OPE for Conformal Defects and Holography

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Outline

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Introduction

Defects = Non-local objects in QFTs

- Defined by boundary conditions around them

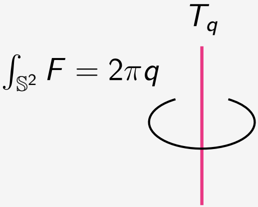
- Many examples:

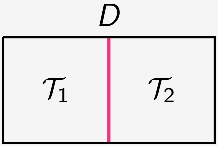
  1-dim : Line operators (Wilson-'t Hooft loops)

  2-dim : Surface operators

  Codim-1 : Domain walls and boundaries

  Codim-2 : Entangling surface for entanglement entropy





Why defects?: classification of QFT phases

- Characterize phases of QFTs: ['t Hooft 78]

$$W: \text{ Wilson loop}, \quad T: \text{ 't Hooft loop}$$

$$\text{Confinement}: \quad \langle W \rangle \sim e^{-\text{Area}}, \qquad \langle T \rangle \sim e^{-\text{Length}}$$

$$\text{Higgs}: \quad \langle W \rangle \sim e^{-\text{Length}}, \quad \langle T \rangle \sim e^{-\text{Area}}$$

- Higher-dimensional generalization:
  - Wilson surface operators,

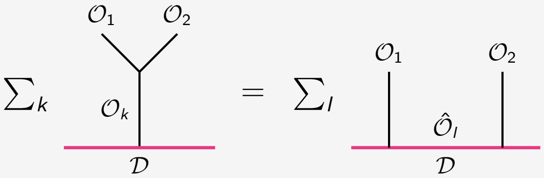
$$W_\Sigma = \exp\left(i \int_\Sigma A \right)$$

for a p-form gauge field A

$\longrightarrow$ higher-form symmetry [Gaiotto-Kapustin-Seiberg-Willett 14]

Possible applications of defects

- Constrain bulk CFT data in defect CFT by conformal bootstrap [Liendo-Rastelli-van Rees 12]



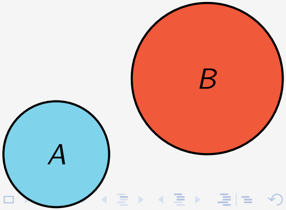
- Understand quantum entanglement in QFT:

e.g. Mutual information

$$I(A,B) \equiv S_A + S_B - S_{A\cup B}$$

as a correlator of two defects,

$$I(A,B) = \log \frac{\langle \mathcal{D}(\partial A)\, \mathcal{D}(\partial B)\rangle}{\langle \mathcal{D}(\partial A)\rangle\, \langle \mathcal{D}(\partial B)\rangle}$$

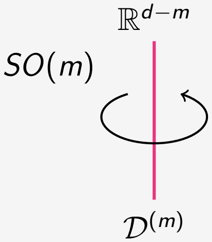


Conformal defects

- In Euclidean CFT$_d$, the conformal group is $SO(d+1,1)$
- Codimension-m conformal defects $\mathcal{D}^{(m)}$ are either flat or spherical, preserving

$SO(d+1-m,1)$: conformal symmetry on defects

$SO(m)$: rotation in the transverse direction

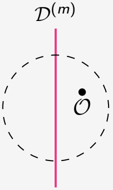


- Defects allow defect local operators $\hat{\mathcal{O}}_n$

OPE for conformal defects

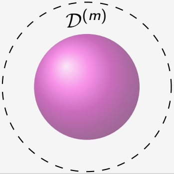
There are two types of OPE in DCFT

- Bulk-to-defect OPE : [Cardy 84, McAvity-Osborn 95]



$$= \sum_n b^{(m)}_{\mathcal{O}\,\hat{\mathcal{O}}_n} \; \mathcal{D}^{(m)} \bullet \hat{\mathcal{O}}_n \;+\; \text{(descendants)}$$

- Defect OPE : [Berenstein et al. 98, Gomis-Okuda 09, Gadde 16]



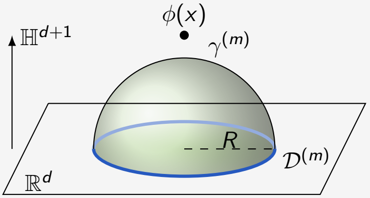
$$= \sum_n c^{(m)}_{\mathcal{O}_n} \; \bullet\, \mathcal{O}_n \;+\; \text{(descendants)}$$

Goal of the talk

1. Understand the structure of defect OPE
   - Decomposition by the irreducible representations

$$\mathcal{D}^{(m)} = \sum_{n \in \text{primaries}} \mathcal{B}^{(m)}[\mathcal{O}_n]$$

2. Probe the bulk AdS information by conformal defects



Overview of our results

1. The integral representation of the defect OPE blocks

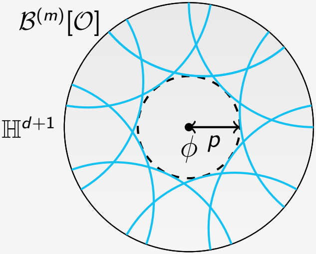
$$\mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \, \langle \mathcal{O}_n(x) \rangle_{\mathcal{D}^{(m)}} \tilde{\mathcal{O}}_n(x)$$

$\tilde{\mathcal{O}}$: shadow operator with $\tilde{\Delta} = d - \Delta$ for $\mathcal{O}$ with Δ

2. The scalar block is the Radon transform of the AdS scalar field

$$\hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}]$$

in parallel with the OPE block
[Czech et al. 16, de Boer et al. 16]



Outline

Embedding space formalism

- Uplift $\mathbb{R}^d$ into the embedding space $\mathbb{R}^{d+1,1}$

$$x^\mu \;\to\; X^A \quad (A = 1, \cdots d+2)$$

- Choose the d-dim hypersurface (projective null cone)

$$X \cdot X \equiv X^A X_A = 0 \;, \qquad X^A \sim \lambda\, X^A \quad (\lambda \in \mathbb{R})$$

- Realize the conformal symmetry $SO(d+1,1)$ linearly

$$X'^A = J^A_{\ B}\, X^B \;, \quad J^A_{\ B} \in so(d+1,1)$$

Correlation functions in embedding space

- Construct scalar invariants under $SO(d+1,1)$ in the embedding space
- Two-point function of scalar fields Φ of dimension Δ:

$$\langle \Phi(X_1)\,\Phi(X_2) \rangle$$

  - Only one scalar invariant: $X_1 \cdot X_2$
  - Scaling dimension fixes the correlator uniquely:

$$\langle \Phi(X_1)\,\Phi(X_2) \rangle = \frac{1}{(X_1 \cdot X_2)^{\Delta}}$$

Spinning correlators

- For a spin l operator $\mathcal{O}_{\Delta,l}(X)$, define the encoding polynomial,

$$\mathcal{O}_{\Delta}(X,Z) \equiv \mathcal{O}_{\Delta,A_1\cdots A_l}(X)\, Z^{A_1}\cdots Z^{A_l}\ , \qquad X\cdot Z = Z^2 = 0$$

Z^A: an auxiliary vector [Costa-Penedones-Poland-Rychkov 11]

- Two-point functions: $\langle \mathcal{O}_{\Delta,l}(X_1,Z_1)\,\mathcal{O}_{\Delta,l}(X_2,Z_2)\rangle$
  - Invariants:

$$X_1\cdot X_2\ , \qquad X_1\cdot Z_2\ , \qquad X_2\cdot Z_1\ , \qquad Z_1\cdot Z_2$$

  - "Gauge" redundancy: $Z\sim Z+\lambda X$
  - Scaling dimensions:

$$\langle \mathcal{O}_{\Delta,l}(X_1,Z_1)\,\mathcal{O}_{\Delta,l}(X_2,Z_2)\rangle = \frac{1}{(X_{12})^{\Delta}}\left[\frac{(Z_1\cdot Z_2)(X_1\cdot X_2)-(Z_1\cdot X_2)(Z_2\cdot X_1)}{X_{12}}\right]^l$$

Conformal defects in embedding space

- Specify a codimension-m defect by the frame vectors P_α^A: [Gadde 16]

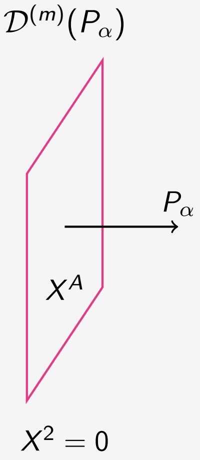
$$X \cdot P_\alpha = 0 \ , \quad (\alpha = 1, \cdots m)$$

- Preserves the symmetry group:

$$SO(d+1-m, 1) \times SO(m)$$

- Correlation function invariant under the frame choice ($P'_\alpha = g_\alpha{}^\beta P_\beta$, $g \in SO(m)$):

$$\langle \mathcal{D}^{(m)}(P_\alpha)\, \mathcal{O}_\Delta(X) \rangle \propto \frac{1}{[(P^\alpha \cdot X)(P_\alpha \cdot X)]^{\Delta/2}}$$



Defect OPE blocks

We expect the defect OPE of the form:

$$\mathcal{D}^{(m)}(P_\alpha) = \langle \mathcal{D}^{(m)}(P_\alpha) \rangle \left[\sum_n c_{\mathcal{O}_n}^{(m)} R^{\Delta_n} \mathcal{O}_n(C) + (\text{descendants}) \right]$$

$$(R : \text{radius}, C : \text{center vector})$$

- The descendant terms are fixed by primaries $\mathcal{O}_n$ and the conformal symmetry,

$$\mathcal{D}^{(m)}(P_\alpha) = \sum_n \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n]$$

Projectors and shadows

- Want to characterize the defect OPE blocks by their irreps

- Spectral decomposition by the irreps of the conformal group:

$$\mathbf{1} = \sum_n |\mathcal{O}_n|$$

- $|\mathcal{O}_n|$: Projector onto the conformal multiplet of the primary $\mathcal{O}_n$
  [Ferrara-Grillo-Parisi-Gatto 72, $\cdots$, Simmons-Duffin 12]

- For a spin-l operator,

$$|\mathcal{O}_{\Delta,l}| = \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, |\mathcal{O}_{\Delta,l}(X, D_Z)\rangle \langle \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z)|$$

$\tilde{\mathcal{O}}_{d-\Delta,l}$: the shadow operator of $\mathcal{O}_{\Delta,l}$

D_Z: Todorov's differential operator ($\sim \partial_Z$)

Integral representation of defect OPE blocks

- Expand the defect by the projectors:

$$
\begin{aligned}
\langle \mathcal{D}^{(m)}(P_\alpha) \cdots \rangle &= \sum_{\Delta,l} \langle \mathcal{D}^{(m)}(P_\alpha) | \mathcal{O}_{\Delta,l} | \cdots \rangle + (\text{other irrep.}) \\
&= \sum_{\Delta,l} \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, \langle \mathcal{D}^{(m)}(P_\alpha) \, \mathcal{O}_{\Delta,l}(X, D_Z) \rangle \, \langle \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z) \cdots \rangle \\
&\quad + (\text{other irrep.})
\end{aligned}
$$

- Can read off the block contribution:

The integral rep of the defect OPE block

$$
\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, \tilde{\mathcal{O}}_{d-\Delta,l}(X, D_Z) \, \langle \mathcal{D}^{(m)}(P_\alpha) \, \mathcal{O}_{\Delta,l}(X, Z) \rangle
$$

Constraint equations

There are two types of equations the defect OPE block satisfies

The conformal Casimir equation

$$\left(L^2(P_\alpha) + \mathcal{C}_{\Delta,l}\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0$$

- $L^2(P_\alpha) \equiv \frac{1}{2} L_{AB}(P_\alpha)\, L^{AB}(P_\alpha)$: quadratic Casimir operator
- $\mathcal{C}_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$: the eigenvalue

"Trivial" equations for scalar primaries $\mathcal{O}_\Delta$

$$C_{ABCD}(P_\alpha)\, \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_\Delta] = 0$$

- $C_{ABCD}(P_\alpha) \equiv \frac{1}{2} L_{A[B}(P_\alpha) L_{CD]}(P_\alpha)$: $\frac{d(d^2-1)(d+2)}{24}$ quadratic operators

Moduli space of conformal defects

- The moduli space has a coset structure:

$$\mathcal{M}^{(d,m)} = \frac{SO(d+1,1)}{SO(m) \times SO(d+1-m,1)}$$

- The quadratic Casimir operator is the Laplacian on $\mathcal{M}^{(d,m)}$:

$$-L^2(P_\alpha) = \Box_{\mathcal{M}^{(d,m)}}$$

- The defect OPE block is a scalar field propagating on $\mathcal{M}^{(d,m)}$

Klein-Gordon equation on $\mathcal{M}^{(d,m)}$

$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0 \ , \quad M^2 = \mathcal{C}_{\Delta,l}$$

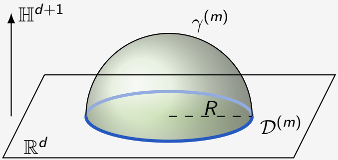
Outline

Conformal defects and submanifolds in AdS

- Associated to a given defect $\mathcal{D}^{(m)}$ is a unique submanifold $\gamma^{(m)}$ in AdS s.t. $\partial\gamma^{(m)} = \mathcal{D}^{(m)}$

- Their moduli spaces are equivalent:

$$\mathcal{M}^{(d,m)} = \frac{\text{Isom}(\text{AdS}_{d+1})}{\text{Stab}(\gamma^{(m)} \in \text{AdS}_{d+1})}$$



- What is the dual description of the defect OPE block in AdS?

Moduli space and AdS

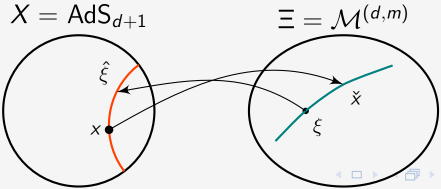
- The defect OPE block $\mathcal{B}^{(m)}$ is a scalar field on $\mathcal{M}^{(d,m)}$

$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0$$

- Coset structures of $\mathcal{M}^{(d,m)}$ and AdS:

$$\mathcal{M}^{(d,m)} = \frac{SO(d+1,1)}{SO(m) \times SO(d+1-m,1)} , \qquad \mathbb{H}^{d+1} = \frac{SO(d+1,1)}{SO(d)}$$

- The Radon transform maps a function from one to the other [e.g. Helgason 10]



Radon transform

From $\mathbb{H}^{d+1}$ to $\mathcal{M}^{(d,m)}$:

$$\hat{\phi}(\xi) = \int_{x\in\xi} d\nu(x)\,\phi(x)$$

- ξ : a codim-m submanifold in $\mathbb{H}^{d+1}$
- $\phi(x)$: a function on $\mathbb{H}^{d+1}$

From $\mathcal{M}^{(d,m)}$ to $\mathbb{H}^{d+1}$:

$$\check{f}(x) = \int_{x\in\xi} d\mu(\xi)\, f(\xi)$$

- ξ : a codim-m submanifold through x
- $f(\xi)$: a function on $\mathcal{M}^{(d,m)}$

Intertwining property

$$\left(\Box_{\mathbb{H}^{d+1}} - M^2\right)\,\phi = 0 \quad \Leftrightarrow \quad \left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right)\,\hat{\phi} = 0$$

Defect OPE blocks as Radon transformed fields

- We may regard the defect OPE blockas the Radon transform of an AdS scalar field ϕ

$$\hat{\phi} = \mathcal{B}^{(m)}$$

- The Radon transform maps ϕ into the manifold with larger dimensions

$$\dim \mathcal{M}^{(d,m)} - \dim \mathbb{H}^{d+1} = (m-1)(d+1-m) \geq 0$$

Hence there should be constraints for $\hat{\phi}$, which are known as the range characterization [Ishikawa 97]

$$C_{ABCD}\, \hat{\phi} = 0$$

- The block $\mathcal{B}^{(m)}[\mathcal{O}_\Delta]$ satisfies them for a scalar primary $\mathcal{O}_\Delta$

Dictionary for the Radon transform

Conformal defect

- Moduli space: $\mathcal{M}^{(d,m)}$
- Scalar block: $\mathcal{B}^{(m)}[\mathcal{O}_\Delta](=\hat{\phi})$

quadratic Casimir equation

$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right) \mathcal{B}^{(m)}[\mathcal{O}_\Delta] = 0$$

constraint equations

$$C_{ABCD}\, \mathcal{B}^{(m)}[\mathcal{O}_\Delta] = 0$$

AdS scalar field

- Spacetime: $\mathbb{H}^{d+1}$
- AdS scalar field: ϕ

Klein-Gordon equation

$$\left(\Box_{\mathbb{H}^{d+1}} - M^2\right) \phi = 0$$

range characterization

$$C_{ABCD}\, \hat{\phi} = 0$$

Construction of AdS scalar fields from defects

- Want to reconstruct an AdS scalar ϕ from the block $\hat{\phi} = \mathcal{B}^{(m)}$
- The inversion formula allows such a construction (note $(\hat{\phi})^{\vee} \neq \phi$):

$$\phi = \mathcal{I} \circ \hat{\phi} = \mathcal{I} \circ \mathcal{B}^{(m)}[\mathcal{O}_\Delta]$$

  $\mathcal{I}$: an integral transform [Helgason 10]

- Reproduce the known bulk reconstruction formula:

$$\phi(Y) = \int D^d X \, K_\Delta(Y|X) \, \mathcal{O}_\Delta(X)$$

  with the Euclidean version of the HKLL kernel [Hamilton-Kabat-Lifschytz-Lowe 06]

$$(\Box_{\mathbb{H}^{d+1}} - M^2) \, K_\Delta(Y|X) = 0 \, , \quad M^2 = \Delta(d - \Delta)$$

Summary

1. Give the integral representation of the defect OPE blocks

$$\mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \, \langle \mathcal{O}_n(x) \rangle_{\mathcal{D}^{(m)}} \tilde{\mathcal{O}}_n(x)$$

$\tilde{\mathcal{O}}$: shadow operator with $\tilde{\Delta} = d - \Delta$ for $\mathcal{O}$ with Δ

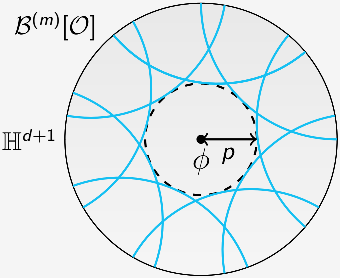
2. Reconstruct the AdS scalar field from the blocks

- $\hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}]$:

  The Radon transform of the AdS scalar field ϕ when $\mathcal{O}$ scalar

- Reproduce the (Euclidean) HKLL formula :

$$\phi(Y) = \phi(\hat{\phi}) = \int d^d x \, K(Y|x) \, \mathcal{O}(x)$$



Future direction

- Constraints on the bulk OPE data from the block?
  - Need the relation to the bulk-to-defect OPE
- Can extend the construction to higher spin fields in AdS?
  - No known Radon transform beyond a scalar field in AdS
  - Spinning defects to incorporate spins [Kobayashi-TN 18]
- Generalized notion of entanglement entropy?
  - Codim-2 deficit angle $\Rightarrow$ codim-m deficit ??
  - Analytic continuation to Lorentzian signature?

Alternative representation and monodromy condition

- The integral rep is invariant under the exchange $\mathcal{O} \leftrightarrow \tilde{\mathcal{O}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = \frac{1}{\mathcal{N}_{d-\Delta,l}} \int D^d X \, \mathcal{O}_{\Delta,l}(X, D_Z) \, \langle \mathcal{D}^{(m)}(P_\alpha) \, \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z) \rangle$$

- The block includes the shadow contribution $g_{\tilde{\mathcal{O}}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = g_{\mathcal{O}} + K \, g_{\tilde{\mathcal{O}}}$$

- To extract the shadow contribution, impose the monodromy condition

$$\begin{aligned} g_{\mathcal{O}} &\rightarrow e^{2\pi i \Delta} \, g_{\mathcal{O}} \\ g_{\tilde{\mathcal{O}}} &\rightarrow e^{2\pi i (d-\Delta)} \, g_{\tilde{\mathcal{O}}} \end{aligned}$$

under the phase rotation $P_\alpha \rightarrow e^{-2\pi i} P_\alpha$