Information metric
for the matrix geometry

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Introduction

– Noncommutative geometry = Quantum structure of space-time ??

[Madore 1991, Connes 1994, Hanada-Kawai-Kimura 2006, Steinacker 2010, etc.]

– Noncommutative coordinates → loss of “locality”

$$[x^\mu, x^\nu] \neq 0 \quad \longrightarrow \quad \Delta x^\mu \Delta x^\nu \gtrsim \Lambda$$

– UV divergences → Cutoff by Λ

Matrix geometry

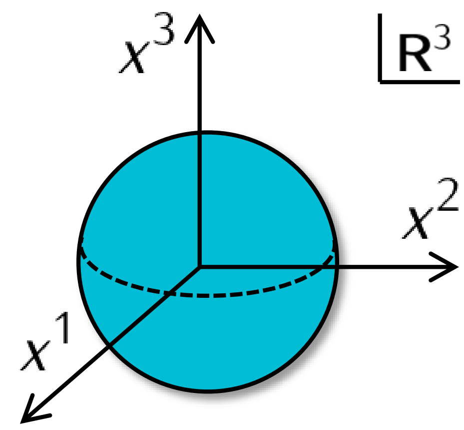
- A typical example of NC geometry
  - Coordinates $\longrightarrow$ Hermitian matrices of finite size
  - Many examples: fuzzy sphere, fuzzy torus, fuzzy CP^N, etc.

Example: fuzzy two sphere

[Madore 1992]

– (Unit) two sphere $S^2 \subset \mathbf{R}^3$

$$\begin{cases} (x^1, x^2, x^3) \in \mathbf{R}^3 : \text{embedding functions} \\ x^i x_i = 1 \\ [x^i, x^j] = 0 \end{cases}$$



– Fuzzy two sphere

$$\begin{cases} X^i = \frac{2}{\sqrt{N^2-1}} L^i \qquad L^i\text{: } N \text{ dim irrep of SU(2) generators} \\ X^i X_i = 1 \\ [X^i, X^j] = \frac{2}{\sqrt{N^2-1}} i\epsilon^{ijk} X^k \longrightarrow 0 \ (N \to \infty) \end{cases}$$

Commutative limit

Functions on two sphere

Two sphere

- Functions

$$f(\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\Omega)$$

- Spherical harmonics

$$Y_{\ell m}(\Omega) = \sum_{k=0}^{\ell} t^{\ell m}_{i_1 i_2 \cdots i_k} x^{i_{i_1}} \cdots x^{i_{i_k}}$$

- Algebra

$$Y_{\ell_1 m_1}(\Omega) Y_{\ell_2 m_2}(\Omega) = \sum_{\ell_3 m_3} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} Y_{\ell_3 m_3}(\Omega)$$

$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$: Clebsch-Gordan coefficients

Fuzzy two sphere

- Matrices

UV cutoff

$$\hat{f} = \sum_{\ell=0}^{N-1} \sum_{m=-\ell}^{\ell} c_{\ell m} \hat{Y}_{\ell m}$$

- Fuzzy spherical harmonics

$$\hat{Y}_{\ell m} = \sum_{k=0}^{\ell} t^{\ell m}_{i_1 i_2 \cdots i_k} X^{i_{i_1}} \cdots X^{i_{i_k}}$$

- Algebra

$$\hat{Y}_{\ell_1 m_1} \hat{Y}_{\ell_2 m_2} = \sum_{\ell_3 m_3} \hat{C}^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} \hat{Y}_{\ell_3 m_3}$$

$$\hat{C}^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} \xrightarrow[N \to \infty]{} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$$

Differential & integral

Two sphere	Fuzzy two sphere
• Poisson bracket $\{x^i, x^j\} = 2\epsilon^{ijk} x_k$	• Commutator $-iN[X^i, X^j] \sim 2\epsilon^{ijk} X_k$
• Integral $\int \frac{d\Omega}{2\pi} Y_{\ell m}(\Omega)$	• Commutator $\frac{1}{N} \mathrm{Tr}\, \hat{Y}_{\ell m} \xrightarrow[N\to\infty]{} \int \frac{d\Omega}{2\pi} Y_{\ell m}(\Omega)$

– Poisson bracket gives angular momentum operator

$$\{x^i, \cdot\} = 2\epsilon^{ijk} x_k \partial_j$$

Dictionary of fuzzy two sphere

– Fuzzy sphere is given by a mapping

$$x^i \;\mapsto\; X^i = \frac{2}{\sqrt{N^2-1}} L^i$$

– Dictionary of functions & differential/integral operators

$$\begin{aligned} Y_{\ell m}(\Omega) &\quad\longleftrightarrow\quad \hat{Y}_{\ell m} \\ \{x^i, \cdot\} &\quad\longleftrightarrow\quad -iN[X^i, \cdot] \\ \int \frac{d\Omega}{2\pi} &\quad\longleftrightarrow\quad \frac{1}{N}\mathrm{Tr} \end{aligned}$$

– This mapping is an example of matrix regularization

Matrix regularization (MR)

— MR is a sequence of linear maps $\{T_N\}$ $(N = 1, 2, \cdots \infty)$ which satisfy

[Goldstone-Hoppe, 1982; Arnlind-Hoppe-Huisken, 2012]

$$
\begin{cases}
T_N : C^\infty(\mathcal{M}) \to N \times N \text{ Hermitian matrices} \\
\lim_{N\to\infty} ||T_N(fg) - T_N(f)T_N(g)|| = 0 \\
\lim_{N\to\infty} ||iN[T_N(f), T_N(g)] - T_N(\{f, g\})|| = 0 \\
\lim_{N\to\infty} \frac{2\pi}{N}\mathrm{Tr}\, T_N(f) = \int_{\mathcal{M}} f\omega
\end{cases}
$$

$\mathcal{M}$: symplectic manifold ω : symplectic form on $\mathcal{M}$

$||\ \cdot\ ||$: norm $\{\ ,\ \}$: Poison bracket on $\mathcal{M}$

Relation to string/M theories

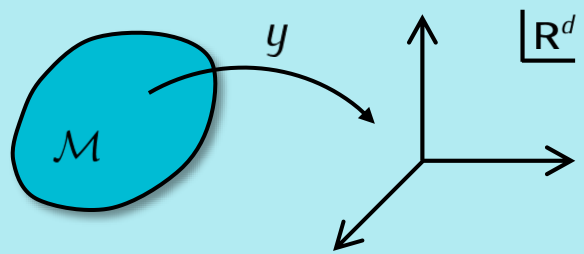
— Matrix models $\leftarrow$ MR of string/M theories

[BFSS 1997, IKKT 1997, DVV 1997]

$$Z = \int \prod_{\mu=1}^{d} DX^\mu e^{-S} \qquad X^\mu : N \times N \text{ Hermitian matrices}$$

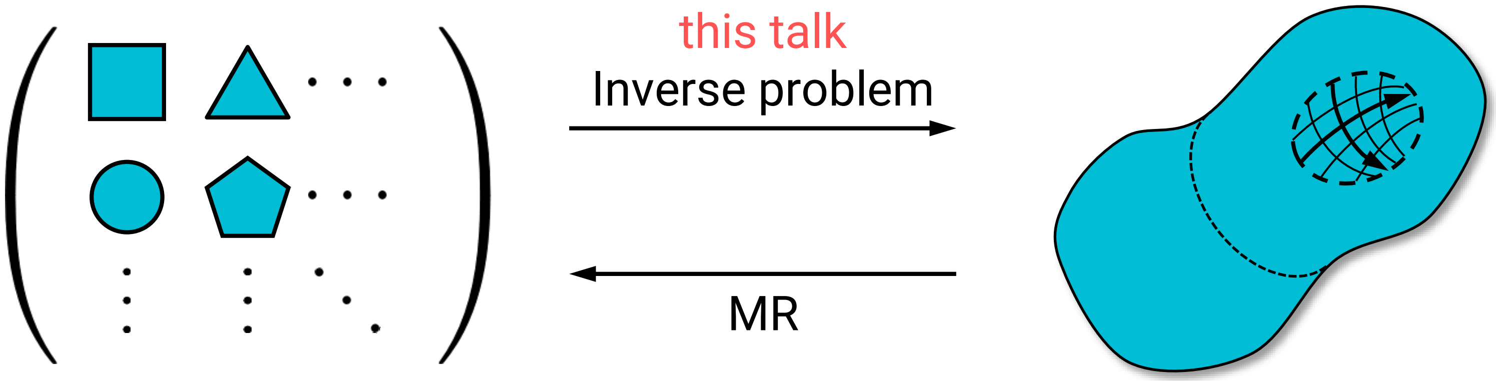
MR of embedding functions $y^\mu : \mathcal{M} \to \mathbf{R}^d$

- String
- D-brane
- Membrane



— Non-perturbative formulation & second quantization

Matrices & Smooth geometry



— Smooth geometry $\rightarrow$ Matrices: MR

[de Wit-Hoppe-Nicolai 1988, Bordemann-Meinrenken-Schlichenmaier 1994]

— Matrices $\rightarrow$ Smooth geometry: inverse problem of MR $\leftarrow$ this talk

Contents

2. Inverse problem of MR

Inverse problem of MR

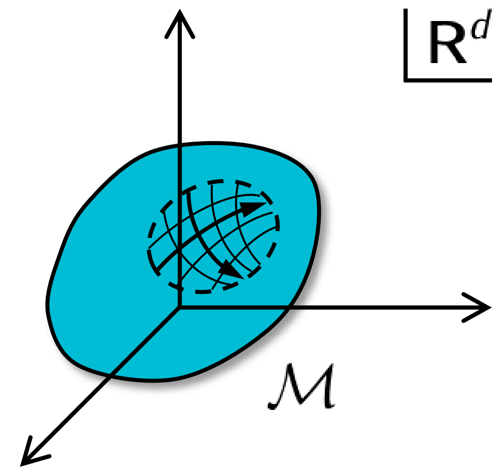
— For given matrices, how can we associate a smooth geometry $\mathcal{M}$??

$N \times N$ Hermitian matrices

$X^\mu \ (\mu = 1, 2, \cdots, d)$

How ??

$\xrightarrow[N \to \infty]{}$



— Many methods

- using Dirac operator, [Asakawa-Sugimoto-Terasima 2002, Berenstein-Dzienkowski 2012]
- using coherent states, [Ishiki 2015, Schneiderbauer-Steinacker 2016] ← this talk
- etc., [Hotta-Nishimura-Tsuchiya 1999, Azeyanagi-Hanada-Hirata-Shimada 2009]

Commutative space for X^μ

[Ishiki 2015]

– d $N \times N$ Hermitian matrices

$$\{(X^1, X^2, \cdots, X^d) \,|\, N = 1, 2, \cdots\}$$

– "Hamiltonian" using given matrices

$$H(y) = \frac{1}{2}\sum_{\mu=1}^{d}(X^\mu - y^\mu)^2 \qquad y^\mu \in \mathbf{R}$$

– Eigenstates & eigenvalues

$$H(y)|n, y\rangle = E_n(y)|n, y\rangle \quad (n = 0, 1, \cdots, N-1)$$

$$E_0(y) \le E_1(y) \le \cdots$$

Commutative space for X^μ

[Ishiki 2015]

– Zeros of $E_0(y)$ = points on commutative space

$$E_0(y) = \langle H(y) \rangle = \frac{1}{2} \left\langle (X^\mu - y^\mu)^2 \right\rangle = \frac{1}{2}\langle (X^\mu)^2 \rangle - \langle X^\mu \rangle y_\mu + \frac{1}{2}(y^\mu)^2$$

$$= \frac{1}{2}\langle (X^\mu)^2 \rangle \underline{- \frac{1}{2}\langle X^\mu \rangle^2 + \frac{1}{2}\langle X^\mu \rangle^2} - \langle X^\mu \rangle y_\mu + \frac{1}{2}(y^\mu)^2$$

$$= (\Delta X^\mu)^2 + (\langle X^\mu \rangle - y^\mu)^2$$

$$E_0(y) \to 0 \quad \Leftrightarrow \quad \begin{cases} \langle 0, y | X^\mu | 0, y \rangle \to y^\mu \\ \Delta X^\mu \to 0 \end{cases}$$

The wave packet shrinks to a point at $y \in \mathbf{R}^d$

Commutative space for X^μ

[Ishiki 2015]

– Commutative space for given matrices = zeros of $E_0(y)$

$$\mathcal{M} = \left\{ y \in \mathbf{R}^d \middle| \lim_{N\to\infty} E_0(y) = 0 \right\}$$

There are no zero points for finite N except trivial cases

– This works for fuzzy sphere, fuzzy torus, etc.

Construction method

[Ishiki 2015]

Given matrices

$N \times N$ Hermitian matrices

$X^\mu \ (\mu = 1, 2, \cdots, d)$

Hamiltonian

$$H(y) = \frac{1}{2}(X^\mu - y^\mu)^2$$

$y^\mu \in \mathbf{R}$

Eigenstates & energies

$H(y)|n, y\rangle = E_n(y)|n, y\rangle$

$E_0(y) \leq E_1(y) \leq \cdots$

Commutative space

Set of zeros of $E_0(y)$

$$\mathcal{M} = \left\{ y \in \mathbf{R}^d \,|\, \lim_{N\to\infty} E_0(y) = 0 \right\}$$

Example: fuzzy two sphere

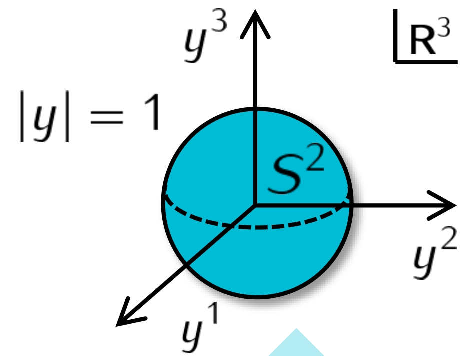
[Ishiki 2015]

Given matrices

$$X^i = \frac{2}{\sqrt{N^2-1}} L^i \ (i = 1, 2, 3)$$

L^i: N dim irrep of SU(2) generators

Commutative space



Hamiltonian

$$H(y) = \frac{1 + |y|^2}{2} - \frac{2 y_i L^i}{\sqrt{N^2 - 1}}$$

Eigenstates & energies

$$E_0(y) = \frac{1 + |y|^2}{2} - \frac{N|y|}{\sqrt{N^2 - 1}} \xrightarrow[N\to\infty]{} \frac{1}{2}(1 - |y|)^2$$

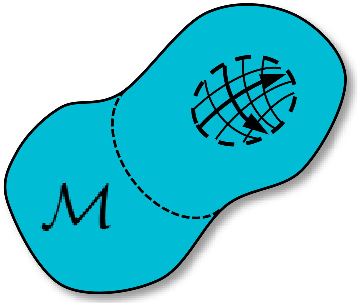
3. Information metric

Geometric structures on $\mathcal{M}$

— In general, smooth space has geometric structures on it

— Do X^μ have information of those geometric structures ??

$X^\mu \longrightarrow$



$(g, \omega, \cdots)$??

— Information metric $\rightarrow$ Riemannian structure on $\mathcal{M}$

Density matrix for X^μ

– In general, $E_0(y)$ is "degenerate" (degree of degeneracy $\equiv k$)

– There are k linearly independent the ground states

$$|0, y\rangle_\alpha \quad \alpha = 1, 2, \cdots, k$$

– The ground states $\rightarrow N \times N$ density matrix

$$\rho(y) = \frac{1}{k} \sum_{\alpha=1}^{k} |0, y\rangle_{\alpha\,\alpha}\langle 0, y| \qquad \text{(weight} = \frac{1}{k}\text{)}$$

– This invariant under changing basis

$$|0, y\rangle_\alpha \;\rightarrow\; c^\beta{}_\alpha(y)|0, y\rangle_\beta \quad (c^\dagger c = 1)$$

Berry phase

[Ishiki-TM-Muraki 2016]

– Berry phase (in the case for degenerate)

$$A_{\alpha\beta} = -i_\alpha\langle 0, y|\frac{\partial}{\partial y}|0, y\rangle_\beta$$

– Under changing basis, it transform as a gauge field

$$\begin{cases} |0, y\rangle_\alpha \;\rightarrow\; c^\beta{}_\alpha(y)|0, y\rangle_\beta \quad (c^\dagger c = 1) \\ A_{\alpha\beta}(y) \;\rightarrow\; (c^\dagger(y)A(y)c(y))_{\alpha\beta} - i(c^\dagger(y)\partial c(y))_{\alpha\beta} \end{cases}$$

Information metric

– Information metric for density matrices ρ

$$\begin{cases} ds^2 = \frac{1}{2}\mathrm{Tr}[G d\rho] \\ d\rho = G\rho + \rho G \end{cases}$$

– This measure the distance between density matrices

– In the case for pure state: $\rho = |\psi\rangle\langle\psi|$

$$ds^2 = \langle d\psi|d\psi\rangle - \langle d\psi|\psi\rangle\langle\psi|d\psi\rangle$$

Riemannian metric via pullback

– The density matrix using the ground states is a mapping

$$\begin{cases} \rho : y \in \mathcal{M} \;\mapsto\; \rho(y) \in \{\text{ Set of all } N \times N \text{ density matrices }\} \\ \rho(y) = \frac{1}{k} \sum_{\alpha=1}^{k} |0, y\rangle_{\alpha\,\alpha}\langle 0, y| \end{cases}$$

– In most cases, it is also an “embedding”

– $\rho(y)$ induces a Riemannian metric on $\mathcal{M}$ via the pullback

$$ds^2 = \frac{1}{2}\mathrm{Tr}[G(y)d\rho(y)]$$

Example 1: fuzzy two sphere

– Fuzzy two sphere: MR of two sphere $S^2 \subset \mathbf{R}^3$

$$\begin{cases} X^i = \frac{2}{\sqrt{N^2-1}} L^i \ (i = 1, 2, 3) \\ L^i : N \text{ dim irrep of SU(2) generators} \end{cases}$$

– Hamiltonian & the ground states

$$\begin{cases} H(y) = U(\theta, \phi) \left(\frac{1+|y|^2}{2} - \frac{2|y| L^3}{\sqrt{N^2-1}} \right) U^\dagger(\theta, \phi) \\ U(\theta, \phi) = e^{-i\phi L_3} e^{-i\theta L_2} e^{i\phi L_3} \\ |\Omega\rangle = U(\theta, \phi) |JJ\rangle \quad (2J+1 = N) \end{cases}$$

Bloch coherent states

Example 1: fuzzy two sphere

– Density matrix & information metric

$$\begin{cases} \rho = |\Omega\rangle\langle\Omega| \\ ds^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2 \end{cases}$$

Riemannian metric on two sphere !!

Example 2: fuzzy “four” sphere

— Fuzzy four sphere: fuzzy surface for $S^4 \subset \mathbf{R}^5$ [Castelino-Lee-Taylor]

$$\begin{cases} X^\mu = \frac{1}{N}(\overbrace{\Gamma^\mu \otimes 1_4 \otimes \cdots \otimes 1_4}^{N \text{ tensor product}} + \cdots + 1_4 \otimes \cdots \otimes 1_4 \otimes \Gamma^\mu)_{\text{sym}} \\ \Gamma^\mu \ (\mu = 1,2,3,4,5) : \text{five dimensional gamma matrices} \end{cases}$$

$$\Gamma^5 = \begin{pmatrix} 1_2 & O \\ O & -1_2 \end{pmatrix}$$

However, fuzzy four sphere satisfies at least
$X^\mu X_\mu = 1 + \mathcal{O}(1/N)$ & $[X^\mu, X^\nu] \to 0 \ (N \to \infty)$

— Four sphere ≠ symplectic manifold i.e. fuzzy four sphere ≠ MR

Example 2: fuzzy "four" sphere

– Hamiltonian & the ground states for fuzzy four sphere

$$\begin{cases} H(y) = U\left(\frac{1}{2}(1+|y|^2) - |y|X^5\right)U^\dagger + \mathcal{O}(1/N) \\ |0,y\rangle_\alpha = U\,|\text{ eigenstates of } X^5 \text{ with } +1\ \rangle_\alpha \quad (\alpha = 1,2,\cdots,N+1) \\ U = e^{-\chi\Gamma^{21}/2}e^{-\psi\Gamma^{32}/2}e^{-\phi\Gamma^{43}/2}e^{-\theta\Gamma^{54}/2} \end{cases}$$

Coherent states are
Degenerate

– Density matrix & information metric

$$\begin{cases} \rho = \frac{1}{N+1}\sum_\alpha |0,y\rangle_{\alpha\,\alpha}\langle 0,y| \\ ds^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2 + \sin^2\theta\sin^2\phi(d\psi)^2 + \sin^2\theta\sin^2\phi\sin^2\psi(d\chi)^2 \end{cases}$$

Riemannian metric on four sphere !!

4. Summary

Summary

– Hamiltonian & its ground states $\rightarrow$ commutative space $\mathcal{M}$

– The ground states $\rightarrow$ mapping from $\mathcal{M}$ to set of all density matrices

– The pullback of the information metric $\rightarrow$ Riemannian metric on $\mathcal{M}$

  – It works for fuzzy four sphere ($\neq$ MR)

– For other examples ?? e.g. fuzzy “three” sphere

[Ramgoolam 2000]

– Apply to matrix models ?? e.g. emergent gravity

[Steinacker 2011]