T-DUALITY AND SCATTERING OF STRINGY STATES

Jnan Maharana

November 1, 2017

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Nucl. Phys. B896, 657.

INTRODUCTION

- Target space Duality (T-duality) is a special attribute of string theory. It has very important roles in our understandings of string dynamics in diverse dimensions. Moreover, when judiciously employed, T-duality is a powerful tool to generate new string vacua from known solutions.
- If we consider evolution of a closed bosonic string in the background of its massless excitations, these backgrounds are required to satisfy 'Equations of Motion'. We construct effective action in the target space from these equations of motion.

INTRODUCTION

- Target space Duality (T-duality) is a special attribute of string theory. It has very important roles in our understandings of string dynamics in diverse dimensions. Moreover, when judiciously employed, T-duality is a powerful tool to generate new string vacua from known solutions.
- If we consider evolution of a closed bosonic string in the background of its massless excitations, these backgrounds are required to satisfy 'Equations of Motion'. We construct effective action in the target space from these equations of motion.
- When we compactify the theory to lower dimensions on a d-dimensional torus, the resulting reduced effective action is expressed in a T-duality invariant form. Solutions to the equations motion are associated with vacuum configurations of the backgrounds.
- Given a set of initial backgrounds, suitable choice of T-duality transformations generate a new set which are also identified as new inequivalent vacua. This technique has proved to be a powerful tool in generating new solutions in black hole physics, in string cosmology,.. This is a perspective from the reduced effective action approach.

- T-duality from the worldsheet perspective:
  Let us consider evolution of a closed bosonic string in the background of its massless excitations, graviton, antisymmetric tensor ... in $\hat{D}$-dimensions. If we compactify on d-dimensional torus T^d, the backgrounds will be decomposed accordingly.

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- The worldsheet action is

$$S = \frac{1}{2}\int d\sigma d\tau \left(\hat{G}_{\hat{\mu}\hat{\nu}}(\hat{X}) \partial_a X^{\hat{\mu}} \partial^a X^{\hat{\nu}} + \epsilon^{ab} \hat{B}_{\hat{\mu}\hat{\nu}}(\hat{X}) \partial_a X^{\hat{\mu}} \partial_b X^{\hat{\nu}} \right)$$

  $\hat{G}_{\hat{\mu}\hat{\nu}}(\hat{X})$ and $\hat{B}_{\hat{\mu}\hat{\nu}}(\hat{X})$ are the backgrounds in $\hat{D}$-dimensional spacetime.
- Consider a scenario where the backgrounds depend only on some of the coordinates, X^μ and are independent of the rest Y^α

$$\hat{X}^{\hat{\mu}} = \left(X^\mu, Y^\alpha \right), \quad \mu = 0, 1..D-1, \ \alpha = D, ..\hat{D}-1$$

  with $\hat{D} = D + d$.
- In a simple compactification scheme (Hassan-Sen) - $\hat{G}$ and $\hat{B}$ are assumed to take the block diagonal form:

- $$\hat{G}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu}(X) & 0 \\ 0 & G_{\alpha\beta}(X) \end{pmatrix}$$
$$\hat{B}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} b_{\mu\nu}(X) & 0 \\ 0 & B_{\alpha\beta}(X) \end{pmatrix}$$
- Introduce a pair of vectors $\mathcal{V}$ and $\mathcal{W}$ of dimensions $2D$ and $2d$ respectively
$$\mathcal{V} = \begin{pmatrix} P_\mu \\ X'^\mu \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} P_\alpha \\ Y'^\alpha \end{pmatrix}$$
- P_μ, P_α are canonical momenta and $\{X'\mu, Y'^\alpha\}$ are σ-derivatives of $\{X^\mu, Y^\alpha\}$.
- The canonical Hamiltonian density is expressed as
$$\mathcal{H}_c = \frac{1}{2}\Big(\mathcal{V}^T \tilde{M} \mathcal{V} + \mathcal{W}^T M \mathcal{W}\Big)$$

- Whereas $\tilde{M}(X)$ is a $2D \times 2D$ matrix, $M(X)$ is another $2d \times 2d$ matrix. These two matrices are

$$\tilde{M} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho} b_{\rho\nu} \\ b_{\mu\rho} g^{\rho\nu} & g_{\mu\nu} - b_{\mu\rho} g^{\rho\lambda} b_{\lambda\nu} \end{pmatrix}$$

-

$$M = \begin{pmatrix} G^{\alpha\beta} & -G^{\alpha\gamma} B_{\gamma\beta} \\ B_{\alpha\gamma} G^{\gamma\beta} & G_{\alpha\beta} - B_{\alpha\gamma} G^{\gamma\delta} B_{\delta\beta} \end{pmatrix}$$

- Let us focus on the second term and define

$$H_2 = \frac{1}{2} \mathcal{W}^T M \mathcal{W}$$

- Note that H_2 is invariant under the global $O(d,d)$ transformations defined below

$$M \to \Omega M \Omega^T,\ \mathcal{W} \to \Omega \mathcal{W},\ \Omega^T \eta \Omega = \eta,\ \Omega \in O(d,d),\ \eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

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- REMARKS:
- We have not used periodicity of $\{Y^\alpha\}$. When we compactify on T^d, Y^α are periodic and are the compact coordinates.
- The moduli G and B parametrize the coset $\frac{O(d,d)}{O(d)\times O(d)}$.

- For compactification on T^d, in general one adopts the prescription

$$\hat{e}_M^A = \begin{pmatrix} e_\mu^r(X) & A_\mu^{(1)\beta}(X)E_\beta^a(X) \\ 0 & E_\alpha^a(X) \end{pmatrix}$$

- $\hat{G}(\hat{X}) = \hat{e}\hat{e}^T$,the D-dimensional spacetime metric is: $g(X) = e(X)e^T(X)$, $G = EE^T$.
- $A_\mu^{(1)\beta}$ are the abelian gauge fields due to the isometries associated with d-dimensional torus. The antisymmetric tensor gets decomposed as

$$\hat{B}_{\hat{\mu}\hat{\nu}}(\hat{X}) = \begin{pmatrix} b_{\mu\nu}(X) & B_{\mu\alpha}(X) \\ B_{\nu\beta}(X) & B_{\alpha\beta}(X) \end{pmatrix}$$

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- Note that there are also d abelian gauge fields $B_{\mu\alpha}(X)$ appearing due to compactification of $\hat{B}$-field.
  The the set of $2d$ gauge fields $A_\mu^{(1)\alpha}$ and $A_{\mu\alpha}^{(2)} = B_{\mu\alpha}(X) + B_{\alpha\beta}A_\mu^{(1)\beta}$ transform as an $O(d,d)$ doublet (see JM+JHS).

T-DUALITY IN WEAK BACKGROUND APPROXIMATION

- Let us recapitulate how we employ weak field approximation
- An example: the graviton vertex operator

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + a\tilde{h}_{\mu\nu}(X)$$

where $g^{(0)}_{\mu\nu}$ is the flat Minkowski space metric and 'a' is a small expansion parameter (is related to Newton's constant in linearized approximation) $\tilde{h}_{\mu\nu}(X)$ is the graviton in the weak field approximation.

- The graviton vertex is

$$\tilde{h}_{\mu\nu}(X)\partial X^{\mu}\bar{\partial}X^{\nu}$$

and the canonical momentum is defined from the free string Lagrangian, $P_{\mu} = \dot{X}^{\mu}$; moreover, $\partial X^{\mu} = \dot{X}^{\mu} + X'^{\mu}$ and $\bar{\partial}X^{\mu} = \dot{X}^{\mu} - X'^{\mu}$.

- The vertex operator is required to be $(1, 0)$ and $(0, 1)$ primary with respect to the free string stress energy momentum tensors, T_{++} and T_{--} respectively and these are given by

$$T_{++} = \frac{1}{2}\partial X^{\mu}\partial X_{\mu}, \qquad T_{--} = \frac{1}{2}\bar{\partial} X^{\mu}\bar{\partial} X_{\mu}$$

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- when we demand the vertex operator to have conformal weight $(1, 1)$, we arrive at

$$\nabla^2 \tilde{h}_{\mu\nu}(X) = 0, \qquad \partial^{\mu}\tilde{h}_{\mu\nu}(X) = \partial^{\nu}\tilde{h}_{\mu\nu}(X) = 0$$

  Here ∇^2 is the flat space D-dimensional Laplacian.

- If we express $\tilde{h}_{\mu\nu} = e^{ik.X}\epsilon_{\mu\nu},\ \epsilon_{\mu\nu}$ being the polarization tensor of graviton. The above constraints translate to the familiar equations

$$k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = k^\nu \epsilon_{\mu\nu} = 0$$

- Similarly, we can write

$$B_{\mu\nu} = b^{(0)}_{\mu\nu} + a\tilde{b}_{\mu\nu}(X)$$

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- However, presence of this constant antisymmetric tensor in the free string worldsheet action, does not contribute to the equations of motion (we are dealing with noncompact string coordinates) and we can still decompose $X^\mu(\sigma,\tau) = X^\mu_L + X^\mu_R$. Thus we can ignore the presence of this constant tensor in all our subsequent discussion. Moreover, $\tilde{b}_{\mu\nu}(X)$ also satisfies similar constraints as $\tilde{h}_{\mu\nu}(X)$.

- For $\tilde{b}_{\mu\nu}(X)$ constraints are

$$\nabla^2 \tilde{b}_{\mu\nu}(X) = 0, \qquad \partial^\mu \tilde{b}_{\mu\nu}(X) = \partial^\nu \tilde{b}_{\mu\nu}(X) = 0$$

- The weak field approximation for the scalar backgrounds $G_{\alpha\beta}(X)$ and $B_{\alpha\beta}(X)$ are

$$G_{\alpha\beta}(X) = \delta_{\alpha\beta} + a\ h_{\alpha\beta}(X), \quad \text{and} \quad B_{\alpha\beta} = a\ b_{\alpha\beta}(X)$$

'a' being the expansion parameter. We might loosely use the term toroidal compactification here and elsewhere. It is understood that we do not account for special features of winding modes etc throughout this discussion.

- The backgrounds $h_{\alpha\beta}(X)$ and $b_{\alpha\beta}(X)$ satisfy

$$\nabla^2 h_{\alpha\beta}(X) = 0, \quad \text{and} \quad \nabla^2 b_{\alpha\beta}(X) = 0$$

If we had considered the gauge fields $A^{\alpha}_{\mu}(X)$ and $B_{\alpha\mu}(X)$ they will couple to the worldsheet coordinates X^{μ} and Y^{α}. The conformal invariance leads to equations of motion

$$\nabla^2 A^{\alpha}_{\mu}(X) = 0, \quad \text{and} \quad \nabla^2 B_{\alpha\mu}(X) = 0$$

and the transversality conditions are

$$\partial^{\mu} A^{\alpha}_{\mu}(X) = 0, \quad \text{and} \quad \partial^{\mu} B_{\alpha\mu}(X) = 0$$

- Note that the Hamiltonian density associated with compact coordinates, the presence of X-dependent backgrounds can be expressed in $O(d,d)$ invariant form. In order to construct the corresponding vertex operators, we have to resort to the weak field approximation.
- Therefore, it is desirable to adopt a weak field expansion prescription for the M-matrix. We propose

$$M = \mathbf{1} + a\tilde{M}$$

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- Therefore, it is desirable to adopt a weak field expansion prescription for the M-matrix. We propose

$$M = \mathbf{1} + a\tilde{M}$$

- $\mathbf{1}$ is the $2d \times 2d$ unit matrix, 'a' is the small expansion parameter - we might omit 'a' in future in such expansion. The backgrounds, G^{-1}, G and B appearing in the definition of M, have weak field expansions as mentioned above. Therefore, $\tilde{M}$ can be expressed in terms of $h^{\alpha\beta}$, $h_{\alpha\beta}$ and $b_{\alpha\beta}$ (note that $h^{\alpha\beta}$ and $h_{\alpha\beta}$ are related and see below).
- Note: $M\eta M = \eta$, since $M \in O(d,d)$ (is symmetric). Therefore, $\tilde{M}$ satisfies

$$\tilde{M}\eta + \eta\tilde{M} = 0$$

$\tilde{M}$ is not an $O(d,d)$ matrix unlike M. Moreover,

- since $G_{\alpha\beta} G^{\beta\gamma} = \delta_\alpha^\gamma$

$$h^{\alpha\beta} = -\delta^{\alpha\gamma} h_{\gamma\sigma} \delta^{\sigma\beta}$$

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- This relation is consistent with the constraint on $\tilde{M}$.
- Moreover, by retaining terms to order 'a', we notice that block the diagonal elements of $\tilde{M}$ are $h^{\alpha\beta}$ and $h_{\alpha\beta}$ and the two off diagonal block elements are $-b$ and b. Thus $\tilde{M}$ is also a symmetric matrix like M-matrix; however, it satisfies a different condition.

- STEPS TO GENERATE NEW BACKGROUNDS FROM GIVEN ONES

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- Let us start with initial backgrounds G_{in} and B_{in} and define the matrix M_{in} and corresponding $\tilde{M}_{in}$, defined in the weak field approximation. Let Ω be the desired $O(d,d)$ transformation. Then the new matrix, M_f is given by

$$M_{in} \to M_f = \Omega M_{in} \Omega^T$$

- The Prescription:
- Just as M_{in} has weak field approximation $M_{in} = \mathbf{1} + \tilde{M}_{in}$; having obtained M_f, it will have its weak field expansion $M_f = \mathbf{1} + \tilde{M}_f$. In order to decompose M_f in this manner we might have to rescale the new metric appearing in the definition of M_f obtained through the $O(d,d)$ transformation in some cases.

- Let us assume that the new backgrounds generated by T-duality are denoted by 'primes' so that

$$\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \to \begin{pmatrix} G'^{-1} & -G'^{-1}B' \\ B'G'^{-1} & G' - B'G'^{-1}B' \end{pmatrix}$$

REMARKS:

- (i) We get G' by inverting top left block matrix G'^{-1}. (ii) Next we extract antisymmetric B' from the upper right element. Obviously G' and B' will be expressed in terms of G and B once we specify an $O(d,d)$ transformation.

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- (i) We get G' by inverting top left block matrix G'^{-1}. (ii) Next we extract antisymmetric B' from the upper right element. Obviously G' and B' will be expressed in terms of G and B once we specify an $O(d,d)$ transformation.
- (iii) We resort to the weak field approximation for G' and B'. Even if we initially have $B = 0$, we can end up with a nontrivial B' with suitable T-duality transformation. However, this method is not very efficient.
- We have to go back and forth to construct vertex operators. The transformation properties of S-matrix is not so easy to see. JM:arXiv:1310.4627.
  We seek a different approach

T-DUALITY INVARIANCE OF S-MATRIX

- The T-duality transformation is implemented on the vertex operators. Therefore, if we compute the S-matrix elements with a set of vertex operators, a T-duality transformation acting on one or more vertex operators will generate a new S-matrix element. We noted that implementing $O(d,d)$ transformation on $\tilde{M}$-matrix and generating new amplitude is a two step process and it is not very efficient.

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- It is known for while that the space of solutions of backgrounds enjoys an $O(d) \otimes O(d)$ symmetry (Sen). Moreover, $O(d) \otimes O(d) \in O(d,d)$. The set of matrices

$$\Omega_{RS} = \frac{1}{2}\begin{pmatrix} S+R & R-S \\ R-S & S+R \end{pmatrix}$$

- implement $O(d) \otimes O(d)$ transformations. The matrices R and S belong to $O(d) \otimes O(d)$. When we resort to linearized approximation for the backgrounds

$$G_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}, \quad \text{and} \quad B_{\alpha\beta} = b_{\alpha\beta}$$

- The linearized backgrounds transform as (Sen)

$$(h' + b') = S(h + b)R^T$$

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- Let us examine the structure of the vertex operator associated with the moduli $G_{\alpha\bar{\beta}}$ or $B_{\alpha\bar{\beta}}$ in linearized, plane-wave approximations:

$$V_i(\epsilon_i, k_i, X_i, Y_i) = \epsilon_{\alpha_i\bar{\beta}_i} : \exp[ik_i.X(z,\bar{z})]\partial Y^{\alpha_i}(z)\bar{\partial}Y^{\bar{\beta}_i}(\bar{z}) :$$

with $k_i^2 = 0$. Where $\epsilon_{\alpha_i\bar{\beta}_i}$ stands for polarization tensor of $G_{\alpha_i\bar{\beta}_i}$ or $B_{\alpha_i\bar{\beta}_i}$ depending on the choice we make and then it will be symmetric or antisymmetric under $\alpha_i \leftrightarrow \bar{\beta}_i$.

NOTE: The plane wave part, $\exp[ik_i.X_i]$, is inert under T-duality.

- We need the following correlation functions ($\alpha' = 2$) for closed string.

$$< \partial Y^{\alpha_i}(z_i)\partial Y^{\alpha_j}(z_j) >= -\frac{\delta^{\alpha_i\alpha_j}}{(z_i - z_j)^2},$$

$$< \bar{\partial}Y^{\bar{\beta}_i}(\bar{z}_i)\bar{\partial}Y^{\bar{\beta}_j}(\bar{z}_j) >= -\frac{\delta^{\bar{\beta}_i\bar{\beta}_j}}{(\bar{z}_i - \bar{z}_j)^2}$$

- NOTE: The products of $\epsilon_{\alpha_1\bar{\beta}_1}, \epsilon_{\alpha_2\bar{\beta}_2}, ...\epsilon_{\alpha_N\bar{\beta}_N}$ get contracted with various combinations of $\delta^{\alpha_i\alpha_j}, \delta^{\bar{\beta}_i\bar{\beta}_j}, ...$ which will come from pairwise contractions of holomorphic parts and antiholomorphic parts within the vertex operators.
- The plane wave contractions are: $<: \exp[ik_i.X_i(z_i, \bar{z}_i)] :: \exp[ik_j.X_j(z_j, \bar{z}_j)] :>= |z_i - z_j|^{2k_i.k_j}$.

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- The polarization tensors transform as follows under $O(d) \otimes O(d)$:

$$(\epsilon'^{G}_{\alpha_i\bar{\beta}_i} + \epsilon'^{B}_{\alpha_i\bar{\beta}_i}) = [S(\epsilon^G + \epsilon^B)R^T]_{\alpha_i\bar{\beta}_j}$$

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- The vertex operator for a gauge boson, arising from compactification on T^d is

$$V_i^A(\epsilon_i, k_i, X_i, Y_i) = \epsilon_{\mu_i\bar{\alpha}_i} : \exp[ik_i.X(z_i,\bar{z}_i)]\partial X^{\mu_i}(z_i)\bar{\partial}Y^{\bar{\alpha}_i}(\bar{z}_i)$$

The constraints are: (i) $k_i^2 = 0$ and (ii)$\epsilon_{\mu_i\bar{\alpha}_i}k^{\mu_i} = 0$. Recall $(\epsilon^G + \epsilon^B) \to S(\epsilon^G + \epsilon^B)R^T$ under $O(d) \otimes O(d)$. The gauge boson polarization transforms as $\epsilon^{(1)}_{\mu_i\bar{\alpha}_i} \to (S+R)\epsilon^{(1)}_{\mu_i\bar{\alpha}_i}$. Since $A^{(1)\alpha}_\mu$ and $A^{(2)}_{\mu\alpha}$ are $O(d,d)$ vectors.

- We are interested in *tree level* amplitudes for scattering of G and B and the amplitude with gauge bosons as external legs.

$$A_{G,B}^{(N)} = \int d^2z_1 d^2z_2..d^2z_N < \mathbf{\Pi}_{i=1}^{N} V_i(\epsilon_i, k_i, X_i, Y_i) > \mathcal{D}$$

and

$$T_A^{(N)} = \int d^2z_1 ... d^2z_N < \mathbf{\Pi}_{i=1}^{N} V_i^A(\epsilon_1, k_i, X_i, Y_i) > \mathcal{D}$$

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- The $\mathcal{D}$ is introduce the gauge fix residual worldsheet symmetry and allows us to choose three arbitrary coordinates and assign fixed values - so called Koba-Nielsen factor. Thus the integration is over $N-3$ complex variables.
- We focus on tree level amplitude. The computations are considerably simplified if one resort to the technique of KLT. We briefly recapitulate essential features of KLT formalism.

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- The principal goal of KLT was to exhibit relationship between closed string and open string tree level amplitudes. The N-point closed string amplitude is factorized into product of open string amplitudes with certain prefactors. The correspondence is valid for vertex operators associated with the leading Regge trajectories.
- For free closed bosonic string, we decompose the coordinates to left and right moving sectors. Moreover, vertex operators for leading Regge trajectories is a product of equal number of holomorphic and antiholomorphic operators.
- One set is $\mathbf{\Pi}_{i=1}^{M}\partial X^{\mu_i}(z_i)$ and the other one is $\mathbf{\Pi}_{i=1}^{N}\bar{\partial} X^{\nu_i}(\bar{z}_i)$. To satisfy level matching condition: $M = N$ at each level. Moreover, the factorization property of three point and four point amplitudes for closed string in terms of corresponding open string amplitudes were was explicitly computed by KLT.

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- The entire KLT mechanism is not utilized here for our purpose as we shall see later.

- To recollect: consider the open string vertex operators for tachyon and gauge boson.

$$V^T(k, X) =: \exp[ik.X] :, \quad \text{and} \quad V^A = \epsilon_\mu : \exp[ik.X]\partial X^\mu :$$

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- The tachyon and gauge boson are on-shell and polarization vector is transverse $k.\epsilon = 0$.
- In order to establish the correspondence between closed and open string amplitudes, one has to contruct vertex operators for all excited levels. KLT introduce a trick which is quite economical for this purpose.
- Instead of considering tachyon and gauge boson open string vertex operator, envisage the following vertex operator

$$V^{open}_{KLT}(\epsilon,k,X) =: \exp[ik.X + i\epsilon_\mu \partial X^\mu] :$$

- Expand above equation in powers of ϵ_μ. The linear term in the polarization vector reproduces gauge boson vertex operator. To compute the N-gauge boson amplitude, $T_A^{(N)}$, compute the correlation function of products of V_{KLT} and isolate the coefficient of the product $\mathbf{\Pi}^N_{i=i}\epsilon_{\mu_i}$.

- The vertex operator for generic excited level of open string is

$$V^{EX} \simeq \epsilon_{\mu_1\mu_2....\mu_m} : \exp[ik.X]\partial X^{\mu_1}\partial X^{\mu_2}...\partial X^{\mu_m} :$$

The excited state momentum k_μ has to be on-mass-shell and the polarization tensor, $\epsilon_{\mu_1\mu_2....\mu_m}$, is required to satisfy some transversality, tracelessness conditions as a consequence of conformal invariance. The KLT prescription is:

$$V^{EX}_{KLT} \simeq: \exp[i(k.X + \epsilon_{\mu_1}\partial X^{\mu_1} + \epsilon_{\mu_2}\partial X^{\mu_2}... + \epsilon_{\mu_m}\partial X^{\mu_m})] :$$

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- Expanding the exponential, keeping the multilinear term the desired vertex operator is derived.

$$\epsilon_{\mu_1}\partial X^{\mu_1}\epsilon_{\mu_2}\partial X^{\mu_2}...\epsilon_{\mu_m}\partial X^{\mu_m}$$

- The procedure outlined for construction of open string vertex operators is generalized for corresponding closed string case. For example, usual the graviton vertex operator is

$$V^{graviton} = \epsilon_{\mu\bar{\nu}} : \exp[ik.X]\partial X^{\mu}\bar{\partial} X^{\bar{\nu}} :$$

- with $k^2 = 0$ and $\epsilon_{\mu\bar{\nu}}k^{\mu} = 0 = \epsilon_{\mu\bar{\nu}}k^{\nu}$.

- That of antisymmetric tensor state is similar, except that $\epsilon_{\mu\bar{\nu}}$ is antisymmetric. The KLT prescription is to have both holomorphic and antiholomorphic part in their vertex operator to reproduce $V^{graviton}$.

$$V_{KLT}^{closed} =: \exp[ik.X + i\epsilon_\mu \partial X^\mu + i\bar{\epsilon}_{\bar{\nu}}\bar{\partial} X^{\bar{\nu}}] :$$

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- (a) Collect the coefficients of the bilinear term $\epsilon_\mu \bar{\epsilon}_{\bar{\nu}}$ in the expansion of the exponential. (b) Note that symmetric product of $\epsilon_\mu \bar{\epsilon}_{\bar{\nu}}$ is graviton polarization tensor and antisymmetric part is that for the B-field. (c) The excited level closed string vertex operators can be obtained by generalizing this prescription.

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- Our goal is to construct vertex operators associated with the moduli and gauge fields.
  (i) Start with the theory in $\hat{D}$-dimensions. We compactify the theory to lower dimension - D. (ii) The states belong to the representations of the rotation group. The moduli are massless scalars under $SO(D-1)$ rotations and Lorentz transformations. (iii) The vertex operator for moduli is:

$$V_{KLT}^{\mathcal{M}} =: \exp[ik.X + i\epsilon_\alpha \partial Y^\alpha + i\bar{\epsilon}_{\bar{\beta}} \bar{\partial} Y^{\bar{\beta}}] :$$

- REMARKS:
- (i) The scalars propagate D-dimensional spacetime and thus the plane wave part is $\exp[ik.X]$. (ii) We do not consider effects of *winding modes*. (iii) All the states are on-shell. (iv) The vertex operators for G and B are symmetric and antisymmetric combinations of $\epsilon_\alpha \bar{\epsilon}_{\bar{\beta}}$ respectively. (v) To compute correlation functions: start from $V_{KLT}^{\mathcal{M}}$. There will be products of equal number of ϵ_{α_i}'s and $\bar{\epsilon}_{\bar{\beta}_i}$ - *from the level matching condition*. (vi)The products of ϵ_{α_i} will mutually contract themselves due the $\delta^{\alpha_i \alpha_j}$ coming from contractions of $\partial Y^{\alpha_i}(z_i)$ and $\partial Y^{\alpha_j}(z_j)$, $i \neq j$. Same contractions occur the antiholomorphic. (vii) The contraction, $< \partial Y^{\alpha_i}(z_i) \bar{\partial} Y^{\bar{\beta}_n}(\bar{z}_n) >= 0$ - no contraction of the 'polarizations' ϵ_{α_i} and $\bar{\epsilon}_{\bar{\beta}_1}$.

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- (a) Recall the Z_2 duality: $P \leftrightarrow Y'$ under $\tau \leftrightarrow \sigma$ interchange. (b) Notice: $P_\pm = (P \pm Y')$, $P_\pm \to \pm P_\pm$. (c) The $O(d,d)$ vector $\mathbf{Z}$ can be decomposed as $\mathbf{Z} = (\mathbf{Z}_+ + \mathbf{Z}_-)$ with.

$$\mathbf{Z}_+ = \frac{1}{2} \begin{pmatrix} P_+ \\ P_+ \end{pmatrix}, \quad \text{and} \quad \mathbf{Z}_- = \frac{1}{2} \begin{pmatrix} P_- \\ -P_- \end{pmatrix}$$

- The $O(d,d)$ symmetry is exhibited in the phase approach.

- SCATTERING OF MODULI AND GAUGE BOSONS

- SCATTERING OF MODULI AND GAUGE BOSONS
- Let us consider scattering of the moduli and gauge bosons in compactified closed bosonic string. The KLT method of constructing vertex operator will be useful now.
- We define ϵ_{α_i} and ∂Y^{α_i} to transform as vectors under $O(d)_R$. For the antiholomorphic part we declare $\{\bar{\epsilon}_{\bar{\beta}_i}$ and $\bar{\partial} Y^{\bar{\beta}_i}\}$ also to transform as $O(d)_L$ vectors. Thus in the expression for $V^{\mathcal{M}}_{KLT}$, $\epsilon_\alpha.\partial Y^\alpha$ and $\bar{\epsilon}_{\bar{\beta}}.\bar{\partial} Y^{\bar{\beta}}$ are $O(d)_R$ and $O(d)_L$ invariant. The N-point amplitude for scattering of moduli is

$$A^{(N)}_{G,B} = \int \mathbf{\Pi}^N_{i=1} d^2 z_i \, \mathcal{D} \, \mathbf{\Pi}_{i>j} |z_i - z_j|^{2k_i.k_j} \exp \mathbf{\Sigma}^N_{i>j} \frac{\epsilon_i.\epsilon_j}{(z_i - z_j)^2} \exp \mathbf{\Sigma}^N_{i>j} \frac{\bar{\epsilon}_i.\bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2}$$

- and

$$\mathcal{D} = \frac{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2}{d^2 z_a d^2 z_b d^2 z_c}$$

- The variables $\{z_a, z_b, z_c\}$ and $\{\bar{z}_a, \bar{z}_b, \bar{z}_c\}$ can be chosen arbitrarily which gauge fixes the underlying $SL(2, C)$ invariance - leaving only $(N-3)$ integrations over $\Pi d^2 z_i$.
- We have two independent products $\Pi\epsilon_{\alpha_i}$ and $\Pi\bar{\epsilon}_{\bar{\beta}_i}$. They contract amongst themselves. We can implement $O(d)_R$ and $O(d)_L$ transformations respectively on each of these products.

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- We eventually have scalar products of ϵ_{α_i} and those of $\bar{\epsilon}_{\bar{\beta}_i}$ and consequently the amplitude is T-duality invariant (now the T-duality group is $O(d)_R \otimes O(d)_L$).

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- Gauge Boson Vertex Operator
  Gauge bosons appear from dimensional reduction of metric and B-field

$$V^A =: \exp[ik.X + i\epsilon_\mu \partial X^\mu + i\bar{\epsilon}^{(j)}_{\bar{\beta}} \bar{\partial} Y^{\bar{\beta}}] :, \quad j = 1, 2$$

- We retain the bilinear $\epsilon_\mu \bar{\epsilon}^{(j)}_{\bar{\beta}}$ in expansion of exponential. $\bar{\beta} = 1, 2..d$ and $j = 1, 2$ for two sets of gauge bosons coming from $\hat{G}$ and $\hat{B}$.

- There are a set of polarization tensors associated with gauge bosons $A_\mu^{(1)\alpha}, \alpha = 1, 2...d$ and another set associated with $B_{\alpha\mu}, \alpha = 1, 2, ...d$ - altogether $2d$ gauge bosons. We define $\epsilon_{\alpha\mu}^{(j)}, \alpha = 1, 2..d;\ j = 1, 2$. T-duality group acts linearly on polarization $\epsilon_{\alpha\mu}^{(i)}$.The corresponding N-point amplitude is ($\mathcal{D}$ is the same as before)

$$T_A^{(N)} \simeq \int d^2 z_1 d^2 z_2 ... d^2 z_N\ \mathcal{D}\ \mathbf{\Pi}_{i>j} |z_i - z_j|^{2k_i.k_j}$$

$$\exp[\mathbf{\Sigma}_{i>j} \frac{\epsilon_i.\epsilon_j}{(z_i - z_j)^2} - \mathbf{\Sigma}_{i\neq j} \frac{\epsilon_i.k_j}{(z_i - z_j)}] \exp[\mathbf{\Sigma}_{i\neq j} \frac{\bar{\epsilon}_i.\bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2}]$$

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- Look at the two exponentials: first term in the first comes from $\mathbf{\Pi}\ \partial X^{\mu_i}(z_i)$ contractions, second term from contraction with $e^{ik.X}$. Second exponential has terms coming from contraction of $\mathbf{\Pi}\ \bar{\partial} Y^{\bar{\beta}_i}$. This is how $\epsilon_i.\epsilon_j$, $\epsilon_i.k_j$ (from holomorphic part) and $\bar{\epsilon}_i.\bar{\epsilon}_j$ appear in exponentials. Note: $\epsilon_i.\epsilon_j$ and $\epsilon_i.k_j$ are inert under T-duality operation. Thus *only* $\{\bar{\epsilon}_{\alpha_i}\}$ transform under T-duality as vectors.

- EXAMPLE I: FOUR POINT AMPLITUDE FOR MODULI, d=3
- Consider the 4-point amplitude for moduli $G_{\alpha\bar{\beta}}$. The vertex operator and the amplitude are

$$V_i = \epsilon_{\alpha_i}\bar{\epsilon}_{\bar{\beta}_i} : \exp[ik.X(z,\bar{z})]\partial Y^{\alpha_i}(z_i)\bar{\partial} Y^{\bar{\beta}_i}(\bar{z}_i) :, \quad i = 1,2,3,4$$

$$A^{(4)} \simeq \int \mathbf{\Pi}_{i=1}^{4} d^2 z_i \; \mathcal{D} \; \mathbf{\Pi}_{i>j} |z_i - z_j|^{2k_i.k_j} \exp[\mathbf{\Sigma}_{i>j} \frac{\epsilon_i.\epsilon_j}{(z_i - z_j)^2}] \exp[\mathbf{\Sigma}_{i>j} \frac{\bar{\epsilon}_i.\bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2}]$$

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- We choose Koba-Nielsen variables as follows:
  $z_1 = 0,\ z_3 = 1,\ z_4 \to \infty$ and make same choice for the correspondingly set $\bar{z}_1, \bar{z}_2, \bar{z}_3$. The only integration is over z_2 and $\bar{z}_2$.

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- Expand the two exponentials in power series and pick up the following terms which is a product $\epsilon_{\alpha_1}\epsilon_{\alpha_2}\epsilon_{\alpha_3}\epsilon_{\alpha_4}$ and $\bar{\epsilon}_{\bar{\beta}_1}\bar{\epsilon}_{\bar{\beta}_2}\bar{\epsilon}_{\bar{\beta}_3}\bar{\epsilon}_{\bar{\beta}_4}$. Choose $1-2$ as the incoming particles and $3-4$ as outgoing ones. The three Mandelstam variables are: $s = -(k_1+k_2)^2,\ t = -(k_1+k_3)^2,\ u = -(k_1+k_4)^2;\quad k_1+k_2+k_3+k_4 = 0,\quad s+t+u = 0$

- Interprete $\epsilon_{\alpha_1}\epsilon_{\alpha_2}\bar{\epsilon}_{\bar{\beta}_1}\bar{\epsilon}_{\bar{\beta}_2}$ as the initial 'wave function' in the internal space. Similarly, $\epsilon_{\alpha_3}\epsilon_{\alpha_4}\bar{\epsilon}_{\bar{\beta}_3}\bar{\epsilon}_{\bar{\beta}_4}$ as the final state 'wave function'. NOTE: ϵ_{α_i} is an $O(3)_R$ vector and $\bar{\epsilon}_{\bar{\beta}_i}$ is a $O(3)_L$ vector.
- We express the products of pair of ϵ_α's as

$$\epsilon_{\alpha_1}\epsilon_{\alpha_2} = (\epsilon_{\alpha_1}\epsilon_{\alpha_2} - \frac{1}{3}\delta_{\alpha_1\alpha_2}\epsilon_1.\epsilon_2) + \frac{1}{3}\delta_{\alpha_1\alpha_2}\epsilon_1.\epsilon_2$$

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- The same decomposition can be adopted for $\bar{\epsilon}_{\bar{\beta}_1}\bar{\epsilon}_{\bar{\beta}_2}$. For future conveniences and to see $O(3)_{R,L}$ rotational properties define

$$Q^I_{\alpha_1\alpha_2} = (\epsilon_{\alpha_1}\epsilon_{\alpha_2} - \frac{1}{3}\delta_{\alpha_1\alpha_2}\epsilon_1.\epsilon_2), \quad \text{and} \quad S^I_{\alpha_1\alpha_2} = \frac{1}{3}\delta_{\alpha_1\alpha_2}\epsilon_1.\epsilon_2$$

- Similarly, $\bar{Q}^I_{\bar{\beta}_1\bar{\beta}_2}$, $\bar{S}^I_{\bar{\beta}_1\bar{\beta}_2}$ 'two initial state.' The 'final states' are $\{Q^F_{\alpha_3\alpha_4},\ \bar{Q}^F_{\bar{\beta}_3\bar{\beta}_4}\}$ and $\{S^F_{\alpha_3\alpha_4},\ \bar{S}^F_{\bar{\beta}_3\bar{\beta}_4}\}$. $Q^I_{\alpha_1\alpha_2}$ transforms as quadrupole and $S^I_{\alpha_1\alpha_2}$ like scalar under $O(3)_R$

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- The four point amplitude assumes the form:

- $$A^{(4)}(s,u) \simeq= \int d^2 z_2 |z|^{2k_1.k_2} |1-z_2|^{2k_2.k_3} (\mathbf{\Sigma}_{a=1}^{3} T_R^a)(\mathbf{\Sigma}_{a=1}^{3} \bar{T}_L^b) \quad (1)$$

- Terms T_R^a and $\bar{T}_L^b$ are

$$T_R^1 = \frac{1}{(z_2)^2}\left(Q^I_{\alpha_1\alpha_2} + S^I_{\alpha_1\alpha_2}\right)\left(Q^F_{\alpha_3\alpha_4} + S^F_{\alpha_3\alpha_4}\right)\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}$$

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- Now we give an example how T-duality operation generates new matrix elements

- Consider the moduli given by the following 'polarization' vectors

$$\epsilon_\alpha = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \bar{\epsilon}_{\bar{\beta}} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

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- The corresponding moduli are

$$G_{\alpha\bar{\beta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ and } B_{\alpha\bar{\beta}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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- We choose following T-duality transformation: The $O(3)_R$ is on $2-3$ plane and the rotation operator is R. The other operator, S is for $O(3)_L$ and also rotates on $2-3$ plane. We identify $S = R^T$ and for simplicity choose rotation angle $\theta = \frac{\pi}{4}$ implying $cos\theta = sin\theta = \frac{1}{\sqrt{2}}$

- The transformed vectors ${\epsilon'}_\alpha$ and $\bar{\epsilon}'_{\bar{\epsilon}'}$ and the moduli G' and B' are given by

$$\epsilon'_\alpha = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \bar{\epsilon}'_{\bar{\beta}} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$G' = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad \text{and} \quad B' = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \end{pmatrix}$$

- The transformed vectors ${\epsilon'}_{\alpha}$ and $\bar{\epsilon}'_{\bar{\epsilon}'}$ and the moduli G' and B' are given by

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- This is a simple example of T-duality transformation. We could parametrize $O(3)_R$ with three Euler angles and $O(3)_L$ with another three independent Euler angles and implement general T-rotations. Then compute amplitudes with new backgrounds. The two amplitudes will be initial (old) and new amplitudes will be related through $O(3) \otimes O(3)$ rotations..

- APPENDIX: Gauge Boson Scatterings

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- Let us consider 4-point amplitude involving gauge bosons. We consider two cases: (I) Tachyon-gauge boson 4-point amplitude and (II) four gauge boson amplitude. The gauge boson is depicted by $A_{\mu}^{\bar{\alpha}_i},\ i = 1, 2, 3, 4.5, 6$ - three gauge bosons from the metric and three from B-field since we deal with $d = 3$ case. The initial and final tachyon momenta are k_1 and k_3 and those of the gauge bosons are k_3 and k_4. The amplitude is

$$
\begin{aligned}
T^{(4)} \sim \quad & \int \mathbf{\Pi}_{i=1}^{4} d^2 z_i \mathcal{D} \mathbf{\Pi}_{i>j} |z_i - z_j|^{2k_i.k_j} \\
& \exp[\frac{\epsilon_2.\epsilon_4}{(z_2 - z_4)^2} - \mathbf{\Sigma}_{i \neq j} \frac{\epsilon_i.k_j}{(z_i - z_j)}] \exp[\frac{\bar{\epsilon}_2.\bar{\epsilon}_4}{(\bar{z}_2 - \bar{z}_4)^2}]
\end{aligned}
$$

- We expand the exponentials and collect the relevant terms for our amplitude: $\epsilon_2.\epsilon_4$ and $\epsilon_i.k_j$ and these two terms are passive under T-duality. Only $\bar{\epsilon}_i$'s transform under duality but the inner product $\bar{\epsilon}_2.\bar{\epsilon}_4$ is also T-duality invariant. We discuss later how to generate new amplitudes in such scattering processes.

- Consider four gauge boson scattering amplitude
- $$A^{(4)}_{gauge} \sim \int \Pi_{i=1}^{4} d^2 z_i \; \mathcal{D} \; \Pi_{i>j} |z_i - z_j|^{2k_i.k_j} \mathcal{E}_1 \mathcal{E}_2$$

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- and $\mathcal{E}_1$ and $\mathcal{E}_2$ and for the two exponentials defines below.

$$\mathcal{E}_1 = \exp[\Sigma_{i>j} \frac{\epsilon_i.\epsilon_j}{(z_i - z_j)^2} - \Sigma_{i \neq j} \frac{\epsilon_i.k_j}{(z_i - z_j)}]$$

$$\mathcal{E}_2 = \exp[\Sigma_{i>j} \frac{\bar{\epsilon}_i.\bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2}]$$

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- We choose the three coordinates $\{z_1, z_3, z_4\}$ as before. Look at the structure of the amplitude: (i) Polarizations $\{\epsilon_{\mu_i}\}$ are spacetime vectors. (ii) $\{\bar{\epsilon}_i\}$ are 'polarizations' from internal directions. We need to carefully discuss how these two sets appear in the amplitude.

- $\{\bar{\epsilon}_i\}$ will mutually contract amongst theme selves. However, these $\{\epsilon_{\mu_i}\}$ polarizations will contract amongst themselves and will also contract with momenta. The structure will be $(\epsilon_i.\epsilon_j)(\epsilon_m.k_n)$ and so on. We keep in mind the transversality conditions: $\epsilon_{\mu_i}.k^{\mu_i} = 0$ (no sum over μ_i). This amplitude has been computed before. However, in order to bring out the essential features, focus on $\mathcal{E}_2$ first.

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- We have to retain up to quadratic terms in the expansion of this exponential. These terms are $\bar{\epsilon}_{\bar{\alpha}_i}.\bar{\epsilon}_{\bar{\alpha}_j}\bar{\epsilon}_{\bar{\alpha}_m}.\bar{\epsilon}_{\bar{\alpha}_n}$ the indices have to be different in the product where i, j, m, n take values $1, 2, 3, 4$. Recall the case of scattering of the moduli. Here we can identify $\bar{\epsilon}_{\bar{\alpha}_1}\bar{\alpha}_{\bar{\alpha}_2}$ as the initial 'wave function' and $\bar{\epsilon}_{\bar{\alpha}_3}\bar{\alpha}_{\bar{\alpha}_4}$ as the final 'wave function'. Then our earlier arguments go through. We conclude that the amplitude is duality invariant.

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- Now we turn to $\mathcal{E}_1$. As noted earlier, we collect terms of different types in the exponential, keeping in mind that four polarization vectors be present. (I) There is a term like $(\epsilon_i.\epsilon_j)(\epsilon_m.\epsilon_n)$, $i \neq j, m \neq n$ and i, j, m, n are all different. The origin lies from the quadratic terms in expansion of the exponential. (II) Another type $(\epsilon_i.\epsilon_j)(\epsilon_m.k_l)(\epsilon_n.k_s)$, coming from cubic term in expansion of $\mathcal{E}_1$.

- (III) There is a term $\epsilon_i.k^j$ and product of four such terms has to appear. They come from certain combination of terms when we expand the exponential up to the fourth power. One such combination is

$$\left(\frac{\epsilon_1.k_2}{(z_1 - z_2)} + \frac{\epsilon_2.k_1}{(z_2 - z_1)} + \frac{\epsilon_3.k_4}{(z_3 - z_4)} + \frac{\epsilon_4.k_3}{(z_4 - z_3)}\right)^4$$

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- There will be one interference term: $\epsilon_1.k_2\epsilon_2.k_1\epsilon_3.k_4\epsilon_4.k_3$ and similar other terms (we have suppressed the numerical factor).
- More comments: (i) Since these are contractions of spacetime polarization vectors and momenta, the T-duality invariance is automatic. (ii) The power of α' as coefficient in these terms is higher than the previous ones which can be seen if we bring back the presence of α' in $< X^\mu(z_1, \bar{z}_1)X^\nu(z_2, \bar{z}_2) >$.

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- Moreover, $A^{(1)\bar{\beta}}_{\mu}$ and $A^{(2)}_{\mu\bar{\beta}}$, denoted by $\mathcal{A}_\mu$, transform as a doublet under $O(3,3)$: $\mathcal{A}_\mu \to \Omega\mathcal{A}_\mu$

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- The polarizations $\bar{\epsilon}^{(1)}$ and $\bar{\epsilon}^{(2)}$ will transform according to the above rules. If we start only $A^{(1)\bar{\beta}}_{\mu}$ and $A^{(2)}_{\mu\bar{\beta}} = 0$ then under $O(3_R)\otimes O(3)_L$ we generate: $A'^{(1)\bar{\beta}}_{\mu} = (R+S)A^{(1)\bar{\beta}}_{\mu}$ and $A'^{(2)}_{\mu\bar{\beta}} = (R-S)_{\bar{\beta}\bar{\alpha}}A^{(1)\bar{\alpha}}_{\mu}$. The former amplitude gets related to amplitude with admixture of two types of gauge fields.

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- To facilitate construction of vertex operators for scattering, we have to resort to the weak field approximation and express $M = \mathbf{1} + \tilde{M},\ M \in O(d,d)$. However, $\tilde{M}$ does not belong to $O(d,d)$. We proposed a prescription to generate new T-duality transformed vertex operators from given initial vertex operator. However, this method turns out to be not so efficient.

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- If we consider T-duality transformations $O(d) \otimes O(d) \in O(d,d)$, then the vertex operators have nice transformation properties.

- For tree level amplitudes, the KLT prescription is very efficient to compute the amplitude. Moreover, $O(d) \otimes O(d)$ transformation is implemented elegantly.
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- The scattering amplitudes for gauge bosons coming from compactification can be cast in a T-duality invariant form.
- It is not essential to adopt KLT formulation to implement T-duality transformation on amplitude - KLT accounts for vertex operators of the first excited states. However, higher massive levels contains states lying on nonleading Regge trajectories. KLT technique is not applicable in such cases. It is possible to implement T-duality transformation on amplitudes for scattering of excited massive states in compactified closed string.