Wave Propagation & Shock formation in The most general scalar-tensor theories

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Wave Propagation and Shock formation in The most general scalar-tensor theories

- Scalar & gravitational wave in modified gravity theories

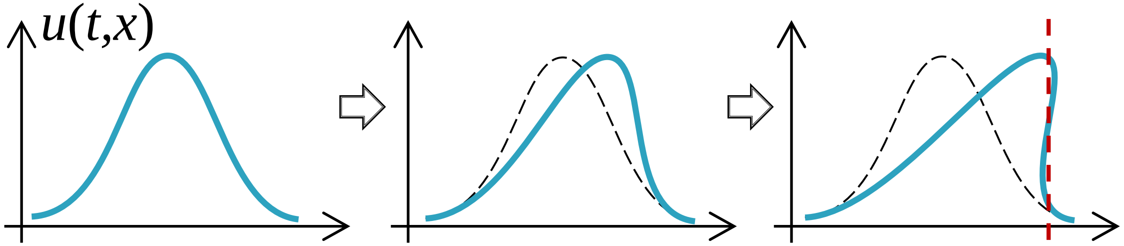
General Relativity: GW propagates at the light speed c

Modified gravity : (GW speed) $\neq c$, environment dependent

Causal structure may be modified

- Waveform distortion & Shock formation

ex.) Burgers' eq. $$\partial_t u + u\,\partial_x u = 0$$



Q: Does this occur for scalar & gravitational waves in modified gravity?

If it occurs, it may be observationally important.

- Study these phenomena in the Horndeski theory.

Summary

- Wave propagation & Shock formation in Horndeski theory
  - Result 1: Wave propagation in Horndeski theory

    On the plane wave solution in Horndeski theory, wave propagation obeys the effective metric, and the causality in this theory can be defined in accord with them.

  - Result 2: Shock formation in Horndeski theory

    In Horndeski theory, and for fluctuations on plane wave and 2D maximally-symmetric background,

    - Shock formation occurs for scalar field wave
    - Shock formation does not occur for gravitational wave

Contents

Horndeski theory

- Horndeski theory [Horndeski 1974]
  - One scalar field ϕ & gravity in 4-dim. spacetime
  - The most general covariant theory with 2nd-order EoM

$$\mathcal{L} = K(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$

$$+ G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

$$\left(X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\right)$$

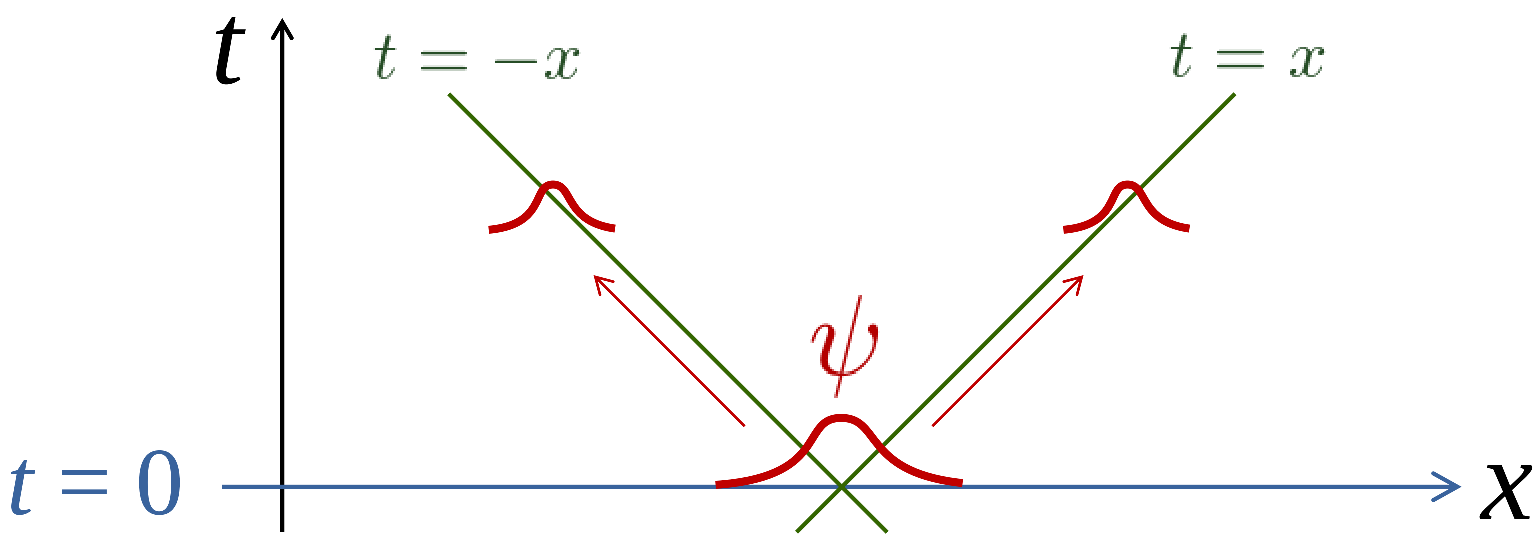
  - 4 arbitrary functions in Lagrangian
  - Applications to cosmology (e.g. inflation, …)

Wave propagation surface = Characteristic surface

- Massless scalar in flat space

$$0 = g^{\mu\nu}\partial_\mu\partial_\nu\psi = \left(-\partial_t^2 + \partial_x^2\right)\psi$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x)$$

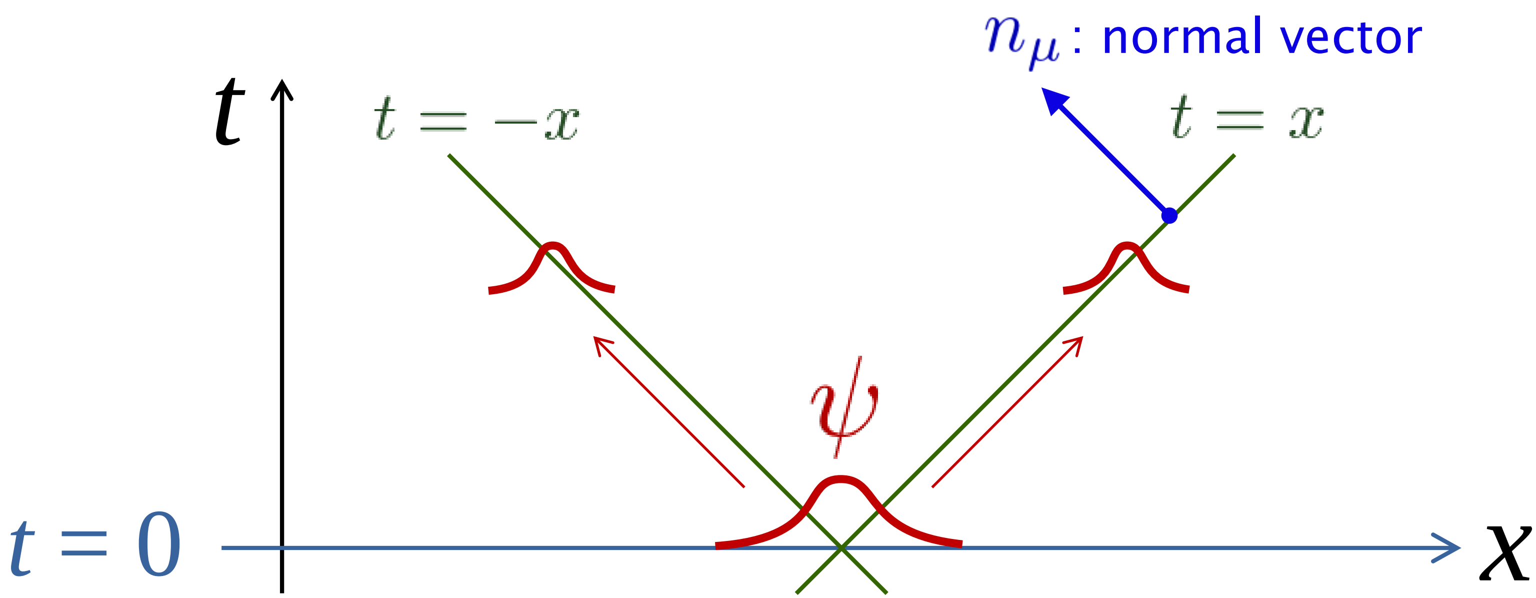


Wave propagation surface = Characteristic surface

- Massless scalar in flat space $\left(\psi = e^{in_\mu x^\mu}\tilde{\psi}_k\right)$

$$0 = g^{\mu\nu}\partial_\mu\partial_\nu\psi = -g^{\mu\nu}n_\mu n_\nu e^{in_\mu x^\mu}\tilde{\psi}_k$$

$$\Rightarrow \quad \psi = f_1(t-x) + f_2(t+x) \;\Leftrightarrow\; g^{\mu\nu}n_\mu n_\nu = 0$$



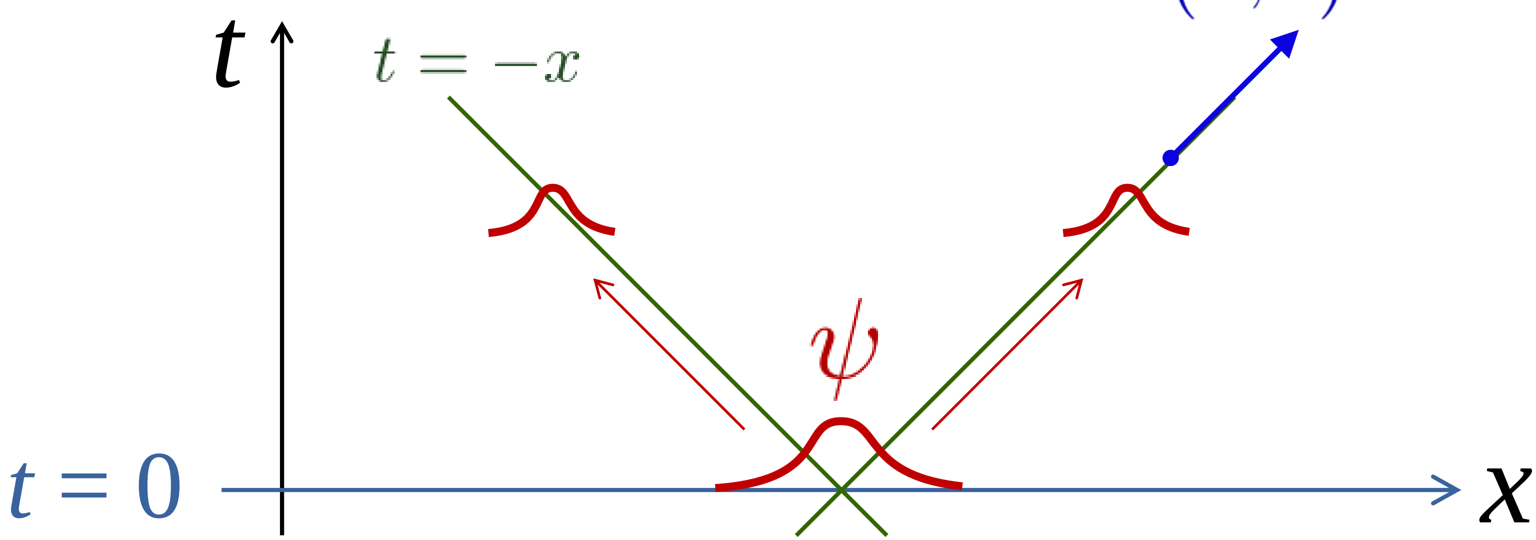
Wave propagation surface = Characteristic surface

- Massless scalar in flat space $\left[\psi = e^{i n_\mu x^\mu} \tilde{\psi}_k\right]$

$$0 = g^{\mu\nu}\partial_\mu \partial_\nu \psi = -g^{\mu\nu} n_\mu n_\nu e^{i n_\mu x^\mu} \tilde{\psi}_k$$

$$\Rightarrow \quad g^{\mu\nu} n_\mu n_\nu = 0 \quad \Leftrightarrow \quad -\omega^2 + \vec{k}^2 = 0$$: dispersion relation

$n^\mu = (\omega, \vec{k})$: wave vector



Wave propagation surface = Characteristic surface

- EoM E of dynamical variable v :

$$0 = E\left(v, \partial v, \partial^2 v\right) = \frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} \partial_\mu \partial_\nu v + \cdots$$

$$= \frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} n_\mu n_\nu v + \cdots$$

- Characteristic equation

$$\frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} n_\mu n_\nu = 0$$

- Surface w/ normal n_μ = Characteristic surface

Physically, a characteristic surface is a wave propagation surface.

Wave propagation surface = Characteristic surface

- EoM E of dynamical variable v :

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- Characteristic equation

$$\frac{\partial E}{\partial\left(\partial_\mu \partial_\nu v\right)} n_\mu n_\nu = 0$$ ← dispersion relation

"effective metric"

- Surface w/ normal n_μ = Characteristic surface

Physically, a characteristic surface is a wave propagation surface.

Wave propagation surface = Characteristic surface

- EoM of scalar-tensor theory

$$\begin{cases} 0 = E_{ab} = \dfrac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu g_{cd})} \partial_\mu \partial_\nu g_{cd} + \dfrac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu \phi_J)} \partial_\mu \partial_\nu \phi_J + \cdots \\ 0 = E_I = \dfrac{\partial E_I}{\partial(\partial_\mu \partial_\nu g_{cd})} \partial_\mu \partial_\nu g_{cd} + \dfrac{\partial E_I}{\partial(\partial_\mu \partial_\nu \phi_J)} \partial_\mu \partial_\nu \phi_J + \cdots \end{cases}$$

Write them collectively as

$$0 = E = \begin{pmatrix} \frac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu g_{cd})} & \frac{\partial E_{ab}}{\partial(\partial_\mu \partial_\nu \phi_J)} \\ \frac{\partial E_I}{\partial(\partial_\mu \partial_\nu g_{cd})} & \frac{\partial E_I}{\partial(\partial_\mu \partial_\nu \phi_J)} \end{pmatrix} n_\mu n_\nu \begin{pmatrix} \tilde{g}_{cd} \\ \tilde{\phi}_J \end{pmatrix} + \cdots$$

$$= P(n_\mu) \cdot r + \cdots$$

- Characteristic equation

propagating mode

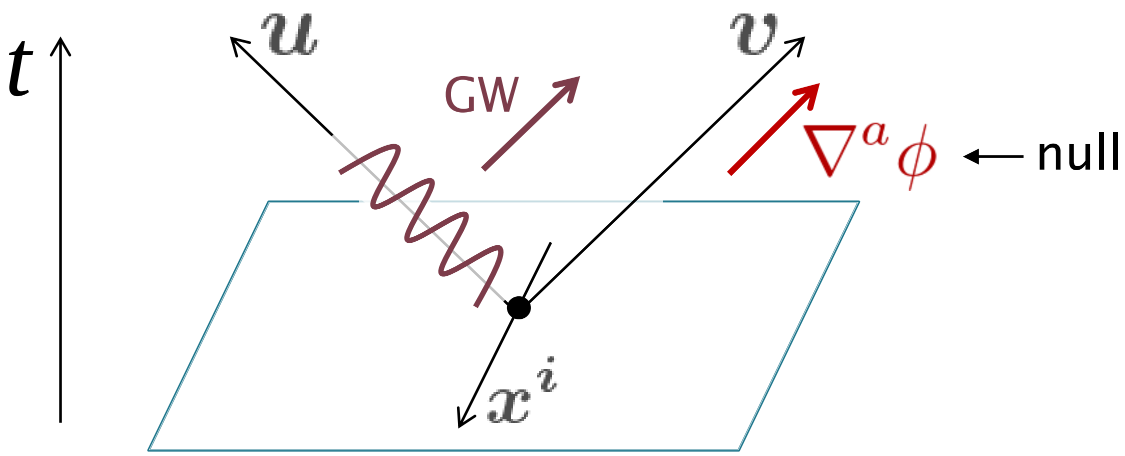
$$\det P(n_\mu) = 0 \iff P(n_\mu) \cdot r = 0 \quad \text{with} \quad r = \begin{pmatrix} \tilde{g}_{cd} \\ \tilde{\phi}_J \end{pmatrix} \neq 0$$

Contents

Wave Propagation on Plane wave background

- Focus on Horndeski with single ϕ for simplicity
- Background = Plane wave solution in Horndeski [Babichev 2012]

$$ds^2 = a_{ij}x^i x^j du^2 + 2dudv + \delta_{ij}dx^i dx^j, \quad \phi = \phi(u)$$



Wave Propagation on Plane wave background

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- Background = Plane wave solution in Horndeski [Babichev 2012]

$$ds^2 = a_{ij}x^i x^j du^2 + 2dudv + \delta_{ij}dx^i dx^j, \qquad \phi = \phi(u)$$

Wave propagation surface (normal: n_a) $\Leftrightarrow$ Characteristic equation $\det P(n_a) = 0$ $\Leftrightarrow$ $P(n_a)\cdot r = 0$ with $r = (r_{ab}, r_\phi) \neq 0$

GR: $(P\cdot r)_{ij} \sim g^{ab}n_a n_b \hat{r}_{ij} = 0 \quad \Longrightarrow \quad g^{ab}n_a n_b = 0$

2D traceless tensor

$\Longrightarrow$ $\hat{r}_{ij}$ propagates along null

(corresponds to GW)

Wave Propagation on Plane wave background

Horndeski: $P\cdot r = \begin{pmatrix}(P\cdot r)_{ij}\\(P\cdot r)_\phi\end{pmatrix} = \begin{pmatrix}G^{ab}_{\rm GW}n_a n_b\,\hat{r}_{ij}\\G^{ab}_\phi\,n_a n_b\,r_\phi\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Longrightarrow \begin{cases}G^{ab}_{\rm GW}n_a n_b = r_\phi = 0\\G^{ab}_\phi\,n_a n_b = \hat{r}_{ij} = 0\end{cases}$

where G^{ab} are the "effective metrics" defined by

$$\begin{cases}G^{ab}_{\rm GW} = g^{ab} + \omega\,\nabla^a\phi\nabla^b\phi \quad \text{with} \quad \omega = \dfrac{G_{4X}}{G_4}\\[2ex] G^{ab}_\phi = g^{ab} + \omega\,\nabla^a\phi\nabla^b\phi \quad \text{with} \quad \omega = -\dfrac{K_{XX}}{K_X} - \dfrac{G_{4X}}{G_4} + \dfrac{2G_{3X}}{K_X}\dfrac{\phi''}{\phi'^2}\end{cases}$$

- ✓ Wave propagation surface is null w.r.t. "effective metric": $G^{ab}n_a n_b = 0$
- ✓ $G^{ab}_{\rm GW}n_a n_b = r_\phi = 0 \;\Rightarrow\; r = (\hat{r}_{ij}, 0)$: Gravitational wave
- ✓ $G^{ab}_\phi\,n_a n_b = \hat{r}_{ij} = 0 \;\Rightarrow\; r = (0, r_\phi)$: Scalar field wave

Wave Propagation on Plane wave background

Horndeski: $P \cdot r = \begin{pmatrix} (P\cdot r)_{ij} \\ (P\cdot r)_\phi \end{pmatrix} = \begin{pmatrix} G^{ab}_{\rm GW} n_a n_b \, \hat{r}_{ij} \\ G^{ab}_\phi \, n_a n_b \, r_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} G^{ab}_{\rm GW} n_a n_b = r_\phi = 0 \\ G^{ab}_\phi \, n_a n_b = \hat{r}_{ij} = 0 \end{cases}$

where G^{ab} are the "effective metrics" defined by

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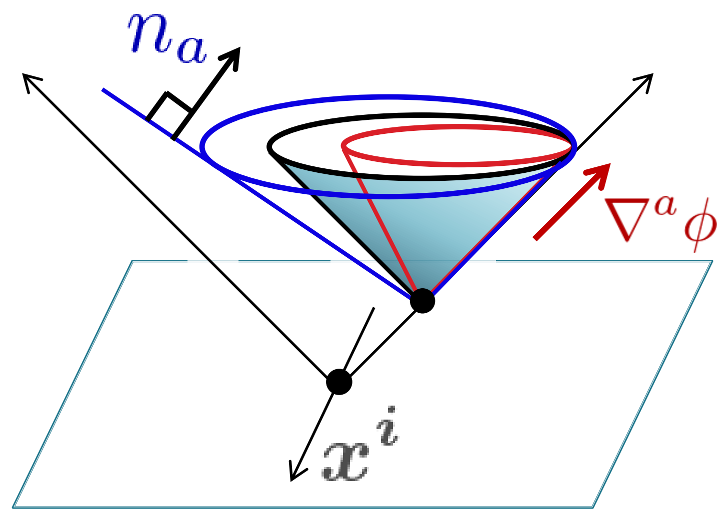
- ✓ Nested characteristic cones: all cones // $\nabla^a \phi$
- ✓ Causality w.r.t. the largest cone: larger $\omega \Leftrightarrow$ larger cone

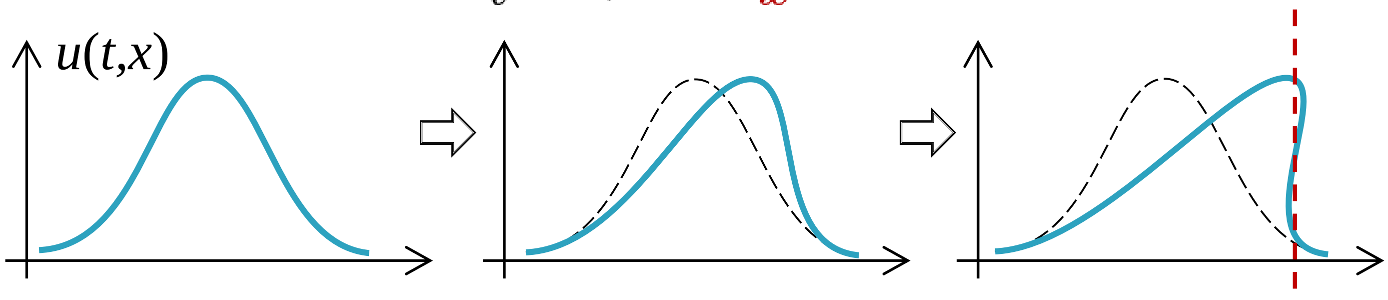


Contents

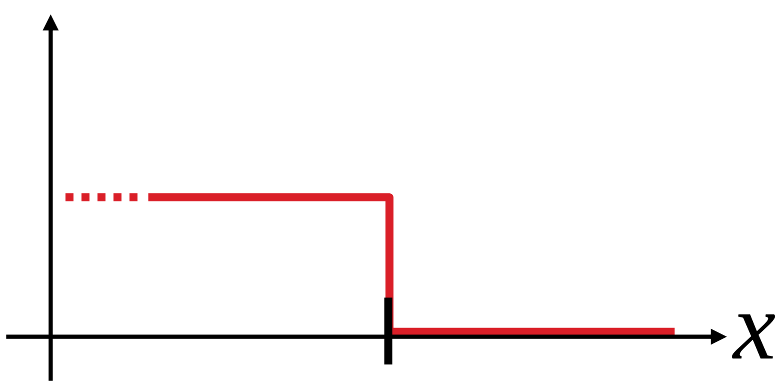
Shock formation in Horndeski theory

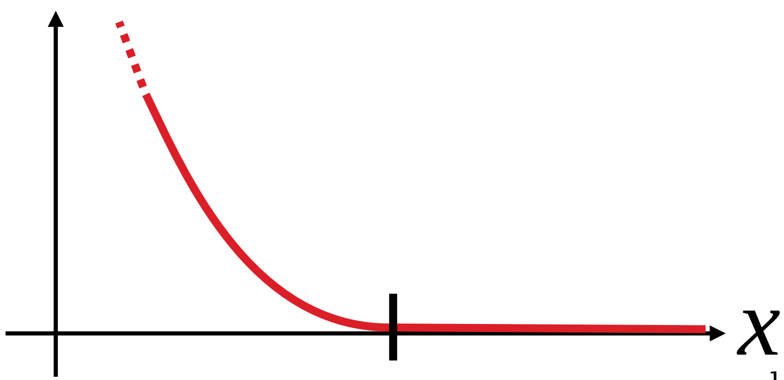
- Shock formation = Divergence in gradient of waveform

Ex.) Burgers' eq. $\partial_t u + u\,\partial_x u = 0$



- For simplicity, we look at wave with discontinuity in second derivative:





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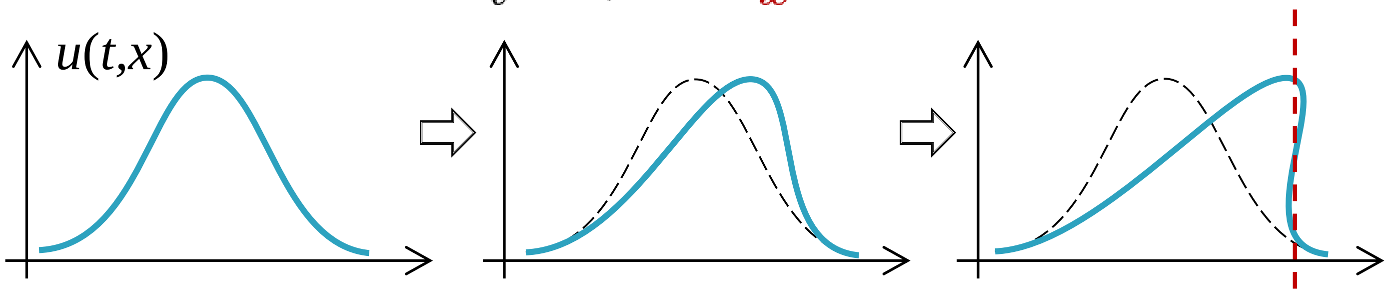

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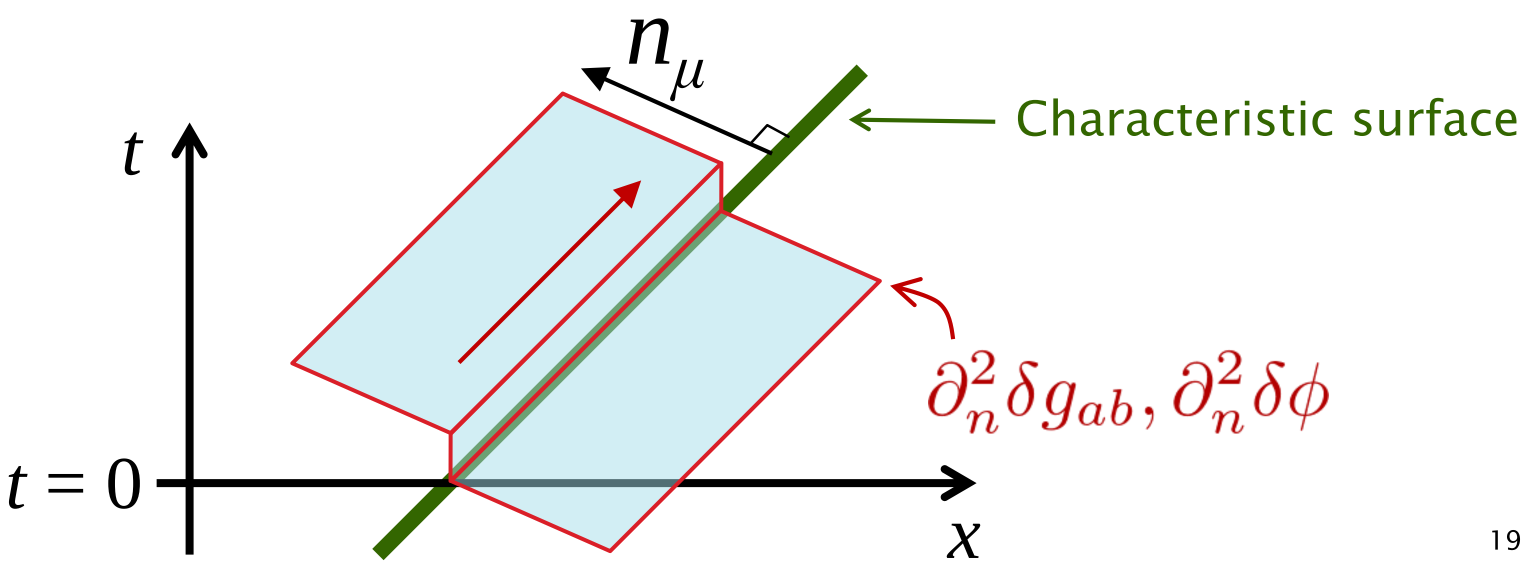


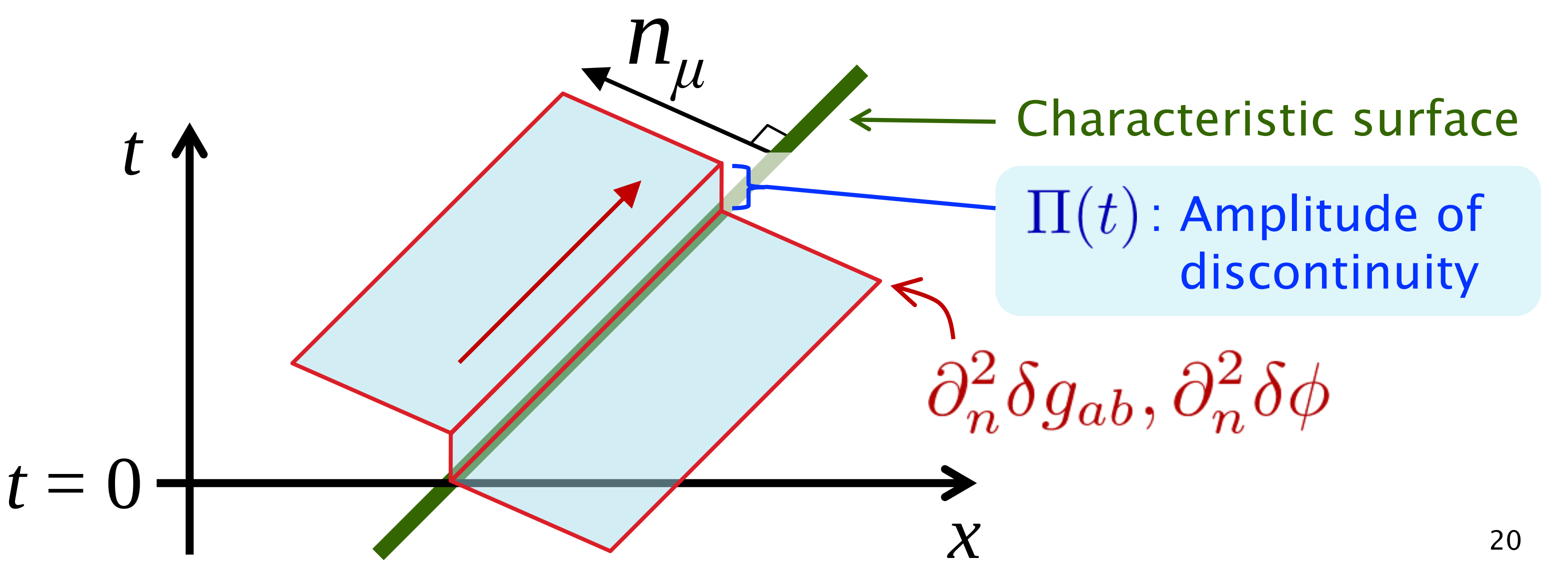
Shock formation in Horndeski theory

- Look at amplitude of discontinuity $\Pi(t) = \left[\partial_n^2 g_{ab}\right]\,,\ \left[\partial_n^2 \phi\right]$

- Transport equation of $\Pi(t)$:

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

- For simplicity, we look at wave with discontinuity in second derivative:



1. EoM:

$$\begin{cases} E_{ab} = \frac{\partial E_{ab}}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \frac{\partial E_{ab}}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \\ E_I = \frac{\partial E_I}{\partial(\partial_n^2 g_{cd})}\partial_n^2 g_{cd} + \frac{\partial E_I}{\partial(\partial_n^2 \phi_J)}\partial_n^2 \phi_J + \cdots = 0 \end{cases}$$

$$P \equiv \begin{pmatrix} \frac{\partial E_{ab}}{\partial g_{cd,nn}} & \frac{\partial E_{ab}}{\partial \phi_{J,n}} \\ \frac{\partial E_I}{\partial g_{cd,nn}} & \frac{\partial E_I}{\partial \phi_{J,n}} \end{pmatrix}$$

Write them collectively as

$$E_a = P_a{}^b\, \partial_n^2 v_b + \cdots = 0$$

2. Take discontinuous part

$$[E_a] = P_a{}^b\, [\partial_n^2 v_b] = 0 \quad \Rightarrow \quad [\partial_n^2 v_b] = \Pi(t)\, r_b$$

(discontinuous part)

For r_b s.t. $P \cdot r = 0$

3. Transport equation of amplitude $\Pi(t)$

$$[\,\partial_n E_a\,] = 0 \quad \Rightarrow \quad \dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

where

$$N = \frac{\partial P^{ab}}{\partial(\partial_n v_c)} r_a r_b r_c$$

What happens when $N \neq 0$?

$$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$$

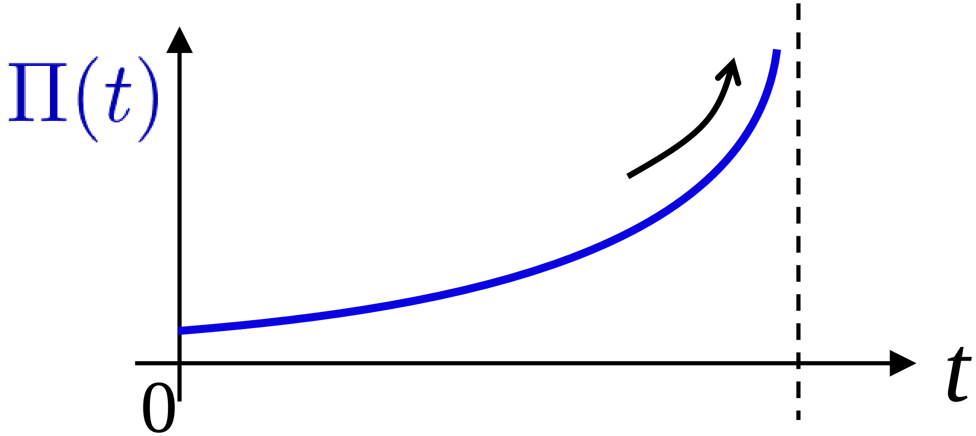
$$\left(\Phi(t) = \int_0^t M(t')dt'\right)$$

$$\Rightarrow \quad \Pi(t) = \frac{\Pi(0)e^{-\Phi(t)}}{1 + \Pi(0)\int_0^t N(t')e^{-\Phi(t')}dt'} \sim \frac{\Pi(0)}{1 + \Pi(0)\,N\,t}$$

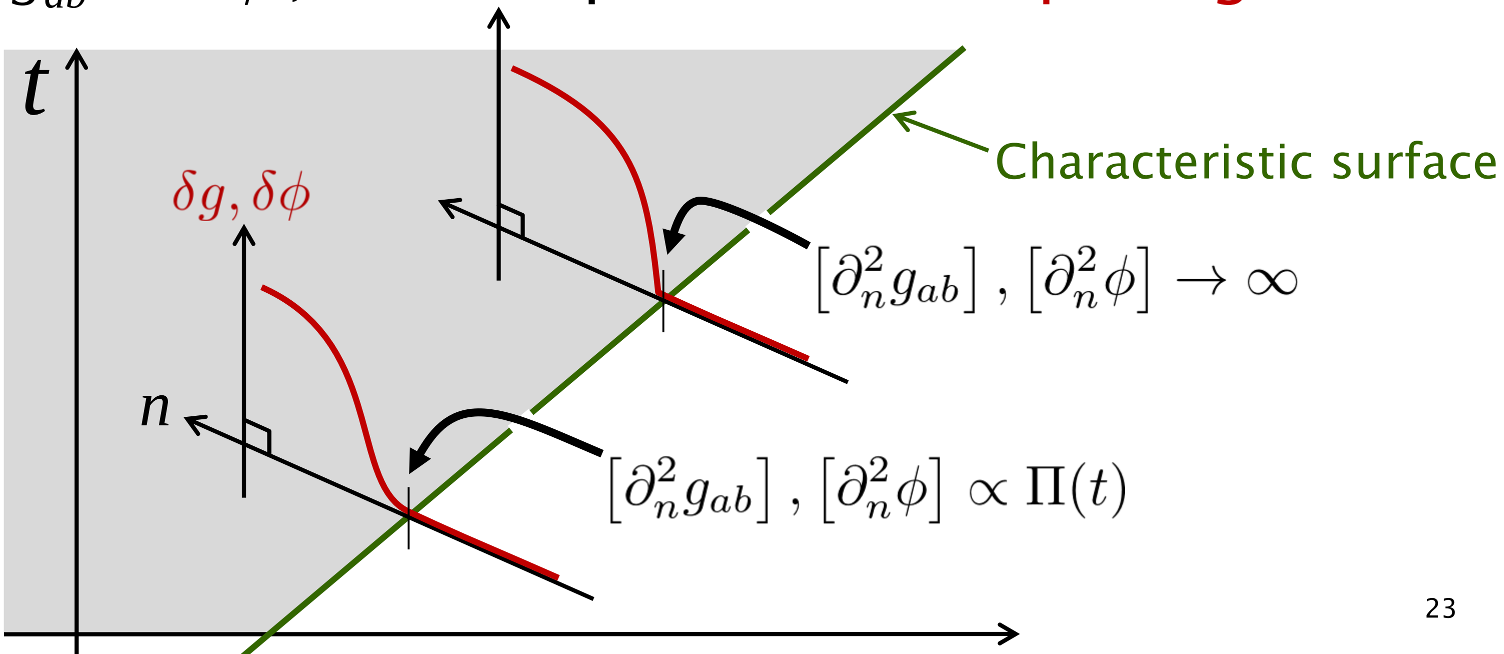
- GR : $N = 0$ $\Rightarrow$ $\Pi(s)$ stays finite
- Modified grav : $N \neq 0$ $\Rightarrow$ Denominator may vanish at $t \sim -1/\Pi(0)N$

  $\Rightarrow$ Amplitude $\Pi(t)$ diverges "Shock formation"

- Amplitude of discontinuity: $\Pi(t) = \left[\partial_n^2 g_{ab}\right], \left[\partial_n^2 \phi\right]$
- Our "shock formation" = 2[nd] derivative $\Pi(t) \to \infty$ at finite t

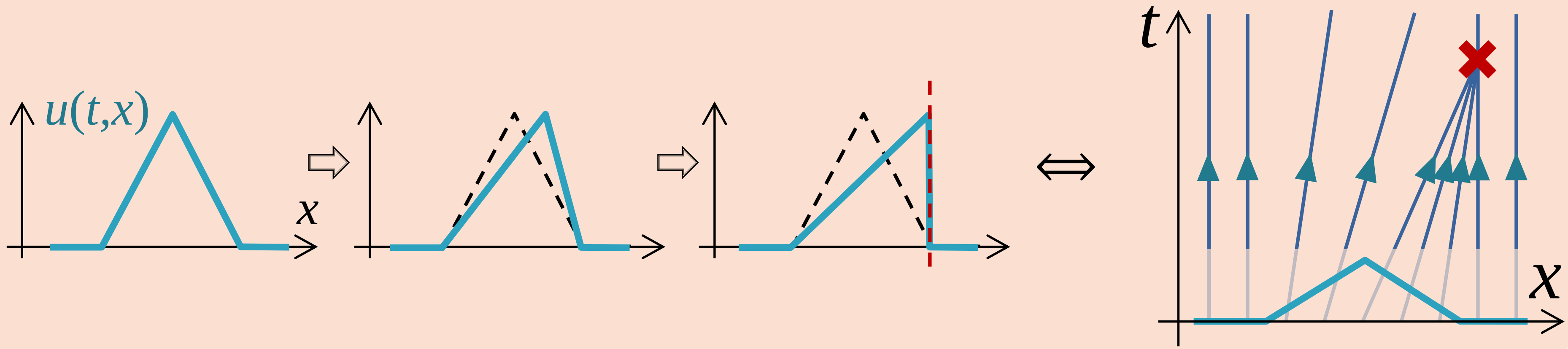


- In terms of g_{ab} and ϕ, it corresponds to "sharpening":



Shock formation caused by variable sound speed

Burgers' eq. $\partial_t u + u\,\partial_x u = 0 \Rightarrow$ propagation speed $c(u) \propto u$



In our case,

$\det P(n_\mu) = 0 \;\Rightarrow\;$ propagation speed $c(u,\partial u)$

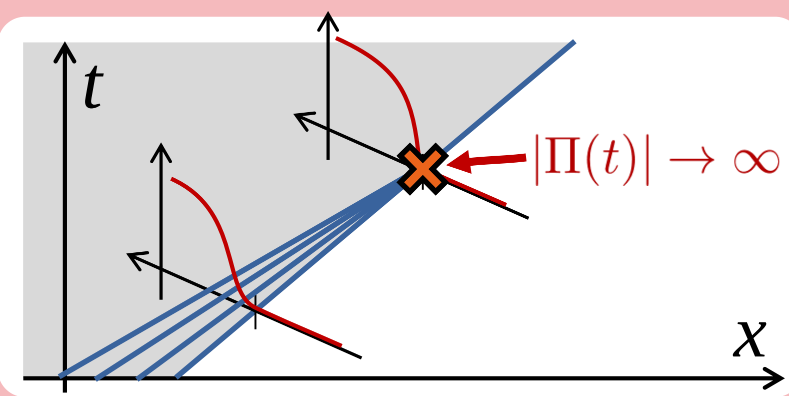
$\dot{\Pi} + M\,\Pi + N\,\Pi^2 = 0$ where $N = \dfrac{\partial P^{ab}}{\partial(\partial_n v_c)} r_a r_b r_c \sim \dfrac{\partial\, c(u,\partial u)}{\partial(\partial u)}$

$$N \sim \frac{\partial\, c(u,\partial u)}{\partial(\partial u)} \neq 0$$

$\Updownarrow$

$c(u,\partial u)$ depends on amplitude

$\Rightarrow$

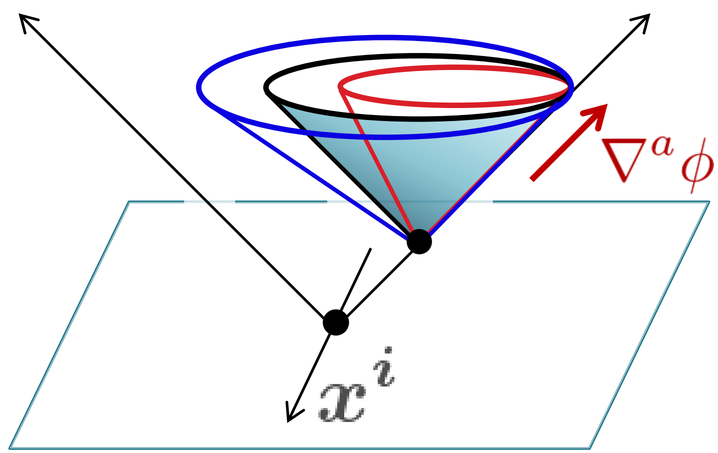


Shock formation on Plane wave background

- Example: Wave on Plane wave solution

$$ds^2 = a_{ij}x^i x^j du^2 + 2dudv + \delta_{ij}dx^i dx^j, \quad \phi = \phi(u)$$

[Babichev 2012]



- ✓ Scalar field & Gravitational wave propagate at different speeds
- ✓ Scalar : $N \neq 0$ → Shock formation
- ✓ GW : $N = 0$ → No shock formation

[Babichev 2016]
[Mukohyama, Namba, Watanabe 2016]
[de Rham, Motohashi 2016]

Shock formation on Plane wave background

$$N = C_+ f_+(t) + C_- f_-(t) + C_0$$

$$C_+ = \phi'^2 \left\{ -\frac{1}{2G_4} \left(2G_{3X}G_{4X} + K_X G_{5X} \right) + \frac{2K_{XX}G_{3X}}{K_X} - G_{3XX} \right\}$$

$$C_- = -2a\, G_{5X}$$

$$C_0 = \phi'^3 \left\{ \frac{3}{G_4} \left(-G_{3X}{}^2 - K_X G_{4XX} + K_{XX} G_{4X} \right) + \frac{3K_{XX}{}^2}{K_X} - K_{XXX} \right\}$$

- ✓ $N \neq 0$ generically, hence shock forms for scalar mode.
- ✓ $N = 0$ identically for scalar DBI model, hence free from shock formation.

$$K = \sqrt{1 + (\nabla\phi)^2}$$

- ✓ Scalar field & Gravitational wave propagate at different speeds
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[Babichev 2016]
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Shock formation on Plane wave background

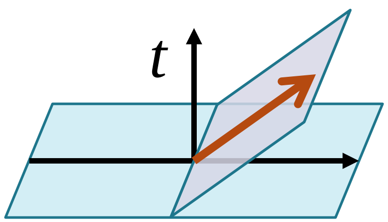
- Example: 2D maximally-symmetric dynamical spacetime

$$ds^2 = f(\tau,\chi)\left(-d\tau^2 + d\chi^2\right) + \rho(\tau,\chi)d\Omega^2, \quad \phi = \phi(\tau,\chi)$$

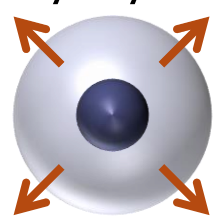
2-dim. flat or S^2 or H^2

and consider waves propagating in (τ, χ) direction.

- Plane wave in FRW universe



- Spherical wave around spherically-sym. star/BH



✓ Scalar & Gravitational perturbations propagate at different speeds

✓ Scalar: $N \neq 0$ → Shock formation

[Babichev 2016]
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  - Result 2: Shock formation in Horndeski theory

    In Horndeski theory, and for fluctuations on plane wave and 2D maximally-symmetric background,

    - Shock formation occurs for scalar field wave
    - Shock formation does not occur for gravitational wave

✓ Shock formation DOES OCCUR for gravitational wave in Gauss-Bonnet gravity in higher dimensions.

[Tomimatsu-Ishihara 1987]
[Choquet-Bruhat 1989]
[Reall, NT, Way 2015]

$$\mathcal{L} = \mathcal{L}_{\mathrm{GR}} + R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$$

(curvature)² term crucial for GW shock?

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- More complicated background in Horndeski
- Bi-Horndeski theory

Shock formation occurs even for gravitational wave?