Soft theorem and large gauge transformation

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Soft theorem

- Pion(axion), 60's

$$\lim_{\omega\to 0}\langle \text{out}|a^{(\pi)}_{\omega\hat{q}}\,\mathcal{S}|\text{in}\rangle = J^{(1)}(q)\langle \text{out}|\mathcal{S}|\text{in}\rangle$$

Soft factor $$J^{(1)}(q) = \sum_k \frac{-iy\,\eta_k}{2m\,p_k\cdot q}\epsilon_{\mu\nu\rho\sigma}q^\mu p_k^\nu J_k^{\rho\sigma}$$

q: soft momentum
p: hard momentum

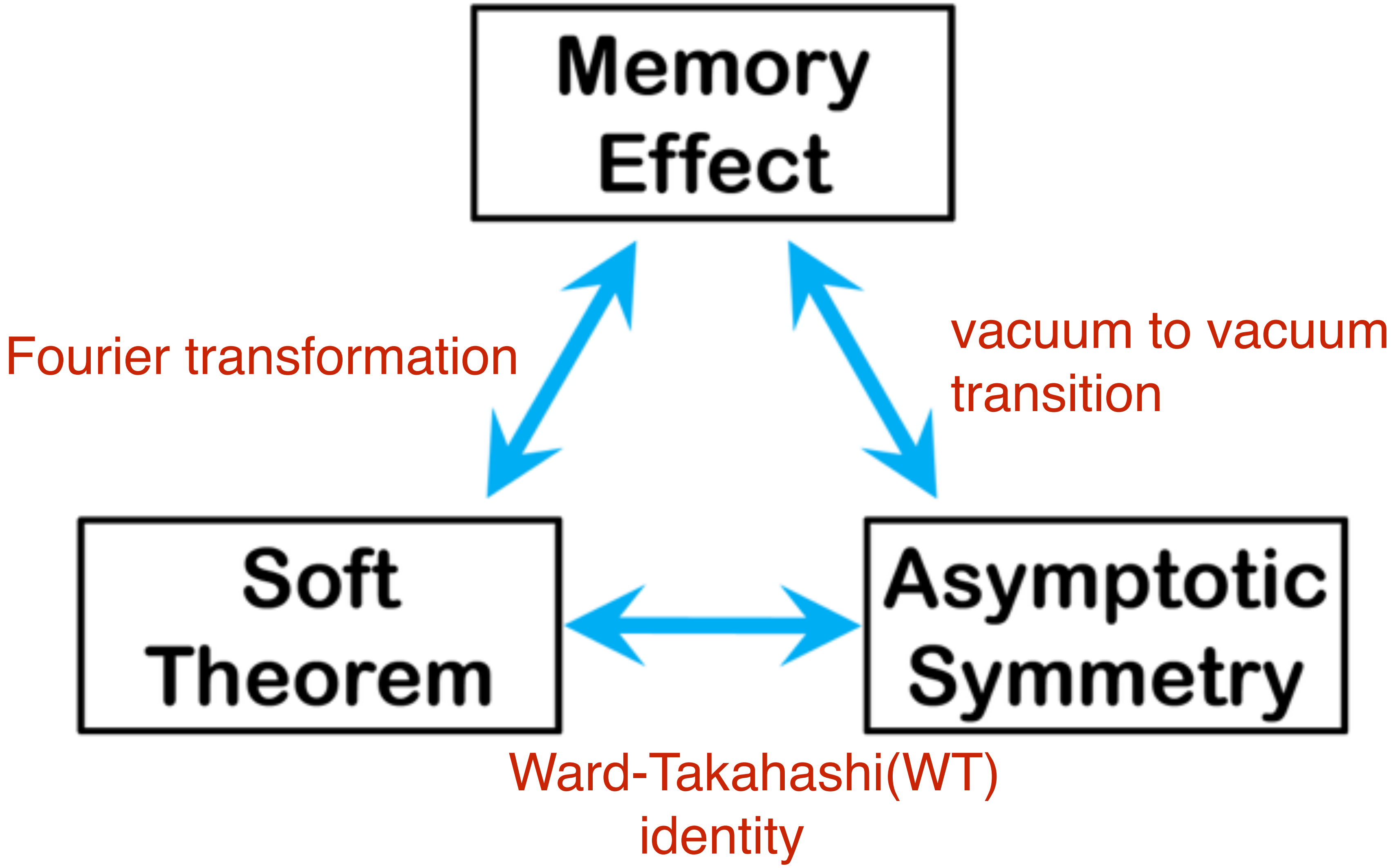
- Cosmology [Maldacena '02]

$$\lim_{\vec{q}\to 0}\frac{1}{P_\zeta(q)}\langle\zeta_{\vec{q}}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle' = -\vec{k}_1\cdot\partial_{\vec{k}_1}\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle'$$

density perturbation

Triangular relation

[Strominger '13, …]



Talk Plan

1. Triangular relation (review)
2. Large gauge transformation and soft theorem [YH, Shiu '18]
3. Memory effect in curved background [YH, Seo, Shiu '17]

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[YH, Shiu ’18]

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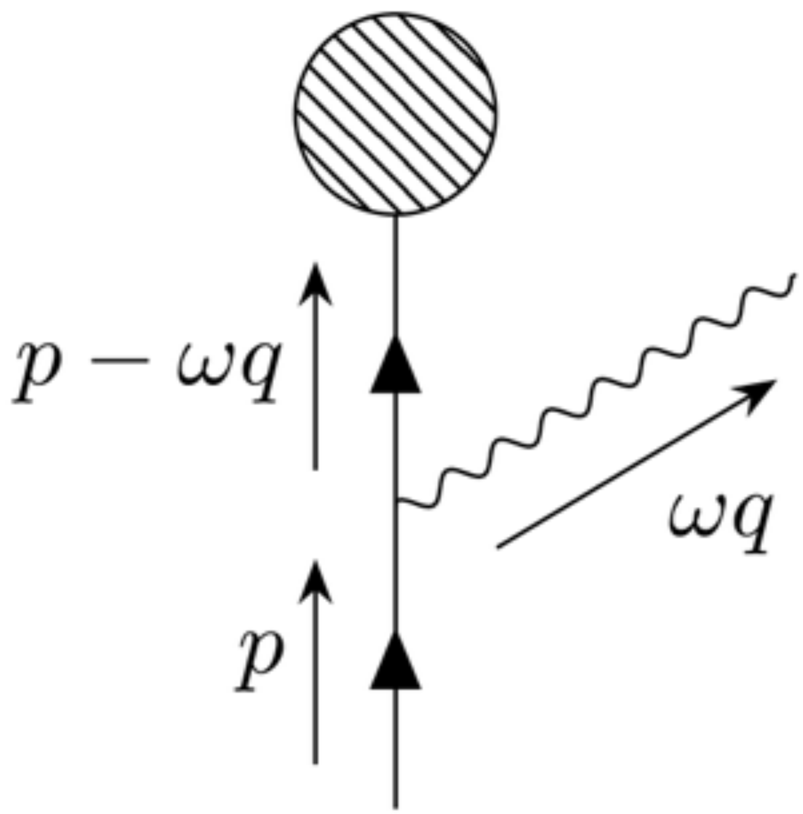
Diagram w/ soft photon

- massless QED, $\omega \to 0$.



$$\epsilon_\nu \frac{i}{p_\mu \gamma^\mu - \omega q_\mu \gamma^\mu} \gamma^\nu u(p)$$

$$\sim i\epsilon_\nu \frac{p_\mu \gamma^\mu}{-2\omega p \cdot q} \gamma^\nu u(p)$$

$$= -i\epsilon_\nu \frac{p^\nu}{\omega p \cdot q} u(p)$$

Soft factor

Leading contribution comes from diagram where soft photon attaches to external line.

Soft photon theorem

[Low '58]

- leading and subleading

$$\lim_{\omega\to 0}\langle \mathrm{out}|a^{(\gamma)}_{\omega\hat{q}}\,\mathcal{S}|\mathrm{in}\rangle = (J^{(1)} + J^{(2)})\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

$$J^{(1)} = \sum_k Q_k \eta_k \frac{p_k\cdot\epsilon}{p_k\cdot q} \qquad J^{(2)} = -i\sum_k Q_k\eta_k \frac{q_\mu \epsilon_\nu J_k^{\mu\nu}}{p_k\cdot q}$$

Leading   Subleading

For spin 1/2

$$J_{\mu\nu} := -i\left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\right) - \frac{i}{2}\gamma_{\mu\nu}$$

Soft graviton theorem

- leading, subleading and subsubleading

$$\lim_{\omega\to 0}\langle \text{out}|a^{(\text{grav})}_{\omega\hat{q}}\,\mathcal{S}|\text{in}\rangle = (J^{(1)}+J^{(2)}+J^{(3)})\langle \text{out}|\mathcal{S}|\text{in}\rangle$$

$$J^{(1)} = \sum_k \eta_k \frac{e_{\mu\nu}p_k^\mu p_k^\nu}{p_k\cdot q}$$

[Weinberg '65] Leading

$$J^{(2)} = -i\sum_k \eta_k \frac{e_{\mu\nu}p_k^\mu q_\rho J_k^{\rho\nu}}{p_k\cdot q}$$

Subleading

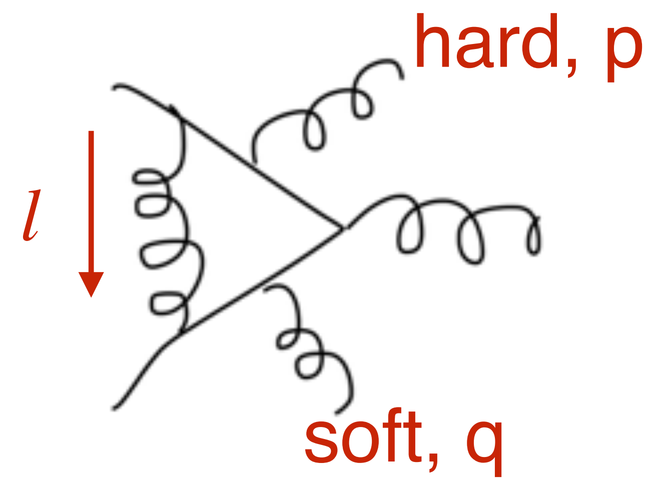
[Cachazo, Strominger '14]

$$J^{(3)} = -\frac{1}{2}\sum_k \eta_k \frac{e_{\mu\nu}q_\rho J_k^{\rho\mu} q_\sigma J_k^{\sigma\mu}}{p_k\cdot q}$$

Subsubleading

Loop correction?

- Beyond leading order, there is loop correction to the soft theorem. related to IR div.



Although factorization may work for integrands, loop integral may spoil it.

$$\mathcal{M}_{n+1} \sim \int \frac{d^4 l}{(2\pi)^4} \left[(\text{soft factor}) \times F(p) \right]$$

Invalid for $l \lesssim q$

- Subleading theorem suffers from correction only from 1-loop diagram.

[Bern, Davies, Nohle '14, He, Huang, Wen '14]

Asymptotic sym.

- Asymptotically flat

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}$$
$$+\mathcal{O}\left(\frac{1}{r}\right)du^2 + \mathcal{O}\left(\frac{1}{r^2}\right)dudr + \mathcal{O}(1)\,dudz + \mathcal{O}(1)\,dud\bar{z}$$
$$+\mathcal{O}(r)\,dz^2 + \mathcal{O}(r)\,d\bar{z}^2 + \mathcal{O}(r)\,dzd\bar{z},$$

- gauge choice (Bondi gauge)

$$g_{rA} = g_{rr} = 0, \qquad \det\left(\frac{g_{AB}}{r^2}\right) = (\gamma_{z\bar{z}})^2$$

Supertranslation/rotation

- The transformation which preserves gauge&boundary condition.

f: real function

$$u \to u + \epsilon_u(u, r, z, \bar{z}), \qquad r \to r + \epsilon_r(u, r, z, \bar{z})$$

$$z \to z + \epsilon_z(u, r, z, \bar{z}), \qquad \bar{z} \to \bar{z} + \epsilon_{\bar{z}}(u, r, z, \bar{z})$$

$$\epsilon_u = -f(z, \bar{z}) + \int du \left[\partial^{\bar{z}}\left(\frac{a_z(z)}{(1+|z|^2)^2}\right) + \partial^z\left(\frac{a_{\bar{z}}(\bar{z})}{(1+|z|^2)^2}\right)\right],$$

$$\epsilon_r = D^z D_z \epsilon_u - r\left[\partial^{\bar{z}}\left(\frac{a_z}{(1+|z|^2)^2}\right) + \partial^z\left(\frac{a_{\bar{z}}}{(1+|z|^2)^2}\right)\right],$$

$$\epsilon_z = -\frac{1}{r}\partial^z \epsilon_u + a_z(z),$$

$$\epsilon_{\bar{z}} = -\frac{1}{r}\partial^{\bar{z}} \epsilon_u + a_{\bar{z}}(\bar{z}).$$

Asymptotic charge

- For gravity, Only leading and subleading theorems can be understood as WT identity of asymptotic transformations.

- Leading soft theorem $\longleftrightarrow$ Supertranslation

  Subleading soft theorem $\longleftrightarrow$ Superrotation

  Subsubleading soft theorem $\longleftrightarrow$ ？？？

- For U(1), only leading.

Memory effect

- Solving classical equation of motion with fixed trajectory of charged particle.

$$y_n^\mu(\tau) = \frac{p_n^\mu}{m_n}\tau + x_0^\mu \quad (\tau < 0)$$

$$y_{n'}^\mu(\tau) = \frac{p_{n'}^\mu}{m_{n'}}\tau + x_0^\mu \quad (\tau > 0)$$

kick of the charged particle.

Leading, subleading,$\cdots$

$$u = t - r$$

permanent change of $O(r^{-1})$

$$A^\mu(x) = \Theta(u) \sum_{n'} \frac{e_{n'} p^\mu_{n'}}{4\pi p^u_{n'} r} + \Theta(-u) \sum_n \frac{e_n p^\mu_n}{4\pi p^u_n r} - \delta(u) \left[\sum_{n'} \frac{e_{n'} J^{u\mu}_{n'}}{4\pi p^u_{n'} r} - \sum_n \frac{e_n J^{u\mu}_n}{4\pi p^u_n r} \right]$$

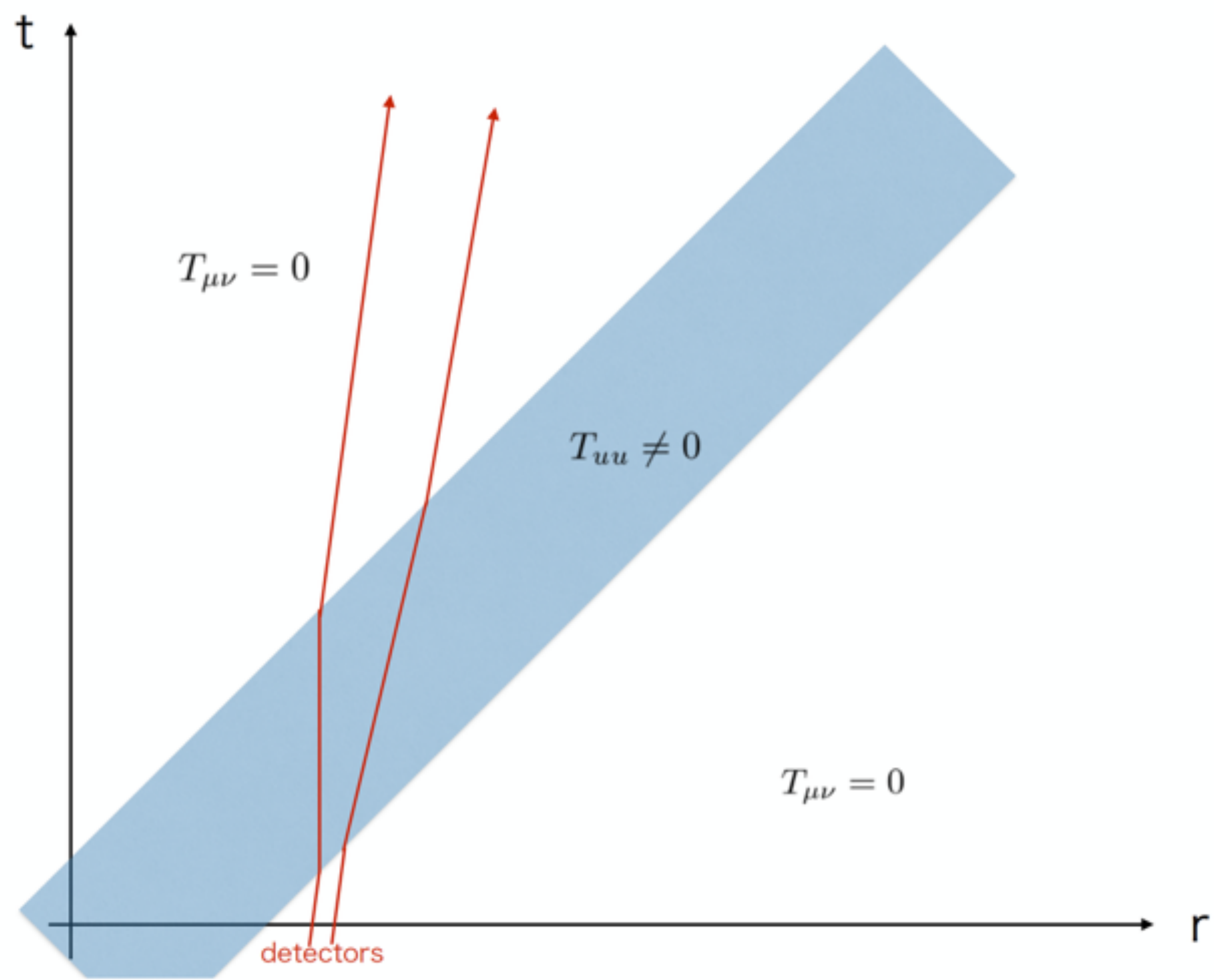
Leading memory effect

subleading memory effect

$$\frac{1}{16\pi G_N} \bar{h}^{\mu\nu}(x) = \Theta(u) \sum_{n'} \frac{p^\mu_{n'} p^\nu_{n'}}{4\pi p^u_{n'} r} + \Theta(-u) \sum_n \frac{p^\mu_n p^\nu_n}{4\pi p^u_n r} - \delta(u) \left[\sum_{n'} \frac{p^\nu_{n'} J^{u\mu}_{n'}}{4\pi p^u_{n'} r} - \sum_n \frac{p^\nu_n J^{u\mu}_n}{4\pi p^u_n r} \right]$$
$$+ \frac{1}{2} \delta'(u) \left[\sum_{n'} \frac{J^{u\mu}_{n'} J^{u\nu}_{n'}}{4\pi p^u_{n'} r} - \sum_n \frac{J^{u\mu}_n J^{u\nu}_n}{4\pi p^u_n r} \right]$$

Gravitational memory effect

The proper distance between 2 particles changes.



Talk Plan

1. Triangular relation (review)
2. Large gauge transformation and soft theorem [YH, Shiu ’18]
3. pion/axion memory effect [YH, Sugishita ’17]
4. Memory effect in curved background [YH, Seo, Shiu ’17]

Our work

- We present different point of view.
- We consider the WT identity of $O(x^n)$ gauge parameter, and show that this corresponds to $O(q^n)$ soft photon/graviton theorem.
- We focus on the tree level formula.

Large gauge tr.(U(1))

- After Lorenz gauge fixing, residual gauge tr. is

$$\partial^2 \chi = 0$$

- Explicitly,

$$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \ldots i_M} x^{i_1} x^{i_2} \cdots x^{i_M}, \qquad \eta_{iii_3 \ldots i_n} = 0,$$

- Large gauge tr. which does not fall off at infinity.

WT identity

- Starting from

$$\lim_{R\to\infty}\langle 0|(Q_{R,\alpha}E_1B - BE_1Q_{R,\alpha})|0\rangle = \lim_{R\to\infty}\langle 0|[Q_{R,\alpha}, B]|0\rangle$$

E_1: projection on the zero-mass one-particle state

$Q_{R,\alpha}$: charge of large gauge transformation

B: arbitrary operator

- LSZ reduction will be performed to obtain relation in terms of amplitude.

$$Q_{R,\alpha} := \int d^3x\, f_R(\vec{x})J_0$$

$f_R(x)=1$, 0 for $|x|<R$ and $|x|>R$

LHS

$$f(\vec{x}) = \begin{cases} 1 & \text{for } |\vec{x}| < 1, \\ 0 & \text{for } |\vec{x}| > 1. \end{cases}$$

$$f_R(\vec{x}) := f(\vec{x}/R)$$

$$\begin{aligned}
\lim_{R\to\infty} \int d^3x\, f_R(\vec{x}) j_0(\vec{x}) &= \lim_{R\to\infty} \int d^3k\, \tilde{f}_R(-\vec{k}) \tilde{j}_0(\vec{k}) = \lim_{R\to\infty} \int d^3k\, R^3 \tilde{f}(R\vec{k}) \tilde{j}_0(\vec{k}) \\
&= \lim_{R\to\infty} \int d^3k\, \tilde{f}(\vec{k}) \tilde{j}_0(\vec{k}/R) = \int d^3k\, \tilde{f}(\vec{k}) \lim_{\vec{q}\to 0} \tilde{j}_0(\vec{q}) \\
&= \lim_{\vec{q}\to 0} \tilde{j}_0(\vec{q})
\end{aligned}$$

Soft photon insertion

- Second term survives after spatial integration and EoM

$$J_{0,U(1)} = \partial_i(A_0\partial^i\chi) - \dot{A}_i\partial^i\chi + \chi\partial^2 A_0$$

- LHS becomes

$$\mathrm{LHS} = \lim_{q\to 0}(\partial^i\chi)\big|_{x_j = i\partial_{q^j}} \langle\vec{q}, i|B|0\rangle$$

RHS

$$\lim_{q\to 0} \partial^i \chi\Big|_{x_j = i\partial_{q^j}} \langle \vec{q}, i | B | 0 \rangle = \langle 0 | \delta B | 0 \rangle$$

- U(1) field w/ charged scalar

$$A_\mu \to A_\mu + \partial_\mu \chi, \quad \psi \to e^{-ie\chi}\psi.$$

$$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \ldots i_M} x^{i_1} x^{i_2} \cdots x^{i_M}, \qquad \eta_{iii_3\ldots i_n} = 0,$$

- x→ - i ∂_p in momentum space.

  δA_μ does not contribute to amplitude.

Soft photon theorem

U(1) w/ charged scalar

$$\lim_{q\to 0} \eta_{ii_2\ldots i_M} \partial_{q_2} \ldots \partial_{q_M} \mathcal{M}^i_{n+2m+1}(q; p, p', k)$$

$$= \lim_{q\to 0} \sum_{j=1}^{m} \eta_{ii_2\ldots i_M} \partial_{q_{i_2}} \ldots \partial_{q_{i_M}}$$

$$\times \Big[\Gamma^i(-p_j - q, p_j, q) S(p_j + q) \mathcal{M}_{n+2m}(p_j + q)$$

$$+ \mathcal{M}_{n+2m}(p'_j + q) S(-p'_j - q) \Gamma^i(-p'_j - q, p'_j, q)\Big]$$

$$+ \frac{Q_\phi}{M} \eta_{i_1\ldots i_M} [\partial_{p_{i_1}} \ldots \partial_{p_{i_M}} \mathcal{M}_{n+2m} + (-1)^{M-1}(p \leftrightarrow p')]$$

S(p): matter propagator
Γ: matter-matter-photon vertex

Note $$\chi = \sum_{M=1}^{\infty} \eta_{i_1 i_2 \ldots i_M} x^{i_1} x^{i_2} \cdots x^{i_M}, \qquad \eta_{iii_3\ldots i_n} = 0,$$

Note

- M=1 corresponds to leading and subleading soft theorem.

  (amplitude is not fully determined by M_n)

- M≧2 corresponds to the constraint on the subsub⋯leading soft behavior of amplitude.

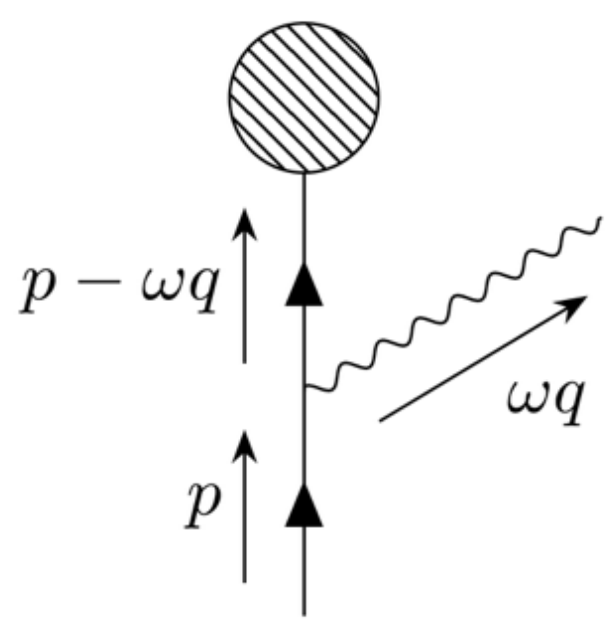
  (amplitude is not fully determined).

[YH, Shiu '18, see also Bern et. al. '14 and Li, Lin, Zhang '18]

Picture

- Leading order, gauge inv. corresponds to charge conservation.

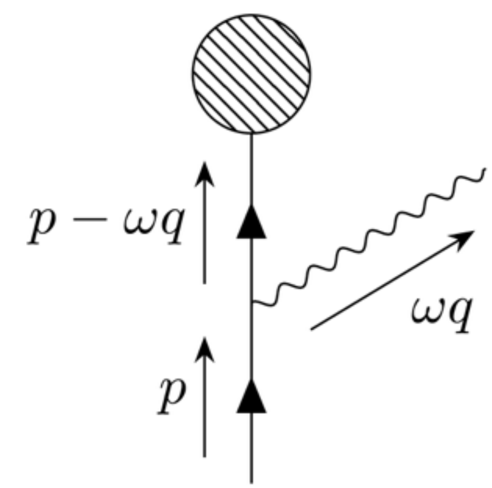
$$J^{(1)} = \sum_k Q_k \eta_k \frac{p_k \cdot \epsilon}{p_k \cdot q} \xrightarrow{\epsilon \to q} \sum_k Q_k \eta_k = 0$$



[YH, Shiu '18, see also Bern et. al. '14 and Li, Lin, Zhang '18]

Picture

- Taylor expansion in q of pole term requires non-pole term for gauge inv.



$$\sum_k Q_k \eta_k \frac{p_k \cdot \epsilon}{p_k \cdot q} \times \mathcal{M}_n(p_k - q)$$

$\epsilon \to q$

$$\Rightarrow \sum_k Q_k \eta_k \times \mathcal{M}_n(p_k - q) = -\sum_k Q_k \eta_k q^\mu \partial_{p_k^\mu} \mathcal{M}_n(p_k) + \dots$$

not gauge inv.

In order to have gauge inv. amplitude, we must add

$$\sum_k Q_k \eta_k \epsilon^\mu \partial_{p_k^\mu} \mathcal{M}_n(p_k) + \dots$$

Non-pole term is determined from pole term (partially).

Graviton

- Similarly, the formula for soft graviton is

$$\lim_{q\to 0}(\eta_{ij\alpha_2\ldots\alpha_M}+\eta_{ji\alpha_2\ldots\alpha_M})\partial_{q_{\alpha_2}}\ldots\partial_{q_{\alpha_M}}\mathcal{M}^{ij}_{n+2m+1}$$

$$=\lim_{q\to 0}\sum_l(\eta_{ij\alpha_2\ldots\alpha_M}+\eta_{ji\alpha_2\ldots\alpha_M})\partial_{q_{\alpha_2}}\ldots\partial_{q_{\alpha_M}}$$

$$\times\Big[\Gamma^{ij}_{\mathrm{matt}}(q,p_l,-p_l-q)S(p_l+q)\mathcal{M}_{n+2m}(p_l+q)$$

$$+\mathcal{M}_{n+2m}(p'_l+q)S(-p'_l-q)\Gamma^{ij}_{\mathrm{matt}}(q,p'_l,-p'_l-q)$$

$$+\Gamma^{(ij,i'_lj'_l,i_lj_l)}(q,k_l,-k_l-q)D^{(i'_lj'_l,i''_lj''_l)}(k_l+q)\mathcal{M}_{n+2m}(k_l+q)_{i''_lj''_l}\Big]$$

$$+(\text{All derivatives act on }M_{n+2m}).$$

S(p): matter propagator

Γ_{matt}: matter-matter-graviton vertex

Γ: 3-graviton vertex

Note

- M=1 corresponds to leading and subleading soft theorem.
- For M=2, starting from ansatz,

$$\begin{aligned}\mathcal{M}_{n+1} \sim \{ & (\text{leading}) + (\text{subleading}) \\ & + A(\epsilon_{\mu\nu} p^\mu p^\nu)(q \cdot p) + B(\epsilon_{\mu\nu} p^\mu p^\nu)(q \cdot \partial_p) + C(\epsilon_{\mu\nu} \partial_p^\mu \partial_p^\nu)(q \cdot p) \\ & + D(\epsilon_{\mu\nu} \partial_p^\mu \partial_p^\nu)(q \cdot \partial_p) + E(\epsilon_{\mu\nu} p_p^\mu \partial_p^\nu)(q \cdot p) + F(\epsilon_{\mu\nu} p_p^\mu \partial_p^\nu)(q \cdot \partial_p) \} \mathcal{M}_n\end{aligned}$$

  we can fix all coefficients.
- M≧3: Constraints on higher order terms.

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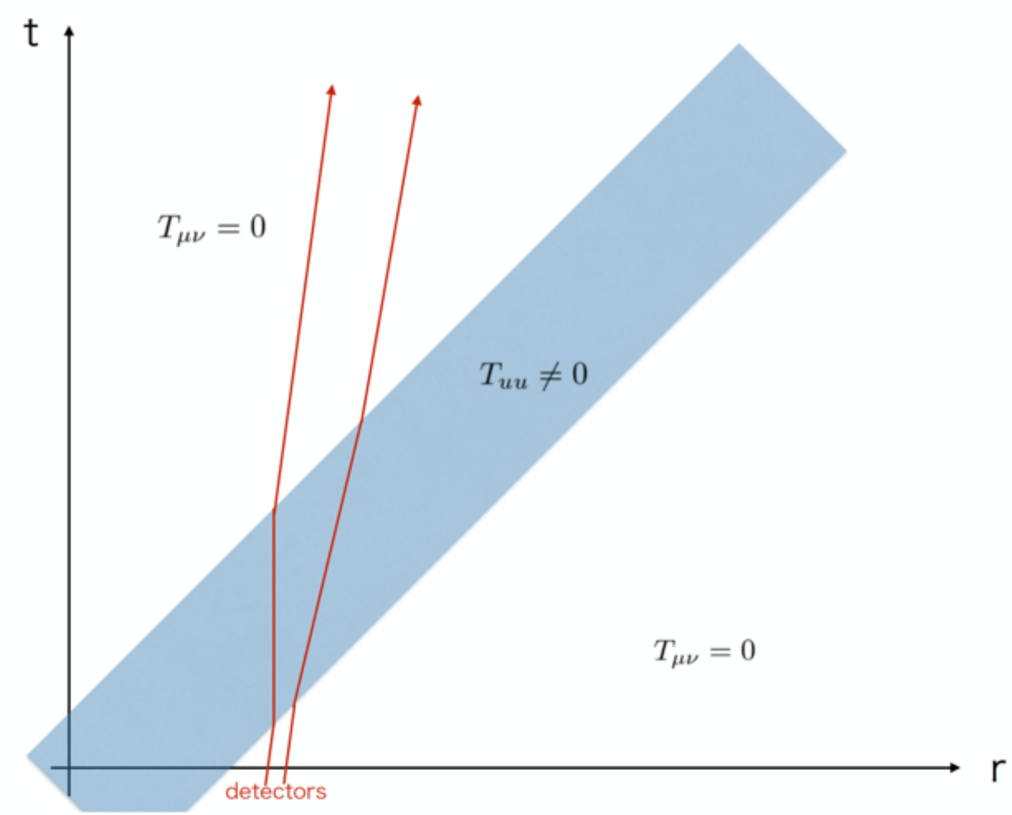
[YH, Shiu '18]

3. Memory effect in curved background

[YH, Seo, Shiu '17]

Memory in dS

- Current universe is close to dS_4.
- The effect of the curved background could be important for the detection of the memory effect.



Strategy

- dS + perturbation (large r expansion)

Static patch

$$
\begin{aligned}
ds^2 &= (-1 + r^2 H^2) du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} \\
&+ \left\{ \frac{m_B(u,z,\bar{z})}{r} - rH^2 h_{uu}(u,z,\bar{z}) \right\} du^2 + rC_{zz}(u,z,\bar{z}) dz^2 + rC_{\bar{z}\bar{z}}(u,z,\bar{z}) d\bar{z}^2 \\
&+ \left\{ U_{z1}(u,z,\bar{z}) + H^2 r^2 U_{z2}(u,z,\bar{z}) \right\} dudz \\
&+ \left\{ U_{\bar{z}1}(u,z,\bar{z}) + H^2 r^2 U_{\bar{z}2}(u,z,\bar{z}) \right\} dud\bar{z} + \frac{1}{r^2} \left(D_{ur1} + H^2 r^2 D_{ur2} \right) dudr \\
&+ \frac{1}{r} \left(\frac{4}{3} N_z + \frac{4}{3} u \partial_z m_B - \frac{1}{4} \partial_z (C_{zz} C^{zz}) \right) dudz + \frac{(1+|z|^2)^2}{4} C_{zz} C_{\bar{z}\bar{z}} dz d\bar{z} + \ldots.
\end{aligned}
$$

- See the consequence of vacuum Einstein eq.

Result

- From vacuum EoM

$$h_{uu} = -\frac{1}{2}\left(\partial^z U_{z2} + \partial^{\bar{z}} U_{\bar{z}2}\right), \qquad C_{zz} = -D_z U_{z2}, \qquad U_{z1} = D^z C_{zz}$$

$$C_{zz} = 2D_z^2 f$$

f=f(z,z̄) is arbitrary real function.

- Vacuum solution is parameterized by f.

- The change of f is given by

$$f_{\mathrm{fin}} = -\int d^2 z' \gamma_{z'\bar{z}'} G(z,\bar{z};z',\bar{z}') \left(m_B\Big|_{u=u_i}^{u=u_f} + \int_{u_i}^{u_f} du T_{uu} \right)$$

$$G(z,\bar{z};z',\bar{z}') := -\frac{1}{\pi}\sin^2\frac{\Theta}{2}\log\sin^2\frac{\Theta}{2}, \quad \sin^2\frac{\Theta}{2} = \frac{|z-z'|^2}{(1+z'\bar{z}')(1+|z|^2)}$$

Comment

- Memory effect in dS is same as the flat one if the proper distance between the source and the detector is the same at the time of detection.

- Change of metric can be understood as BMS-like supertranslation.

$$u \to u - f, \qquad r \to r - D^z D_z f,$$
$$z \to z + \frac{1}{r} D^z f, \qquad \bar{z} \to \bar{z} + \frac{1}{r} D^{\bar{z}} f.$$

- Compared with previous results ([Yau et. al. '16, Wald et. al. '16]), we can treat general situation (e.g. extended source).

Summary

- We consider the WT identity of the large gauge transformation.
- The $O(q^n)$ soft photon/graviton theorem corresponds to WT identity of $O(x^n)$ gauge parameter.
- The pion/axion memory effect is proposed.
- The memory effect in dS is considered.