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Topological order and zero modes in interacting Kitaev/Majorana chains

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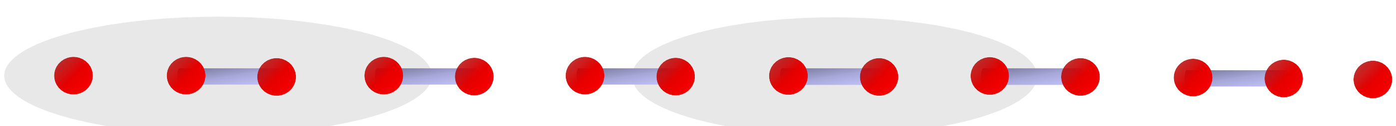
- *Phys. Rev. B*, **92**, 115137 (2015)
- *Phys. Rev. B*, **95**, 195140 (2017)
- *Phys. Rev. B*, **98**, 155119 (2018)



Outline

1. Introduction & Motivation

- Exotic particles in cond-mat and stat. mech.

2. Kitaev/Majorana chain

3. Interacting Kitaev/Majorana chain

4. Energy gap, edge zero modes & topo. Order

5. Generalizations

Much ado about Majorana fermions

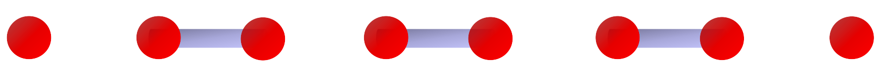
- What are Majorana fermions?

  "*Majorana returns*", F. Wilczek, *Nat. Phys*. **5** (2009).
  Particles that are own antiparticles $\gamma^\dagger = \gamma$
  Real solutions of Dirac eq. → Electrically neutral
  Neutrinos might be Majoranas?

- Emergent Majoranas in condensed matter

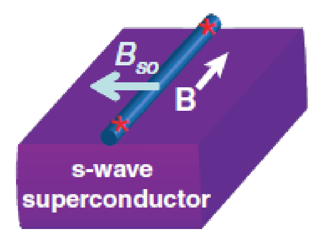
  "*Unpaired Majorana fermions in quantum wires*",
  A. Kitaev, *Phys. Usp*., cond-mat/0010440 (2000).
  Majorana fermions = edge zero modes

$$[H, \gamma_{L/R}] = 0 \qquad \{|0\rangle,\ \gamma_L|0\rangle,\ \cancel{\gamma_R|0\rangle},\ \cancel{\gamma_L\gamma_R|0\rangle}\}$$

γ_L  γ_R

qubit?

Experimental Signatures
Mourik *et al*., *Science* **336**, 1003 (2012).
Nadj-Perge *et al*., Science **346**, 602 (2014), …



Majorana fermions in Stat. Mech.

- Kitaev chain = Quantum Ising chain

P. Fendley, *J. Stat. Mech.* P11020 (2012).

> Surprisingly, the word "**Ising**" does not appear in Kitaev's paper (except inside "Surprisingly") nor in the review … ; often this chain is now referred to as the "Kitaev chain".

- History of the 2d Ising model

1941: Kramers-Wannier, transition temperature
1941-1944: Onsager, exact free energy
1949: Kaufman, Majorana-fermion method, Pffafian
1949-1952: Onsager, Yang, exact magnetization
1970~: McCoy-Wu, Jimbo-Miwa-Sato, …, still ongoing!

Spontaneous magnetization

$$\langle \sigma_j^z \rangle = (1-k^2)^{1/8} \qquad 1/k = \sinh(2\beta J)\sinh(2\beta J')$$

Tower of Babel

Exotic “particles” appear in several seemingly-unrelated fields. People in these fields use different languages. ***Dictionary*** reads

Topological cond-mat.	2d stat-mech.	Math-phys.
Majorana fermions $\psi^\dagger = \psi,\ \psi^2 = 1$	**2d Ising model**	**Onsager algebra**
Parafermions $\psi^\dagger = \psi^{m-1},\ \psi^m = 1$	**Chiral Potts model** superintegrable case	**Yang-Baxter w/o difference property** PLUS Onsager alg.
Fibonacci anyons $\mathbf{1} \times \tau = \tau$ $\tau \times \tau = \mathbf{1} + \tau$	**RSOS (ABF) model**	**Temperley-Lieb alg.**

Their critical points are described by 2d CFTs, correlators of which give FQHE wavefunctions.

What people are doing these days are just a rephrasing of what have been known for many decades…?

Interacting Majorana fermions

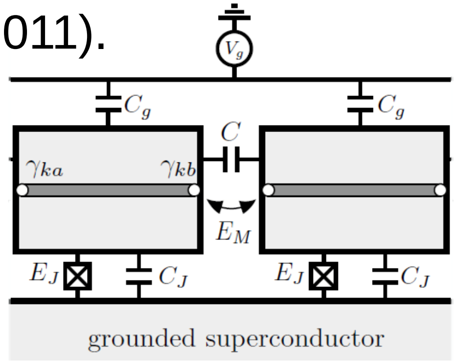
■ Motivation

Do what no one else can do! Let's *solve interacting Majorana fermions exactly*. What is topo. order in interacting systems?

If that's not motivation enough…

1. Interactions narrow or expand the topo. phase?
   Narrow) S. Gangadharaiah *et al*., *PRL* **107**, 036801 (2011).
   Expand) E.M. Stoudenmire *et al*., *PRB* **84**, 014503 (2011).

2. Physical realization of coupled Majorana wires
   Array of Josephson junctions
   F. Hassler & D. Schuricht, *New. J. Phys*. **14** (2012).



■ Dictionary again…

Via Jordan-Wigner transformation,

Kitaev chain	**Quantum Ising chain**
Interacting Kitaev chain	**XYZ chain in a field**

■ E-mail discussion

On May 6th, 2014:

H.K.: I found that the interacting Kitaev/Majorana chain is exactly solvable if the chemical potential is tuned to a particular function of the other parameters (t, Δ, U).

D.S.: That sounds very interesting. But did you know the **Peschel-Emery** line? [*Z. Phys. B*, **43**, 241, (1981).]

H.K.: No, I didn't. But I will look into their paper...

It turns out that I'm also the guy who is just rephrasing the known results in the literature…

Kind of disappointing…
But I will try to add something new to both topological condensed matter physics and mathematical physics.

Outline

1. Introduction & Motivation

2. Kitaev/Majorana chain

- Trivial and topological phases, edge zero modes
- Mapping to quantum Ising chain

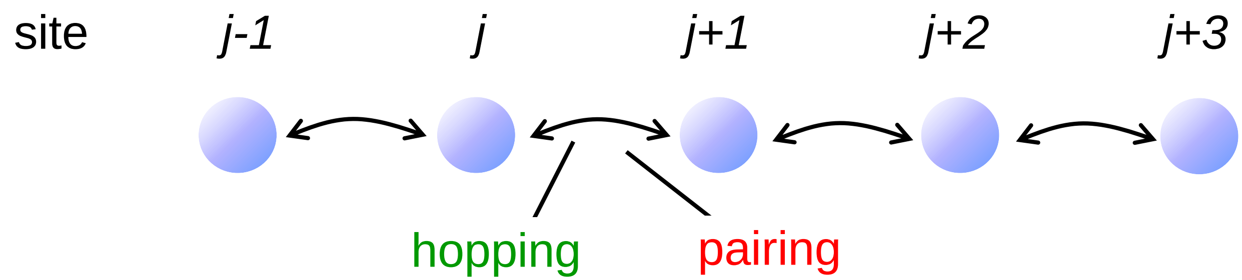
3. Interacting Kitaev/Majorana chain

4. Energy gap, edge zero modes & topo. Order

5. Generalizations

Kitaev/Majorana chain (non-interacting)

- Hamiltonian (complex fermions, with OBC)   A. Kitaev, *Phys. Usp*. (2001).

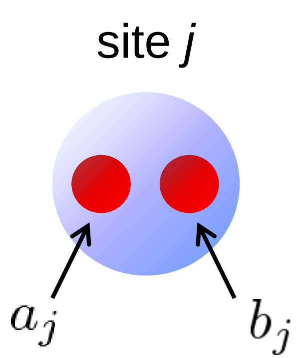


$$\{c_i, c_j^\dagger\} = \delta_{ij}$$

Chemical potential

$$H_0 = -w\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + w\sum_{j=1}^{L-1}(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) - \mu\sum_{j=1}^{L-1}(c_j^\dagger c_j - 1/2)$$

- Complex fermion = pair of real (Majorana) fermions



$$c_j = \frac{1}{2}(a_j + ib_j), \quad c_j^\dagger = \frac{1}{2}(a_j - ib_j)$$

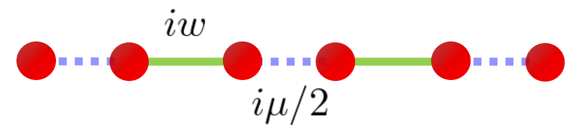
$$a_j = c_j + c_j^\dagger, \quad b_j = (c_j - c_j^\dagger)/i$$
$$a_j^\dagger = a_j, \quad b_j^\dagger = b_j$$
$$\{a_j, a_k\} = \{b_j, b_k\} = 2\delta_{jk}, \quad \{a_j, b_k\} = 0$$

Defining relations

Phases in the Kitaev/Majorana chain

- Hamiltonian (Majorana fermions, with OBC)

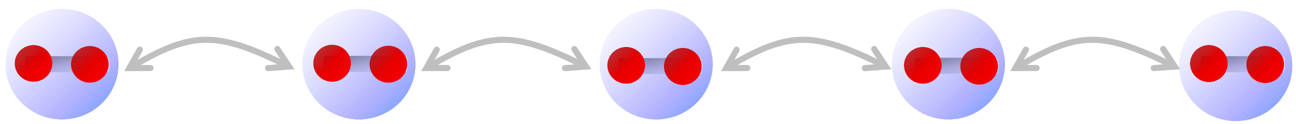
$$H_0 = iw \sum_{j=1}^{L-1} b_j a_{j+1} - \frac{i}{2}\mu \sum_{j=1}^{L} a_j b_j$$



- Trivial phase $(\, w \ll \mu \,)$

$$H_0 \sim -\frac{i}{2}\mu \sum_{j=1}^{L} a_j b_j = -\mu \sum_{j=1}^{L} (c_j^\dagger c_j - 1/2)$$

The fully filled state is the unique g.s. (μ>0)

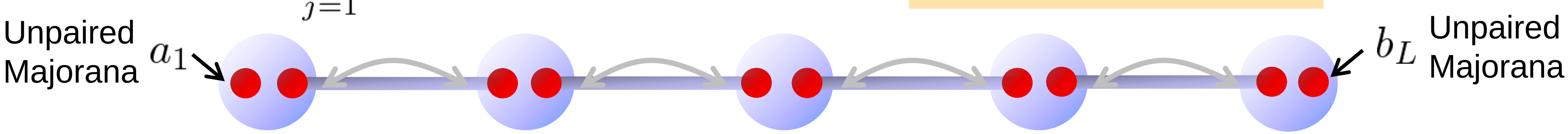


- Topological phase $(\, w \gg \mu \,)$

$$H_0 \sim iw \sum_{j=1}^{L-1} b_j a_{j+1}$$

New, non-local fermion:

$$f = \frac{1}{2}(a_1 + i b_L)$$



presence/absence of *f* ⇔ two-fold degenerate g.s. $|0\rangle, \quad |1\rangle = f^\dagger |0\rangle$

Non-local zero mode commuting with H_0 exists as long as $w>\mu$.
Topological order! Quantum phase transition occurs at $w=\mu$.

Kitaev chain = Quantum Ising chain

- Hamiltonian in terms of spins

  Jordan-Wigner transformation

$$a_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^x, \qquad b_j = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^y$$

Ising model! $J_x = w,\ h = -\frac{\mu}{2}$

$$H_0 = iw \sum_{j=1}^{L-1} b_j a_{j+1} - \frac{i}{2}\mu \sum_{j=1}^{L} a_j b_j = -J_x \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^{L} \sigma_j^z$$

- Ground states of the spin model (J_x >> $|h|$)

$$\sigma^x|\pm\rangle = \pm|\pm\rangle \qquad \sigma^z|\pm\rangle = |\mp\rangle$$

2-fold degenerate g.s.: $|+\rangle_1|+\rangle_2\cdots|+\rangle_L$

$|-\rangle_1|-\rangle_2\cdots|-\rangle_L$

Ferromagnetically ordered in *x* direction

Order parameter

$$\mathcal{O} = \sum_{j=1}^{L} \sigma_j^x = \sum_{j=1}^{L} \exp\left[i\pi \sum_{k=1}^{j-1} n_k\right] (c_j + c_j^\dagger)$$

Local in spin variables, but non-local in fermions!

Bulk topological invariant

- Hamiltonian in *k*-space (PBC assumed, $-\pi<k\leqq\pi$)

$$H = \frac{1}{2}\sum_k \Psi^\dagger(k)\,\mathcal{H}(k)\,\Psi(k), \qquad \Psi(k) = \begin{pmatrix} \psi_k \\ \psi^\dagger_{-k} \end{pmatrix}$$

$$\mathcal{H}(k) = \begin{pmatrix} h(k) & \Delta(k) \\ \Delta^*(k) & -h(k) \end{pmatrix} = h_x(k)\,\sigma^x + h_y(k)\,\sigma^y + h_z(k)\,\sigma^z$$

Symmetry

$$h_{x,y}(k) = -h_{x,y}(-k) \quad h_z(k) = h_z(-k)$$

n-vector $\boldsymbol{n}(k) := \dfrac{\boldsymbol{h}(k)}{|\boldsymbol{h}(k)|}$

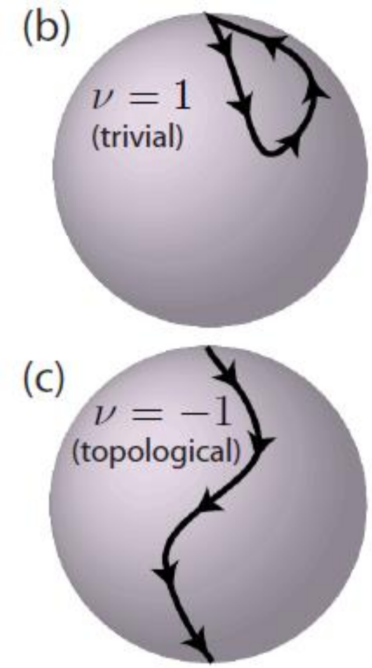
Nonzero gap ⇔ $|\boldsymbol{h}(k)| > 0$

- Topological number

k= 0 and *π* are special!

$$\boldsymbol{n}(0) = s_0\,\hat{z}, \quad \boldsymbol{n}(\pi) = s_\pi\,\hat{z}$$

$$\nu = s_0 s_\pi \in \{+1, -1\}$$



J. Alicea, *Rep. Prog. Phys*. **75**, 076501 (2012)

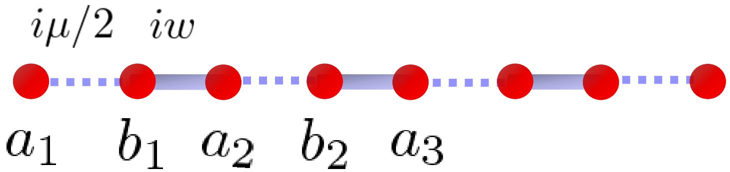
Majorana edge zero modes

■ Implication of topological invariant

Gap closing → zero mode in the spectrum of system with OBC

■ Zero mode in real space

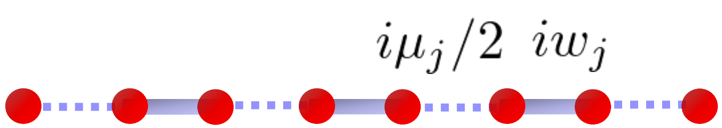
Uniform chain $(\, w \gg \mu \,)$



$$\Gamma_L \propto a_1 - \frac{\mu}{2w} a_2 + \left(\frac{\mu}{2w}\right)^2 a_3 + \cdots$$

$$[H, \Gamma_L] = 0$$ Majorana zero mode!

Disordered chain



$$\Gamma_L \propto a_1 - \frac{\mu_1}{2w_1} a_2 + \frac{\mu_1\mu_2}{4w_1w_2} a_3 + \cdots$$

$$[H, \Gamma_L] = 0$$ Majorana zero mode!

Majorana zero mode at the edge is robust against disorder!

Edge characterization

■ Fermionic (Z_2) Parity

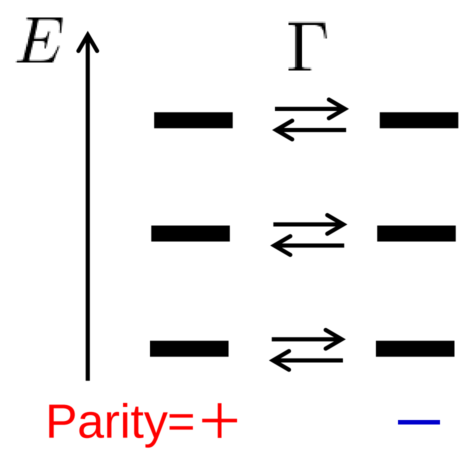
H_0 conserves fermion # mod 2,
→ H_0 commutes with $$(-1)^F = \prod_{j=1}^{L}(-ia_j b_j) = \prod_{j=1}^{L}\sigma_j^z$$

■ Characteristic of topological order

i) Nonvanishing energy gap,
ii) g.s. degeneracy (OBC), and
iii) locally indistinguishable g.s. (in the fermionic basis)

PLUS iv) existence of Majorana edge zero modes s.t.

- $\Gamma^\dagger = \Gamma$
- $[H_0, \Gamma] = 0$
- $\{(-1)^F, \Gamma\} = 0$
- localized near the edge, and normalizable as $\Gamma^2 = 1$ even in the infinite-size limit.



NOTE) i), …, iv) are not totally independent.

Outline

1. Introduction & Motivation
2. Kitaev/Majorana chain
3. Interacting Kitaev/Majorana chain

- Operator inequalities, frustration-free Hamiltonian
- Exact ground states, solvable line in the phase diagram

4. Energy gap, edge zero modes & topo. Order
5. Generalizations

Interacting Kitaev/Majorana chain

- Hamiltonian (complex fermions, with OBC)

$$H = -t\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta\sum_{j=1}^{L-1}(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$

$$-\sum_{j=1}^{L}\mu_j(c_j^\dagger c_j - 1/2) + U\sum_{j=1}^{L-1}\underline{(2c_j^\dagger c_j - 1)(2c_{j+1}^\dagger c_{j+1} - 1)}$$

4-Majorana int. $-a_j b_j a_{j+1} b_{j+1}$

$H \rightarrow H_0$ when $t=\Delta=w$, $\mu_j=\mu$, and U=0. μ_j may depend on j.

- Spin Hamiltonian = XYZ chain in a magnetic field

$$H = \sum_{j=1}^{L-1}(-J_x\sigma_j^x\sigma_{j+1}^x - J_y\sigma_j^y\sigma_{j+1}^y + \underline{J_z\sigma_j^z\sigma_{j+1}^z}) + \frac{1}{2}\sum_{j=1}^{L}\mu_j\sigma_j^z$$

$$J_x = (t+\Delta)/2,\ \ J_y = (t-\Delta)/2,\ \ J_z = U$$

Symmetries: H commutes with fermionic parity $(-1)^F = \prod_{j=1}^{L}\sigma_j^z$

Integrable when μ_j =0 for all j (Baxter, 1971).

Easily solvable for a particular set of μ_j. (Frustration free!!)

A crash course in *inequalities*

- Positive semidefinite operators (H. Tasaki, *PTP* **99**, 489 (1998).)

  $\mathcal{H}$: finite-dim. Hilbert space.

  Def. 1. For a hermitian matrix on $\mathcal{H}$, we write $A \geq 0$ and say *A* is positive semidefinite if $\langle\psi|A|\psi\rangle \geq 0, \ \forall|\psi\rangle \in \mathcal{H}.$

  Def. 2. For two hermitian matrices *A* and *B* on $\mathcal{H}$, we write $A \geq B$ if $A - B \geq 0.$

  Lem. 1. $A \geq 0$ iff all the eigenvalues of *A* are nonnegative.

  Lem. 2. If $A \geq 0$ and $B \geq 0$, then $A + B \geq 0.$

- Min-max theorem (Courant-Fischer-Weyl)

  Let *A* and *B* be two hermitian matrices on $\mathcal{H}$.
  Let a_i and b_i be the *i*-th eigenvalues of *A* and *B*, respectively.
  (Assume the order, $a_1 \leq a_2 \leq \cdots, \ b_1 \leq b_2 \leq \cdots$.)
  If $A \geq B$, then we have $a_i \geq b_i, \ \forall i.$

  For *i*=1, theorem simply implies the variational principle.

Frustration-free Hamiltonian

- Anderson's bound (*Phys. Rev*. **83** , 1260 (1951).

  Suppose Hamiltonian takes the form $H = \sum_j h_j$,
  where each local h_j satisfies $h_j \geq E_j^{(0)} \mathbf{1}$.
  ($E_j^{(0)}$ is the lowest eigenvalue of h_j .)

(The g.s. energy of *H*) =: $E_0 \geq \sum_j E_j^{(0)}$

Gives a lower bound on the g.s. energy of AFM Heisenberg model.

- Frustration-free Hamiltonian

  The case where the *equality* holds.

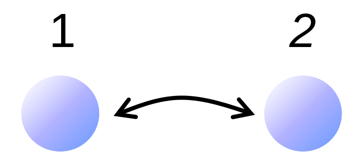
(Pseudo-)Definition. A Hamiltonian $H = \sum_j h_j$ is said to be *frustration-free* when the ground state is obtained by minimizing each term independently.

Examples: AKLT, Kitaev's toric code, RK (quantum dimer) , …
A similar concept, *Bogomolnyi's bound* appears in field theories.

From two to many (1)

Cook up a toy model from 2 sites

$$h_1 = -t(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta(c_1 c_2 + c_2^\dagger c_1^\dagger)$$
$$-\frac{\mu}{2}(n_1 + n_2 - 1) + U(2n_1 - 1)(2n_2 - 1)$$



- Even and odd Hamiltonians

h_1 commutes with fermion parity. → Even and odd sectors.

Even subspace: $|\circ\circ\rangle := |\mathrm{vac}\rangle, \ |\bullet\bullet\rangle := c_1^\dagger c_2^\dagger |\mathrm{vac}\rangle$ $\quad \tan\theta = 2\Delta/\mu$

$$h_{1,\mathrm{even}} = \begin{pmatrix} U + \mu/2 & -\Delta \\ -\Delta & U - \mu/2 \end{pmatrix}$$

g.s.1: $\sin\frac{\theta}{2}|\circ\circ\rangle + \cos\frac{\theta}{2}|\bullet\bullet\rangle$

$$E_{0,\mathrm{even}} = U - \sqrt{\Delta^2 + (\mu/2)^2}$$

Odd subspace: $|\bullet\circ\rangle := c_1^\dagger|\mathrm{vac}\rangle, \ |\circ\bullet\rangle := c_2^\dagger|\mathrm{vac}\rangle$

$$h_{1,\mathrm{odd}} = \begin{pmatrix} -U & -t \\ -t & -U \end{pmatrix}$$

g.s.2: $|\bullet\circ\rangle + |\circ\bullet\rangle$

$$E_{0,\mathrm{odd}} = -U - t$$

g.s.1 and g.s.2 become degenerate if $\mu = \mu^* = 4\sqrt{U^2 + tU + \frac{t^2 - \Delta^2}{4}}$.

From two to many (2)

- Product ground states

When $\mu=\mu^*$, the g.s. of h_1 can be expressed as

$$\left(\alpha^2|\circ\circ\rangle + |\bullet\bullet\rangle\right) \pm \alpha(|\bullet\circ\rangle + |\circ\bullet\rangle) = (\alpha|\circ\rangle_1 \pm |\bullet\rangle_1)(\alpha|\circ\rangle_2 \pm |\bullet\rangle_2)$$

g.s.1: g.s.2:

$$= (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)|\mathrm{vac}\rangle. \quad \alpha^2 = \tan\frac{\theta}{2}$$

Disentangled (product) states are the ground states!

L-site Hamiltonian

$$H = \sum_{j=1}^{L-1} h_j, \quad \begin{aligned} h_j = &-t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \\ &- \frac{\mu}{2}(n_j + n_{j+1} - 1) + U(2n_j - 1)(2n_{j+1} - 1) \end{aligned}$$

is *frustration free* if $\mu=\mu^*$, in which case the *unique* g. s. are

$$|\Psi_\pm\rangle = (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)(\alpha \pm c_2^\dagger)\cdots(\alpha \pm c_L^\dagger)|\mathrm{vac}\rangle.$$

NOTE) The boundary potential is half the bulk one.

A fermionic rephrasing of the known results in XYZ spin chain.

Peschel-Emery (1981), Mueller-Schrock (1985), Giampaolo's works, …

Ground states in individual sectors

Product form

$$|\Psi_\pm\rangle = (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)(\alpha \pm c_2^\dagger)\cdots(\alpha \pm c_L^\dagger)|\text{vac}\rangle.$$

They are NOT eigenstates of fermionic parity $(-1)^F$

$$|\Psi_+\rangle + |\Psi_-\rangle \in \mathcal{H}_{\text{even}}, \qquad |\Psi_+\rangle - |\Psi_-\rangle \in \mathcal{H}_{\text{odd}}$$

Examples:

L=2

$$|\Psi_+\rangle \propto \alpha^2|\circ\circ\rangle + |\bullet\bullet\rangle$$

$$|\Psi_-\rangle \propto \alpha|\circ\bullet\rangle + \alpha|\bullet\circ\rangle$$

The power of *α* counts the number of ○ (empty sites).

L=3

$$|\Psi_+\rangle \propto \alpha^3|\circ\circ\circ\rangle + \alpha|\circ\circ\bullet\rangle + \alpha|\circ\bullet\circ\rangle + \alpha|\bullet\circ\circ\rangle$$

$$|\Psi_-\rangle \propto \alpha^2|\circ\circ\bullet\rangle + \alpha^2|\circ\bullet\circ\rangle + \alpha^2|\bullet\circ\circ\rangle + |\bullet\bullet\bullet\rangle$$

Frustration-free Kitaev chain (t=Δ) often discussed in the literature corresponds to the case with α=1.

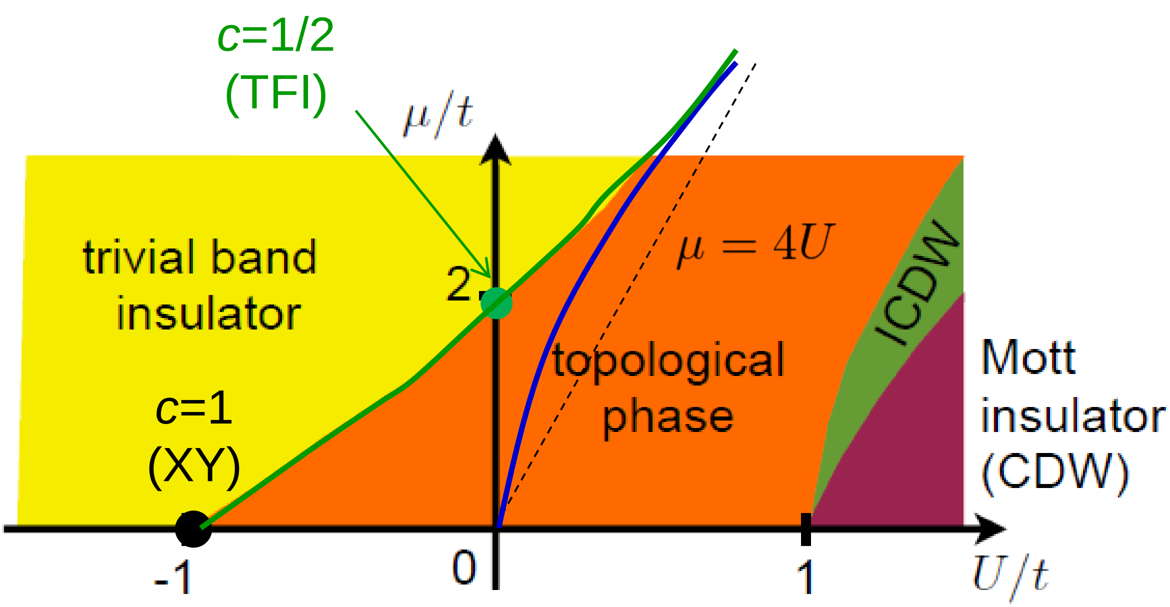
Solvable line in the phase diagram

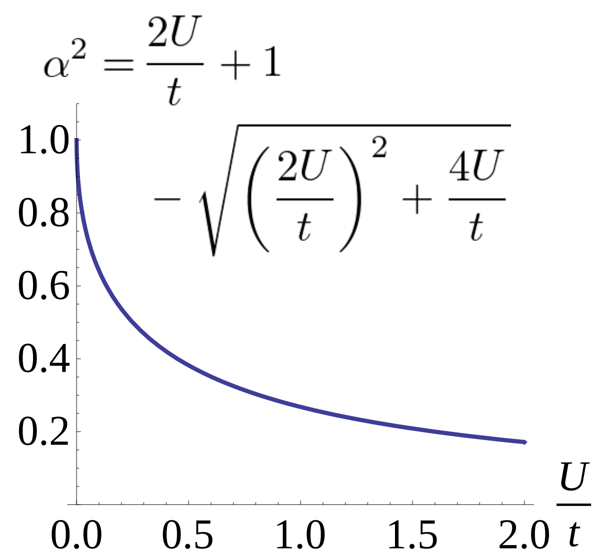
- Phase diagram ($t=\Delta$)

  → Quantum ANNNI model
  Beccaria *et al*., *PRB* (2007); Sela & Pereira, *PRB* (2011); Hassler and Schuricht, *New. J. Phys*. **14** (2012).

Solvable line: $\mu = \mu^* = 4\sqrt{U^2 + tU}$

Exact g.s.: $|\Psi_\pm\rangle = (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)\cdots(\alpha \pm c_L^\dagger)|\mathrm{vac}\rangle.$





Outline

1. Introduction & Motivation
2. Kitaev/Majorana chain
3. Interacting Kitaev/Majorana chain
4. Energy gap, edge zero modes & topo. order

- Poor man's definition of topo. order
- Proof of the gap, exact edge zero modes

5. Generalizations

A poor man's definition of topological order

- Topo. order in 1d interacting Majorana fermions

Fidkowski and Kitaev, *PRB* **81**, 134509 (2010); *PRB* **83**, 075103 (2011).
Z classification reduces to $\boldsymbol{Z}_8$ one.

> Definition (My ver.) The interacting Kitaev chain is said to be in a **topological phase** if it can be adiabatically transformed into a non-interacting Kitaev chain in a topological phase.

We need to check …

Nonvanishing energy gap

Interacting model — Non-interacting model

$$H(s) \to H(0) = H_0$$

⇐ In topo. phase or not?

The energy gap must be nonzero along the entire path.

Proof of the energy gap –outline-

- Hamiltonian (re-parametrization, t =1)

$$H(s,\theta) = \sum_{j=1}^{L-1} h_j(s,\theta) \qquad (0 \le \theta \le \pi,\ -\theta \text{ is achieved by } c_j \to i c_j.)$$

$$h_j(s,\theta) = \left(1+\frac{s}{2}\right) - (c_j^\dagger c_{j+1} + \mathrm{h.c.}) + (1+s)\sin\theta(c_j c_{j+1} + \mathrm{h.c.})$$
$$- (1+s)\cos\theta(n_j + n_{j+1} - 1) + \frac{s}{2}(2n_j - 1)(2n_{j+1} - 1)$$

- Ground states

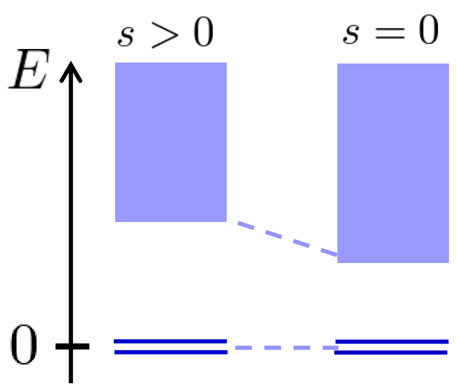
$$|\Psi_\pm\rangle = (\alpha \pm c_1^\dagger)(\alpha \pm c_2^\dagger)\cdots(\alpha \pm c_L^\dagger)|\mathrm{vac}\rangle. \qquad \alpha^2 = \tan\frac{\theta}{2}$$

H(2*U*, *θ*) (interacting) and *H* (0, *θ*) (non-interacting) share the same ground states!

- Unique g.s. and energy gap

For *s*>0, we have *H*(s, *θ*) ≧ *H* (0, *θ*).

From the min-max theorem,
the uniqueness of the g.s. and the energy gap of *H*(2*U*, *θ*) follow from those of *H* (0, *θ*).



Spectrum of $H\,(0, \theta)$

- Single-particle spectrum

$$H(0,\theta)+\text{const.} = \frac{i}{2}\sum_{j,k=1}^{L} B_{j,k}\, a_j b_k = \sum_{k=1}^{L}\epsilon_k\left(f_k^\dagger f_k - \frac{1}{2}\right)$$

Matrix B, may not be diagonalizable, but can be written in SVD form: $B = U\Lambda V^{\mathrm{T}},\ \Lambda = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_L)$, where $U,\ V \in \mathrm{O}(L)$.

- Miraculous properties of $H(0, \theta)$

Special factorization

$$BB^{\mathrm{T}} = C^2$$

BB^T *is pentadiagonal, but*
C is tridiagonal & symmetric!

$$C = \begin{pmatrix} 1-\mathfrak{s} & \mathfrak{c} & & & \\ \mathfrak{c} & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & \mathfrak{c} \\ & & & \mathfrak{c} & 1+\mathfrak{s} \end{pmatrix}$$

$\mathfrak{s} = \sin\theta$

$\mathfrak{c} = \cos\theta$

Exact eigenvalues of C

$$0 \quad \text{and} \quad 2 + 2\mathfrak{c}\cos\left(\frac{n\pi}{L}\right), \qquad n = 1, 2, \ldots, L-1$$

Existence of many-body gap

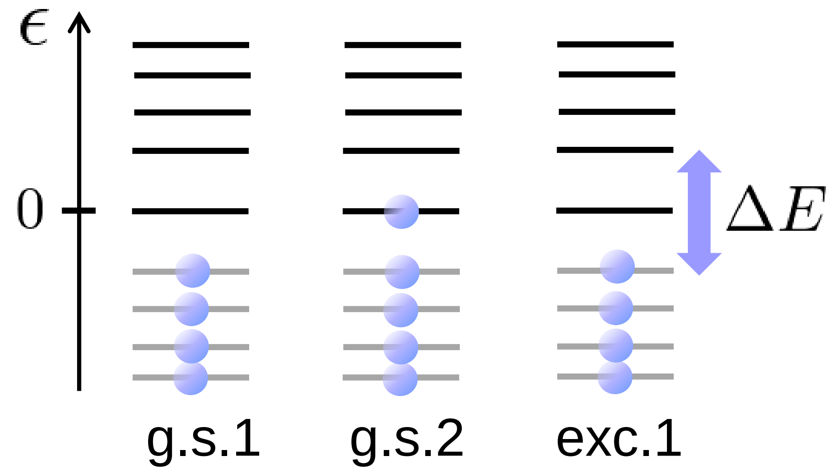
- Many-body eigenstates of $H(0, \theta)$

  I. two g.s. with opposite parities.
  (Unique in each parity sector.)

  II. Many-body gap ($\theta \neq 0, \pi$)

  $$\Delta E \geq 2(1 - |\cos\theta|)$$

  is nonzero in the infinite-L limit.

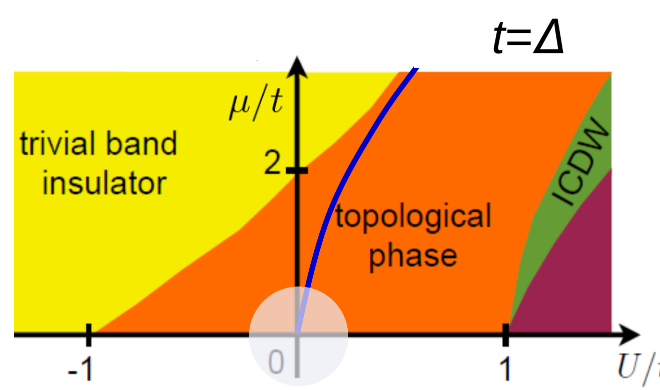


Properties I & II also hold for $H(2U, \theta)$. (min-max thm.)

$H(2U, \theta)$ is adiabatically connected to $H(0, \theta)$ which is in a topological phase! The gap closing does not happen.

- Stability away from frustration-free line?
  1. Kato's theorem ($||V|| < \Delta E/2$)
  2. Cluster expansion (Kennedy-Tasaki)
  3. Lieb-Robinson bound?

  Main difficulty: boundary conditions



Majorana edge zero modes

- Edge zero mode of $H(0, \theta)$ ⇔ *Left* or *Right* null vector of B

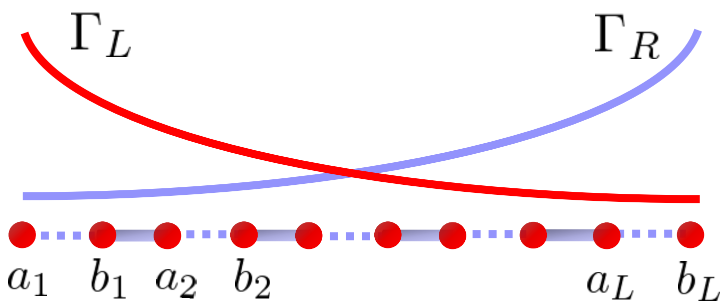
$$B = -\begin{pmatrix} \mathfrak{c} & 1-\mathfrak{s} & & & \\ 1+\mathfrak{s} & 2\mathfrak{c} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2\mathfrak{c} & 1-\mathfrak{s} \\ & & & 1+\mathfrak{s} & \mathfrak{c} \end{pmatrix}$$

$$(1,\, r,\, r^2,\, \cdots,\, r^{L-1})B = 0$$

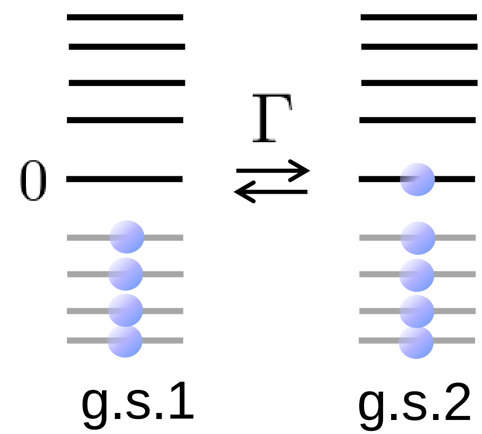
$$r = -\frac{\mathfrak{c}}{1+\mathfrak{s}}, \quad \mathfrak{s}^2 + \mathfrak{c}^2 = 1$$

$$\Gamma_L = \mathcal{N}(\, a_1 + r a_2 + \cdots r^{L-1} a_L \,)$$
$$\Gamma_R = \mathcal{N}(\, b_L + r b_{L-1} + \cdots r^{L-1} b_1 \,)$$

Exact zero modes even for finite L.



They do not exactly commute with $H(2U, \theta)$.
Nevertheless, they map one g.s. to the other.



A significant overlap with many-body Majorana?

M.B. Hastings & X-G. Wen, *PRB* **72**, 045141 (‘05).

Outline

1. Introduction & Motivation

2. Kitaev/Majorana chain

3. Interacting Kitaev/Majorana chain

4. Energy gap, edge zero modes & topo. order

5. Generalizations

- Twisted Kitaev chain
- Staggered case

Twisted Kitaev chain

- Twisted boundary conditions

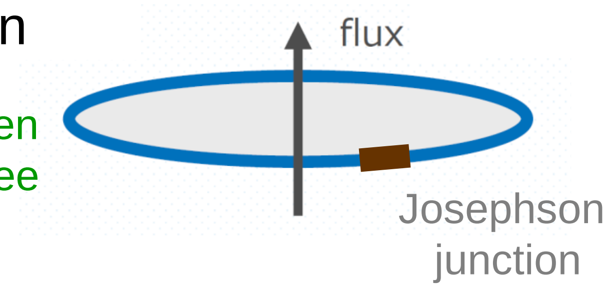
$$H = H_{\rm bulk} + H_{\rm booundary} + H_{\rm int} \qquad H_{\rm bulk} = H_0 \text{ (non-int. Kitaev)}$$

$$H_{\rm boundary} = \mathsf{b}[-t(e^{i\phi_1} c_L^\dagger c_1 + \text{h.c.}) + \Delta(e^{i\phi_2} c_L c_1 + \text{h.c.})]$$

ϕ_1 : magnetic flux, ϕ_2 : Josephson junction

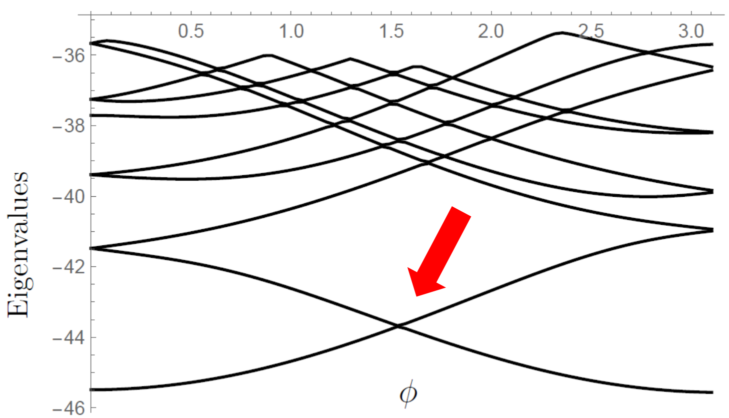
PBC: b=1, $(\phi_1, \phi_2) = (0, 0)$
APBC: b=1, $(\phi_1, \phi_2) = (\pi, \pi)$

Solvable when frustration-free

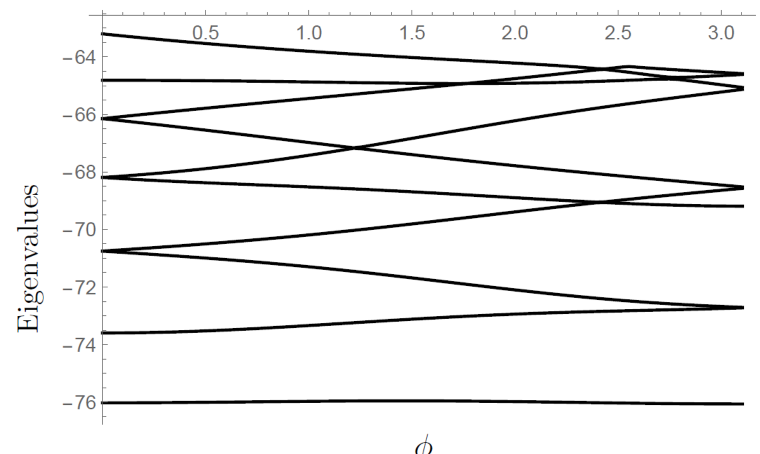


- (Many-body) level crossing

$t = 4,\ \Delta = 4,\ U = 2,\ \mu = 6$



$t = 4,\ \Delta = 4,\ U = 2,\ \mu = 18$



- Level crossing occurs in topo. phase

Staggered case

- Hamiltonian (complex fermions, with OBC)

$$H = -t\sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta\sum_{j=1}^{L-1}(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$
$$-\sum_{j=1}^{L}\mu_j(c_j^\dagger c_j - 1/2) + U\sum_{j=1}^{L-1}(2c_j^\dagger c_j - 1)(2c_{j+1}^\dagger c_{j+1} - 1)$$

  - Staggered potential

$$\mu_j = \begin{cases} q, & \text{if } j \text{ odd} \\ 1/q, & \text{if } j \text{ even} \end{cases}$$

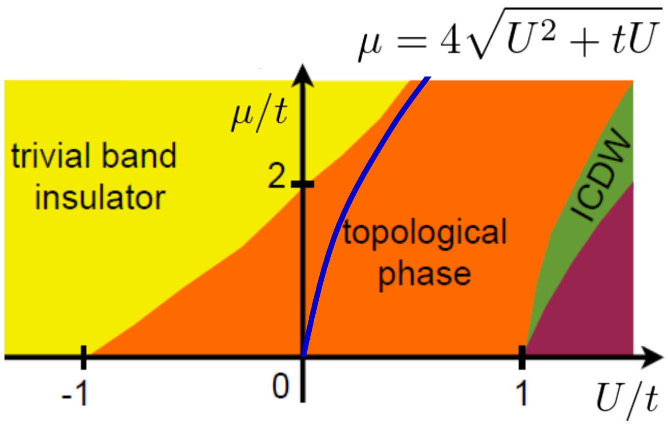
- Frustration-free case
  - Non-interacting (*U*=0) model is so with

$$t = \frac{\eta + \eta^{-1}}{2}, \quad \Delta = \frac{\eta - \eta^{-1}}{2}.$$ (parametrization)

  - Interacting model was worked out by Jurriaan.
    (XYZ spin chain in staggered magnetic field)
  - Upper and lower bound on gap, topo. order …

Conclusions

- Studied effect of interactions on Kitaev/Majorana chains
- Solvable (frustration-free) line
- Exact ground states, proof of the gap
- Exact solution of the BdG equation
- Topological order and edge zero modes
- Some generalizations



What I did not touch on

- Parafermionic generalization $\psi^m = 1 \qquad \psi_i \psi_j = \omega \psi_j \psi_i \quad (i < j)$
  Frustraion-free cases exist
- Interacting Majorana models with *N*=1 SUSY $H = Q^2, \quad Q^\dagger = Q$
  Sannomiya-Katsura [1712.01148], solvable even in higher dim.
- Kitaev-type models with dissipation
  Solvable/Integrable Lindblad (Master) eq., …