Flow equation, conformal symmetry and AdS geometry with general conformally flat boundary

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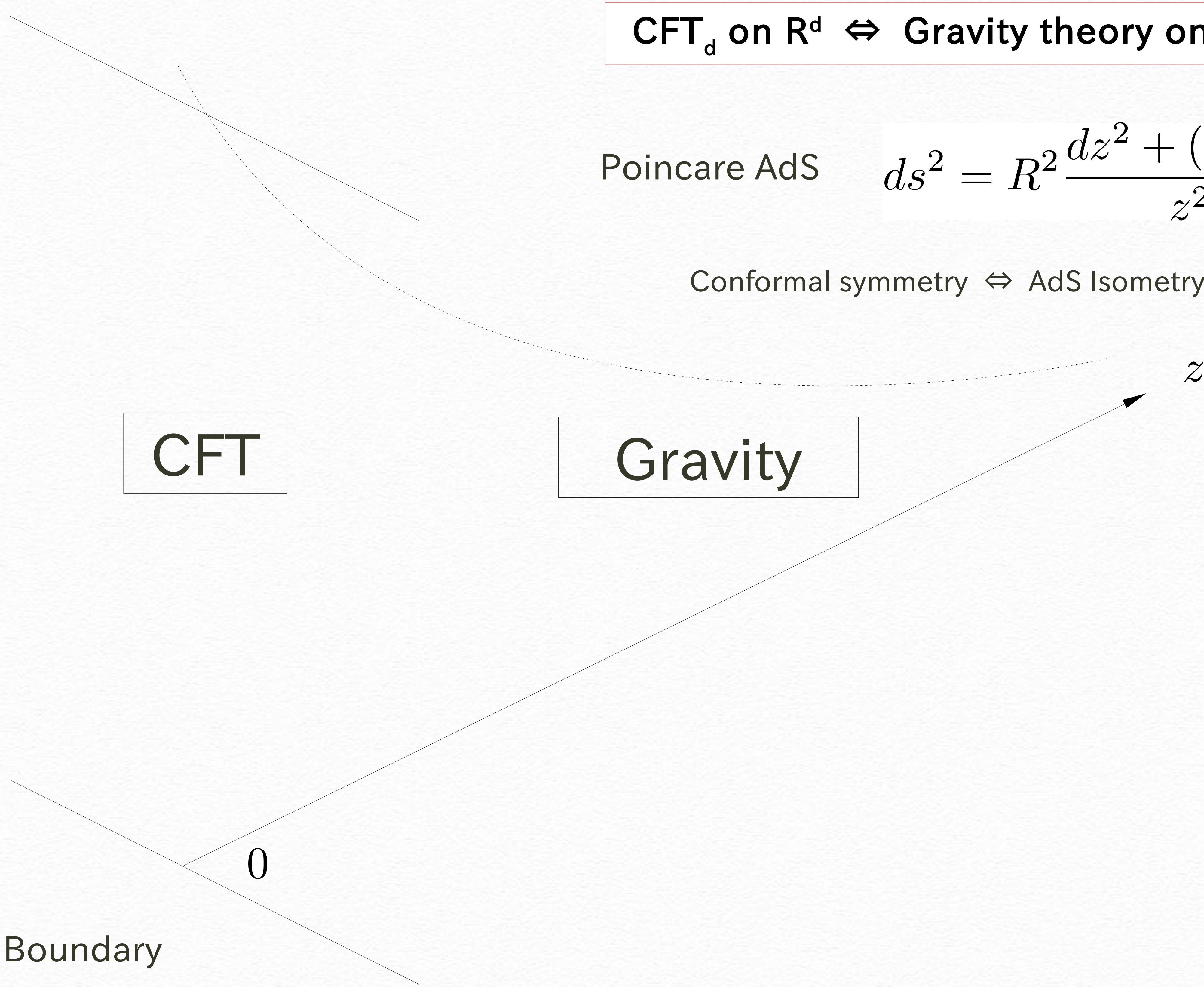
Holography and AdS/CFT

[Maldacena '97]

CFT_d on R^d ⇔ Gravity theory on AdS_{d+1}

Poincare AdS $$ds^2 = R^2 \frac{dz^2 + (dx^\mu)^2}{z^2}$$

Conformal symmetry ⇔ AdS Isometry: SO(2,d)



AdS geometry from CFT

AdS radial direction ⇔ renormalization scale [Maldacena ’97]

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1. Relevant RG flow

Construction of gravity solutions corresponding to UV and IR CFTs in the asymptotic regions.

[Girardello-Petrini-Porrati-Zaffaroni '98] [Distler-Zamora '98] [de-Boer-Verlinde-Verlinde '99] [Skenderis '00]

2. Wilsonian RG flow

The Wilsonian cut-off will correspond to sharp cut-off at the AdS radial direction

[Heemskerk-Polchinski '10]

3. Stochastic quantization

Euclidean path integral ≡ Equilibrium limit of statistical mechanical system coupled to a heat bath.

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4. Entanglement enetropy

continuous multi-scale entanglement renormalization ansatz (cMERA)

[Swingle '09] [Van Raamsdonk '09] [Nozaki-Ryu-Takayanagi '12]

5. Flow equation

Smearing operators so as to resolve a UV singularity in the coincidence limit

[Aoki-Kikuchi-Onogi '15] [Aoki-Balog-Onogi-Weisz '16,'17]

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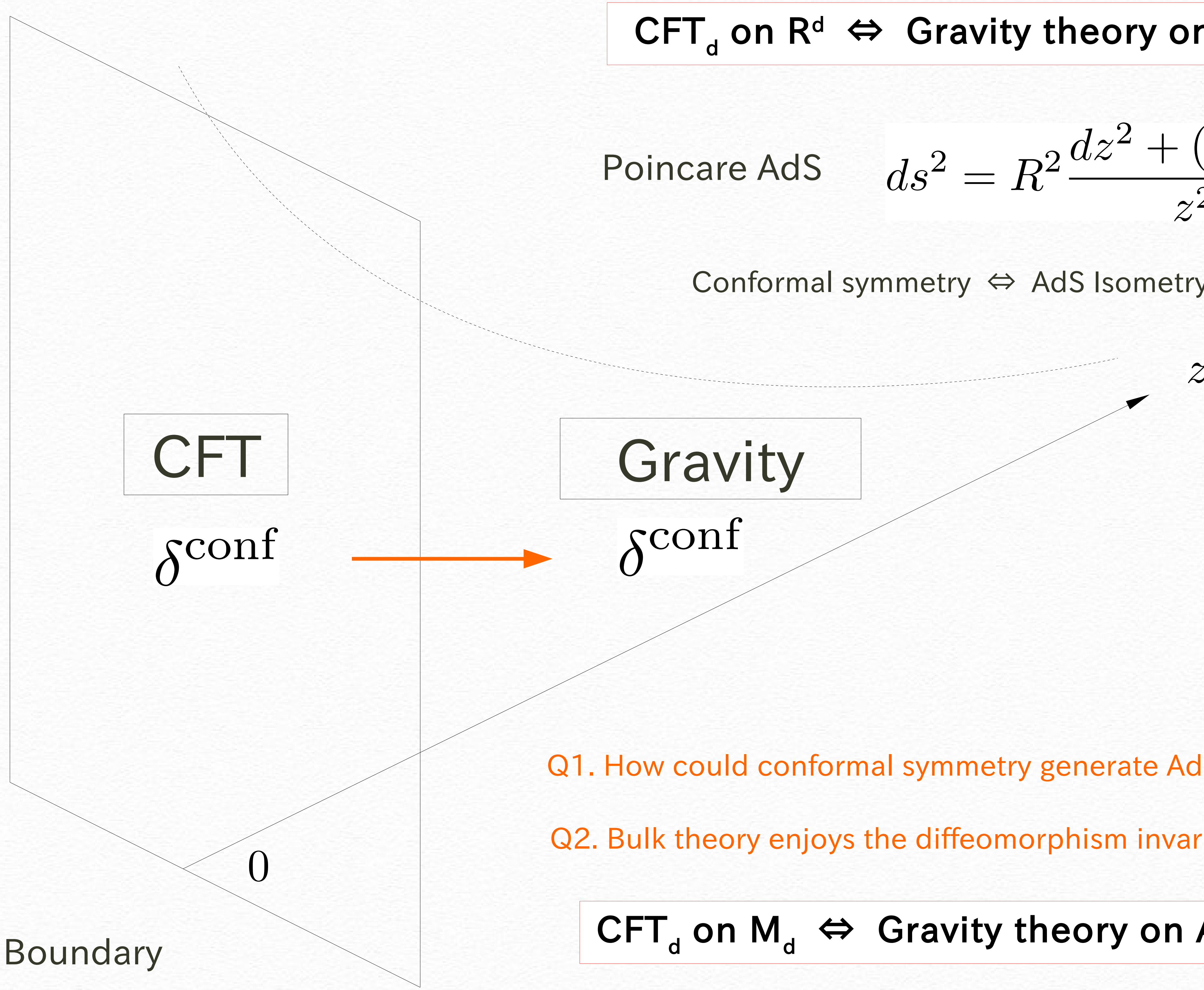

Q1. How could conformal symmetry generate AdS isometry?

Q2. Bulk theory enjoys the diffeomorphism invariance

CFT_d on M_d ⇔ Gravity theory on AdS_{d+1}?

Plan

✓ 1. Introduction

2. Flow equation & induced metric

Induced metric = information metric

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⬅ Answer for Q1

4. Generalization to conformally flat manifolds

4.1. Primary flow equation

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4.2. AdS metric with conformally flat boundary

5. Summary

Flow equation

(Gradient) flow equation

Equation to smear operators to resolve a UV singularity in the coincidence limit

[Albanese et al. (APE) '87] [Narayanan-Neuberger '06] [Luscher '10,'13]

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Consider QFT with n real component scalar field in d dimensions

Flow equation

$$\frac{\partial \phi^a(x;t)}{\partial t} = - \left. \frac{\delta S_f(\varphi)}{\delta \varphi^a(x)} \right|_{\varphi(x)\to\phi(x;t)} \qquad \phi(x;0) = \varphi(x)$$

t: flow time, ϕ (x;t): flowed field

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<u>Comments</u>

(I) If S_f coincide with the action of the theory S
➡ the flow equation is called the gradient one.

(II) If S_f coincide with the free theory
➡ the flow equation becomes the heat equation.
➡ the operator ϕ is smeared similarly as in finite temperature t.

(III) General solution ➡ "flow kernel method".

“Flow kernel method”

1. Rewrite the flow equation in the heat equation form.

$$\frac{\partial O(x;t)}{\partial t} = -\hat{H} O(x;t) \qquad O(x;0) = O(x)$$

2. Introduce the flow kernel (density).

$$\frac{\partial \rho(x,y;t)}{\partial t} = -\hat{H} \rho(x,y;t) \qquad \rho(x,y;0) = \delta^d(x-y)$$

3. A general solution is given by

$$O(x;t) = \int d^d y \rho(x,y;t) O(y)$$

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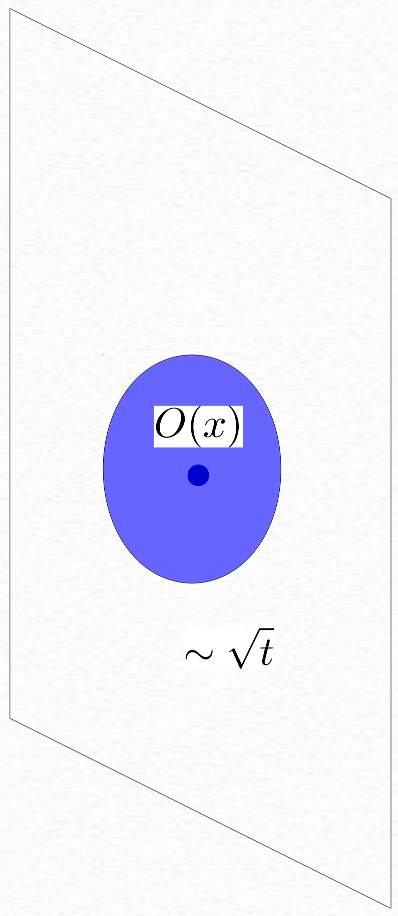
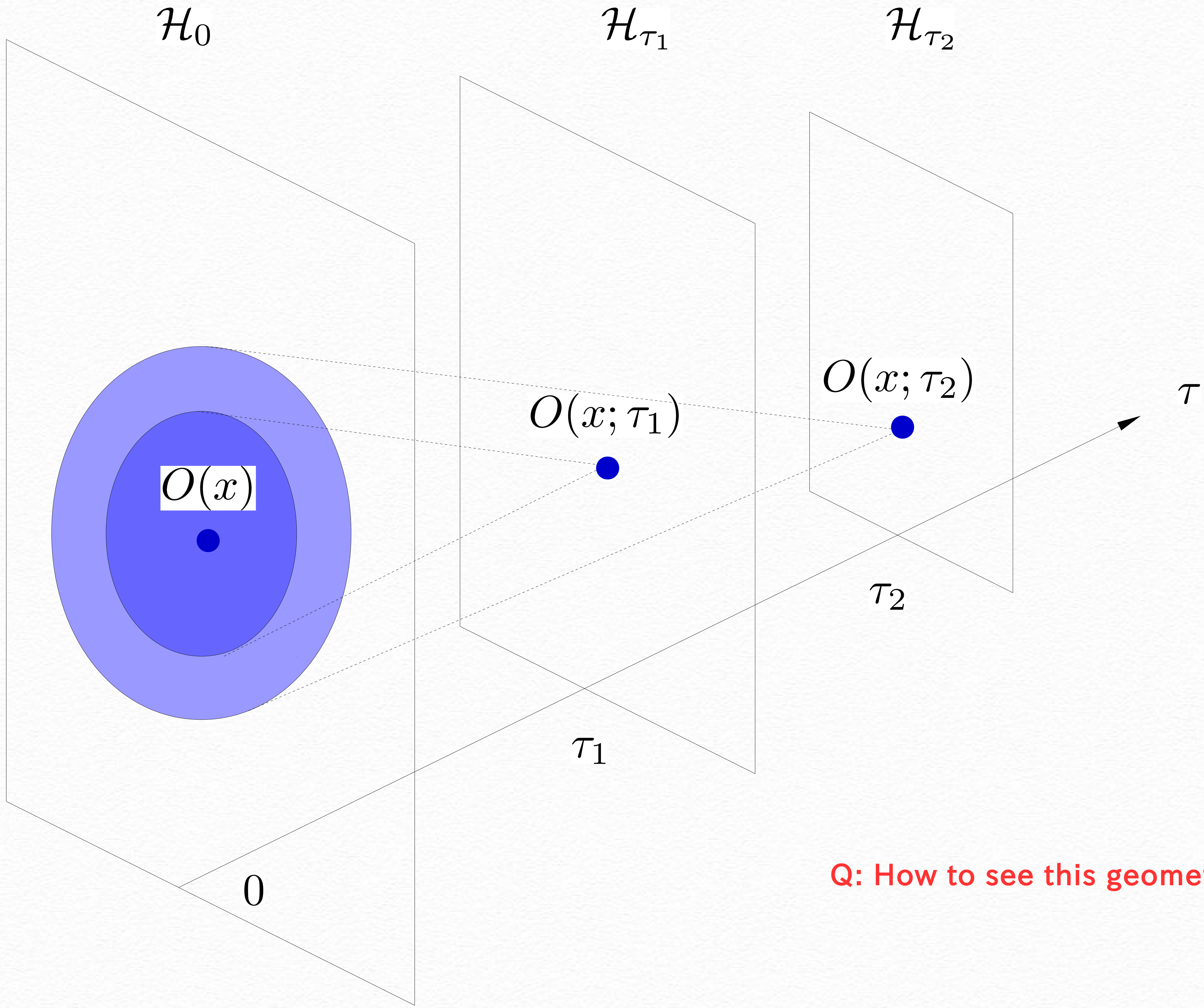


<u>Ex.</u> Free flow

$$\rho(x,y;t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{4t}}$$

Sketch of smearing and extra direction



Q: How to see this geometry?

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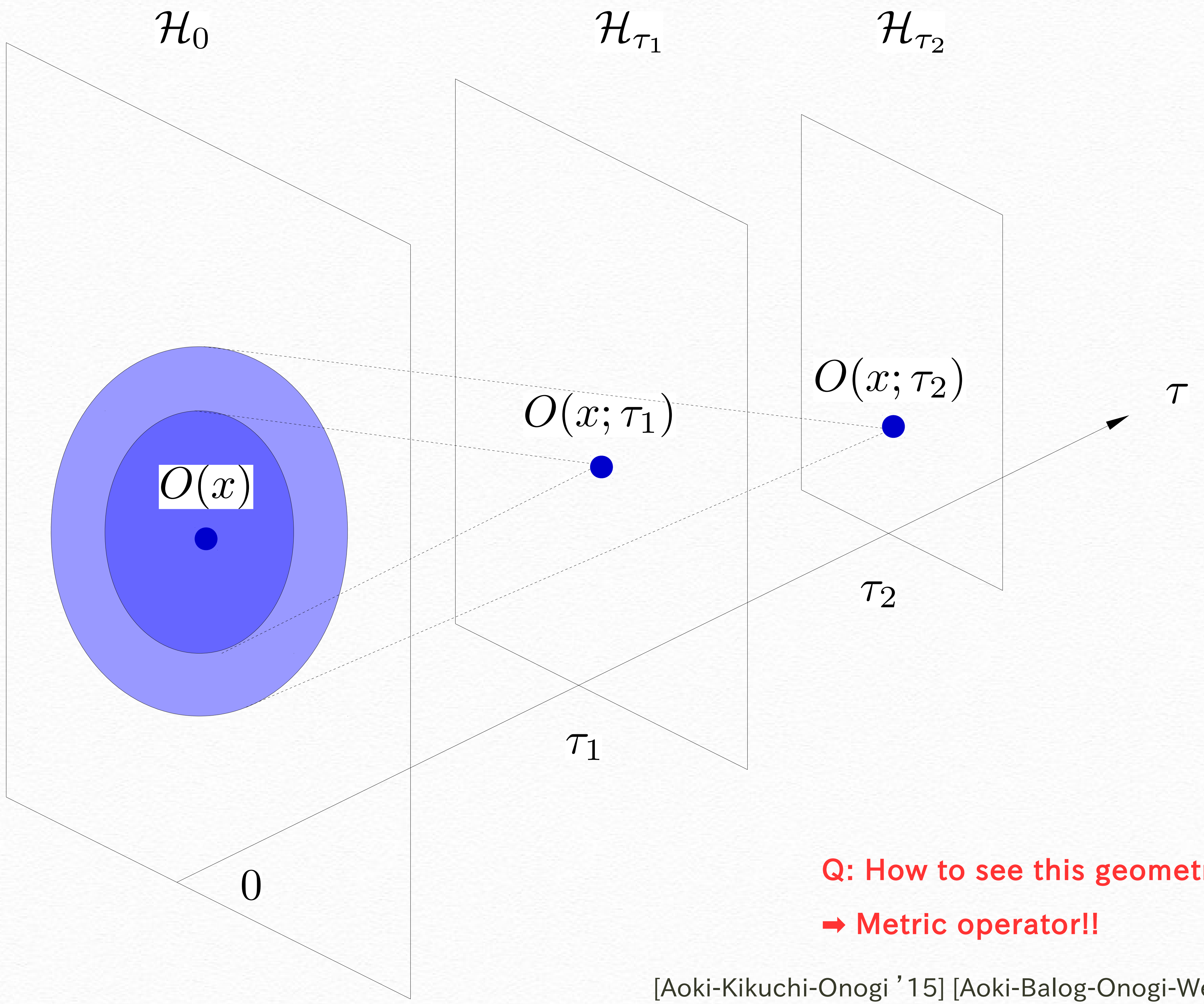

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➡ Metric operator!!

[Aoki-Kikuchi-Onogi '15] [Aoki-Balog-Onogi-Weisz '16,'17]

Metric operator and induced metric

Def. (Dimensionless normalized operator)

$$\sigma^a(x;t) := \frac{\phi^a(x;t)}{\sqrt{\langle \sum_{a=1}^{n} \phi^a(x;t)^2 \rangle_S}}$$

(Normalization)

where

$$\langle O(\varphi) \rangle_S := \frac{1}{Z} \int \mathcal{D}\varphi \, O(\varphi) \, e^{-S(\varphi)} \qquad Z = \int \mathcal{D}\varphi \, e^{-S(\varphi)}$$

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Def. (Metric operator)

$$\hat{g}_{MN}(x;t) := R^2 \sum_{a=1}^n \frac{\partial\sigma^a(x;t)}{\partial z^M}\frac{\partial\sigma^a(x;t)}{\partial z^N}$$

R: constant of length dimension $z^M = (x^\mu, \tau)$ with $\tau = \sqrt{2dt}$

Def. (Induced metric)

$$g_{MN}(z) := \langle\hat{g}_{MN}(x;t)\rangle_S$$

Note. $$\sum_{a=1}^n \langle\sigma^a(x;t)\sigma^a(x;t)\rangle_S = 1$$

Induced metric = information metric

Def. (Bure metric for a density matrix)

$$D(\rho, \rho + d\rho)^2 = \frac{1}{2}\mathrm{tr}(d\rho\, G)$$

ρ : density matrix G: hermitian 1 form operator satisfying $\rho\, G + G\, \rho = d\rho$

For a pure state, G is given by $G = d\rho$

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In our setup, the density matrix for a state localized at z = (x, τ) is

$$\rho_z = |\sigma(x;t)\rangle\langle\sigma(x;t)|, \quad |\sigma(x;t)\rangle := \sum_{a=1}^{n} |\sigma^a(x;t)\rangle \qquad z^M = (x^\mu, \tau) \text{ with } \tau = \sqrt{2dt}$$

Def. (Inner product)

$$\langle\sigma(x;t)|\sigma(w;s)\rangle := \sum_{a,b=1}^{n} \langle\sigma^a(x;t)\sigma^b(w;s)\rangle_S = \sum_{a=1}^{n}\langle\sigma^a(x;t)\sigma^a(w;s)\rangle_S,$$

$$R^2 D(\rho_z,(\rho+d\rho)_z)^2 = \frac{R^2}{2}\mathrm{tr}(d\rho_z d\rho_z) = g_{MN}(z)dz^M dz^N,$$

cf. Fischer information metric $|\langle\Psi_\lambda|\Psi_{\lambda+\delta\lambda}\rangle|^2 = 1 - 2G_{\lambda\lambda}\delta\lambda^2 + \cdots$

$|\Psi_\lambda$>: Vacuum state for $H_0 + \lambda V$

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Correlator of free flowed field

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The 2pt function is $G_0(x;t|y;s) := \langle O(x;t) O(y;s) \rangle_{\mathrm{CFT}} = e^{(t\partial_x^2 + s\partial_y^2)} \langle O(x) O(y) \rangle_{\mathrm{CFT}}$

Poincare symmetry: $G_0(x;t|y;s) = {}^{\exists} G_0((x-y)^2, t+s)$

Scaling property: $G_0(\lambda x; \lambda^2 t | \lambda y; \lambda^2 s) = \lambda^{-2\Delta} G_0(x;t|y;s)$

➡ $G_0(x;t|y;s) = \frac{1}{(t+s)^{\Delta}} F_0\left(\frac{(x-y)^2}{t+s}\right)$ $\quad$ F_0: a smooth function

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The 2pt function of normalized field $$G(x;t|y;s) := \langle \sigma(x;t)\sigma(y;s)\rangle_{\rm CFT} = \left(\frac{2\sqrt{ts}}{t+s}\right)^{\Delta} F\left(\frac{(x-y)^2}{t+s}\right)$$

$$F(x) \equiv F_0(x)/F_0(0),\ F(0) = 1,\ 2dF'(0) = -\Delta$$

Explicit computation: $$F(x) = \frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(d/2-\Delta)} \int_0^1 dv\, v^{\Delta-1}(1-v)^{d/2-\Delta-1} e^{-xv/4}.$$

NOTE: UV singularity is resolved (regularized) ⇔ F is a smooth function for x>0.

Q1:Conformal symmetry ➡ AdS isometry?

cf. [Jevicki-Kazama-Yoneya '98]

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[Aoki-SY '17]

Conformal transformation:

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b_\nu x^\nu),$$

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Vanish by taking VEV!!

⬇

$$\left\langle \delta^{\rm conf}\hat g_{MN}(x;t)\right\rangle = \left\langle \delta^{\rm diff}\hat g_{MN}(x;t)\right\rangle.$$

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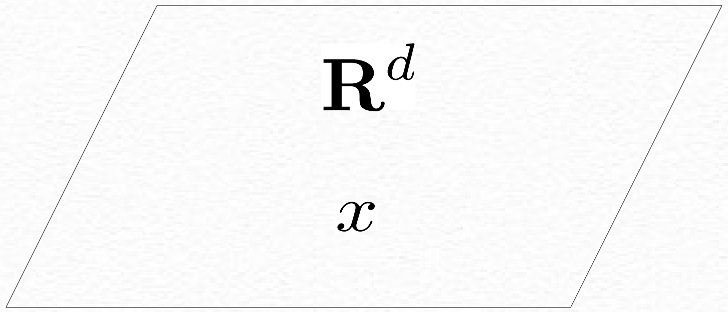
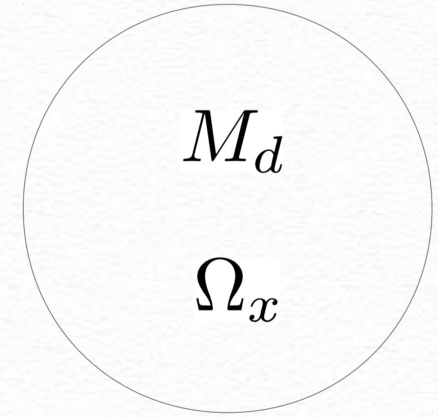


Ω : conformal map
$\longrightarrow$



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$$(ds^2)_{M_d} = g^{\frac{1}{d}}(x)\delta_{\mu\nu} dx^\mu dx^\nu$$

the conformal factor

$O(x)$

$\Omega*$: pullbuck
$\longleftarrow$

$O(\Omega_x)$

$$\Omega * O(\Omega_x) = U_\Omega^{-1} O(\Omega_x) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x)$$

Primary flow

$\mathbf{R}^d$

x

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M_d

Ω_x

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flow $\downarrow$

$\downarrow$ flow

$O(x;t)$

$???$

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We request the following 2 conditions:

① There exists a flow time ˜t associated with M_d corresponding to the flow time t on R^d such that the flowed operator inserted at Ω_x is related to the flowed one at x by the pullback of a conformal map Ω:

$$\Omega * O(\Omega_x;\tilde{t}) = U_\Omega^{-1} O(\Omega_x;\tilde{t}) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x;t)$$

② The flow equation is invariant under the scale transformation.

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We request the following 2 conditions:

① There exists a flow time ˜t associated with M_d corresponding to the flow time t on R^d such that the flowed operator inserted at Ω_x is related to the flowed one at x by the pullback of a conformal map Ω:

$$\Omega * O(\Omega_x;\tilde{t}) = U_\Omega^{-1} O(\Omega_x;\tilde{t}) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x;t)$$

② The flow equation is invariant under the scale transformation.

Primary flow equation

⬇

$$\tilde{t} = g^{\frac{1}{d}}(x) t$$

$$\frac{\partial}{\partial \tilde{t}} O(\Omega_x;\tilde{t}) = g^{-\frac{1}{d}}(x) g^{-\frac{\Delta}{2d}}(x) \partial^2 \cdot g^{\frac{\Delta}{2d}}(x) O(\Omega_x;\tilde{t}), \qquad O(\Omega_x;0) = O(\Omega_x)$$

Primary flow

$\mathbf{R}^d$

x

Ω : conformal map

$\longrightarrow$

M_d

Ω_x

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$$

$$(ds^2)_{M_d} = g^{\frac{1}{d}}(x)\delta_{\mu\nu} dx^\mu dx^\nu$$

the conformal factor

$O(x)$

$\Omega*$: pullbuck

$\longleftarrow$

$O(\Omega_x)$

$$\Omega * O(\Omega_x) = U_\Omega^{-1} O(\Omega_x) U_\Omega = g^{-\frac{\triangle}{2d}}(x) O(x)$$

flow $\downarrow$

$\downarrow$ flow

$\Omega*$: pullbuck

$\longleftarrow$

$O(x;t)$

$O(\Omega_x;\tilde{t})$

$$\frac{\partial O(x;t)}{\partial t} = \partial^2 O(x;t)$$

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Primary flow equation

Q2: Generalization to non-trivial curved boundary?

Generalization to curved background

A curved manifold needs to admit CFT to live. ➡ Restrict ourselves to a conforamlly flat manifold.

We need to construct a flow equation associated with the curved manifold.

"Canonical" free flow equation **(Primary flow equation)**

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⬇

The induced metric:

$$\begin{aligned}
\tilde{g}_{\tilde{\tau}\tilde{\tau}}(z) =& R^2\frac{\Delta}{\tilde{\tau}^2}, \\
\tilde{g}_{\tilde{\tau}\mu}(z) =& g_{\mu\tilde{\tau}}(z) = -R^2\frac{\Delta}{\tilde{\tau}}\frac{\partial}{\partial x^\mu}\log\{g^{\frac{1}{2d}}(x)\}, \\
\tilde{g}_{\mu\nu}(z) =& R^2\Delta\left[\frac{\partial}{\partial x^\mu}\log\{g^{\frac{1}{2d}}(x)\}\frac{\partial}{\partial x^\nu}\log\{g^{\frac{1}{2d}}(x)\} + \frac{\delta_{\mu\nu}g^{\frac{1}{d}}(x)}{\tilde{\tau}^2}\right],
\end{aligned}$$

This is the (local) AdS metric whose radius is $R\sqrt{\Delta}$ with the boundary M_d!!

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\end{aligned}$$

This is the (local) AdS metric whose radius is $R\sqrt{\Delta}$ with the boundary M_d!!

This AdS metric can be obtained from the usual Poincare AdS by a bulk diffeormorphism

$$ds^2(\tilde{t},x) = ds^2_{\mathrm{PAdS}}(t,x)|_{t=g^{-1/d}(x)\tilde{t}}$$

Plan

✓ 1. Introduction

✓ 2. Flow equation & induced metric

Induced metric = information metric

✓ 3. Conformal symmetry ➡ AdS isometry

⬅ Answer for Q1

✓ 4. Generalization to conformally flat manifolds

4.1. Primary flow equation

⬅ Answer for Q2

4.2. AdS metric with conformally flat boundary

5. Summary

Summary

- An induced metric defined from the normalized flowed field generally corresponds to the **quantum information metric**.

- For a general CFT, we explicitly showed that the induced metric with the free flow equation always becomes the **AdS** metric.

- Conformal symmetry converts to AdS isometry **after quantum averaging**.

- We have constructed a canonical flow equation for a primary scalar operator on a conformally flat manifold, called the **primary flow equation**.

- The induced metric associated with the primary flow equation becomes AdS with **the conformally flat boundary**.

- AdS with the conformally flat boundary is obtained from the usual Poincare AdS **by a simple bulk diffeomorphism transformation**.

Future works

- How to encode dynamics beyond geometry?

[Aoki-Balog-SY] To appear 180x.xxxxx.

- How to construct bulk operator?

- Any application of the metric operator to other formalism?

- Spin 1,2 field? Fermion?

- de Sitter construction? Application to the real world?

⋮

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Thank you!!