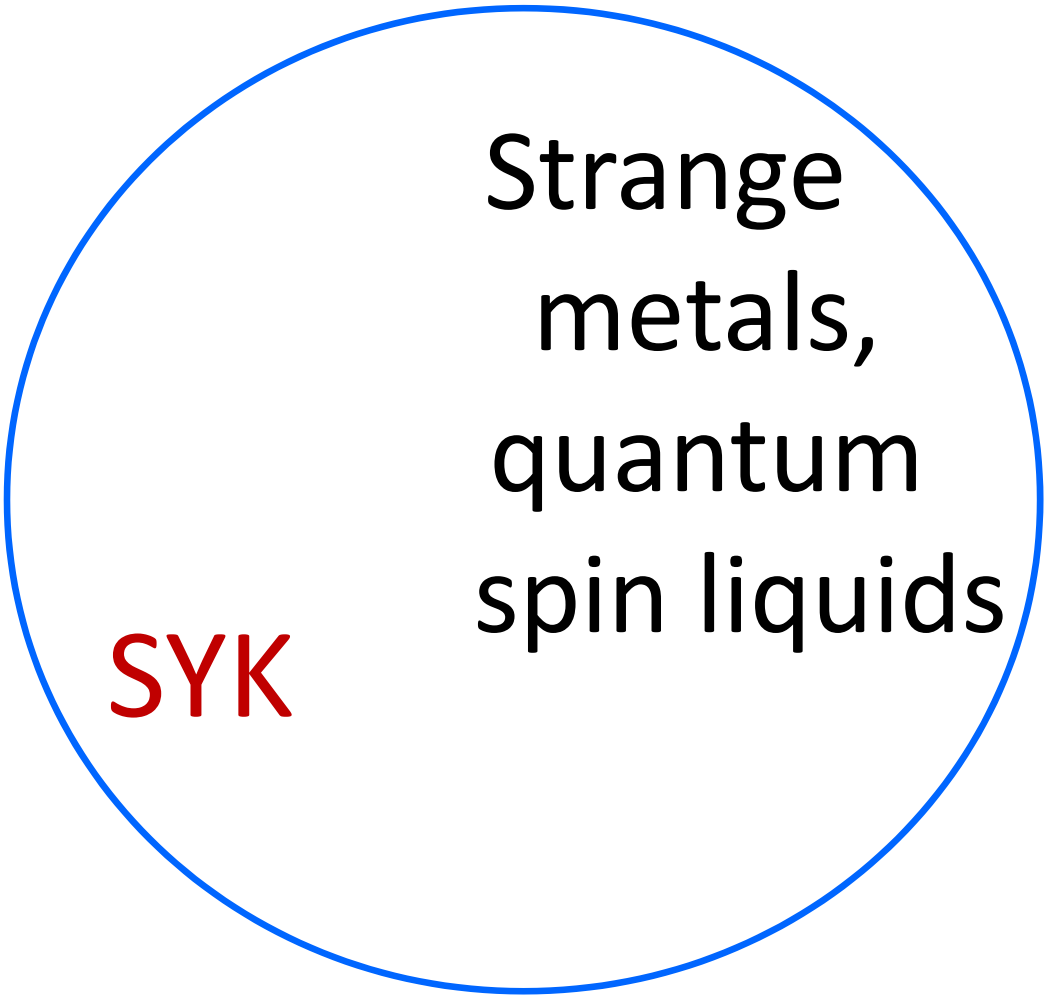
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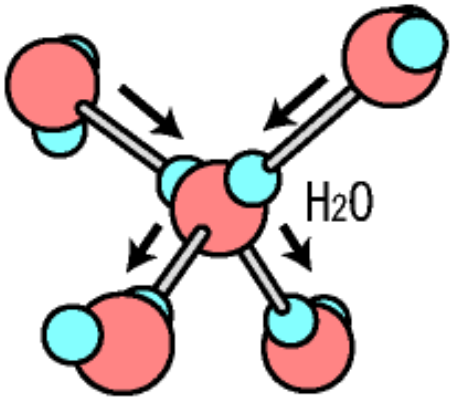
Macroscopic residual entropy ??

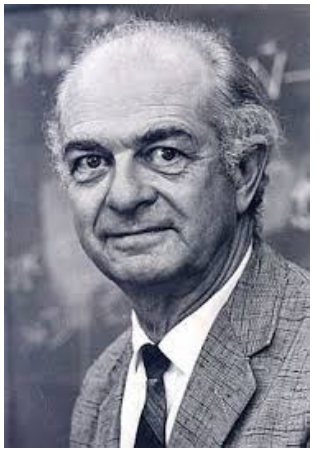
- Example 1) water ice



Entropy per Oxygen:

$$\lim_{T\to 0} \frac{S}{k_{\mathrm{B}} N} \simeq \frac{1}{2} \log \frac{3}{2}$$

Pauling, 1935



- Example 2) spin glass (frustration)

Metals with magnetic impurities

e.g. $Cu_{1-x}Fe_x$, $Au_{1-x}Mn_x$ $(x \ll 1)$

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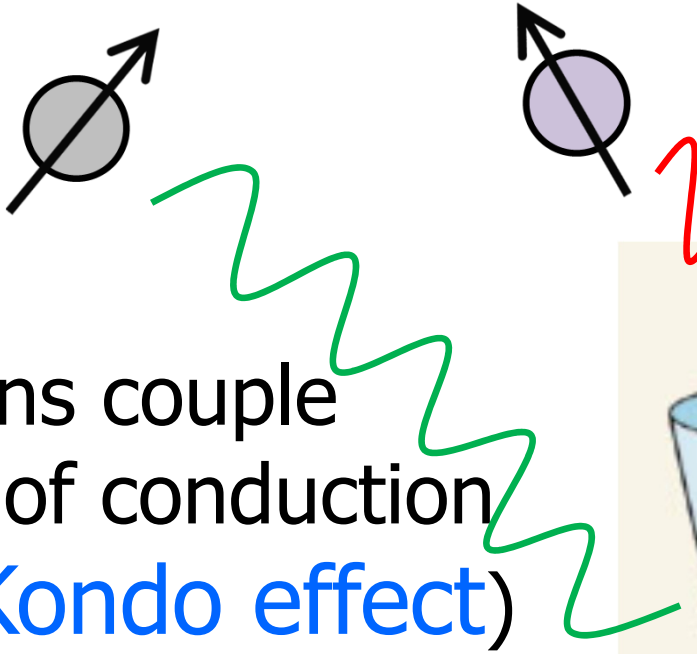
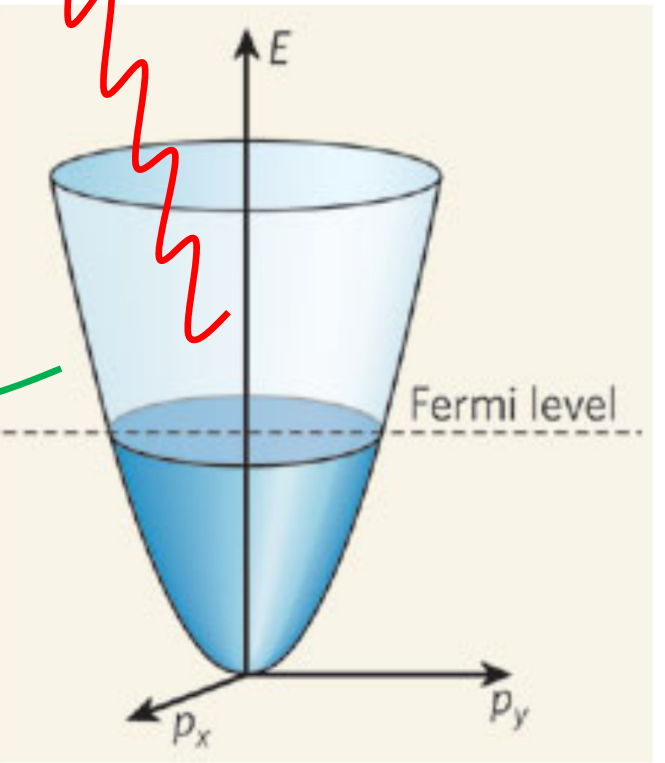


(1) Impurity spins couple to the spins of conduction electrons. (Kondo effect)



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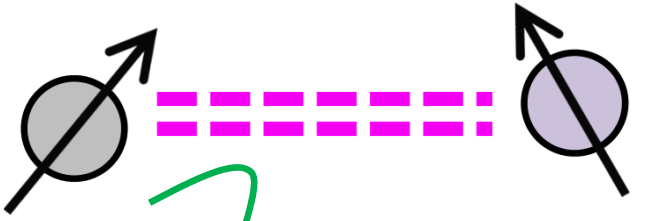

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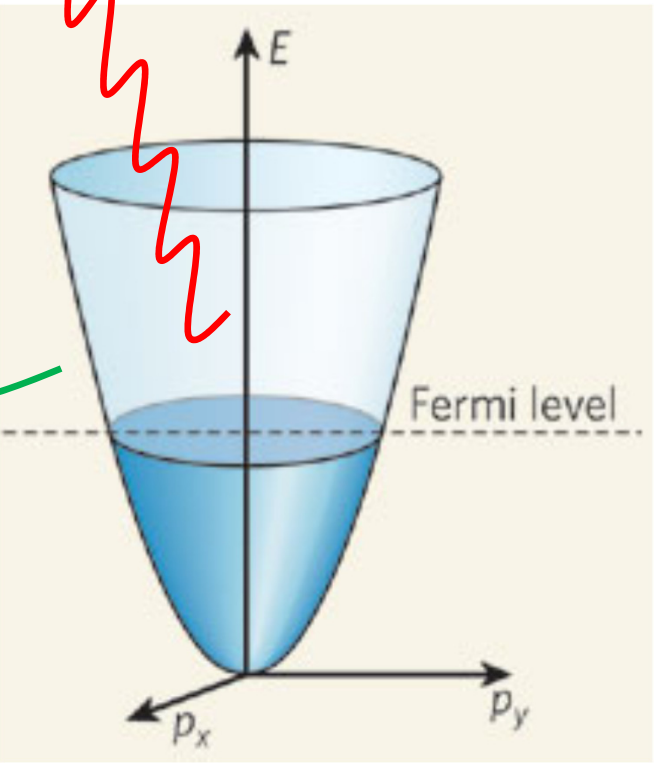
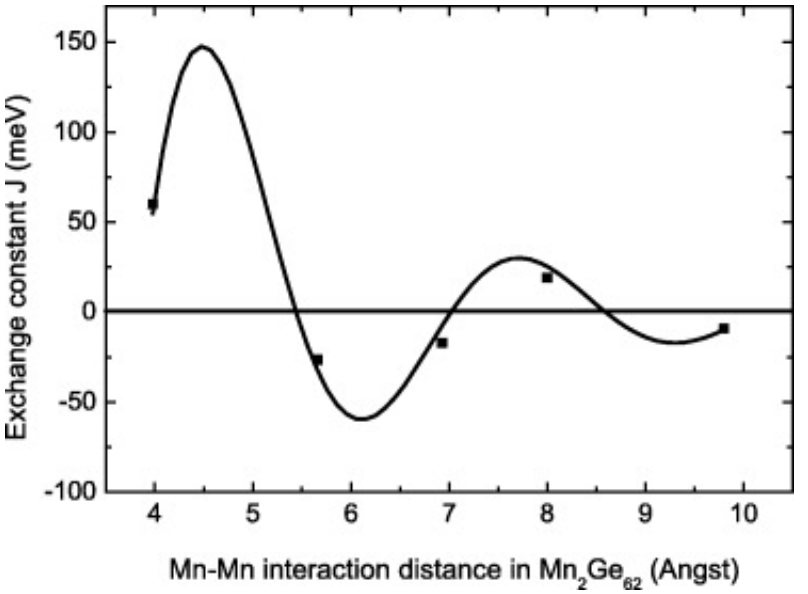
(2) RKKY interaction acts between impurities.



RKKY interaction

$$J_{ij} \sim \frac{\cos(2k_{\mathrm{F}} r_{ij})}{r_{ij}^3}$$



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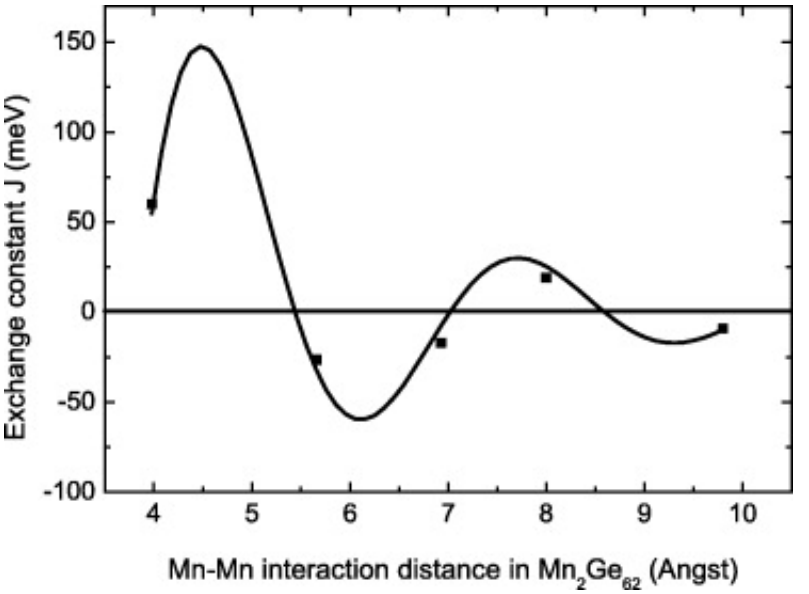


Let's replace it with a **random** variable!
(quenched disorder)

$$\exp[-\mathscr{F}(\mathscr{J})/kT] = \int \exp[+\sum \mathscr{J}_{ij} \boldsymbol{s}_i . \boldsymbol{s}_j] \Pi(\mathrm{d}s_i)$$

$$P(\mathscr{J}_{ij}) = \exp(-\mathscr{J}_{ij}^2 / 2J^2 \rho_0^2)$$

Edwards & Anderson, 1975

Classical theory

Infinite-range version (← solvable)

・・・ "SK model"

All pairs

$$\mathcal{H} = -\frac{1}{2}\sum_{i \neq i} J_{ij} S_i S_j, \quad S_i = \pm 1,$$

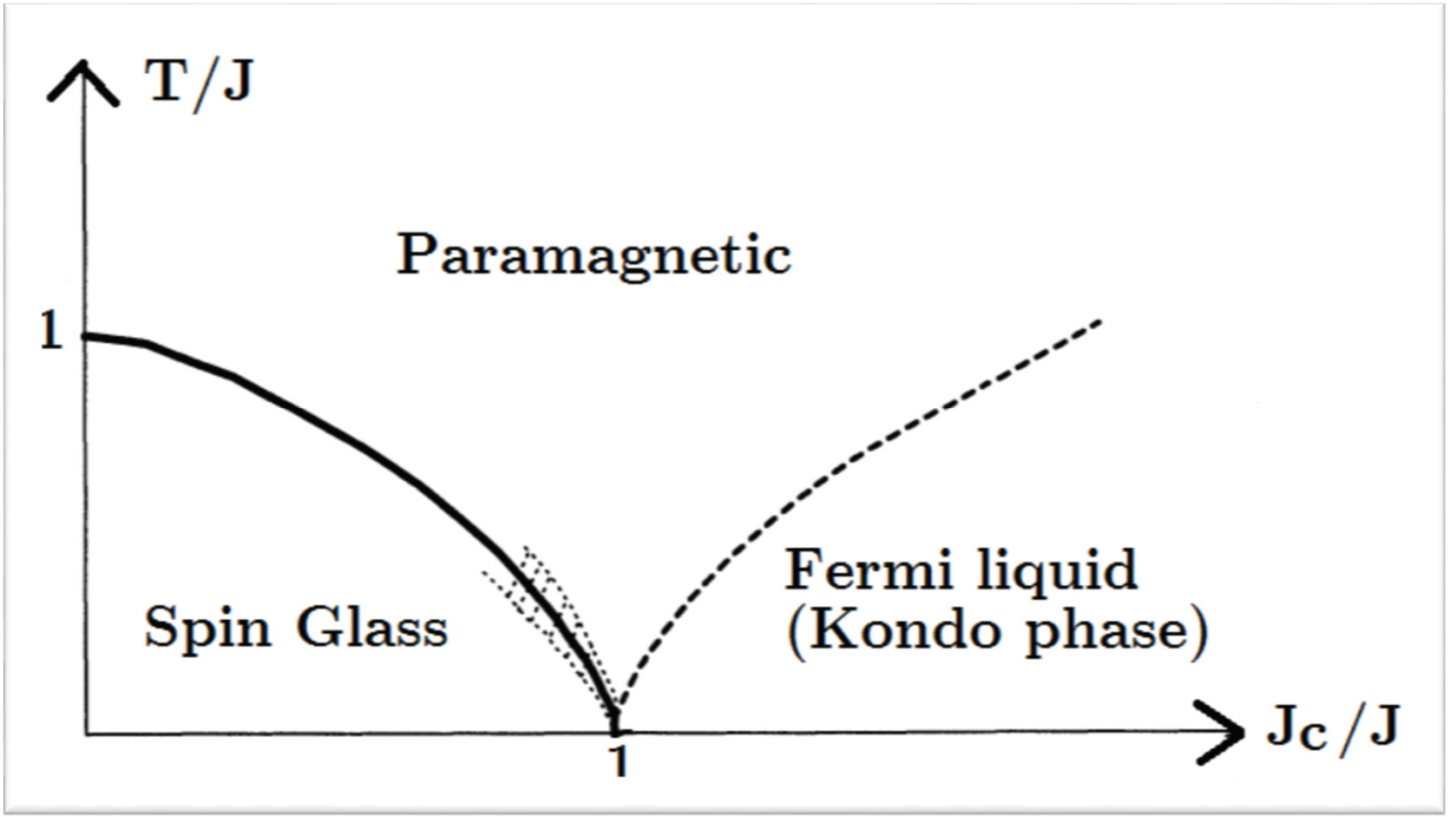
J_{ij} distributed according to

$$p(J_{ij}) = [(2\pi)^{1/2} J]^{-1} \exp[-(J_{ij} - J_0)^2 / 2J^2]$$

Classical theory Sherrington & Kirkpatrick 1975

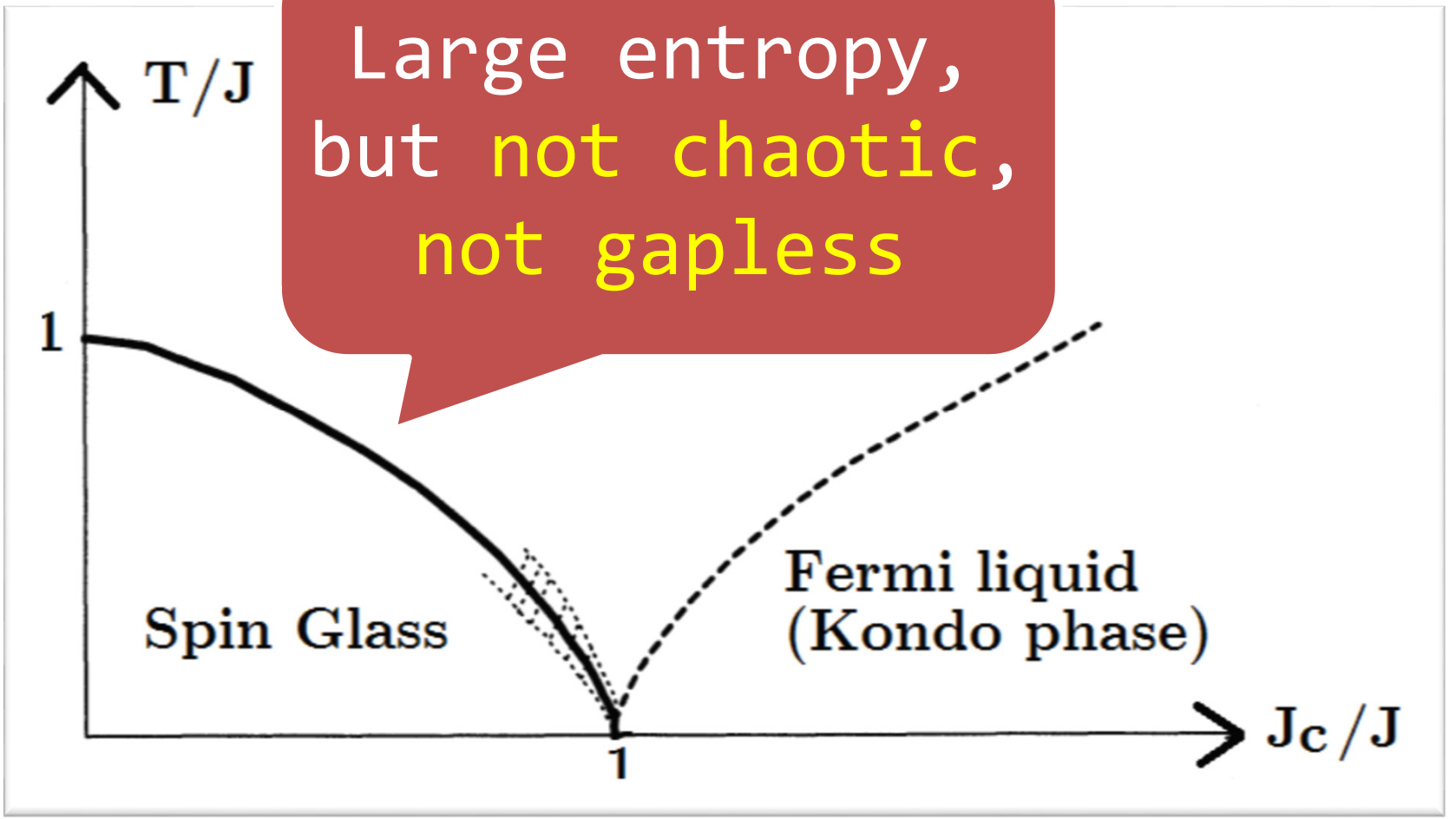
It competes with the Kondo effect.

Typical phase diagram



cf. Sengupta and Georges, PRB 52 (1995) 10295

Typical phase diagram



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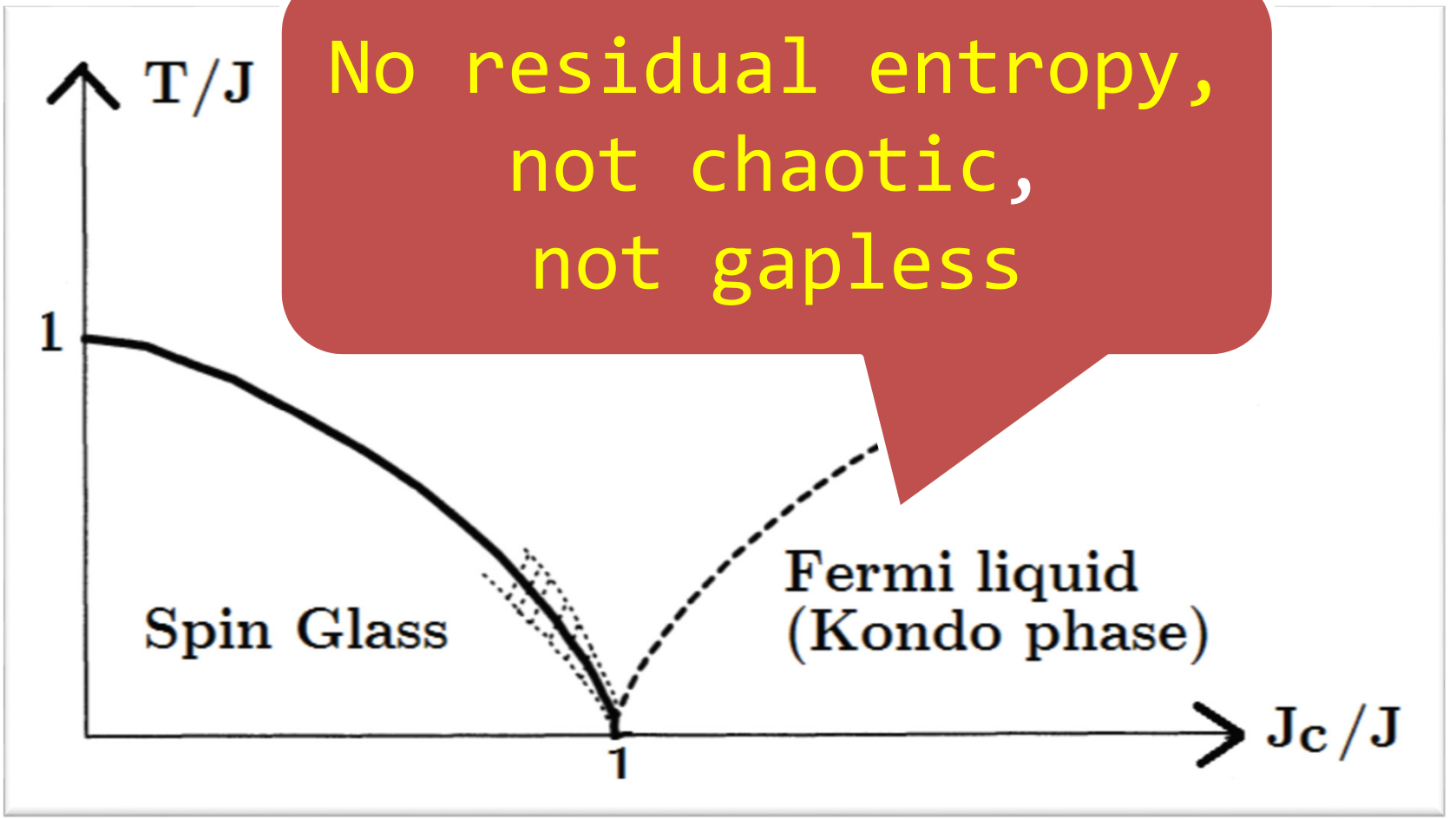
Typical phase diagram



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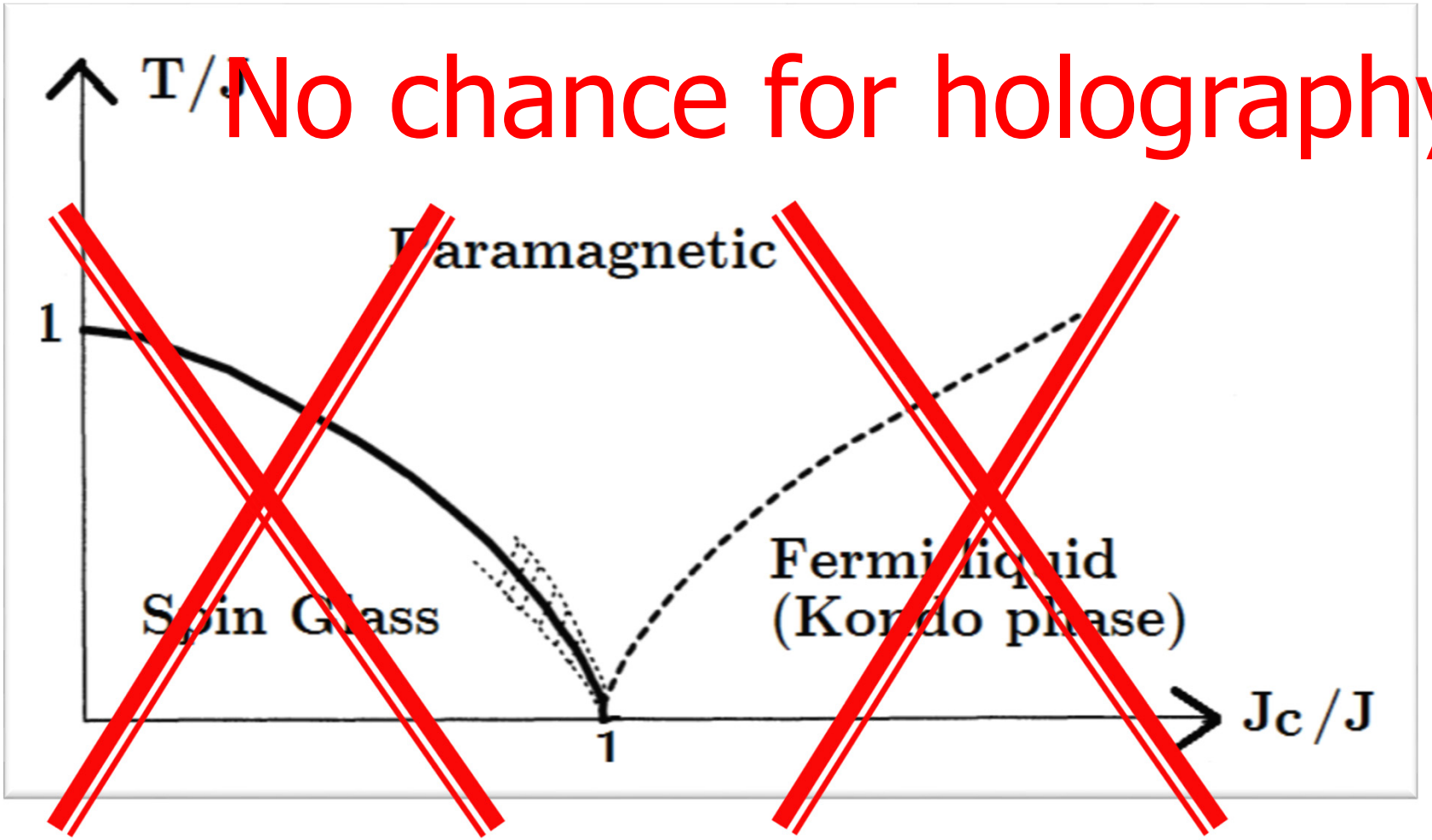
Typical phase diagram



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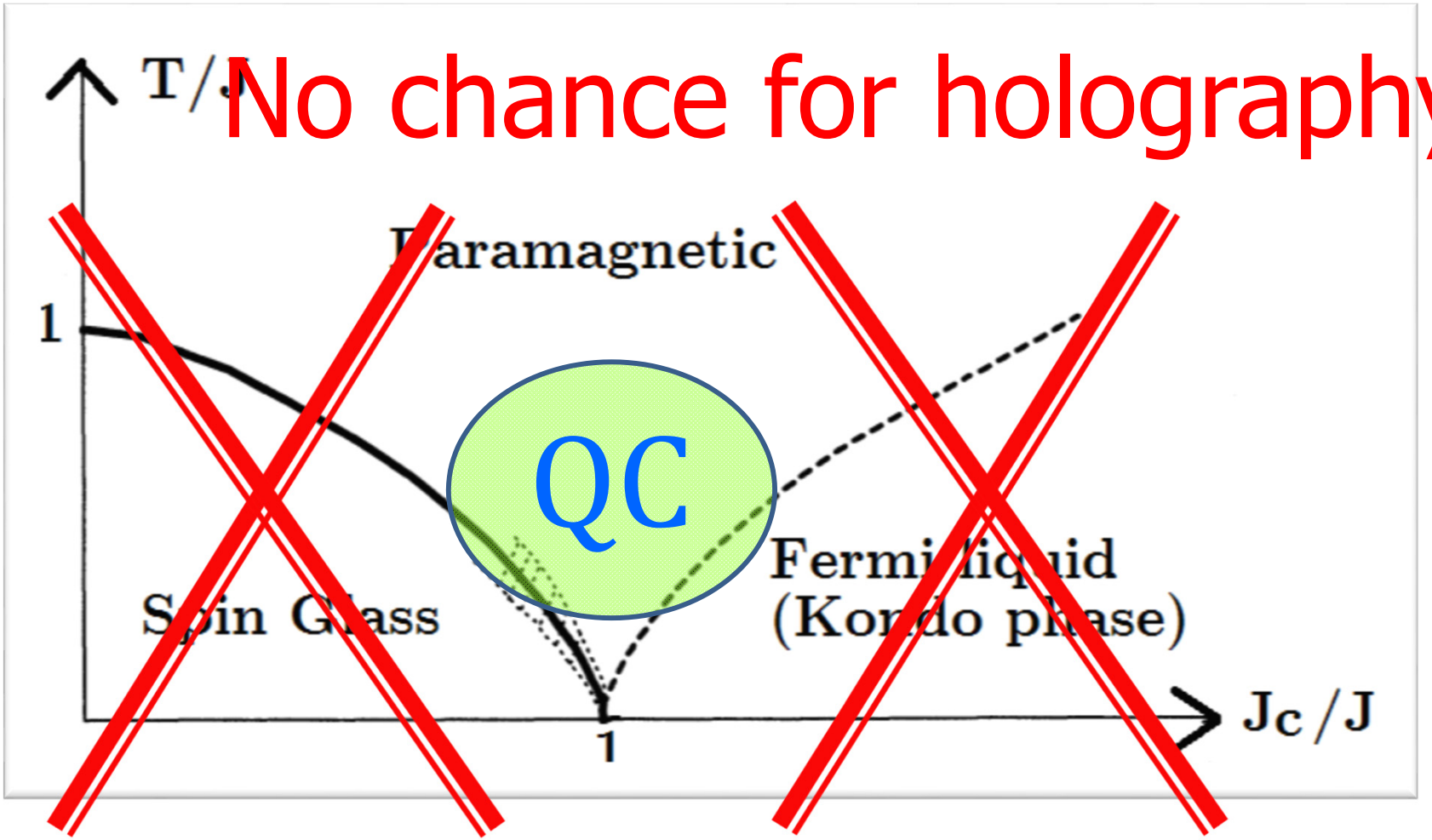
Typical phase diagram

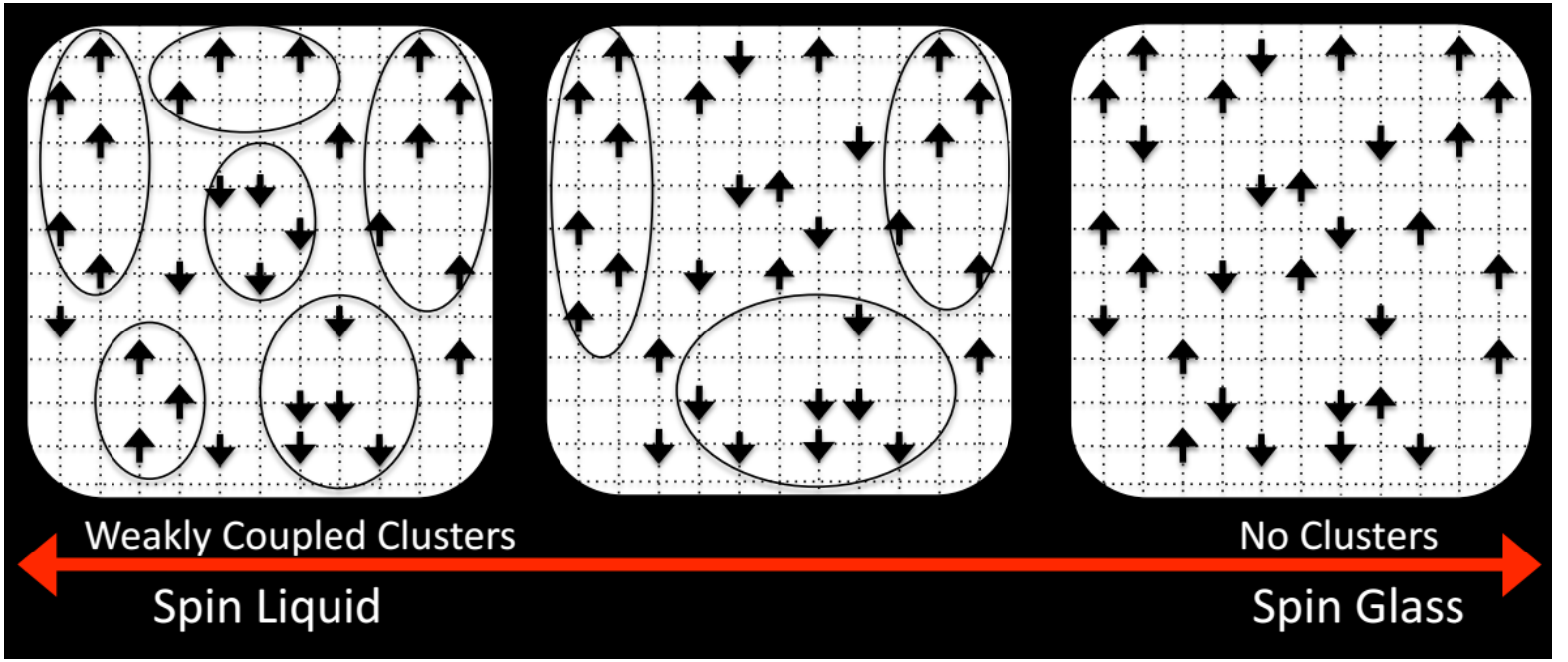


cf. Sengupta and Georges, PRB 52 (1995) 10295

Quantum Spin Liquid

Anderson 1973

- No magnetic order down to $T = 0$
- Large residual entropy
- Spins not frozen (fluctuation persist)
- Gapless excitation possible
- Short-range entanglement



T. F. Rosenbaum

Quantum Spin Liquid

Anderson 1973

Candidate materials (Balents 2010)

Table 1 | Some experimental materials studied in the search for QSLs

Material	Lattice	S	Θ_{CW} (K)	R*	Status or explanation
κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$	Triangular†	½	−375‡	1.8	Possible QSL
EtMe$_3$Sb[Pd(dmit)$_2$]$_2$	Triangular†	½	−(375–325)‡	?	Possible QSL
Cu$_3$V$_2$O$_7$(OH)$_2$•2H$_2$O (volborthite)	Kagomé†	½	−115	6	Magnetic
ZnCu$_3$(OH)$_6$Cl$_2$ (herbertsmithite)	Kagomé	½	−241	?	Possible QSL
BaCu$_3$V$_2$O$_8$(OH)$_2$ (vesignieite)	Kagomé†	½	−77	4	Possible QSL
Na$_4$Ir$_3$O$_8$	Hyperkagomé	½	−650	70	Possible QSL
Cs$_2$CuCl$_4$	Triangular†	½	−4	0	Dimensional reduction
FeSc$_2$S$_4$	Diamond	2	−45	230	Quantum criticality

BEDT-TTF, bis(ethylenedithio)-tetrathiafulvalene; dmit, 1,3-dithiole-2-thione-4,5-ditholate; Et, ethyl; Me, methyl. *R is the Wilson ratio, which is defined in equation (1) in the main text. For EtMe$_3$Sb[Pd(dmit)$_2$]$_2$ and ZnCu$_3$(OH)$_6$Cl$_2$, experimental data for the intrinsic low-temperature specific heat are not available, hence R is not determined. †Some degree of spatial anisotropy is present, implying that $J' \neq J$ in Fig. 1a. ‡A theoretical Curie–Weiss temperature (Θ_{CW}) calculated from the high-temperature expansion for an $S = ½$ triangular lattice; $\Theta_{CW} = 3J/2k_B$, using the J fitted to experiment.

Still hypothetical (?)

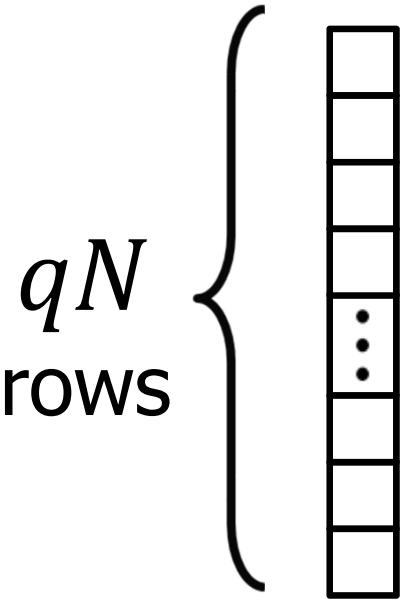
Sachdev-Ye (SY) model (1993)

= Quantum SK model (infinite-range)

$$\mathcal{H} = \frac{1}{\sqrt{NM}} \sum_{i>j} J_{ij} \hat{\mathcal{S}}_i \cdot \hat{\mathcal{S}}_j,$$

$\hat{S}_i$: SU(N) spin

Young tableau



$0 < q < 1$

Sachdev-Ye (SY) model (1993)

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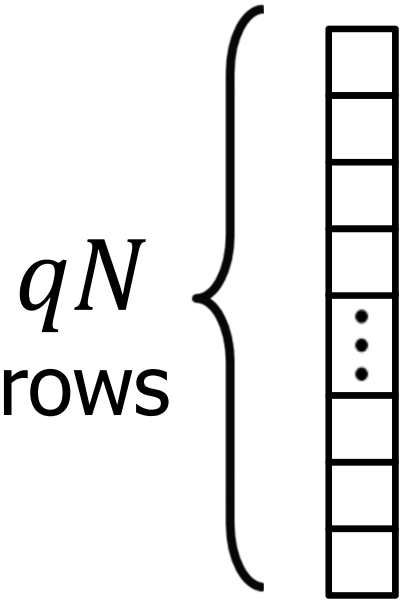

$0 < q < 1$

Gapless spin liquid is realized in the large-N limit !

◆ List of qualifications (tentative)

1. Conformally invariant (gapless).	YES
2. Large-N limit can be taken.	YES
3. Strongly coupled.	YES
4. Extensive entropy at $T = 0$.	YES
5. Fastest information scrambler. (New)	?
(6. Analytically solvable.)	YES

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1. Conformally invariant (gapless).	YES
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5. Fastest information scrambler. (New)	?
(6. Analytically solvable.)	YES

SY model is almost perfect.

➔ Gravity dual exists! Sachdev 2010

SY+Kitaev (SYK) model (2015)

$$H = \sum_{1 \le i < j < k < l \le N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l$$

Majorana fermions $\{\chi_i, \chi_j\} = \delta_{ij}$

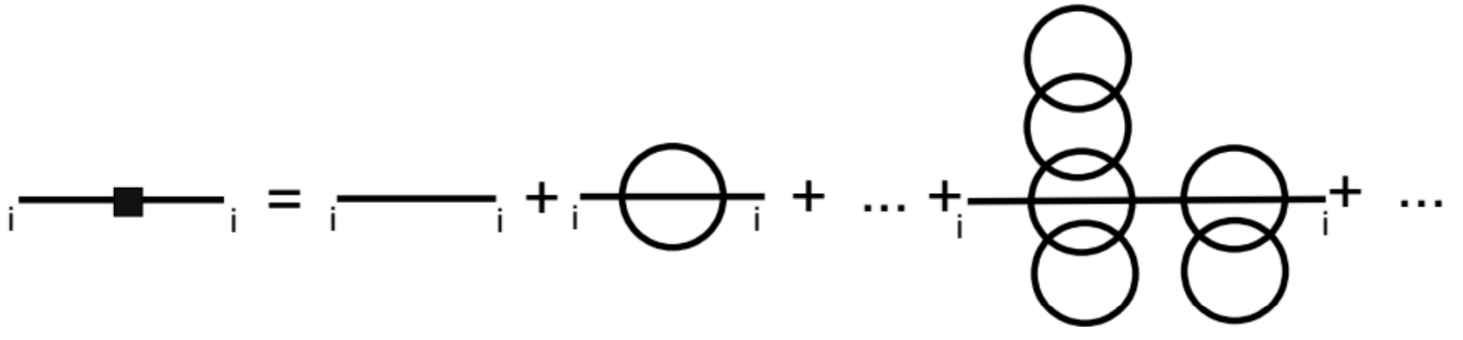
Quenched disorder

$$P(J_{ijkl}) \sim \exp\left(-12 N^3 J_{ijkl}^2 / J^2\right)$$

Spatial index removed; (0+1)d QM model.

→ Numerical analysis easier

Large-N limit is solvable



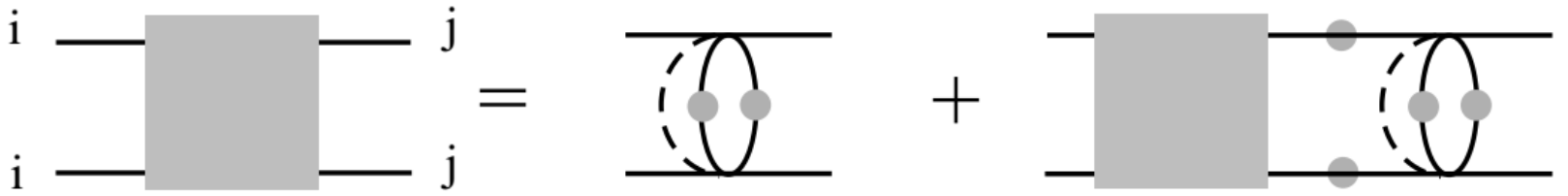
SD eq.

Kitaev found a sweet spot between O(N) vector models and Matrix models.

Polchinski & Rosenhaus 2016

SYK is a fast scrambler

4-point function at $N \gg 1$



$$t_* \sim \frac{\beta}{2\pi} \log N \quad : \quad \text{exactly same as BH}$$

A unique model which is chaotic (non-integrable) but analytically solvable.

◆ List of qualifications (tentative)

1. Conformally invariant (gapless).	YES
2. Large-N limit can be taken.	YES
3. Strongly coupled.	YES
4. Extensive entropy at $T = 0$.	YES
5. Fastest information scrambler. (New)	YES
(6. Analytically solvable.)	YES

SYK model is perfect.

➔Gravity dual Maldacena, Stanford, Sachdev, Gross, Polchinski, …

RMT and Supersymmetric SYK

T. Kanazawa **(speaker)**

T. Wettig

JHEP 09 (2017) 050
arXiv:1706.03044

Osaka University,
Oct 31, 2017

NOTATION

- $\{A, B\} := AB + BA$
- $[A, B] := AB - BA$

- N_c ⋯ # Complex fermions
- N_m ⋯ # Majorana fermions

- q ⋯ Even integer (# fermions in the Hamiltonian H)
- $\hat{q}$ ⋯ Odd integer (# fermions in the Supercharge Q)

"SPACETIME" SUPERSYMMETRY

- Extension of the Poincaré group

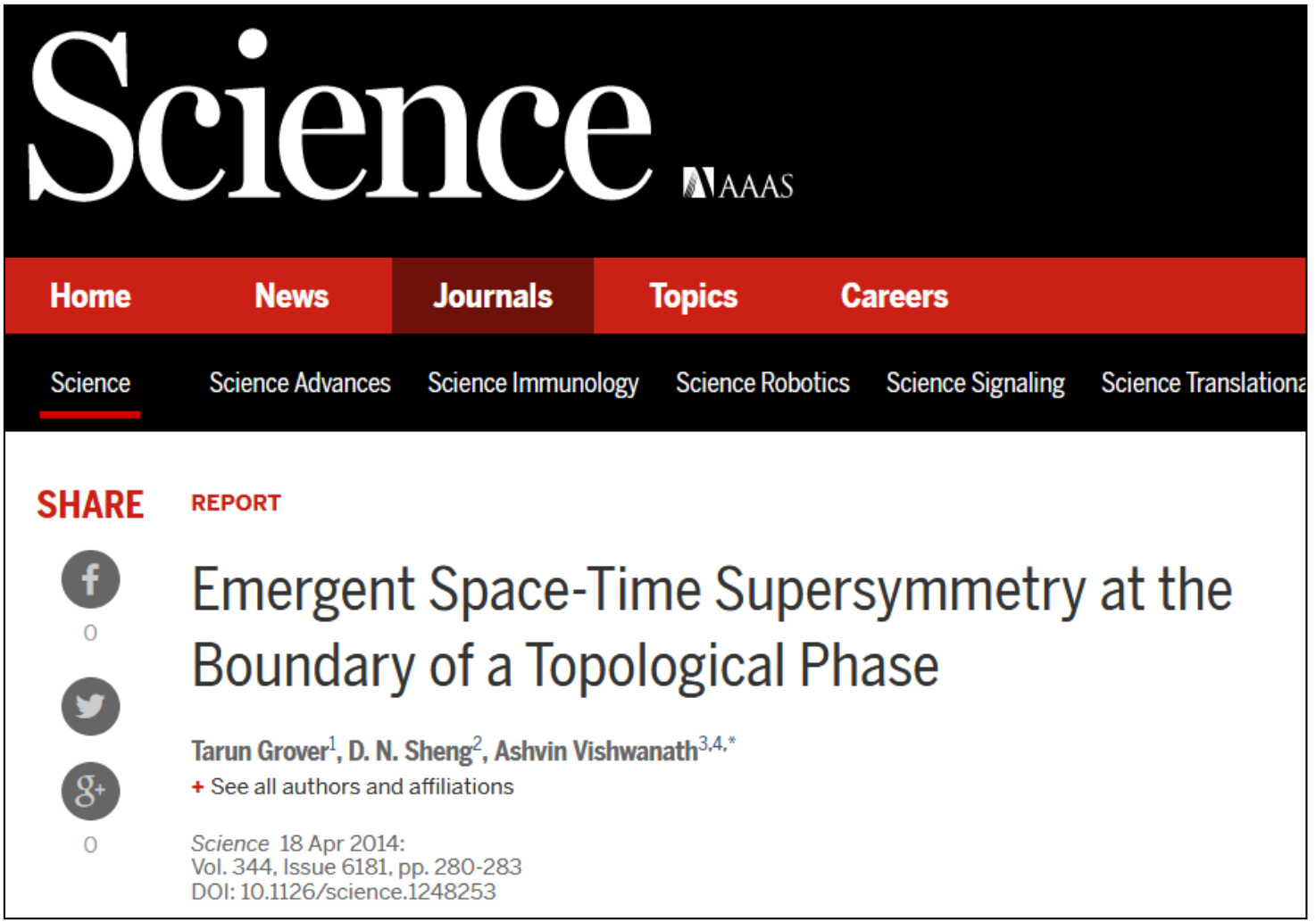
$$\mathbb{R}^{1,3} \rtimes \mathrm{SO}(1,3)$$

- Transformation between bosons and fermions
- Supercharge algebra

$$\{Q_\alpha, \overline{Q}_\beta\} = (\sigma^\mu)_{\alpha\beta} P_\mu$$

- Cancellation between bosons and fermions
  ➔ quantum fluctuation suppressed
- Hard to realize on a lattice: $Q \sim \sqrt{\partial_\mu}$
- May solve deep problems in HEP

Condensed matter physicists are making progress…



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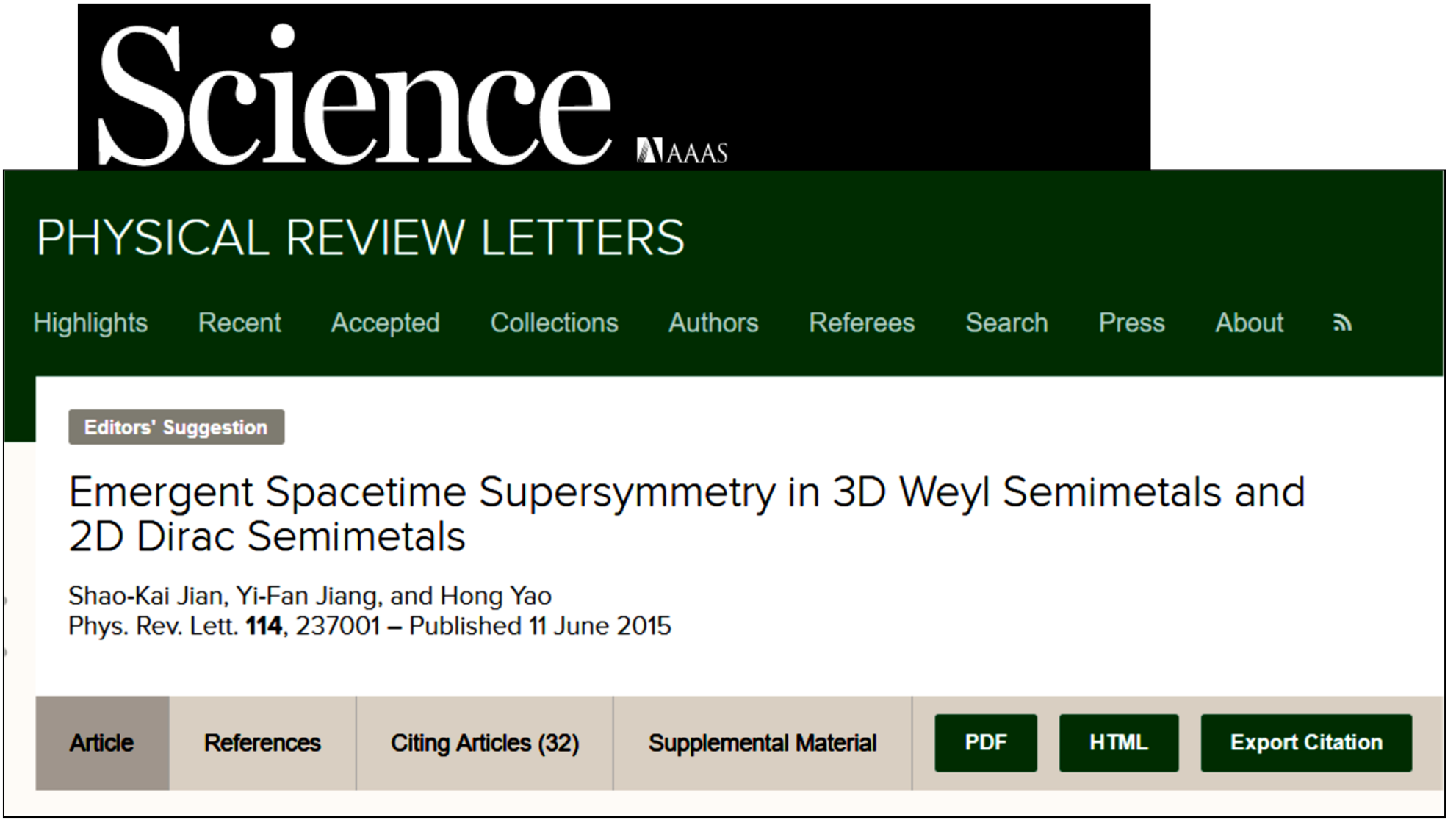


SUSY QUANTUM MECHANICS

- Let H a Hamiltonian acting on some Hilbert space V
- Assume the existence of Hermitian operators

$$Q_n = Q_n^\dagger \ , \qquad n = 1,2,\dots,N$$

that also act on V

- N-extended SUSY is defined by the algebra

$$\{Q_i, Q_j\} = 2\delta_{ij} H$$

- $N = 1$:

$$H = (Q_1)^2 .$$

$\mathrm{Spec}(H) \geq 0.$

SUSY QUANTUM MECHANICS

- $N = 2$: $\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2 = H$

$\downarrow$

Define $\quad Q \equiv \frac{Q_1 + iQ_2}{2} \;, \; \bar{Q} \equiv \frac{Q_1 - iQ_2}{2}$

$\downarrow$

$$H = \{Q, \bar{Q}\}, \quad Q^2 = \bar{Q}^2 = 0.$$

$\text{Spec}(H) \geq 0.$

- The first SUSY-QM model ($N = 2$) on a one-dimensional lattice

H. Nicolai, J. Phys. A 9, 1497 (1976)

SUSY QUANTUM MECHANICS

- $N = 2$: $\{Q_1, Q_2\} = 0, \quad Q_1^2 = Q_2^2 = H$

$\downarrow$

Define $\quad Q \equiv \frac{Q_1 + iQ_2}{2} \quad , \quad \bar{Q} \equiv \frac{Q_1 - iQ_2}{2}$

Supersymmetry and spin systems

H Nicolai

Institut für Theoretische Physik, Universität Karlsruhe, 75 Karlsruhe 1, Germany

Received 21 April 1976

Abstract. It is shown that supersymmetry may be applied to spin systems. A simple algebra is proposed and various examples are discussed. It is argued that certain correlation functions must vanish on account of supersymmetry.

OPERATORS

- Complex fermions $(a, b = 1, \cdots, N_c)$

$$\{c_a, c_b\} = \{\overline{c}_a, \overline{c}_b\} = 0\,, \quad \{c_a, \overline{c}_b\} = \delta_{ab}$$

- Fermion number operator $F \equiv \sum_{a=1}^{N_c} \overline{c}_a c_a\,.$

- Majorana fermions $(N_m = 2N_c)$

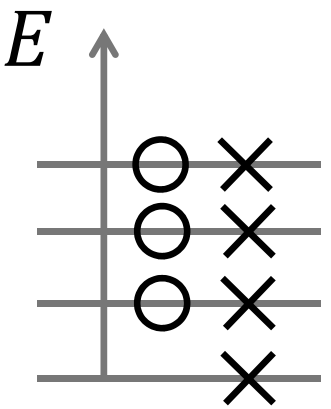
$$\chi_{2k-1} = \frac{c_k + \overline{c}_k}{\sqrt{2}}\,, \quad \chi_{2k} = \frac{c_k - \overline{c}_k}{\sqrt{2}\,i}\,, \quad \{\chi_i, \chi_j\} = \delta_{ij}\,.$$

- Complex conjugation operator K

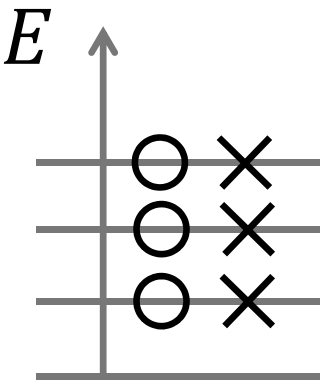
BASICS

- $N = 1$:

  SUSY-preserving ground state: $Q\psi = 0$.
  All excited states are boson-fermion paired $\left(\psi_n\,,\ \frac{1}{\sqrt{E_n}} Q\psi_n\right)$ since $[H, Q] = 0$.



  If $Q\psi = 0$ has no (normalizable) solution, SUSY is spontaneously broken.



  (In QFT, this means that, in the infinite-volume limit the charge Q is not well-defined due to IR divergence associated with the massless Goldstino.)

SYK WITH N=1 SUSY

Fu–Gaiotto–Maldacena–Sachdev 2016

$H = Q^2$ $(Q = Q^\dagger)$ with

$$Q = i^{(\hat{q}-1)/2} \sum_{1 \le i_1 < \cdots < i_{\hat{q}} \le N_m} C_{i_1 i_2 \cdots i_{\hat{q}}} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_{\hat{q}}}$$

Gaussian disorder

$$\langle C_{i_1 i_2 \cdots i_{\hat{q}}} \rangle = 0 \text{ and } \langle C^2_{i_1 i_2 \cdots i_{\hat{q}}} \rangle = \frac{(\hat{q}-1)!}{N_m^{\hat{q}-1}} J \text{ for some } J > 0 .$$

- H includes $(2\hat{q}-2), (2\hat{q}-6)$, ...-body interactions.
  (Simplest case ・・・ $\hat{q} = 3$)

SYK WITH N=1 SUSY

Fu–Gaiotto–Maldacena–Sachdev 2016

Lagrangian formulation of the same model

$$\mathcal{L} = \sum_i \left[\frac{1}{2} \psi^i \partial_\tau \psi^i - \frac{1}{2} b^i b^i + i \sum_{1 \le j < k \le N} C_{ijk} b^i \psi^j \psi^k \right]$$

SUSY transformation

$$Q\psi^i = b^i, \quad Qb^i = \partial_\tau \psi^i$$

N=0 SYK

N=1 SYK

cf. Maldacena-Stanford 2016 / Garcia-Verbaarschot 2016 / Stanford-Witten 2017

N=0 SYK

N=1 SYK

$$\log Z = -\beta E_0 + S_0 + \frac{c}{2\beta}$$

E_0: ground state energy,
S_0: zero-temperature entropy,
c/β: specific heat

cf. Maldacena-Stanford 2016 / Garcia-Verbaarschot 2016 / Stanford-Witten 2017

N=0 SYK

N=1 SYK

$$\log Z = -\beta E_0 + S_0 + \frac{c}{2\beta}$$

E_0: ground state energy,
S_0: zero-temperature entropy,
c/β: specific heat

- ✓ $\frac{E_0}{N_m} \simeq 0.044\text{J}$
- ✓ $\frac{S_0}{N_m} = \frac{1}{2}\log 2 - \int_0^{1/q} dx\, \pi \left(\frac{1}{2} - x\right) \tan \pi x$

  $= 0.2324 \ldots$ for $q = 4$
- ✓ IR: Schwarzian EFT

  → Spectral density

$$\rho(E) \propto \sinh\left(\#\sqrt{N_m \frac{E - E_0}{J}}\right)$$

cf. Maldacena-Stanford 2016 / Garcia-Verbaarschot 2016 / Stanford-Witten 2017

N=0 SYK

N=1 SYK

$$\log Z = -\beta E_0 + S_0 + \frac{c}{2\beta}$$

E_0: ground state energy,
S_0: zero-temperature entropy,
c/β: specific heat

✓ $\frac{E_0}{N_m} \simeq 0.044\mathrm{J}$

✓ $\frac{S_0}{N_m} = \frac{1}{2}\log 2$

$$- \int_0^{1/q} dx\, \pi \left(\frac{1}{2} - x\right) \tan \pi x$$

$$= 0.2324 \ldots \text{ for } q = 4$$

✓ IR: Schwarzian EFT

→ Spectral density

$$\rho(E) \propto \sinh\left(\#\sqrt{N_m \frac{E - E_0}{J}}\right)$$

✓ $\frac{E_0}{N_m} \to 0$

as $N_m \to \infty$

✓ $\frac{S_0}{N_m} = \frac{1}{2}\log\left[2\cos\frac{\pi}{2\hat{q}}\right]$

$$= 0.2746 \ldots \text{ for } \hat{q} = 3$$

✓ IR: Super-Schwarzian EFT

→ Spectral density

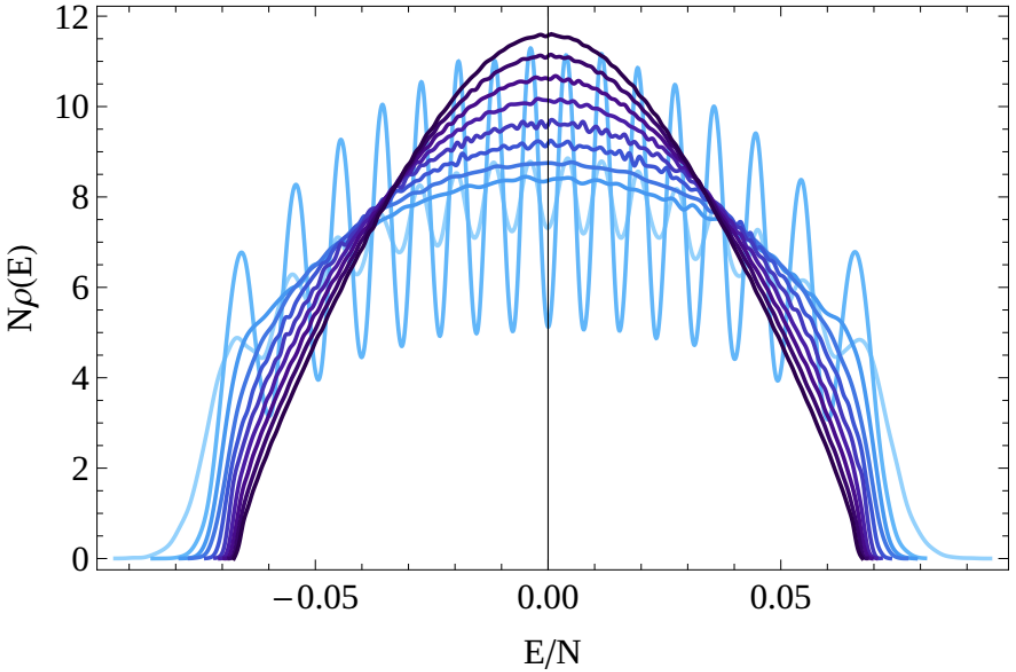
$$\rho(E) \propto \frac{\cosh\left(\#\sqrt{N_m E/J}\right)}{\sqrt{E}}$$

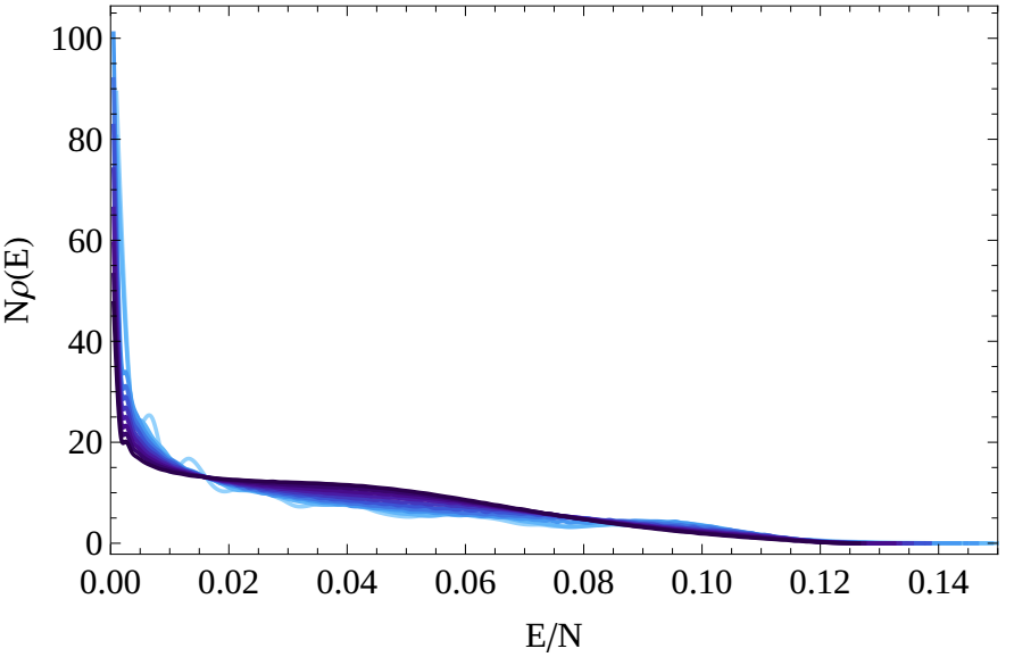
cf. Maldacena-Stanford 2016 / Garcia-Verbaarschot 2016 / Stanford-Witten 2017

N=0 SYK

N=1 SYK

Density of states from $N_m = 10$ to $N_m = 28$





$q = 4$

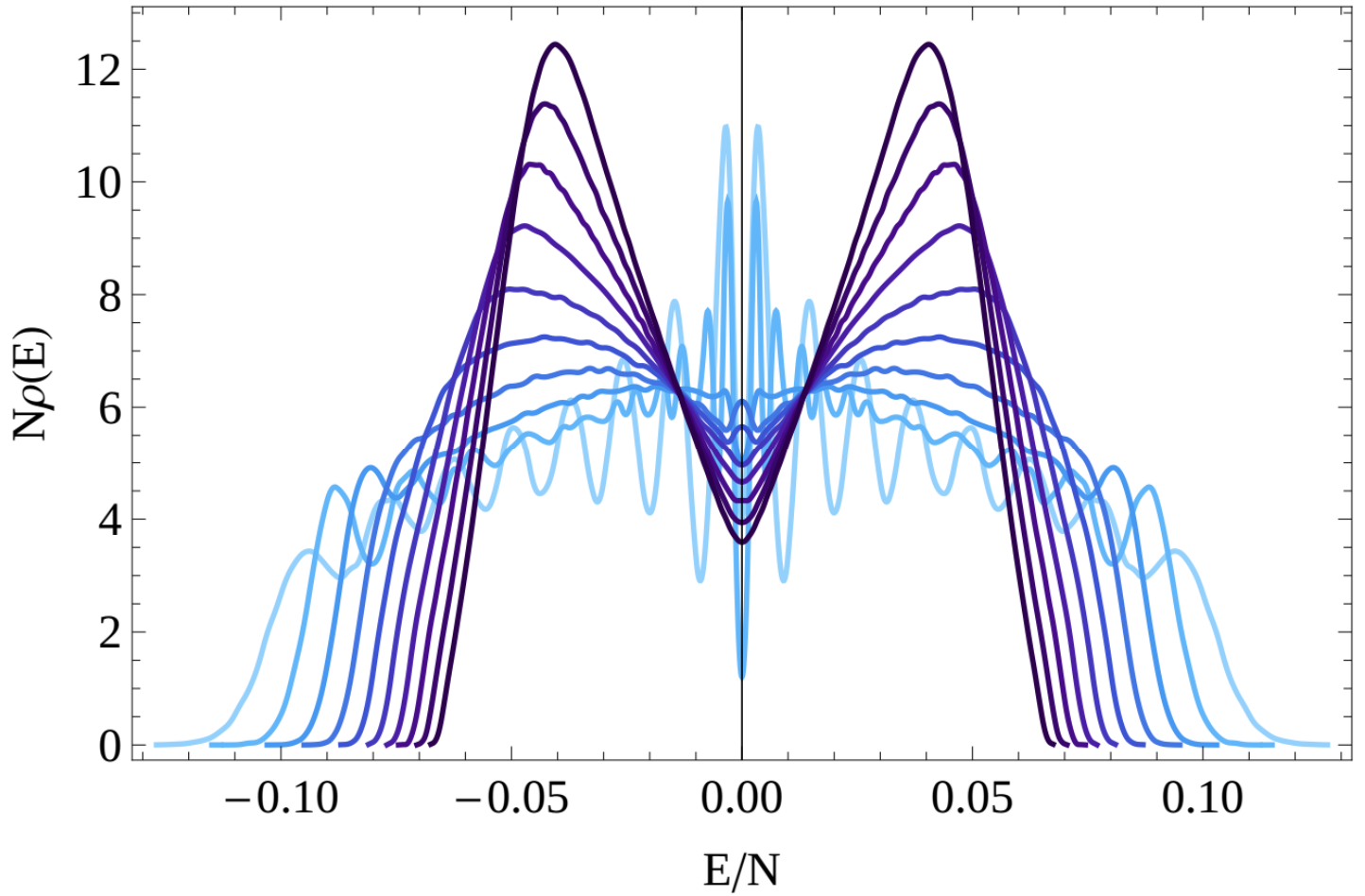
$\hat{q} = 3$

Taken from: Li et al. arXiv:1702.01738 [hep-th]

N=1 SYK

Density of states from $N_m = 10$ to $N_m = 28$

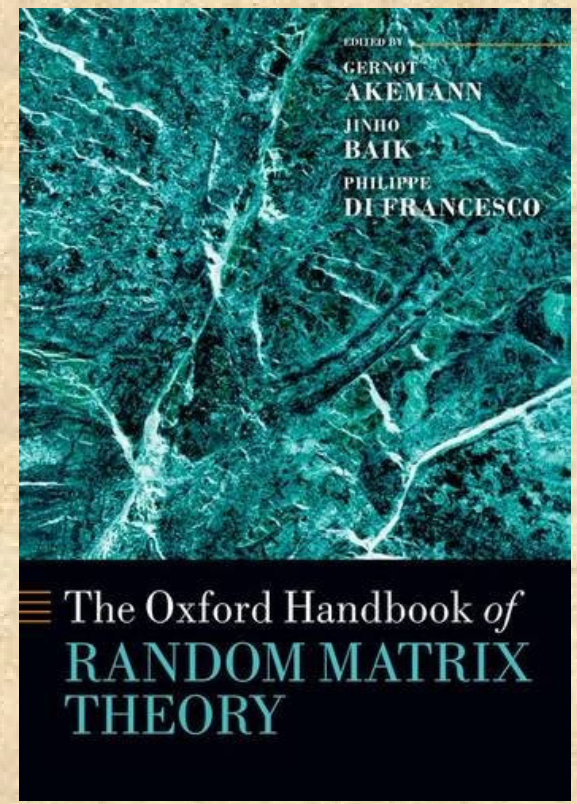
Spectral density of the supercharge Q $(\hat{q} = 3)$



Taken from: Li et al. arXiv:1702.01738 [hep-th]

Symmetry Classes in Random Matrix Theory

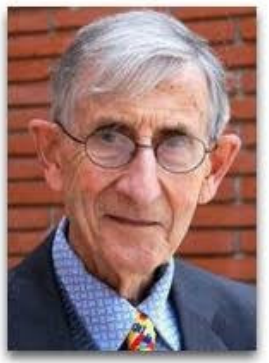
(applicable to a general Hamiltonian of free fermions)



RMT	Cartan name	β	α	T_+^2	T_-^2	Λ^2	Block structure	#Zero modes
GUE	A	2	—	—	—	—	$H = H^\dagger$ complex	0
GOE	AI	1	—	+1	—	—	$H = H^T$ real	0
GSE	AII	4	—	−1	—	—	$H = H^\dagger$ quaternion	0

Wigner-Dyson classes

$$P(\lambda) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_n e^{-V(\lambda_n)}$$



RMT	Cartan name	β	α	T_+^2	T_-^2	Λ^2	Block structure	#Zero modes
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GOE	AI	1	—	+1	—	—	$H = H^T$ real	0
GSE	AII	4	—	−1	—	—	$H = H^\dagger$ quaternion	0
chGUE	AIII	2	$2\nu+1$	—	—	1	$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$, W : complex $n \times m$	$\lvert n-m\rvert$ $(\equiv \nu)$
chGOE	BDI	1	ν	+1	+1	1	$\begin{pmatrix} 0 & W \\ W^T & 0 \end{pmatrix}$, W : real $n \times m$	
chGSE	CII	4	$4\nu+3$	−1	−1	1	$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$, W : quaternion $n \times m$	2ν

Chiral (Wishart-Laguerre) classes

$$P(\lambda) \propto \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^\beta \prod_n |\lambda_n|^\alpha \, e^{-V(\lambda_n)}$$

$$P(\lambda) \propto \prod_{i<j} |\lambda_i^2 - \lambda_j^2|^\beta \prod_n |\lambda_n|^\alpha \, e^{-V(\lambda_n)}$$

RMT	Cartan name		β	α	T_+^2	T_-^2	Λ^2	Block structure		#Zero modes
GUE	A		2	—	—	—	—	$H = H^\dagger$ complex		0
GOE	AI		1	—	+1	—	—	$H = H^T$ real		0
GSE	AII		4	—	−1	—	—	$H = H^\dagger$ quaternion		0
chGUE	AIII		2	$2\nu+1$	—	—	1	$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$, W : complex $n \times m$		$\lvert n-m \rvert$ $(\equiv \nu)$
chGOE	BDI		1	ν	+1	+1	1	$\begin{pmatrix} 0 & W \\ W^T & 0 \end{pmatrix}$, W : real $n \times m$		
chGSE	CII		4	$4\nu+3$	−1	−1	1	$\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$, W : quaternion $n \times m$		2ν
BdG	C		2	2	—	−1	—	$\begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix}$, A : Hermitian, B : complex symmetric		0
	CI		1	1	+1	−1	1	$\begin{pmatrix} 0 & Z \\ \overline{Z} & 0 \end{pmatrix}$, Z : complex symmetric		0
	BD	D	2	0	—	+1	—	H pure imaginary and skew-symmetric	$\dim[H]$ = even	0
		B		2					$\dim[H]$ = odd	1
	DIII	DIII even	4	1	−1	+1	1	$\begin{pmatrix} 0 & Z \\ -\overline{Z} & 0 \end{pmatrix}$, Z : complex and skew-symmetric	$\dim[Z]$ = even	0
		DIII odd		5					$\dim[Z]$ = odd	2

SYMMETRY CLASS

- N=0 SYK
  - ✓ The case $q = 4$ was classified in:
    You et al., PRB 95, 115150 (2017)
    Garcia & Verbaarschot, PRD 94, 126010 (2016)
    Cotler et al., JHEP 1705 (2017) 118
  - ✓ General even q : this work

- N=1 SYK
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  - ✓ General odd $\hat{q}$ + Microscopic limit: this work

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→ Different symmetry classes from those for N=0 SYK

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KEY OPERATORS ($N = 0$)

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- T_+: Antiunitary symmetry that commutes with H
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If T_+ and T_- are present, one can choose $\Lambda = T_+T_-$.

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)			
$T_-^2 = +1$			
$T_-^2 = -1$			

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$			
$T_-^2 = -1$			

$\mathcal{N} = 0$ SYK

$q = 0 \pmod 4$

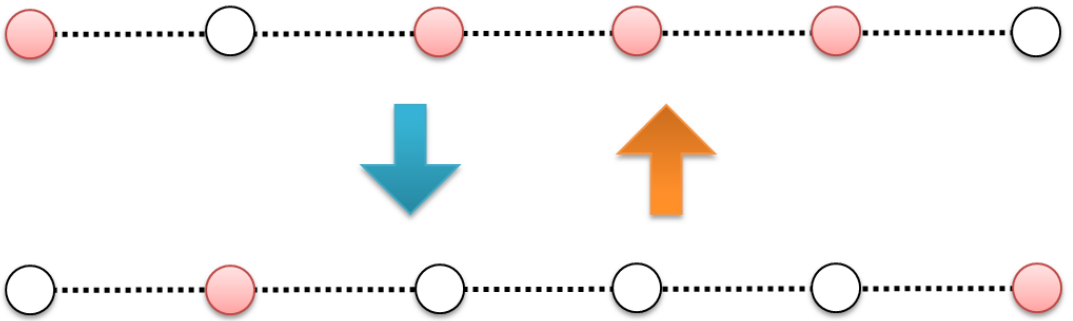
PARTICLE-HOLE TRANSFORMATION OPERATOR

$$P = K \prod_{a=1}^{N_c} (c_a + \bar{c}_a) \equiv K(c_1 + \bar{c}_1) \cdots (c_{N_c} + \bar{c}_{N_c})$$

$$P c_a P = \eta \bar{c}_a$$
$$P \bar{c}_a P = \eta c_a$$
$$P \chi_i P = \eta \chi_i$$

$$\eta = (-1)^{\lfloor \frac{N_c - 1}{2} \rfloor},$$
$$P^2 = (-1)^{\lfloor N_c/2 \rfloor}$$

$N_c = N_m/2$



$q = 2 \pmod 4$ in N=0 SYK

- H is block-diagonal ... $\begin{pmatrix} H_{\text{boson}} & 0 \\ 0 & H_{\text{fermion}} \end{pmatrix}$
- $\{H, P\} = 0$ ➔ P corresponds to $T_$
- For $N_m = 0$ and $4 \pmod 8$, P is bosonic.
  - $N_m = 0 \pmod 8$ ➔ $P^2 = +1$
  - $N_m = 4 \pmod 8$ ➔ $P^2 = -1$
- For $N_m = 2$ and $6 \pmod 8$, P is fermionic.

P does not constrain $H_{\text{boson/fermion}}$. ➔ GUE

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$			
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	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$	BdG (BD)		
$T_-^2 = -1$	BdG (C)		

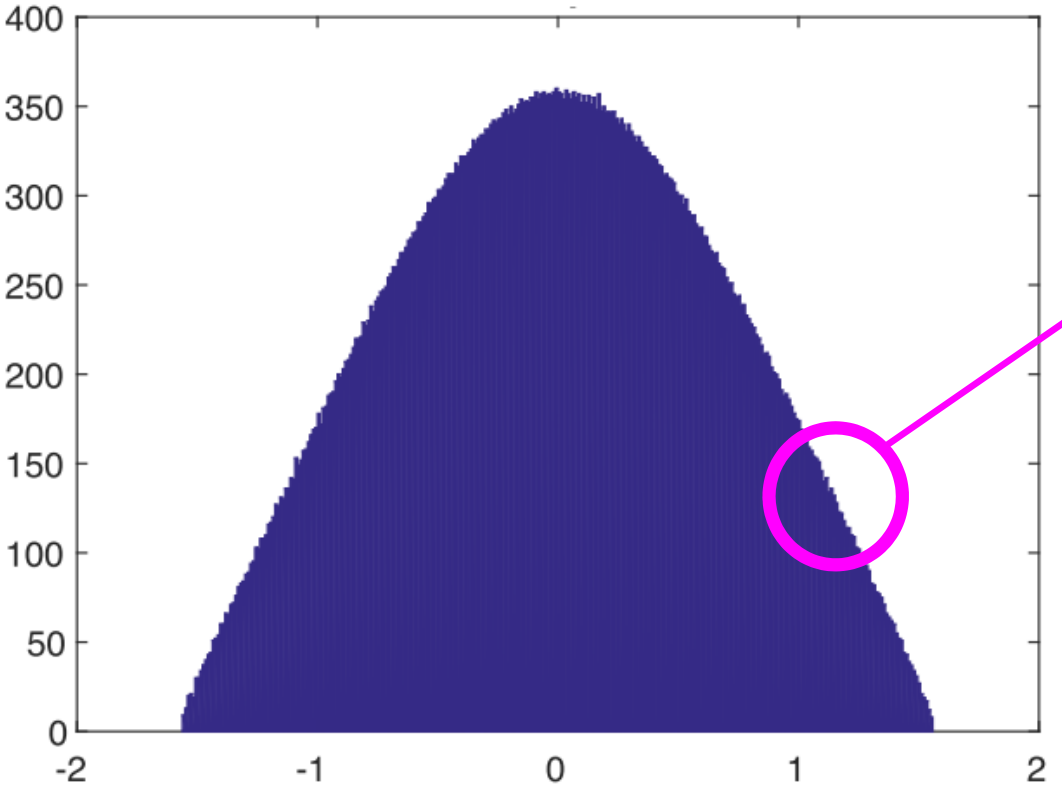
$\mathcal{N} = 0$ SYK

$q = 0 \pmod 4$

$\mathcal{N} = 0$ SYK

$q = 2 \pmod 4$ [All $\beta = 2$]

For large N_m , the global spectral density looks like this, irrespective of the symmetry class.



Magnify
local correlations!

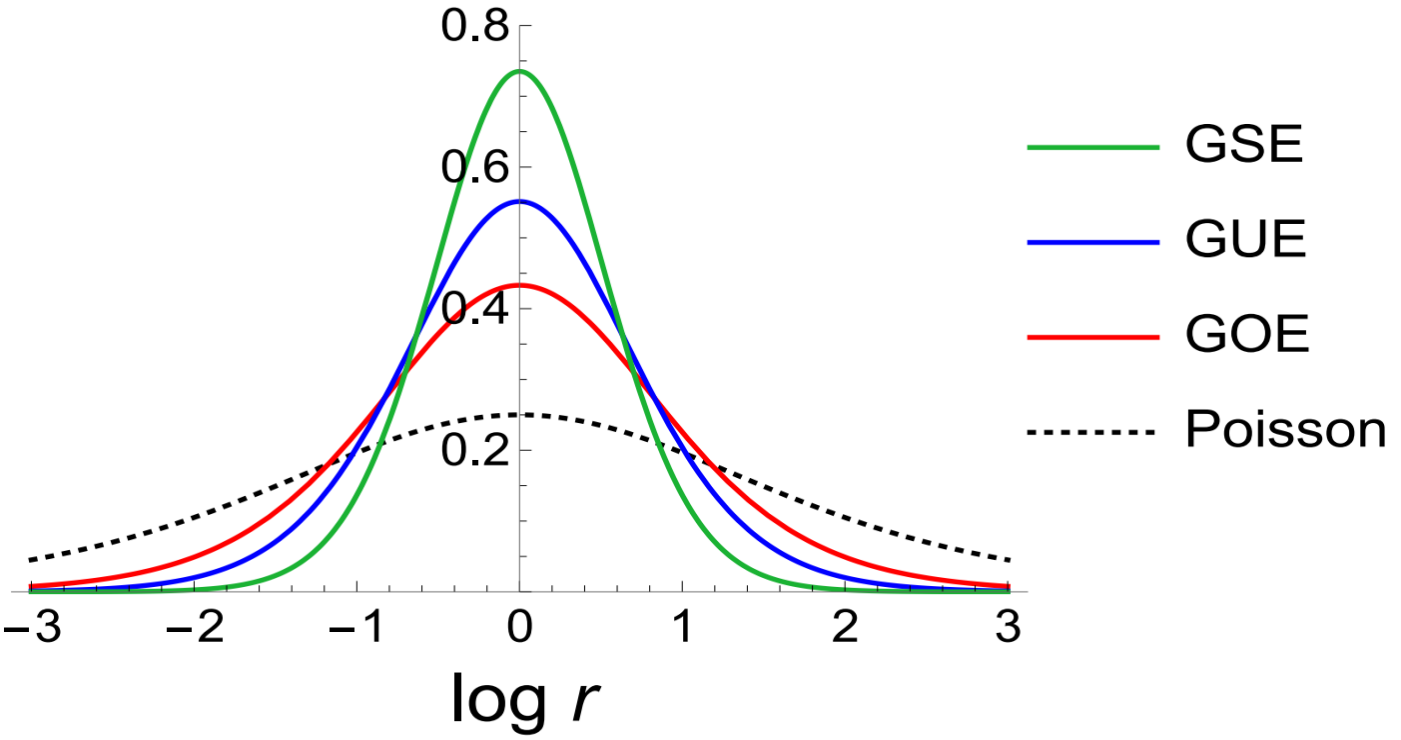
NUMERICS

- We measure the degree of chaos by accumulating the ratio $r = \frac{E_{n+2} - E_{n+1}}{E_{n+1} - E_n}$ that requires no unfolding

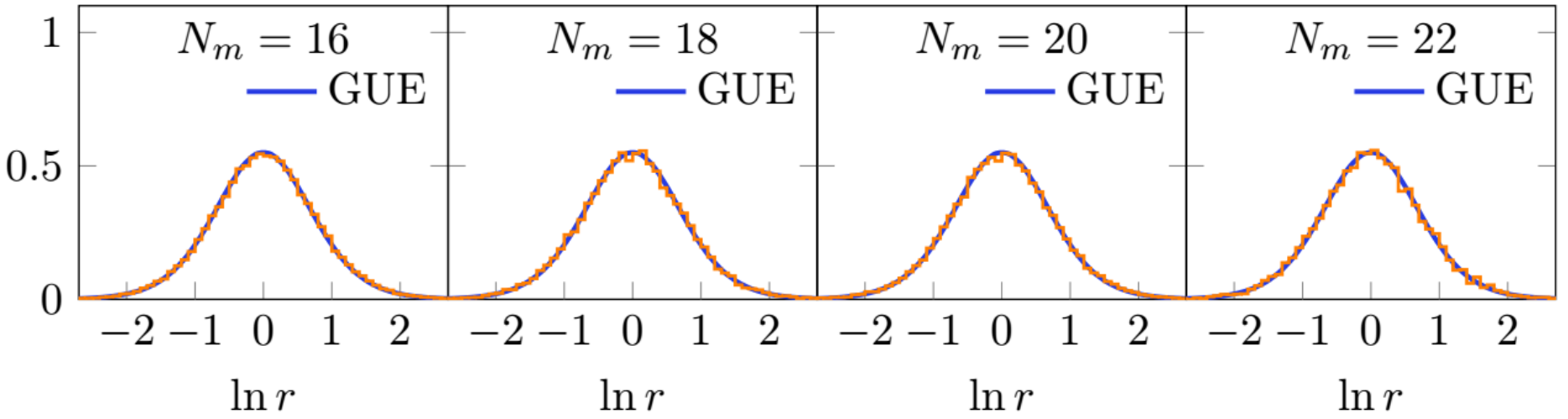
Oganesyan and Huse, PRB 75, 155111 (2007)

- Wigner-like surmises

Atas et al., PRL 110, 084101 (2012)



RESULT (N=0, $q = 6$)



- Orange — Histogram
- Blue — Wigner Surmise

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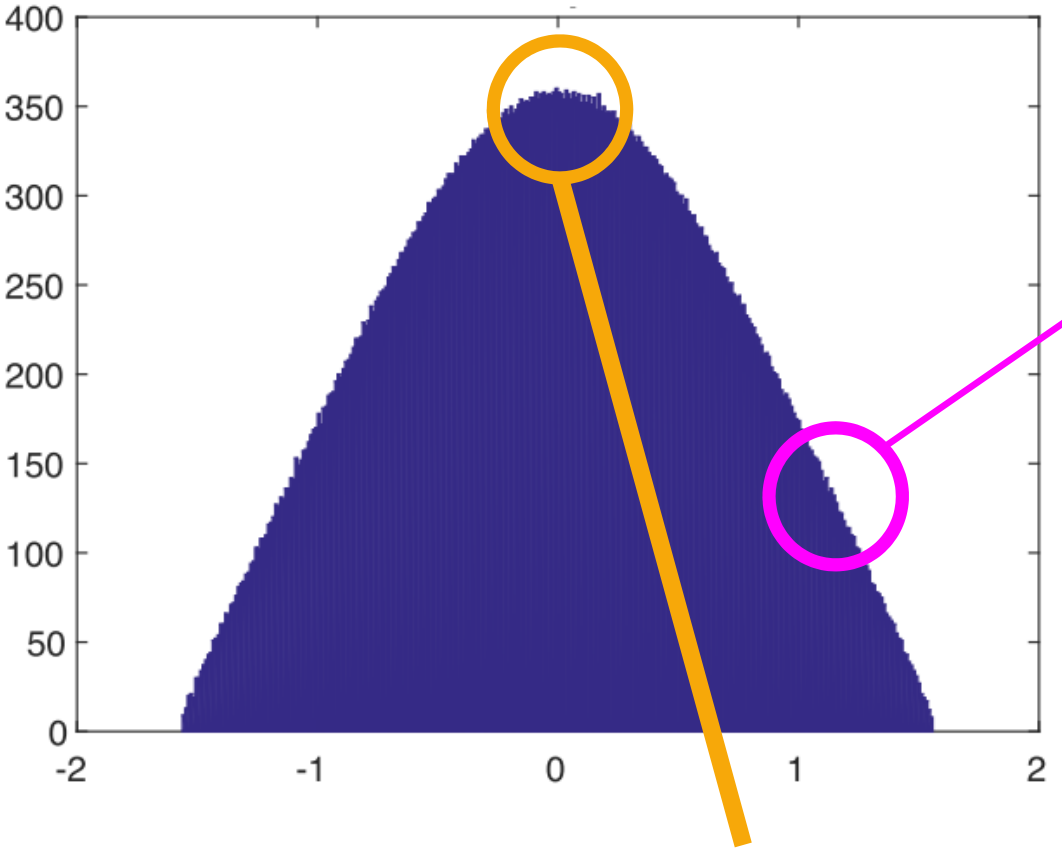
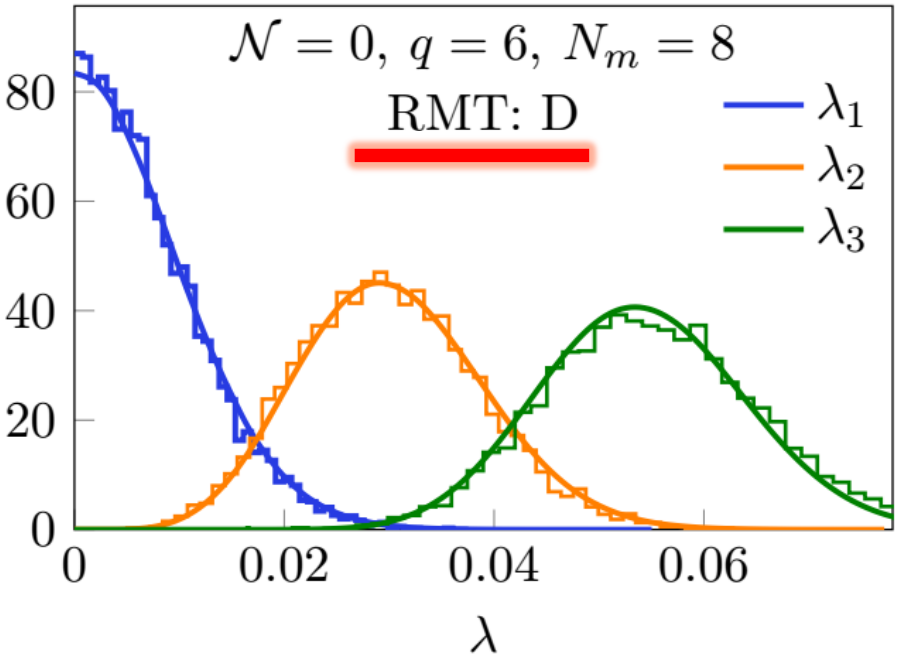


Magnify
local correlations!

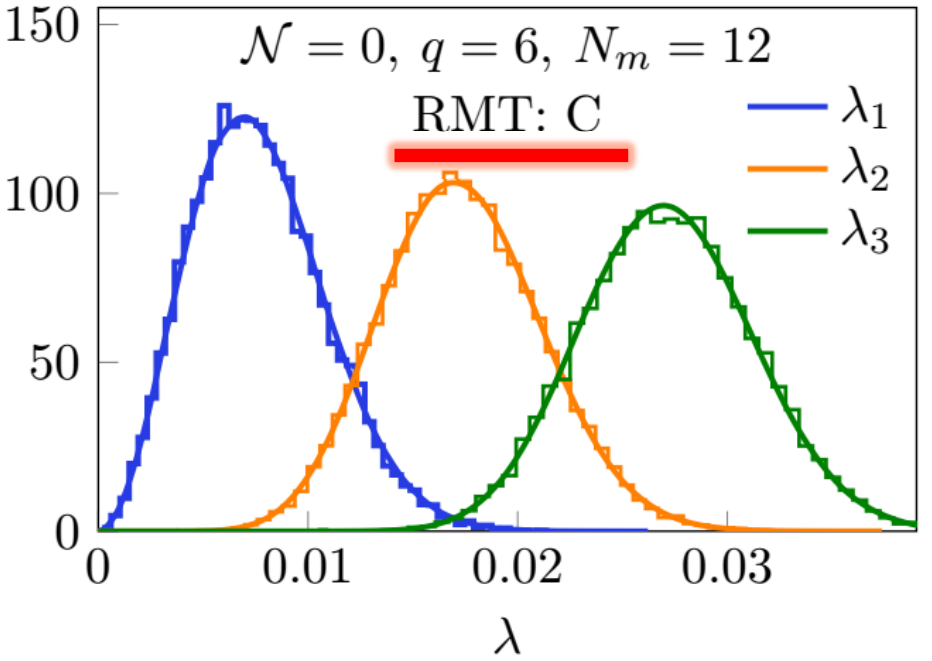
Magnify the hard-edge for class C & BD

RESULT (N=0, $q = 6$)

✓ **Agree**



✓ **Agree**



The number of independent random samples is 10^4 for each plot.

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$$\{(-1)^F, Q\} = 0.$$

Define

$$R = P(-1)^F.$$

CLASSES (N=1, $q = 1 \bmod 4$)

$\mathcal{N}=1$ SYK $\hat{q}=1 \pmod 4$	P^2	R^2	(anti-) commutators	β
$N_m = 0 \pmod 8$	$+1$	$+1$	$\{P,Q\}=0$ $[R,Q]=0$	1
$N_m = 2 \pmod 8$	$+1$	-1	$[P,Q]=0$ $\{R,Q\}=0$	1
$N_m = 4 \pmod 8$	-1	-1	$\{P,Q\}=0$ $[R,Q]=0$	4
$N_m = 6 \pmod 8$	-1	$+1$	$[P,Q]=0$ $\{R,Q\}=0$	4

CLASSES (N=1, $q = 1 \bmod 4$)

$\mathcal{N}=1$ SYK $\hat{q}=1 \pmod 4$	P^2	R^2	(anti-) commutators	β	class of Q
$N_m = 0 \pmod 8$	$+1$	$+1$	$\{P,Q\}=0$ $[R,Q]=0$	1	chGOE (BDI) $\nu=0$
$N_m = 2 \pmod 8$	$+1$	-1	$[P,Q]=0$ $\{R,Q\}=0$	1	BdG (CI)
$N_m = 4 \pmod 8$	-1	-1	$\{P,Q\}=0$ $[R,Q]=0$	4	chGSE (CII) $\nu=0$
$N_m = 6 \pmod 8$	-1	$+1$	$[P,Q]=0$ $\{R,Q\}=0$	4	BdG (DIII-even)

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$	BdG (BD)		
$T_-^2 = -1$	BdG (C)		

$\mathcal{N} = 0$ SYK

$q = 0 \pmod 4$

$\mathcal{N} = 0$ SYK

$q = 2 \pmod 4$

All

$\beta = 2$

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$	BdG (BD)	chGOE	BdG (DIII)
$T_-^2 = -1$	BdG (C)	BdG (CI)	chGSE

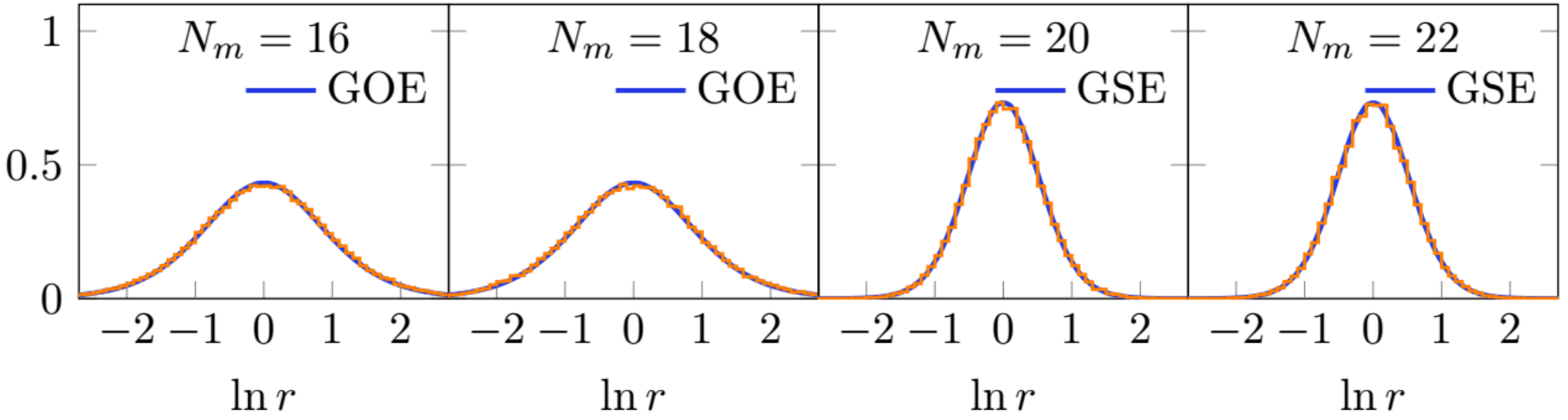
$\mathcal{N} = 0$ SYK
$q = 0 \pmod 4$

$\mathcal{N} = 0$ SYK
$q = 2 \pmod 4$

All
$\beta = 2$

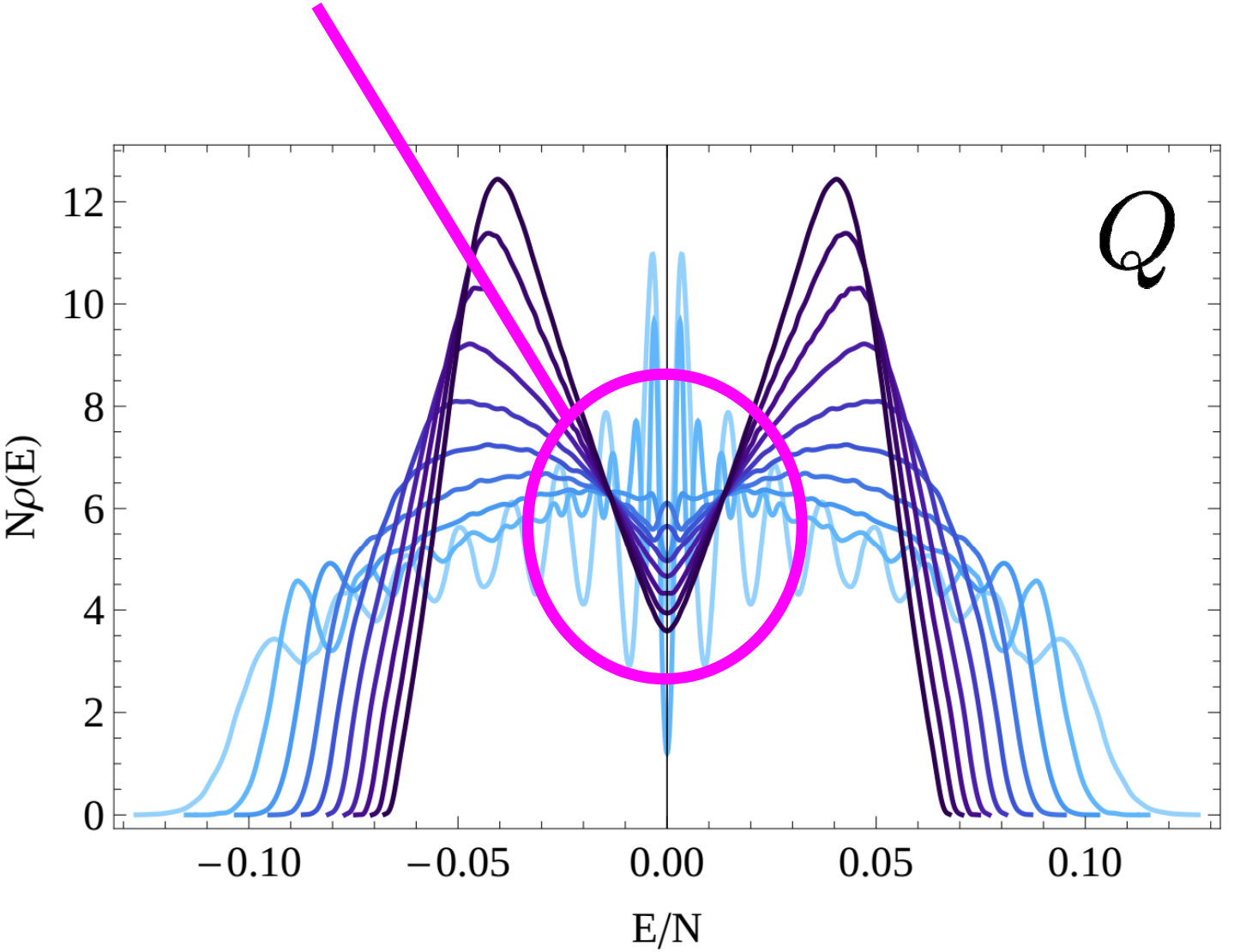
$\mathcal{N} = 1$ SYK

NUMERICS (N=1, $\hat{q} = 5$)



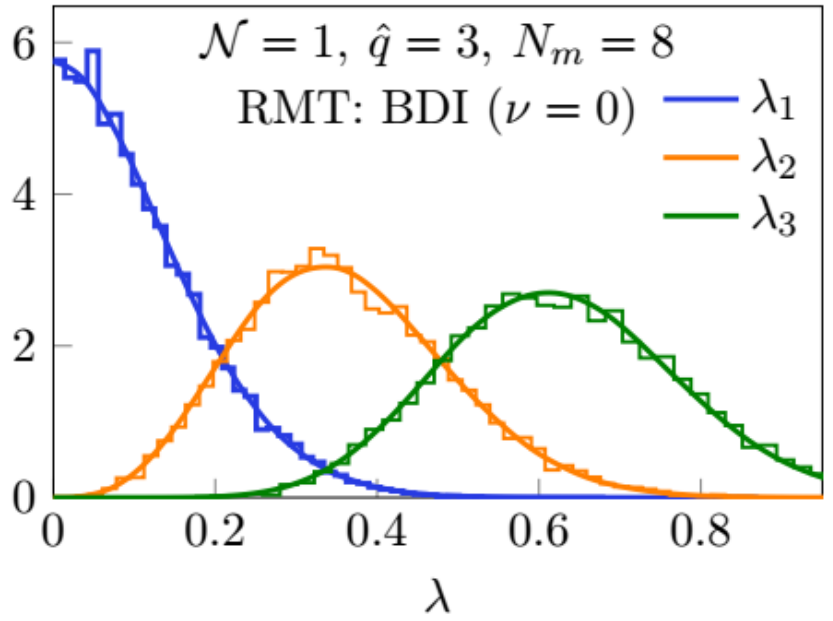
Quantum chaos is confirmed.

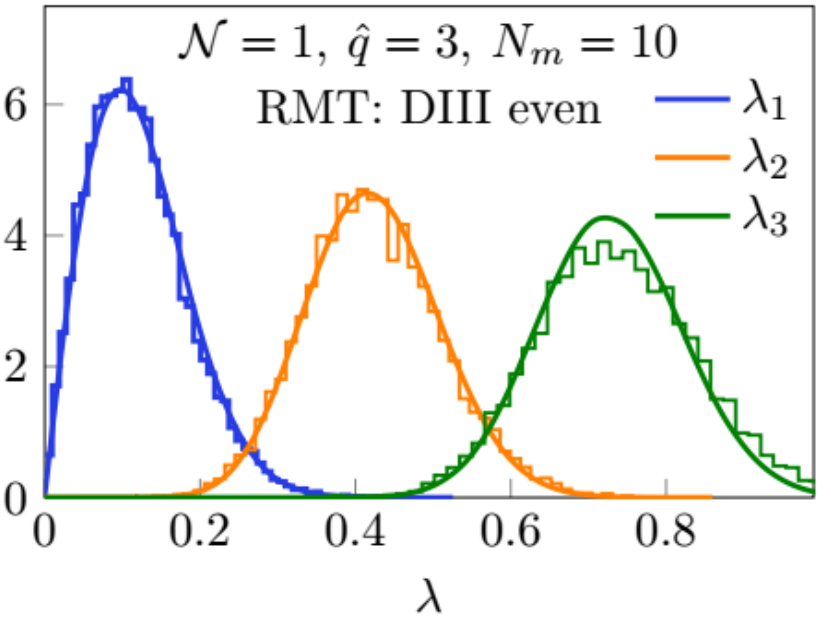
Magnify the hard-edge!
(Relevant to low-T thermodynamics)

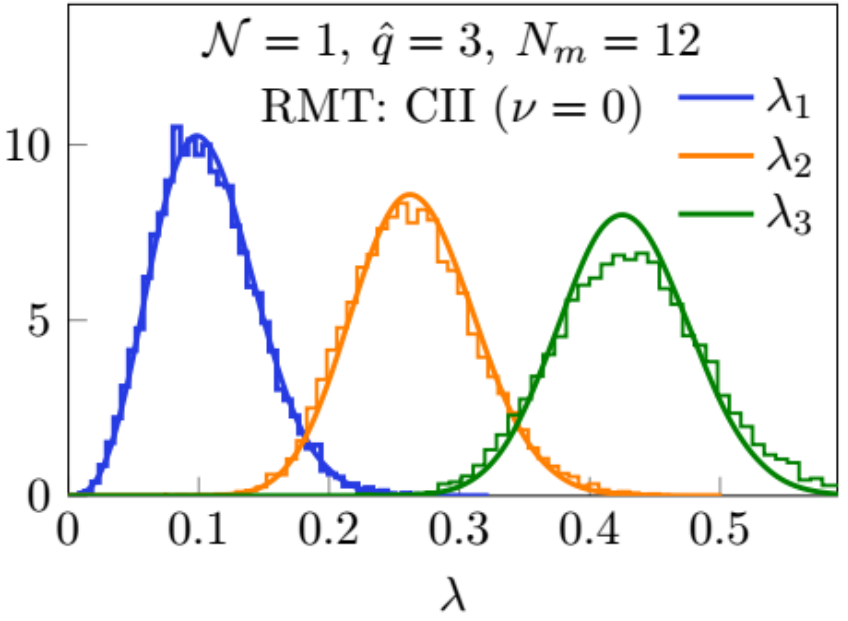


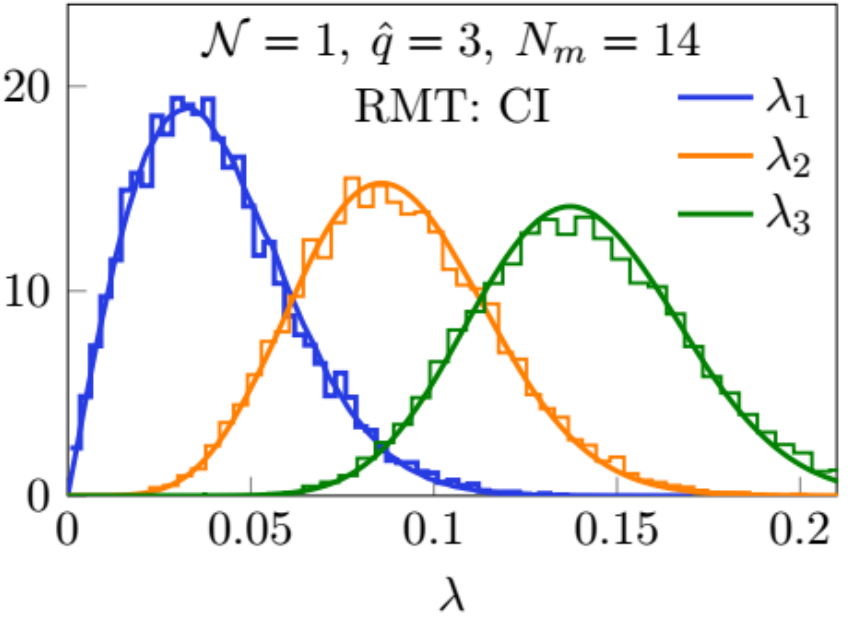
$\hat{q} = 3$

✓ **Agree**


$\mathcal{N} = 1,\ \hat{q} = 3,\ N_m = 8$
RMT: BDI ($\nu = 0$)
λ_1
λ_2
λ_3
0
2
4
6
0
0.2
0.4
0.6
0.8
λ

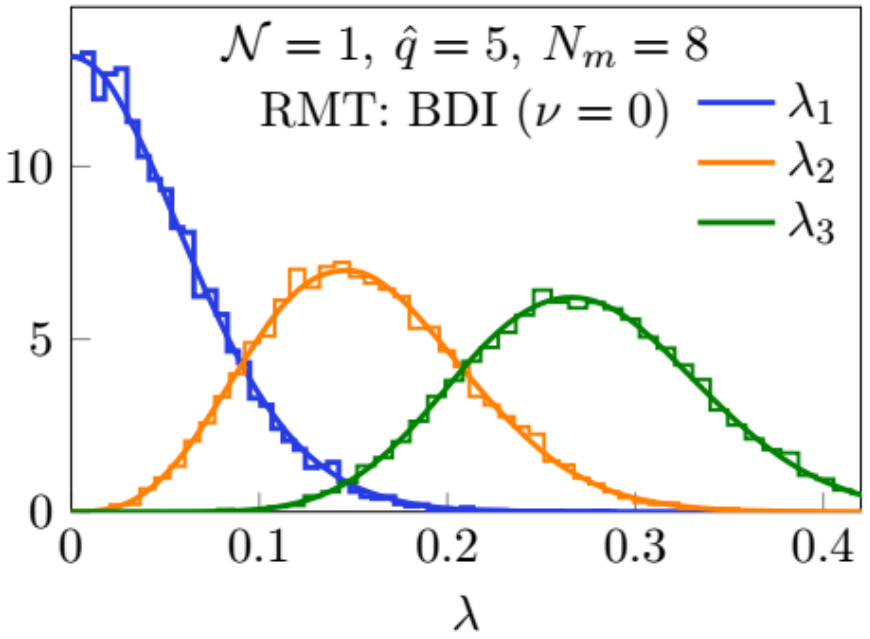

$\mathcal{N} = 1,\ \hat{q} = 3,\ N_m = 10$
RMT: DIII even
λ_1
λ_2
λ_3
0
2
4
6
0
0.2
0.4
0.6
0.8
λ

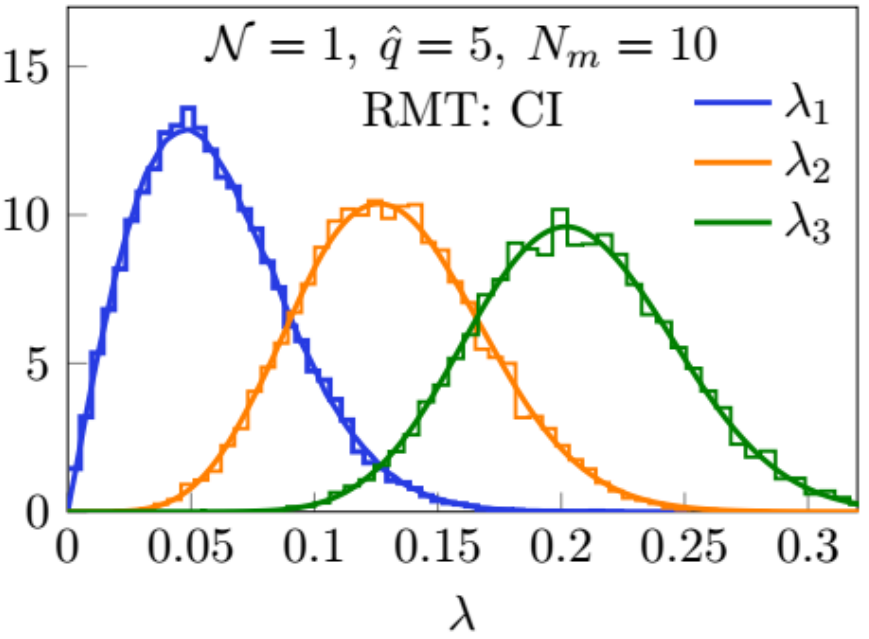

$\mathcal{N} = 1,\ \hat{q} = 3,\ N_m = 12$
RMT: CII ($\nu = 0$)
λ_1
λ_2
λ_3
0
5
10
0
0.1
0.2
0.3
0.4
0.5
λ

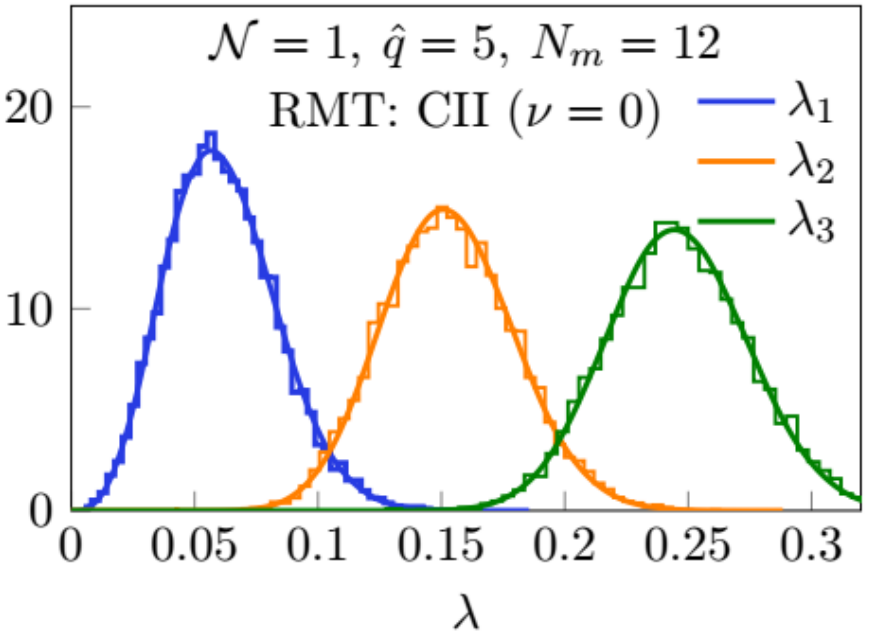

$\mathcal{N} = 1,\ \hat{q} = 3,\ N_m = 14$
RMT: CI
λ_1
λ_2
λ_3
0
10
20
0
0.05
0.1
0.15
0.2
λ

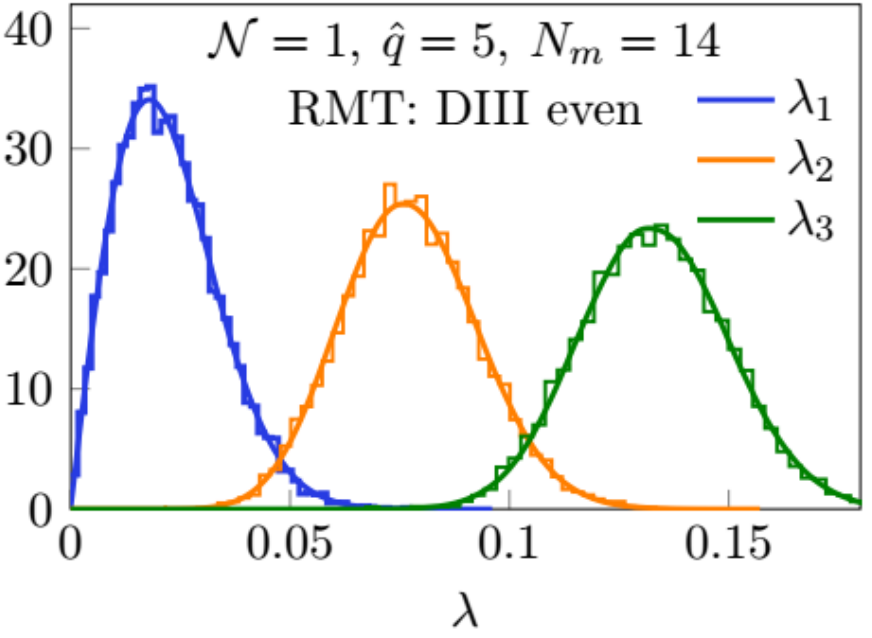
$\hat{q} = 5$

✓ **Agree**


$\mathcal{N} = 1,\ \hat{q} = 5,\ N_m = 8$
RMT: BDI ($\nu = 0$)
λ_1
λ_2
λ_3
0
5
10
0
0.1
0.2
0.3
0.4
λ


$\mathcal{N} = 1,\ \hat{q} = 5,\ N_m = 10$
RMT: CI
λ_1
λ_2
λ_3
0
5
10
15
0
0.05
0.1
0.15
0.2
0.25
0.3
λ

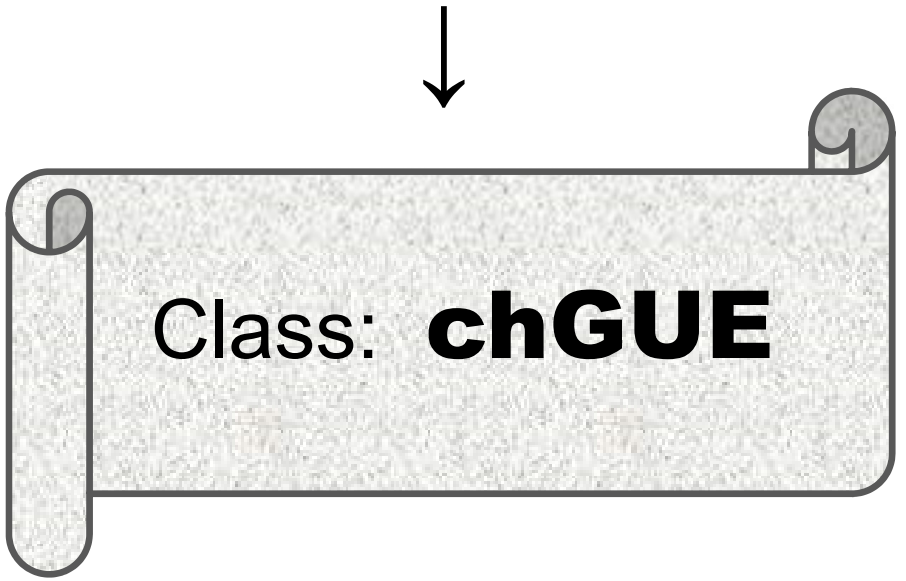

$\mathcal{N} = 1,\ \hat{q} = 5,\ N_m = 12$
RMT: CII ($\nu = 0$)
λ_1
λ_2
λ_3
0
10
20
0
0.05
0.1
0.15
0.2
0.25
0.3
λ


$\mathcal{N} = 1,\ \hat{q} = 5,\ N_m = 14$
RMT: DIII even
λ_1
λ_2
λ_3
0
10
20
30
40
0
0.05
0.1
0.15
λ

If Q contains terms with $(4k+1)$ fermions and $(4\ell+3)$ fermions with $k, \ell \in \mathbb{Z}$ simultaneously,

e.g., $Q = i\sum_{ijk} C_{ijk}\chi_i\chi_j\chi_k + \sum_i D_i\chi_i$,

then anti-unitary symmetries are lost.



	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	GUE	GOE	GSE
$T_-^2 = +1$	BdG (BD)	chGOE	BdG (DIII)
$T_-^2 = -1$	BdG (C)	BdG (CI)	chGSE

$\mathcal{N} = 0$ SYK

$q = 0 \pmod 4$

$\mathcal{N} = 0$ SYK

$q = 2 \pmod 4$

All

$\beta = 2$

$\mathcal{N} = 1$ SYK

	T_+^2 (None)	$T_+^2 = +1$	$T_+^2 = -1$
T_-^2 (None)	chGUE / GUE	GOE	GSE
$T_-^2 = +1$	BdG (BD)	chGOE	BdG (DIII)
$T_-^2 = -1$	BdG (C)	BdG (CI)	chGSE

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All

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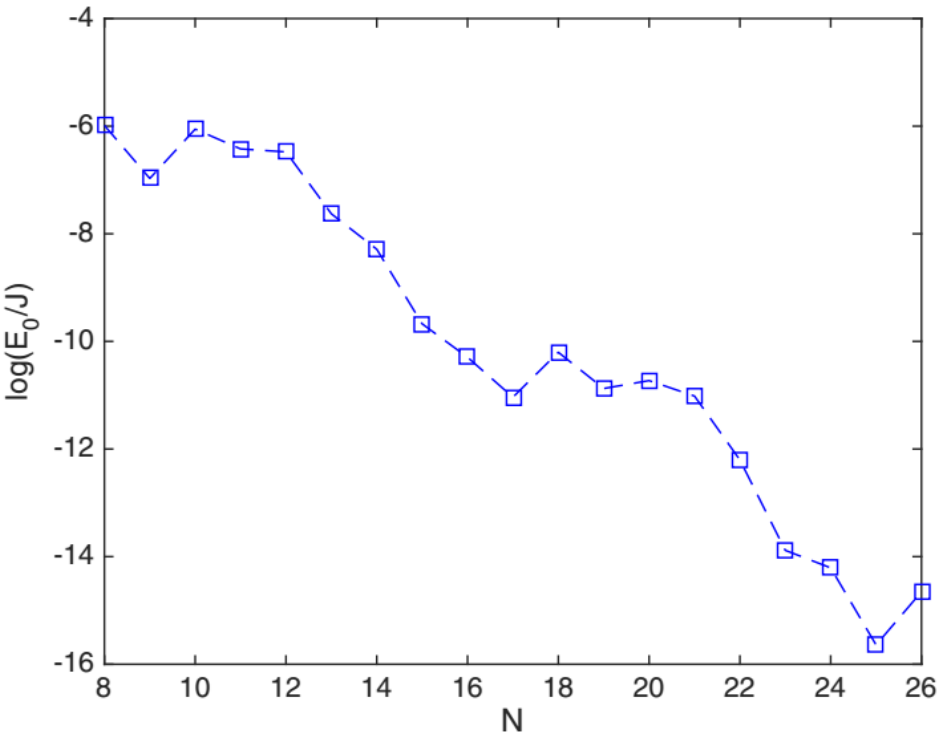
NO EXACT ZERO MODE $(\nu = 0)$

$\Downarrow$

SUSY is restored only in the limit $N_m \to \infty$.

(Finite temperature also breaks SUSY.)



Taken from: Fu–Gaiotto–Maldacena–Sachdev 2016

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$\Downarrow$

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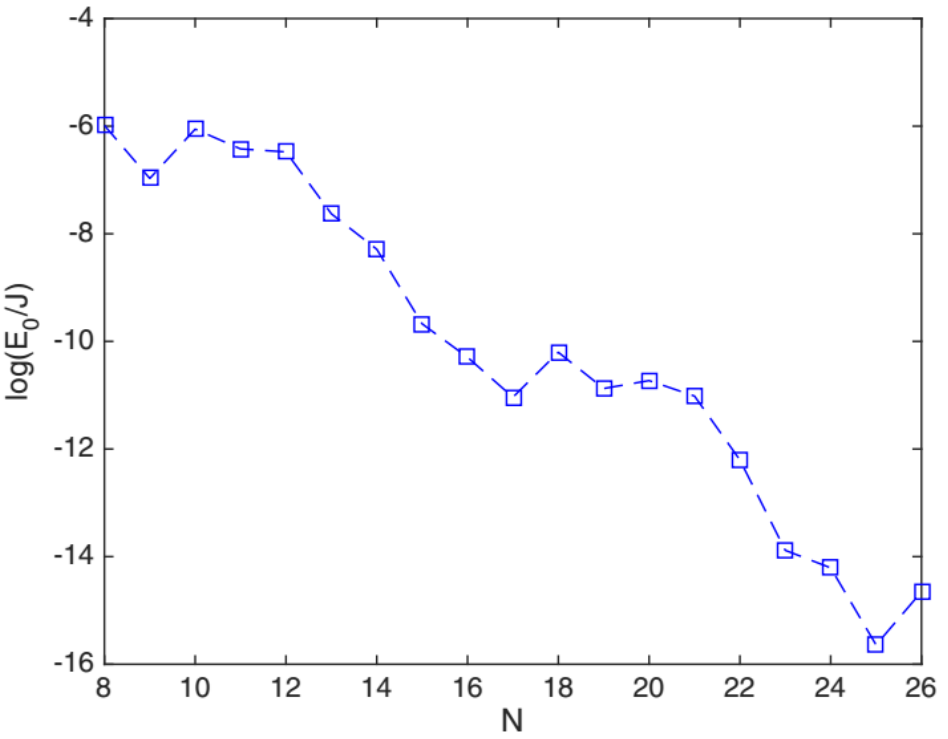
$N = 0$ & 1 finished.



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ROAD TO $N = 2$ SYK

SYK WITH N=2 SUSY $(\hat{q}=3)$

Fu–Gaiotto–Maldacena–Sachdev 2016

$H=\{Q,\bar{Q}\}$ $(Q^2=\bar{Q}^2=0)$, with

$$Q=i\sum_{1\le i<j<k\le N_c} X_{ijk}c_ic_jc_k \qquad \text{and} \qquad \overline{Q}=i\sum_{1\le i<j<k\le N_c} \overline{X_{ijk}}\,\bar{c}_i\bar{c}_j\bar{c}_k$$

Gaussian disorder $\langle X_{ijk}\overline{X_{ijk}}\rangle = 2J/N_c^2$.

➔ $$H=E_c+\sum_{ijab} J_{ij}^{ab}c_ic_j\bar{c}_a\bar{c}_b$$

$$E_c=\sum_{i<j<k}|X_{ijk}|^2,$$

$$J_{ij}^{ab}=\sum_p X_{ijp}\overline{X_{abp}}$$

SYK WITH N=2 SUSY $(\hat{q}=3)$

Fu–Gaiotto–Maldacena–Sachdev 2016

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Gaussian disorder $\langle X_{ijk}\overline{X_{ijk}}\rangle = 2J/N_c^2$.

➔ $$H=E_c+\sum_{ijab} J_{ij}^{ab}c_ic_j\bar{c}_a\bar{c}_b$$

$$E_c=\sum_{i<j<k}|X_{ijk}|^2,$$

$$J_{ij}^{ab}=\sum_p X_{ijp}\overline{X_{abp}}$$

If J were Gaussian, we recover the complex SYK model considered in

Sachdev PRX 5, 041025 (2015),
Fu-Sachdev PRB 94, 035135 (2016)

NUMERICAL EXPERIMENT

N_c	0	2	4	Sum ($= 2^{N_c}$)
4	12	0	1	16
5	20	0	3	32
6	36	0	7	64
7	54	21	8	128
8	108	0	37	256
9	168	0	86	512
10	324	0	175	1024
11	486	319	231	2048

NUMERICAL EXPERIMENT

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Ground states grows with N_c

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N_c mod 4 ?

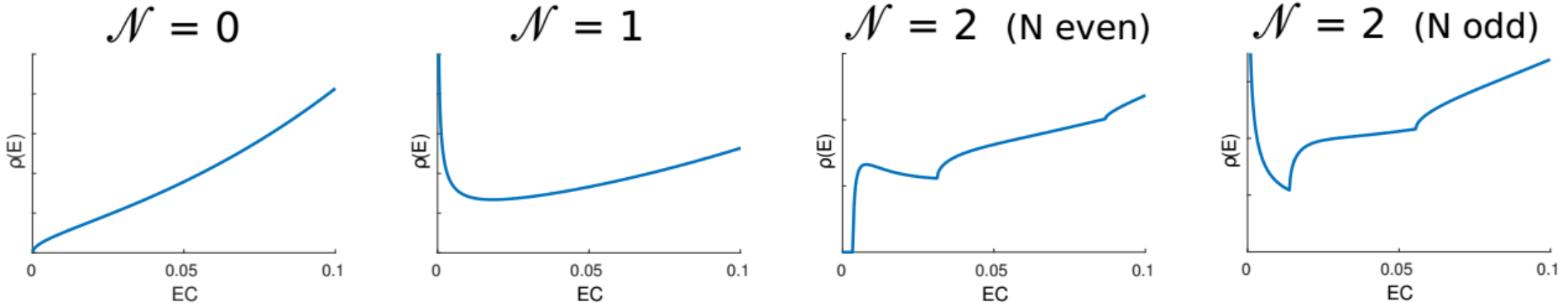
Ground states grows with N_c

Stanford & Witten, 1703.04612
Mertens & Turiacia & Verlinde, 1705.08408

DOS FROM N=2 SCHWARZIAN EFT

$$\rho(E) = \sum_{n\in\mathbf{Z}} \frac{2\cos\pi n}{1-4n^2}\left(\frac{\sqrt{1-4n^2}\,I_1\left(2\pi\sqrt{(1-4n^2)E}\right)}{\sqrt{E}} + \delta(E)\right)$$

Exact zero mode



TK & Wettig, 1706.03044

Let

$$M \equiv Q + \bar{Q} \quad \rightarrow \quad H = M^2.$$

M preserves F mod 3 and changes the fermion parity of the state.

$$M = \underbrace{\begin{pmatrix} 0 & A_0 \\ A_0^\dagger & 0 \end{pmatrix}}_{\overline{V}_0 \oplus \overline{V}_3} \oplus \underbrace{\begin{pmatrix} 0 & A_1 \\ A_1^\dagger & 0 \end{pmatrix}}_{\overline{V}_1 \oplus \overline{V}_4} \oplus \underbrace{\begin{pmatrix} 0 & A_2 \\ A_2^\dagger & 0 \end{pmatrix}}_{\overline{V}_2 \oplus \overline{V}_5}$$

Hilbert space

$$V = \bigoplus_{f=0}^{5} \overline{V}_f \quad \text{with} \quad \overline{D}_f \equiv \dim(\overline{V}_f)$$

TK & Wettig, 1706.03044

Then

$$N^z \geq \sum_{f=0,1,2} |\overline{D}_f - \overline{D}_{f+3}|$$

$$= \begin{cases} 4 \cdot 3^{N_c/2-1} & \text{for } N_c \text{ even}\,, \\ 2 \cdot 3^{(N_c-1)/2} & \text{for } N_c \text{ odd}\,. \end{cases}$$

The inequality is saturated for $N_c = 0,2,3 \pmod 4$, while $O(1)$ discrepancy arises for $N_c = 1 \pmod 4$ due to a special combinatorial reason.

SYMMETRY CLASS

TK & Wettig, 1706.03044

- Distinctive property of the N=2 SYK:

$$[H, F] = 0.$$

For classification of symmetries, nonzero eigenvalues must be collected in each V_f separately.

$$V = \bigoplus_{f=0}^{N_c} V_f \quad \text{with} \quad \dim(V_f) = \binom{N_c}{f}.$$

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$$V = \bigoplus_{f=0}^{N_c} V_f \quad \text{with} \quad \dim(V_f) = \binom{N_c}{f}.$$

- $[H, P] = 0$ for all N_c ··· particle-hole symmetry
- P maps V_f to $V_{N_c - f}$

  ➔ Spec(H) on V_f and $V_{N_c - f}$ are identical.

A NAÏVE GUESS

$N_c \pmod 4$	P^2
0	$+1$
1	$+1$
2	-1
3	-1

- Spec(H) on V_f with $f \neq \frac{N_c}{2}$ would be all **GUE**
- Spec(H) on $V_{N_c/2}$ would be

$$\begin{cases} \text{GOE} & \text{if } N_c = 0 \pmod 4 \\ \text{GSE} & \text{if } N_c = 2 \pmod 4 \end{cases}$$

A NAÏVE GUESS

DATA

- Spec(H) on $V_{N_c/2}$ is two-fold degenerate regardless of $N_c = 0$ or 2 (mod 4).

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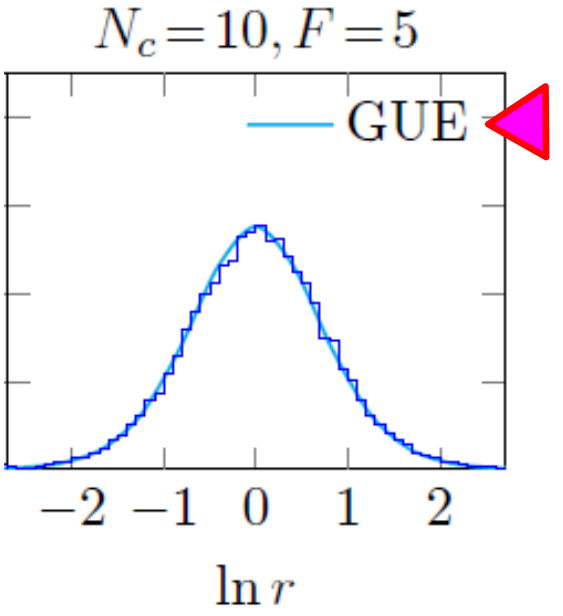
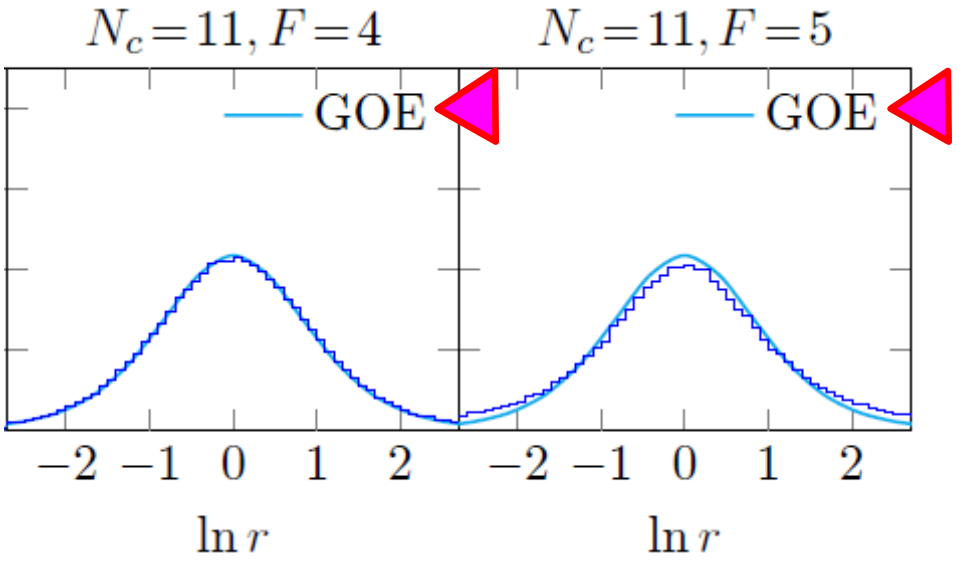
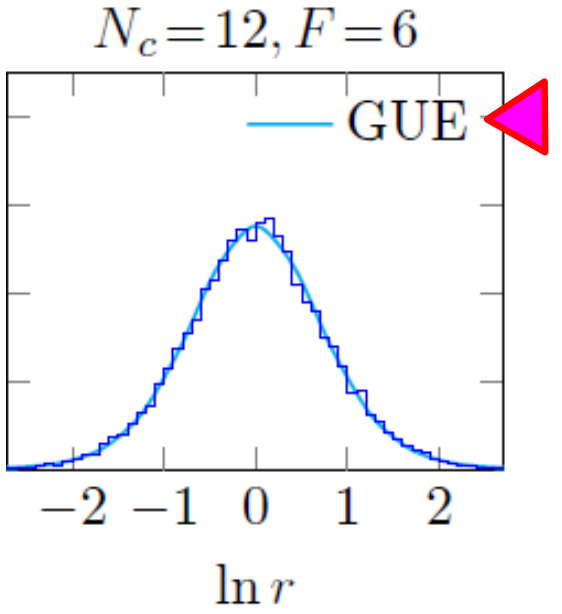






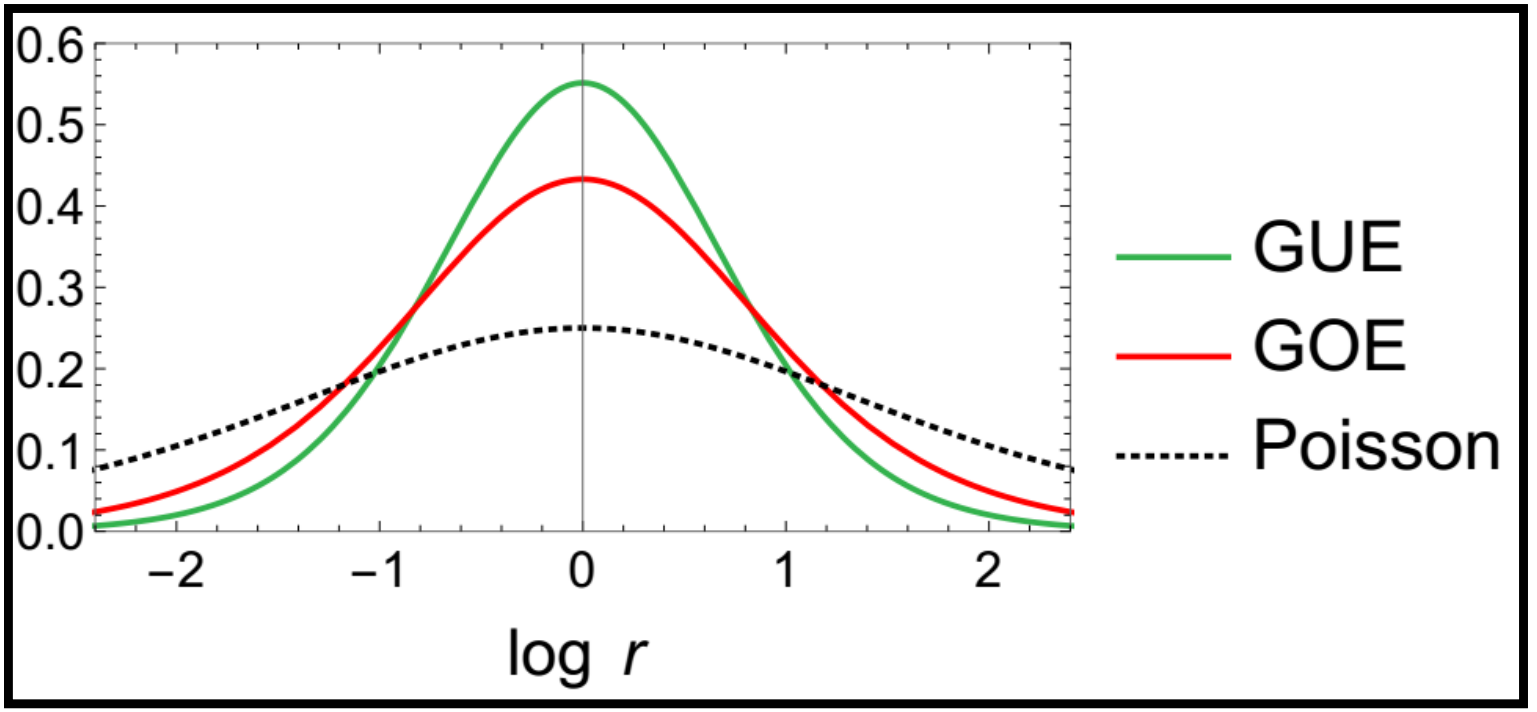
FAKE STATISTICS

- TEST:

  Create 10×10 and 40×40 **GUE** matrices of the same variance

  → mix their eigenvalues into 1 sequence

  → record the consecutive spacing ratios



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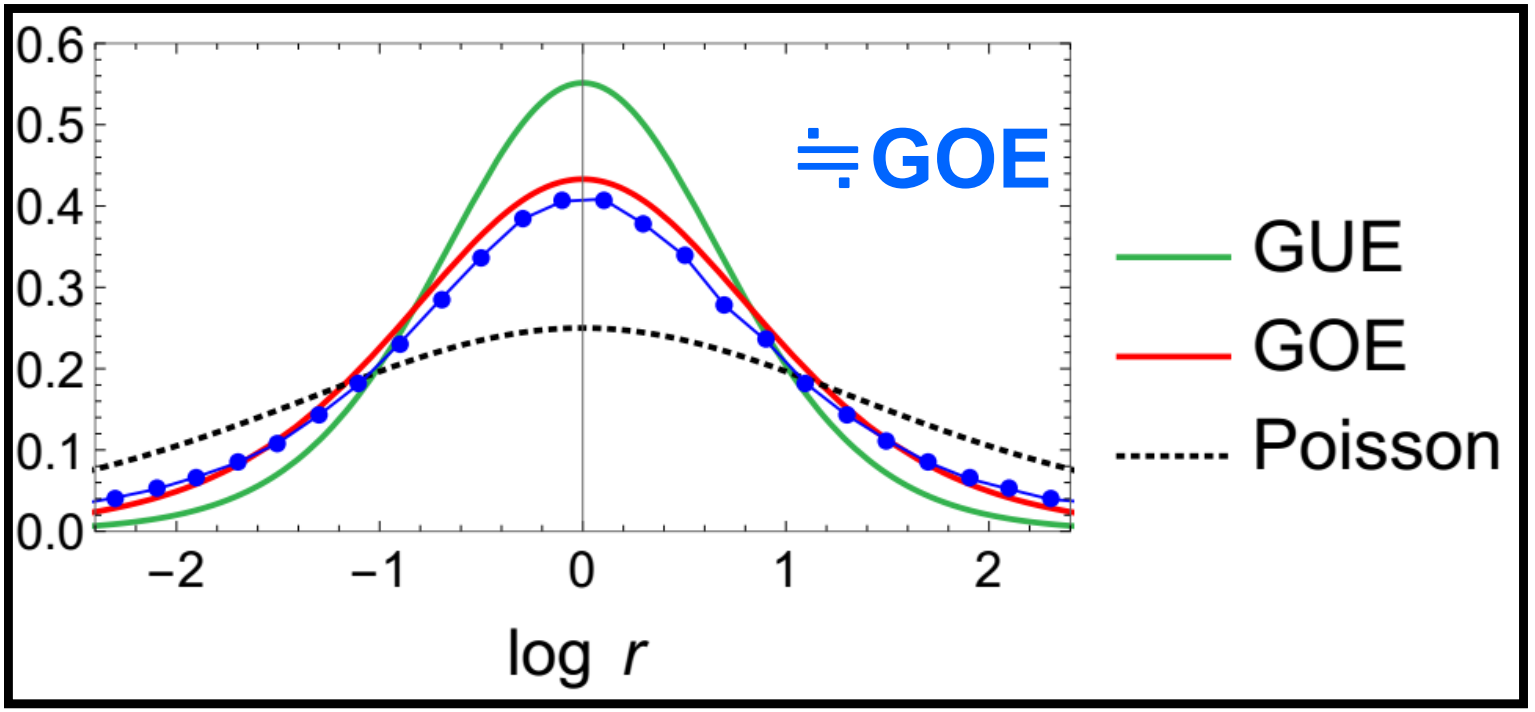


A NAÏVE GUESS

⇕

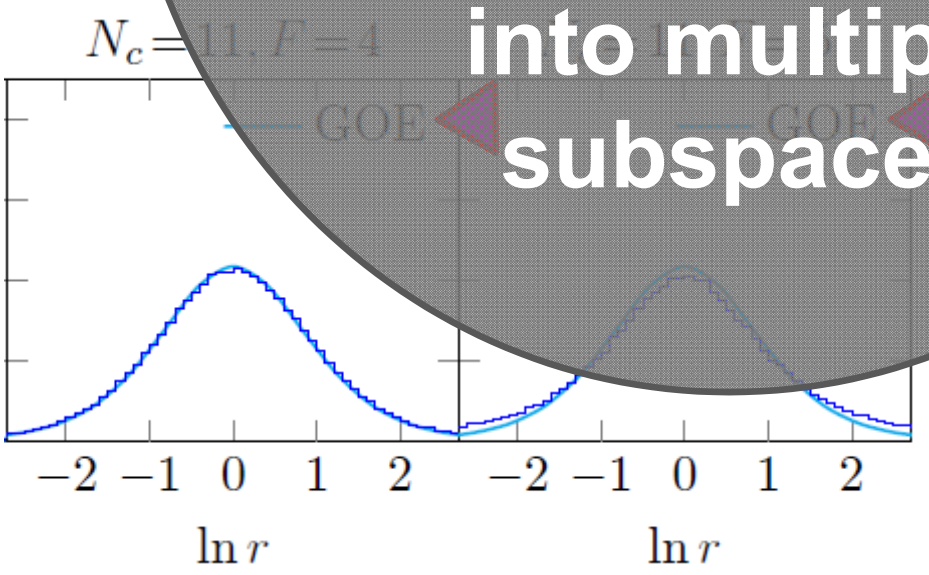
DATA

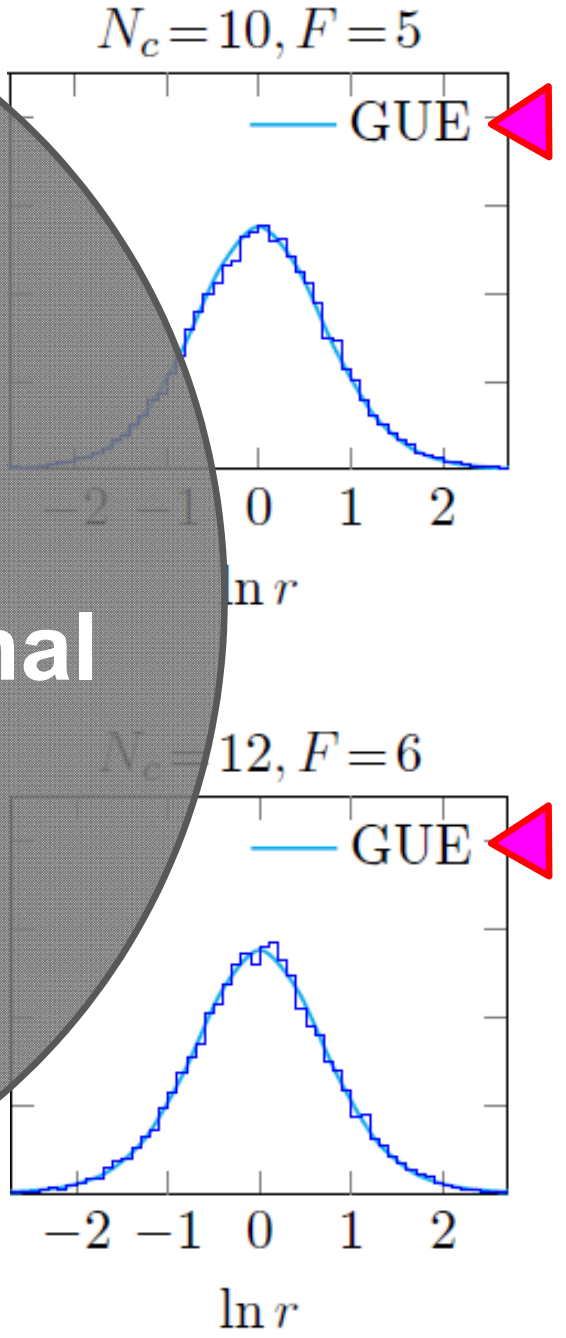
- Spec(H) on $V_{N_c/2}$ is two-fold degenerate regardless of $N_c = 0$ or 2 (mod 4)
- *Unexpected* statistics

What did we overlook?

=

V_f admits orthogonal decomposition into multiple subspaces





$$H=\{Q,\bar{Q}\}=Q\bar{Q}+\bar{Q}Q$$

- $[Q\bar{Q},\bar{Q}Q]=[Q\bar{Q},F]=[\bar{Q}Q,F]=0.$

  $Q\bar{Q}$ and $\bar{Q}Q$ can be diagonalized simultaneously within each V_f .

$$H=\{Q,\bar{Q}\}=Q\bar{Q}+\bar{Q}Q$$

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  $Q\bar{Q}$ and $\bar{Q}Q$ can be diagonalized simultaneously within each V_f .

- Let ψ an eigenvector with

  $$Q\bar{Q}\psi=E_+\psi \quad \text{and} \quad \bar{Q}Q\psi=E_-\psi$$

  for $E_+,E_-\geq 0$. Then $E_+=0$ or $E_-=0$ holds.

- ψ is a zero mode ($Q\psi=\bar{Q}\psi=0$)

  $$\Longleftrightarrow \quad E_+=E_-=0.$$

$$V_f = V_f^+ \oplus V_f^- \oplus V_f^z \,,$$

$V_f^+ =$ Hilbert space spanned by eigenstates ψ with $\overline{Q}\psi \neq 0$ and $Q\psi = 0$,

$V_f^- =$ Hilbert space spanned by eigenstates ψ with $\overline{Q}\psi = 0$ and $Q\psi \neq 0$,

$V_f^z =$ Hilbert space spanned by zero modes $(Q\psi = \overline{Q}\psi = 0)$.

$$N_f^+ \equiv \dim(V_f^+)\,, \quad N_f^- \equiv \dim(V_f^-)\,, \quad N_f^z \equiv \dim(V_f^z)$$

$$\left.\begin{aligned} Q\bar{Q} &= P^{-1}\bar{Q}QP \\ \bar{Q}Q &= P^{-1}Q\bar{Q}P \end{aligned}\right) \Rightarrow \begin{aligned} V_f^+ &\xrightarrow{P} V_{N_c-f}^- \\ V_f^- &\xrightarrow{P} V_{N_c-f}^+ \end{aligned}$$

$$V_f = V_f^+ \oplus V_f^- \oplus V_f^z \,,$$

V_f^+ = Hilbert space spanned by eigenstates ψ with $\overline{Q}\psi \neq 0$ and $Q\psi = 0$,

V_f^- = Hilbert space spanned by eigenstates ψ with $\overline{Q}\psi = 0$ and $Q\psi \neq 0$,

V_f^z = Hilbert space spanned by zero modes $(Q\psi = \overline{Q}\psi = 0)$.

if $\psi \in V_f^+$ with $Q\overline{Q}\psi = \lambda\psi$ for $\lambda > 0$, then

$$\psi' \equiv \frac{1}{\sqrt{\lambda}}\overline{Q}\psi \in V_{f+3}^- \quad \text{with} \quad \overline{Q}Q\psi' = \lambda\psi'.$$

This can be inverted to give $\psi = \frac{1}{\sqrt{\lambda}}Q\psi'$.

Hence

$$\begin{aligned} \overline{Q}(V_f^+) &= V_{f+3}^- \\ Q(V_{f+3}^-) &= V_f^+ \end{aligned} \,.$$

EXAMPLE

- $N_c = 4$

f	0	1	2	3	4
N_f^+	1	1	0	0	0
N_f^-	0	0	0	1	1
N_f^z	0	3	6	3	0

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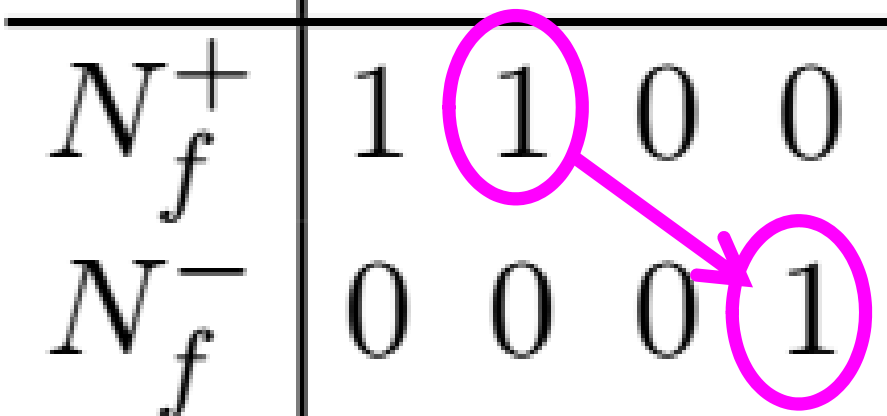

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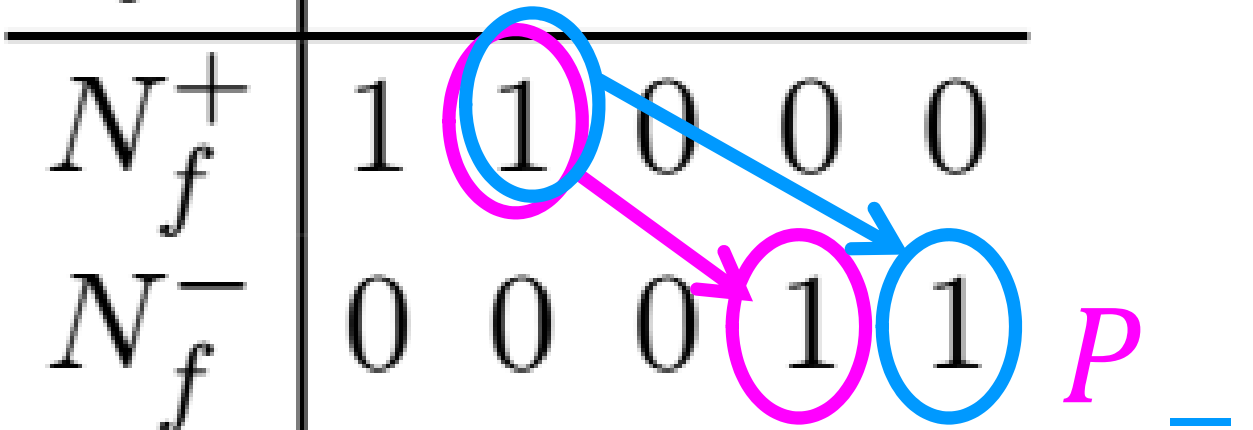


EXAMPLE

- $N_c = 8$

f	0	1	2	3	4	5	6	7	8
N_f^+	1	8	28	28	8	1	0	0	0
N_f^-	0	0	0	1	8	28	28	8	1
N_f^z	0	0	0	27	54	27	0	0	0

EXAMPLE

- $N_c = 8$

P

f	0	1	2	3	4	5	6	7	8
N_f^+	1	8	28	28	8	1	0	0	0
N_f^-	0	0	0	1	8	28	28	8	1
N_f^z	0	0	0	27	54	27	0	0	0

EXAMPLE

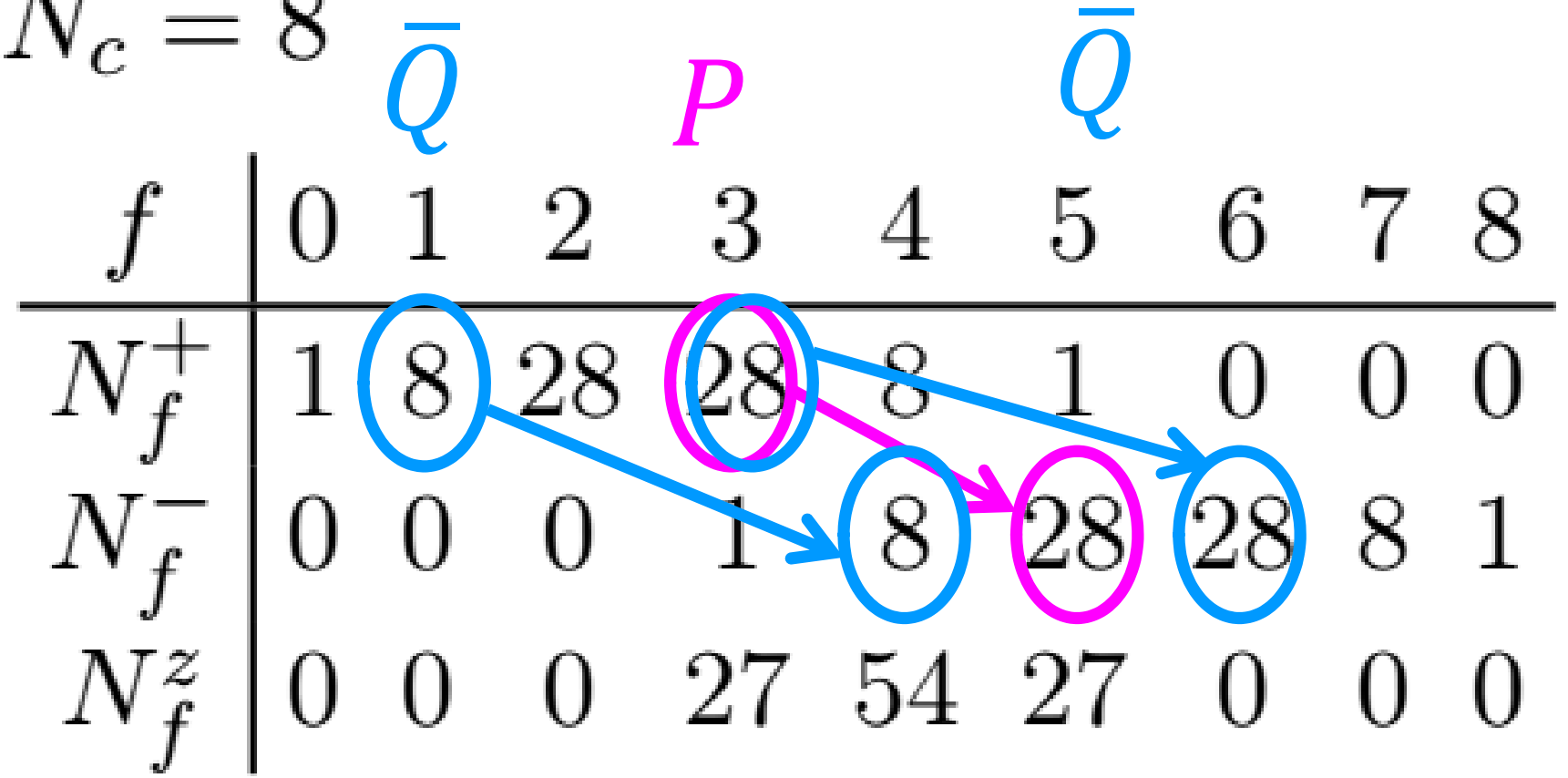
- $N_c = 8$

P $\bar{Q}$

f	0	1	2	3	4	5	6	7	8
N_f^+	1	8	28	28	8	1	0	0	0
N_f^-	0	0	0	1	8	28	28	8	1
N_f^z	0	0	0	27	54	27	0	0	0

EXAMPLE

- $N_c = 8$



f	0	1	2	3	4	5	6	7	8
N_f^+	1	8	28	28	8	1	0	0	0
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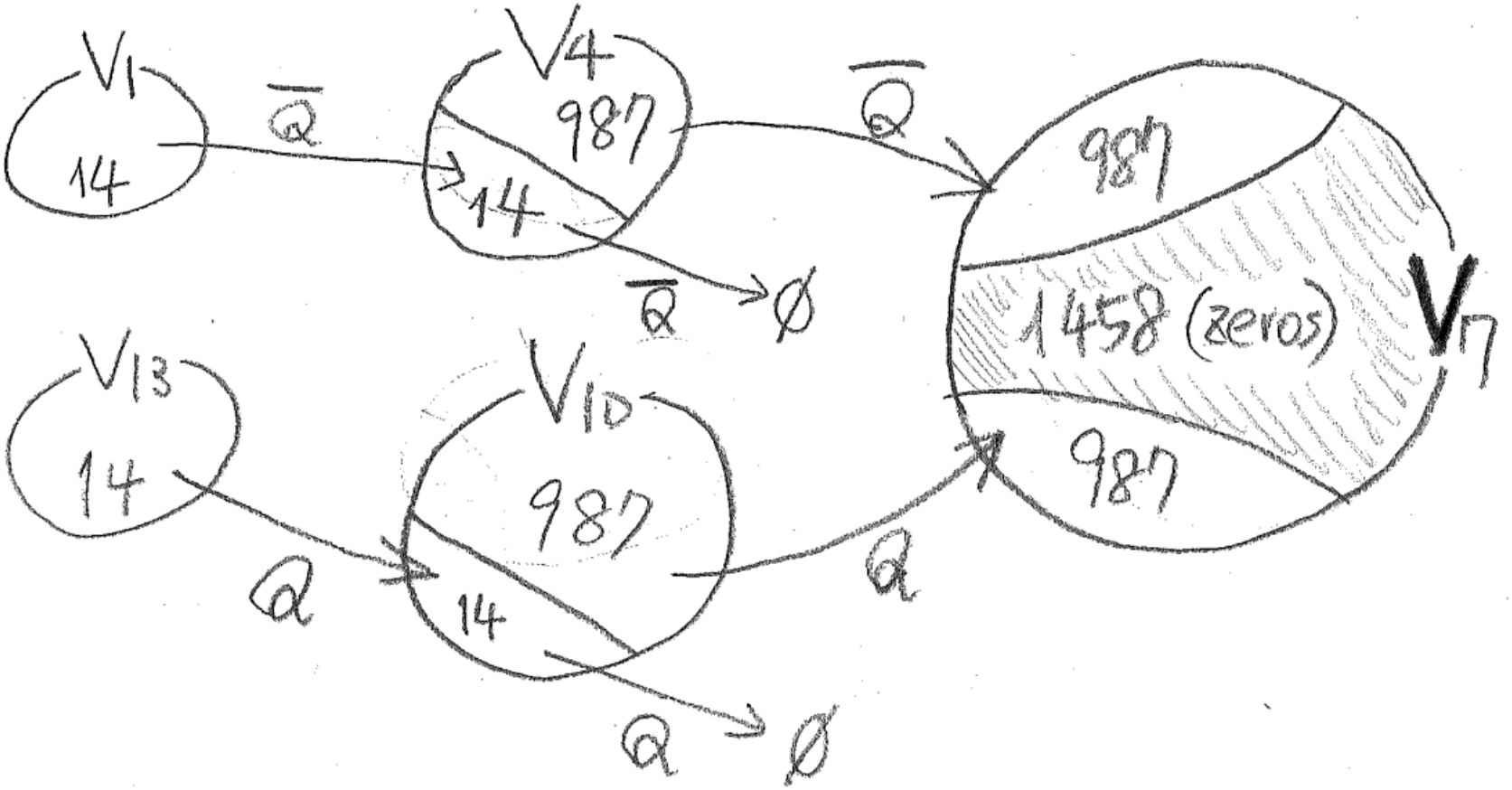
EXAMPLE

- $N_c = 14$



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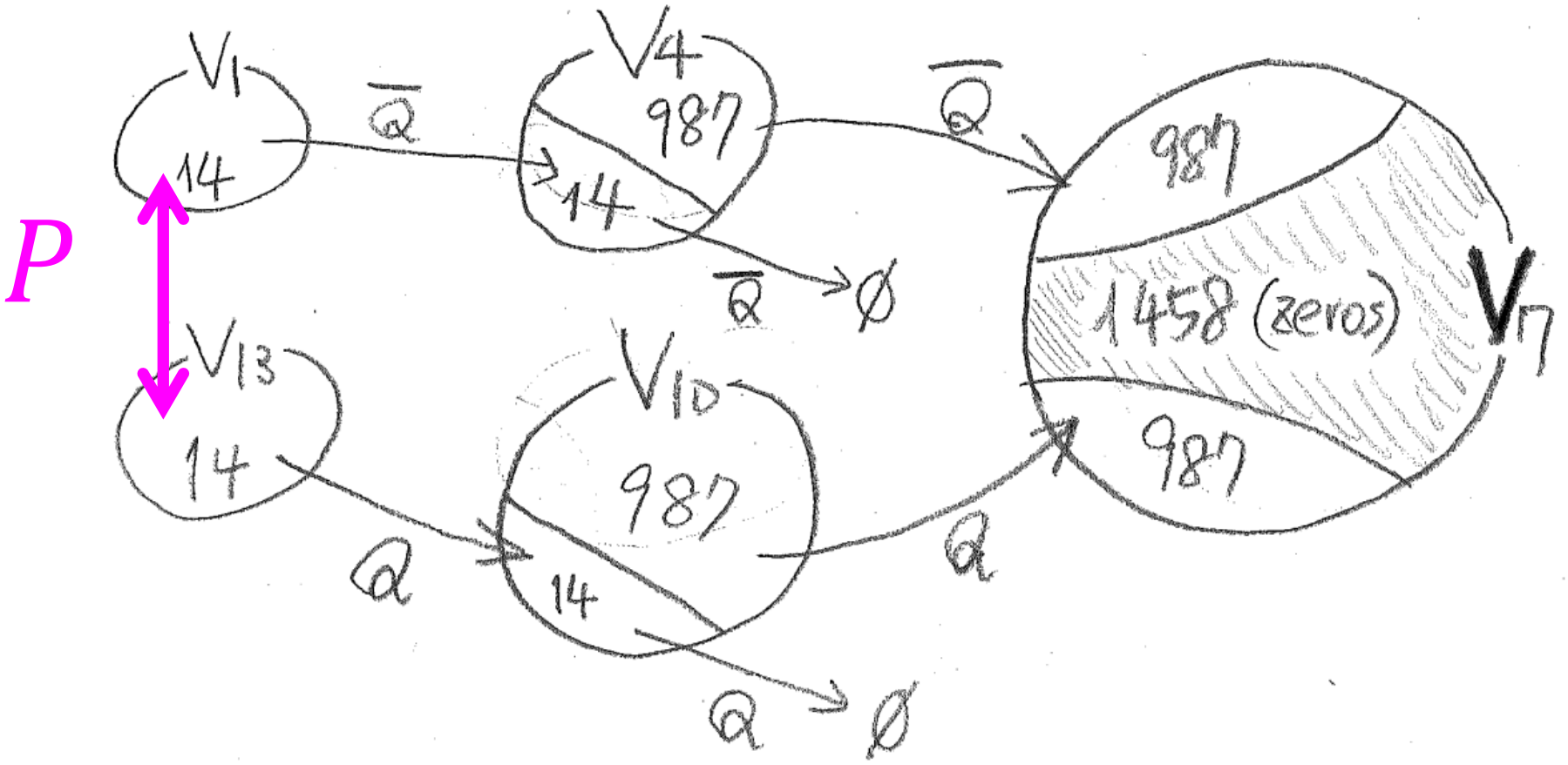

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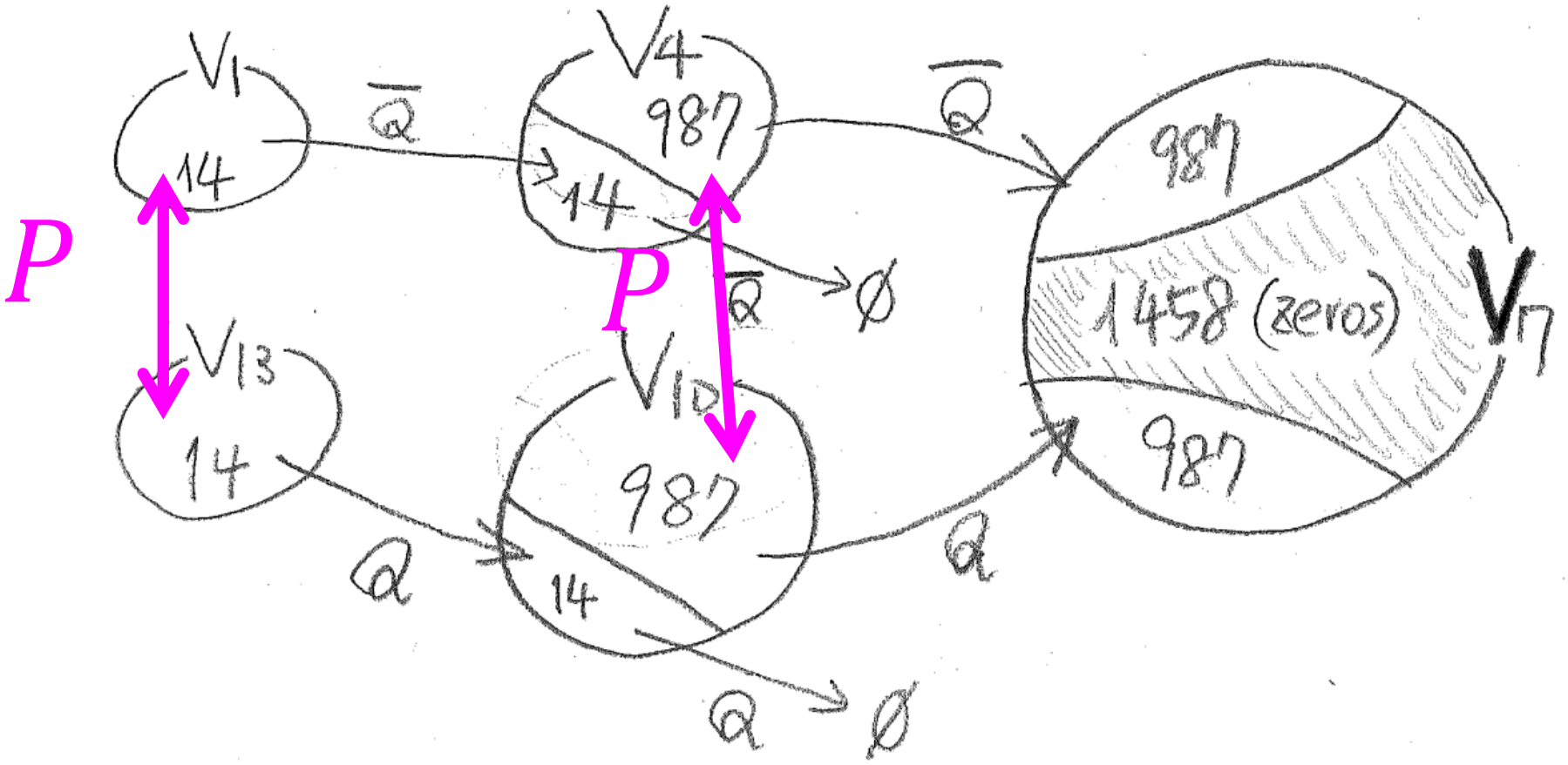

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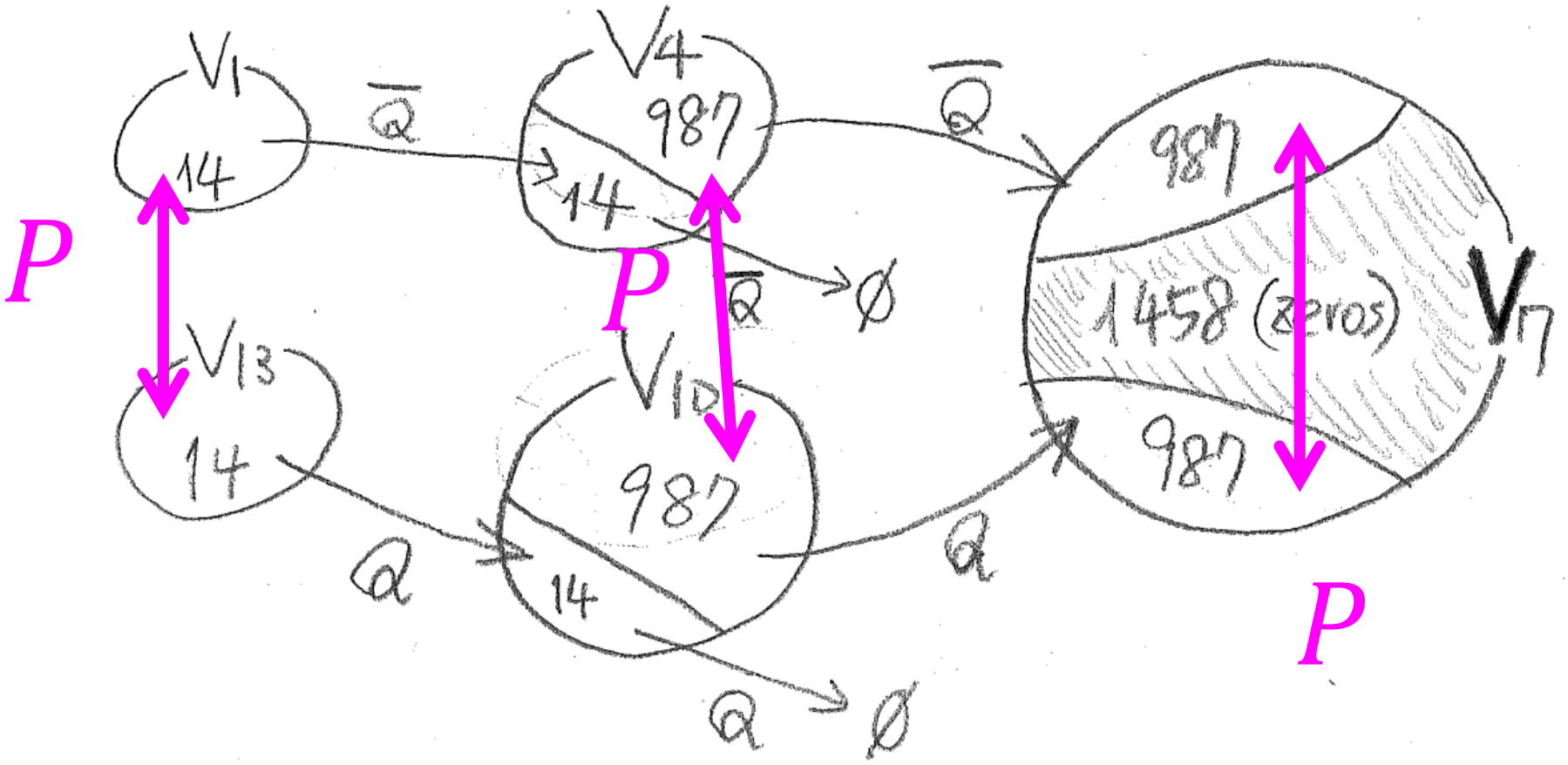

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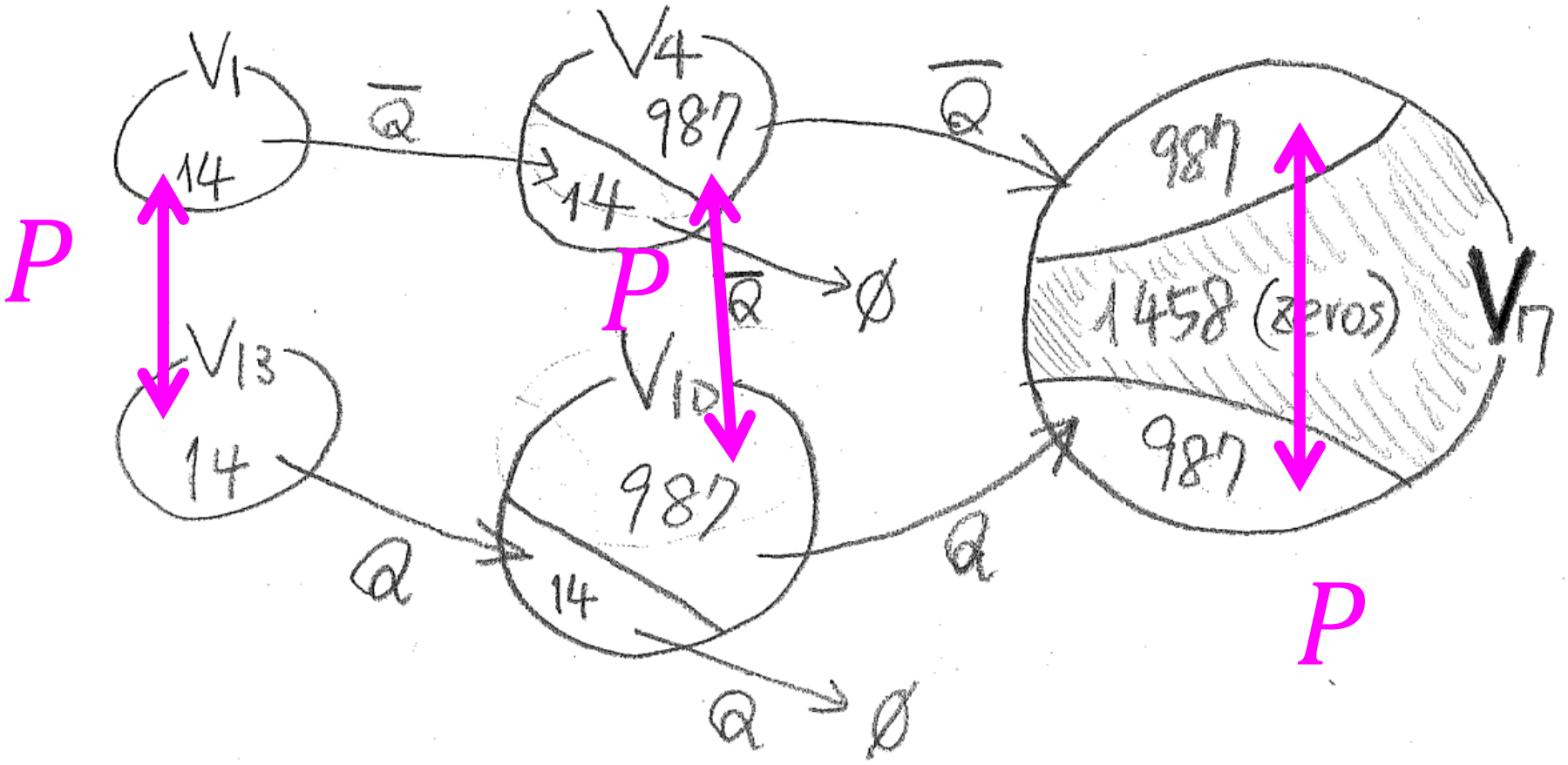

P doesn't lead to GSE/GOE in V_7 because it just exchanges two identical copies of GUEs.

EXAMPLE

- $N_c = 14$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

EXAMPLE

- $N_c = 14$

$\bar{Q}$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

EXAMPLE

- $N_c = 14$

$\bar{Q}$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

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- $N_c = 14$

$\bar{Q}$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

In general, for each N_c there are 3 sequences connected by $\bar{Q}$ (Q).

EXAMPLE

- $N_c = 14$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

Quadruplets are formed by

$$\psi \in V_f^+, \quad \overline{Q}\psi \in V_{f+3}^-, \quad P\psi \in V_{N_c-f}^-, \quad \text{and} \quad P\overline{Q}\psi \in V_{N_c-f-3}^+$$

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f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
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- $N_c = 14$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N_f^+	1	14	91	363	987	1911	1911	987	363	91	14	1	0	0	0
N_f^-	0	0	0	1	14	91	363	987	1911	1911	987	363	91	14	1
N_f^z	0	0	0	0	0	0	729	1458	729	0	0	0	0	0	0

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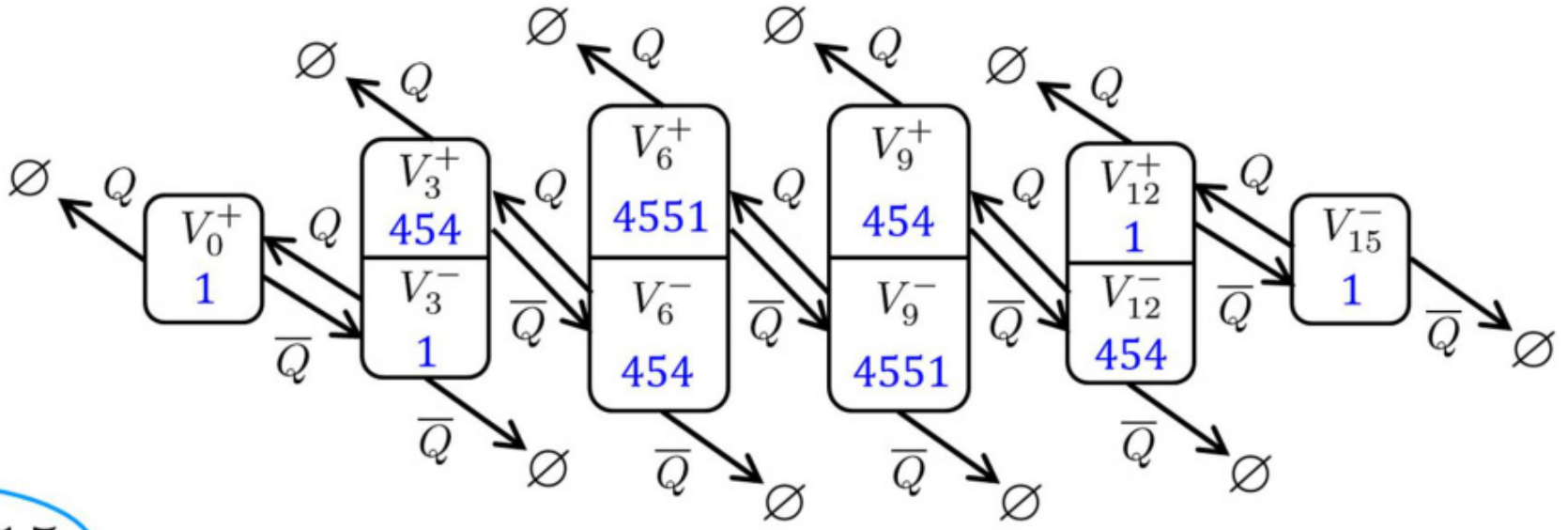
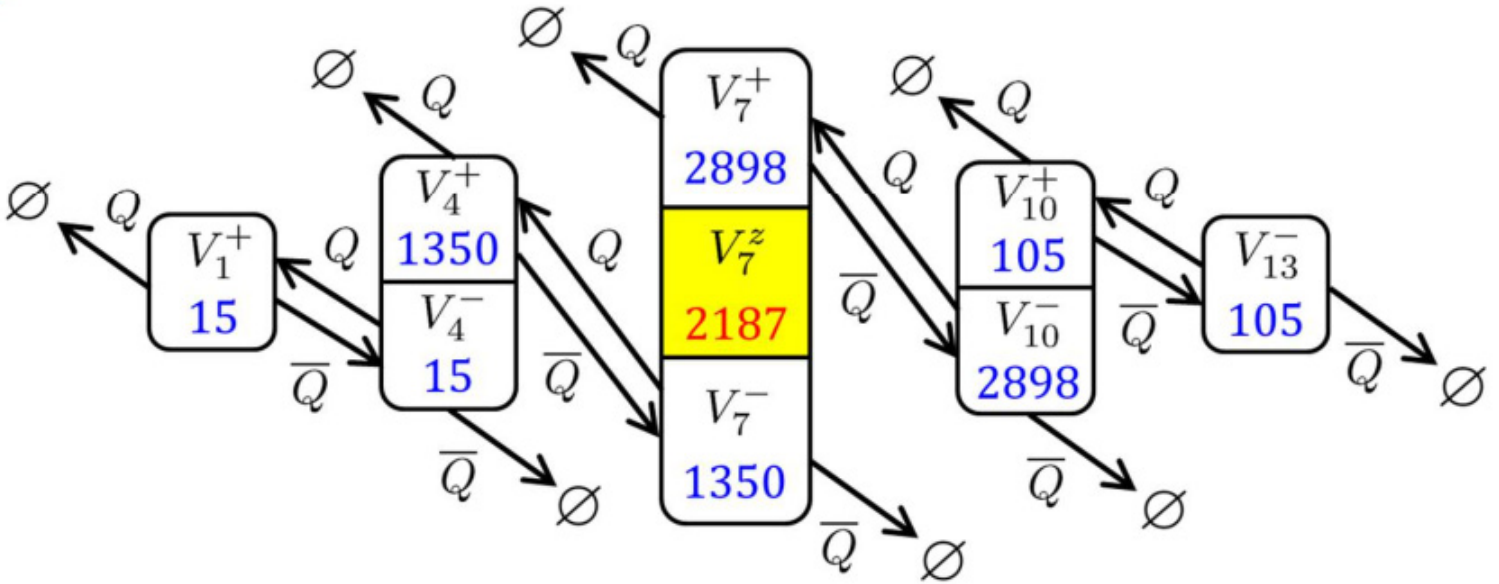
EXAMPLE

$N_c = 15$



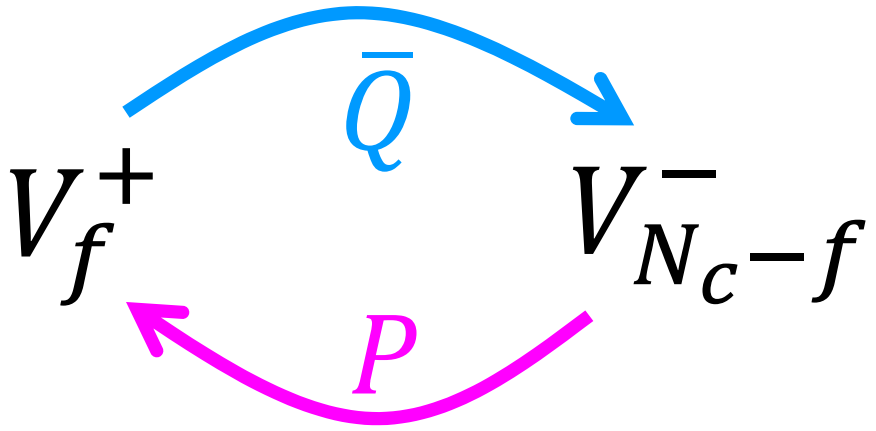


ANTI-UNITARY SYMMETRY

Candidates are $P\bar{Q}$ and PQ.

They act <u>within</u> an irreducible block $V_f^{\pm}$ only if the distance of f and $N_c - f$ is 3.

ex)



This is possible if $N_c - f = f + 3$.

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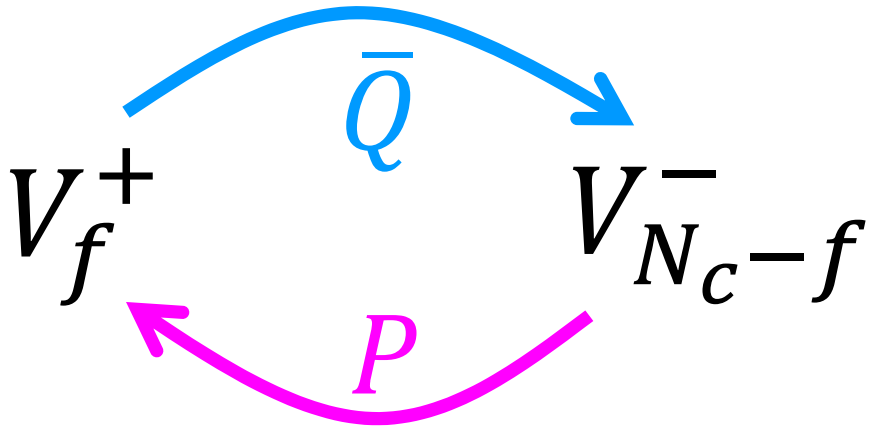


This is possible if $N_c - f = f + 3$. ➔ N_c must be odd

Sign of $(P\bar{Q})^2$ and $(PQ)^2$ can be easily determined from the table below.

$N_c \pmod 4$	P^2		
0	$+1$	$PQ = \bar{Q}P,$	$P\bar{Q} = QP$
1	$+1$	$PQ = -\bar{Q}P,$	$P\bar{Q} = -QP$
2	-1	$PQ = \bar{Q}P,$	$P\bar{Q} = QP$
3	-1	$PQ = -\bar{Q}P,$	$P\bar{Q} = -QP$

EXAMPLE

■: GOE, ■: GSE

■ $N_c = 5$

f	0	1	2	3	4	5
N_f^+	1	4	1	0	0	0
N_f^-	0	0	0	1	4	1
N_f^z	0	1	9	9	1	0

■ $N_c = 7$

f	0	1	2	3	4	5	6	7
N_f^+	1	7	21	7	1	0	0	0
N_f^-	0	0	0	1	7	21	7	1
N_f^z	0	0	0	27	27	0	0	0

■ $N_c = 13$

f	0	1	2	3	4	5	6	7	8	9	10	11	12	13
N_f^+	1	13	78	285	702	1208	702	285	78	13	1	0	0	0
N_f^-	0	0	0	1	13	78	285	702	1208	702	285	78	13	1
N_f^z	0	0	0	0	0	1	729	729	1	0	0	0	0	0

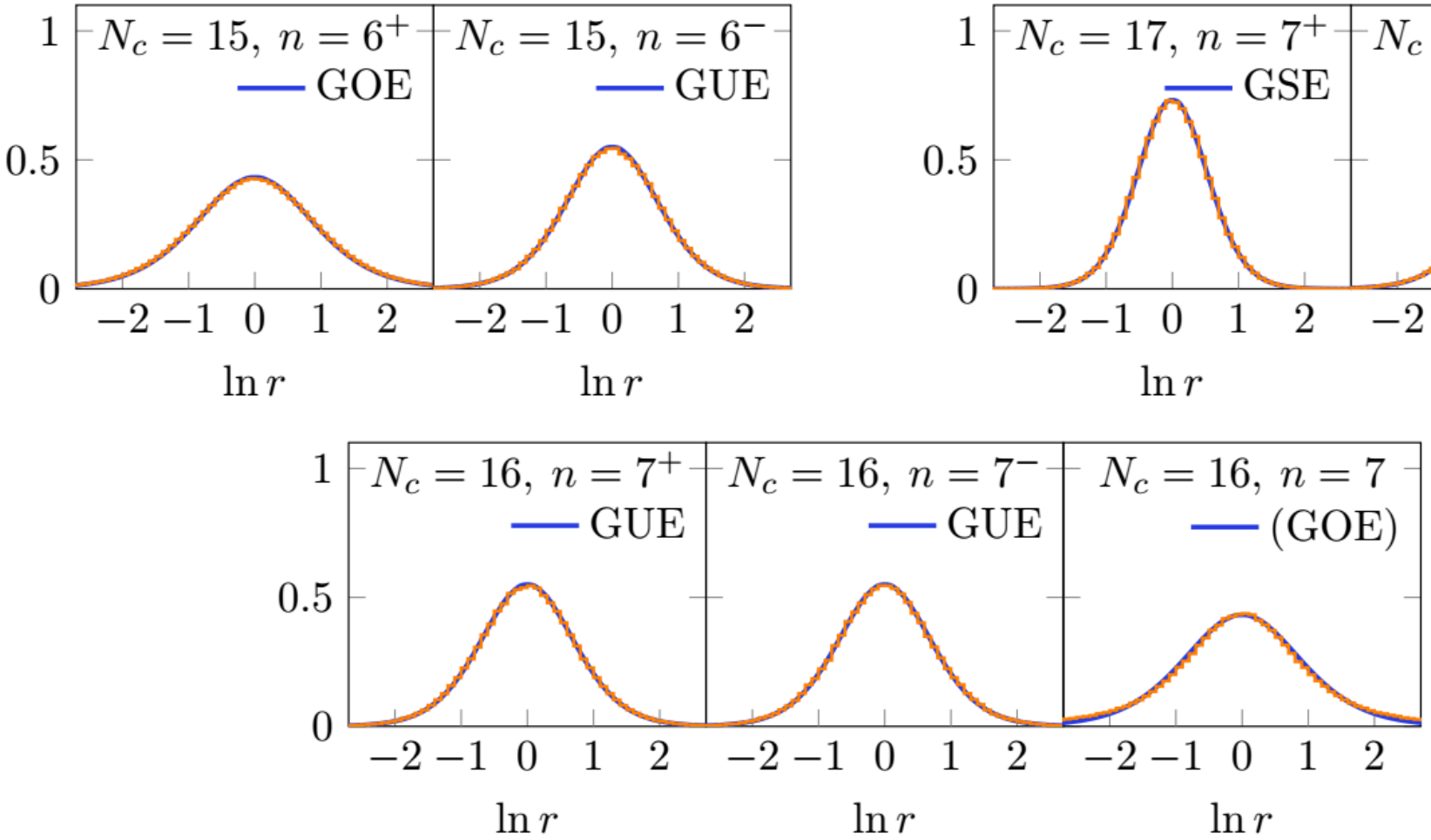
GENERAL RULE

TK & Wettig, 1706.03044

	$N_c = 0, 2 \pmod 4$	$N_c = 1 \pmod 4$	$N_c = 3 \pmod 4$
V_f^+	**GUE** for $\forall f$	**GSE** for $f = \frac{N_c-3}{2}$ **GUE** for $f \neq \frac{N_c-3}{2}$	**GOE** for $f = \frac{N_c-3}{2}$ **GUE** for $f \neq \frac{N_c-3}{2}$
V_f^-	**GUE** for $\forall f$	**GSE** for $f = \frac{N_c+3}{2}$ **GUE** for $f \neq \frac{N_c+3}{2}$	**GOE** for $f = \frac{N_c+3}{2}$ **GUE** for $f \neq \frac{N_c+3}{2}$

NUMERICAL CHECK

✓ **Agree**



The number of realizations used for averaging is 10^3 for $N_c = 14$ and 10^2 for $N_c = 15$ and 16.

GENERALIZED N=2 SYK

Fu–Gaiotto–Maldacena–Sachdev 2016

$H = \{Q, \overline{Q}\}$ with

$$Q = i^{(\hat{q}-1)/2} \sum_{1 \le i_1 < \cdots < i_{\hat{q}} \le N_c} X_{i_1 i_2 \cdots i_{\hat{q}}} c_{i_1} c_{i_2} \cdots c_{i_{\hat{q}}}$$

and

$$\overline{Q} = i^{(\hat{q}-1)/2} \sum_{1 \le i_1 < \cdots < i_{\hat{q}} \le N_c} \overline{X_{i_1 i_2 \cdots i_{\hat{q}}}}\, \overline{c}_{i_1} \overline{c}_{i_2} \cdots \overline{c}_{i_{\hat{q}}} \,.$$

MORE GENERAL RULE FOR ODD $\hat{q}$

TK & Wettig, 1706.03044

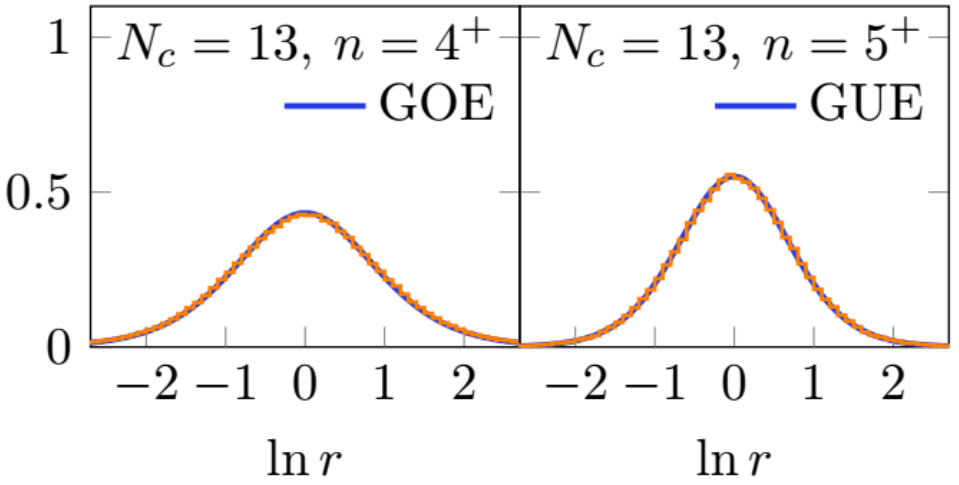
$N_c \pmod 4$	P^2			
0	$+1$	$PQ = (-1)^{\frac{\hat{q}+1}{2}} \overline{Q} P,$	$P\overline{Q} = (-1)^{\frac{\hat{q}+1}{2}} Q P$	
1	$+1$	$PQ = (-1)^{\frac{\hat{q}-1}{2}} \overline{Q} P,$	$P\overline{Q} = (-1)^{\frac{\hat{q}-1}{2}} Q P$	$[P, H] = 0$
2	-1	$PQ = (-1)^{\frac{\hat{q}+1}{2}} \overline{Q} P,$	$P\overline{Q} = (-1)^{\frac{\hat{q}+1}{2}} Q P$	for all N_c .
3	-1	$PQ = (-1)^{\frac{\hat{q}-1}{2}} \overline{Q} P,$	$P\overline{Q} = (-1)^{\frac{\hat{q}-1}{2}} Q P$	

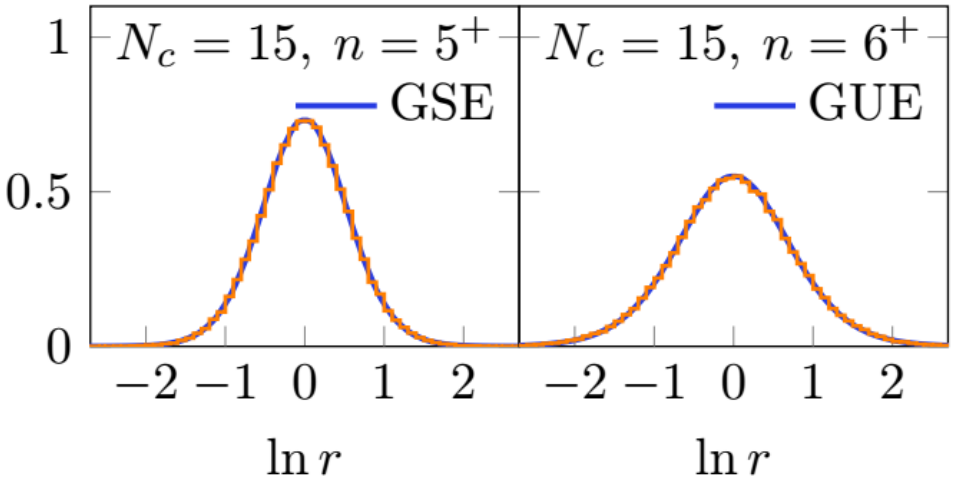
MORE GENERAL RULE FOR ODD $\hat{q}$

TK & Wettig, 1706.03044

	$N_c = 0,\ 2 \pmod 4$	$N_c = 1 \pmod 4$	$N_c = 3 \pmod 4$
V_f^+	**GUE** for $\forall f$	**GOE** if $\hat{q} = 1 \pmod 4$ **GSE** if $\hat{q} = 3 \pmod 4$ for $f = \frac{N_c - \hat{q}}{2}$ **GUE** for $f \neq \frac{N_c - \hat{q}}{2}$	**GSE** if $\hat{q} = 1 \pmod 4$ **GOE** if $\hat{q} = 3 \pmod 4$ for $f = \frac{N_c - \hat{q}}{2}$ **GUE** for $f \neq \frac{N_c - \hat{q}}{2}$
V_f^-	**GUE** for $\forall f$	**GOE** if $\hat{q} = 1 \pmod 4$ **GSE** if $\hat{q} = 3 \pmod 4$ for $f = \frac{N_c + \hat{q}}{2}$ **GUE** for $f \neq \frac{N_c + \hat{q}}{2}$	**GSE** if $\hat{q} = 1 \pmod 4$ **GOE** if $\hat{q} = 3 \pmod 4$ for $f = \frac{N_c + \hat{q}}{2}$ **GUE** for $f \neq \frac{N_c + \hat{q}}{2}$

NUMERICAL CHECK ($\hat{q} = 5$)





The number of realizations used for averaging is 10^3 for $N_c = 13$ and 10^2 for $N_c = 15$.

✓ **Agree**

Thank you for your attention!

BACKUP SLIDES

SCHWARZIAN EFT

Kitaev 2015, Maldacena & Stanford 2016

$$S[f] = -C\int_0^\beta d\tau \{F,\tau\}, \qquad F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right),$$

$$\{f,\tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \qquad f(\tau+\beta) = f(\tau)+\beta$$

$S[f]$ is invariant under $SL(2,\mathbb{R})$ Möbius transformations

$$F \to \frac{aF+b}{cF+d}. \qquad ad-bc=1.$$

Partition function

$$Z(\beta) = \int_{\mathcal{M}} \mathcal{D}f\, e^{-S[f]}, \qquad \mathcal{M} = \mathrm{Diff}(S^1)/SL(2,\mathbb{R}).$$

$\overline{D}_f$ in the $\mathcal{N} = 2$ SYK model

$$\overline{D}_0 = \frac{1}{6}\left[2^N + 2\cdot 3^{N/2}\cos\frac{N\pi}{6} + 2\cos\frac{N\pi}{3}\right],$$
$$\overline{D}_1 = \frac{1}{6}\left[2^N - 2\cdot 3^{N/2}\cos\frac{(N+4)\pi}{6} + 2\cos\frac{(N-2)\pi}{3}\right],$$
$$\overline{D}_2 = \frac{1}{6}\left[2^N + 2\cdot 3^{N/2}\cos\frac{(N-4)\pi}{6} + 2\cos\frac{(N+2)\pi}{3}\right],$$
$$\overline{D}_3 = \frac{1}{6}\left[2^N - 2\cdot 3^{N/2}\cos\frac{N\pi}{6} + 2\cos\frac{N\pi}{3}\right],$$
$$\overline{D}_4 = \frac{1}{6}\left[2^N + 2\cdot 3^{N/2}\cos\frac{(N+4)\pi}{6} + 2\cos\frac{(N-2)\pi}{3}\right],$$
$$\overline{D}_5 = \frac{1}{6}\left[2^N - 2\cdot 3^{N/2}\cos\frac{(N-4)\pi}{6} + 2\cos\frac{(N+2)\pi}{3}\right].$$

$(N = N_c)$

$N_f^{\pm}$

The case of even N_c

For $3 \leq f < N_c/2 - 1$, V_f does not contain zero modes.
Hence

$$N_f^+ = \binom{N_c}{f} - N_f^- = \binom{N_c}{f} - N_{f-3}^+ .$$

initial conditions

$$N_0^+ = 1, \quad N_1^+ = N_c, \quad N_2^+ = N_c(N_c-1)/2.$$

$$N_f^+ = N_{N_c-f}^- = (-1)^{f+f_0} N_{f_0}^+ + (-1)^f \sum_{n=1}^{(f-f_0)/3} (-1)^{3n+f_0} \binom{N_c}{3n+f_0},$$

$$N_f^- = N_{N_c-f}^+ = \binom{N_c}{f} - N_f^+ ,$$

where $f_0 \equiv f - 3\lfloor f/3 \rfloor \in \{0,1,2\}$.
These formulas hold in the range $0 \leq f < N_c/2 - 1$.