Towards Entanglement of Purification for Conformal Field Theories

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What objects in QFT can capture quantum entanglement finer than the entanglement entropy?

Today, I will show an explicit (calculable) example!
What was entanglement entropy (EE)?

EE = # of EPR pair-like correlation

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_{in} |\uparrow\rangle_{out} + |\downarrow\rangle_{in} |\downarrow\rangle_{out}) \]

\[ S_{EE} = -\text{Tr}_{H_{in}} \left( \hat{\rho}_{in} \log \hat{\rho}_{in} \right) = \log 2 \]

where \( \hat{\rho}_{in} = \text{Tr}_{H_{out}} |\Psi\rangle \langle \Psi| \)
How can we learn about gravity from QFT?

The AdS/CFT correspondence

[AdS\textsubscript{d+1} \equiv CFT\textsubscript{d}]

[Heemskerk-Penedones-Polchinski-Sully '09]

How about \textit{metric} in gravity side?

Quantum \textit{entanglement} plays a key role!
How can we learn about **metric** from QFT?

**Entanglement entropy (EE) ↔ Area of minimal surface**

\[ S_{\text{in}} = \frac{\text{Area}}{4G_N} \]

[Ryu-Takayanagi ‘06]

- Evaluate EE in QFT → Explore metric
- To what extent? (Is EE enough?)
Q. Is EE enough?

A. For time slice of AdS3, Yes!

minimal geodesics can pass through the center
Q. Is EE enough?

A. In general, No…

a example: cut & glue the slice (orbifold)
minimal geodesics can NOT pass through the center
How to probe deep inside the bulk?

an example:

**Conjecture**

Bulk: Entanglement Wedge Cross Section (EWCS)

[Umemoto-Takayanagi ’17],
[Nguyen-Devakul-Halbasch-Zaletel-Swingle ’17]
c.f. intensive lecture by Takayanagi-san last year

Boundary: Entanglement of Purification (EoP)

I will explain these definitions later
Our result

CFT quantities which describe EWCS/EoP

(1) EoP in holographic code model $\rightarrow$ “bulk” twist op.

(2) CFT calculation

(@ large-c, reduce to CB with internal twist op.s)

[Hirai, KT, Yokoya ‘18]
Plan

(1) EWCS & EoP (Review)

(2) EoP in holographic code model $\rightarrow$ “bulk” twist op.

(3) CFT calculation

(@ large-c, reduce to CB with internal twist op.s)

(4) Discussion

[Hirai, KT, Yokoya ‘18]
Entanglement Wedge Cross Section

[Umemoto-Takayanagi ’17], [Nguyen-Devakul-Halbasch-Zaletel-Swingle ‘17]

1. Divide $\overline{AB}$ and $\overline{\overline{AB}}$
2. Write minimal surfaces

$E_W(A : B) = \frac{1}{4G_N} \sigma_{min.}$

Region surrounded by $A \cup B \cup (\text{min. surfaces})$
EWCS : a generalization of min. surface

Take the size of $\overline{AB} \rightarrow 0$: original RT formula
Entanglement of Purification (EoP)

a correlation measure for two subregions

\[ \hat{\rho}_{AB} \]

\[ |\psi\rangle_{ABC} \]

\[ \text{s.t. } \rho_{AB} = \text{Tr}_C |\psi\rangle\langle\psi| \]

\[ E_P(A : B) = \min_{|\psi\rangle_{ABA'B'}} S(\rho_{AA'}) \]
Entanglement of Purification (EoP)

Minimal EE for all possible purification

[Terhal-Horodecki-Leung-DiVincenzo '02]

For a pure state: \(|\psi\rangle_{AB}\)

\[ E_P(A : B) = S_A(= S_B) \]

A generalization of EE in this sense
Summary so far

EWCS:

EoP:

- a correlation measure for A vs B
- pure state $\rightarrow$ reduce to EE
(2) EoP in holographic code model
Replica method $\rightarrow$ “bulk” twist op.s

[Hirai, KT, Yokoya ‘18]
EE from twist op.s correlation function (Replica method)

\[ S_{\text{in}} = - \frac{\partial}{\partial n} \left\langle \sigma_n \bar{\sigma}_n \right\rangle \bigg|_{n \to 1} \]

[Cabrease, Cardy '04]

Scaling dim. of twist op.
- from Conformal WT id.

\[ \Delta_n = \frac{c}{12} \left( n - \frac{1}{n} \right) \]
Let us consider EoP for mixed state

\[ \rho_{AB} = \text{Tr}_{\overline{AB}} |0\rangle\langle 0| \]
What is a optimized purification?

Can ask for help to the Tensor Network Model

\[ \text{We used Holographic code model [Pastawski, Yoshida, Harlow, Preskill '15]} \]
Punch line of the following argument

Within the holographic code model,

Twist op. for optimized solution

= “bulk dual of the CFT twist op.”

$$\phi_n |\Psi_{opt.}\rangle \simeq (K\sigma_n) |0\rangle$$

(i.e. CFT twist op. smeared by HKLL kernel $K$)

[Hamilton, Kabat, Lifschytz, Lowe '06]
Holographic code model: duality as isometry

\[ \prod v : |0\rangle_{\text{bulk}} \rightarrow |0\rangle_{\text{bdry}}. \equiv |0\rangle \]

[Pastawski, Yoshida, Harlow, Preskill '15]
Holographic code model: duality as isometry

[Ryu-Takayanagi formula]

\[ S_A = \text{const.} \times \text{Area}(\gamma_A) + S_{\text{bulk}} \]

[HKLL bulk reconstruction]

[Hamilton, Kabat, Lifschytz, Lowe '06]

\[ \phi_{\text{bulk.}} |0\rangle_{\text{bulk.}} \rightarrow \mathcal{O}_{\text{bdry.}} |0\rangle_{\text{bdry.}} \]
Optimized purification within the holographic code model

\[ \rho_{AB} = \text{Tr}_{AB}(|0\rangle\langle 0|) \overset{\text{purify}}{\longrightarrow} |\Psi_{opt.}\rangle \sim V^\dagger |0\rangle \]

Let us assume

“Duality between new boundary and bulk”

c.f. Miyaji-Takayanagi ‘15

Then, new boundary

\[ \rightarrow C=A'B' \]
Replica method can apply to EoP

\[ S_{AA'} |_{min.} \]
\[ = - \frac{\partial}{\partial n} \langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle \bigg|_{n \to 1} \]

\( \phi_n \) : (so far) unknown twist op.

\[ \rho_{AB} = \text{Tr}_C (|0\rangle\langle 0|) \xrightarrow{\text{purify}} |\Psi_{opt.}\rangle \simeq V^\dagger |0\rangle \]
Recap: for pure states, EoP $\rightarrow$ EE

Take the size of $\overline{AB}$ $\rightarrow$ 0: original RT formula
“Bulk twist op.” $\rightarrow$ twist op. in CFT

**Boundary condition**

$\phi_n \xrightarrow{r \to \infty} \sigma_n$

$m_{\phi_n}^2 R_{AdS}^2 = \Delta_n(\Delta_n - 2)$

! Within framework of the Holographic code model, $\phi_n$ is just the HKLL map of $\sigma_n$

$$\Delta_n = \frac{c}{12} \left( n - \frac{1}{n} \right)$$
(3) From CFT correlation function: the channel including internal twist op.

[Hirai, KT, Yokoya ‘18]
Set Up: \( \text{CFT}^n / \mathbb{Z}_n \)

\[
\langle O_1 O_2 O_3 O_4 \rangle = \sum_{O} C_{12} O C_{34} O W_{O}^{12,34}(x_i)
\]

Consider a channel including twist op. \( \sigma_n \in O \)

(at large-c limit, the twist op. will dominate)

\[
W_{O}^{12,34}(x_i) = \langle O_1 O_2 \rangle \langle O_3 O_4 \rangle \times G_{O}\left(u, v\right)
\]

\( u, v: \text{cross ratio} \)

We will consider **sum of Go** rather than \( \langle O_1 O_2 O_3 O_4 \rangle \)
Conformal Block from AdS geodesics

[Hi-Jano–Kraus–Perlmutter–Snively ’15]

\[
\text{Conformal Block} = \int d\lambda \int d\lambda' G_{bb}^{\Delta,0}(X(\lambda), Y(\lambda')) \\
\sim e^{-\Delta \sigma_{\text{min}}}
\]

\[1 \ll \Delta \simeq mR_{\text{AdS}}\]

At the semiclassical limit, only \(\sigma_{\text{min}}\) contribute!

(Now we are treating so-called “light operators” \(1 \ll \Delta i, \Delta \ll c\))
(a 4pt function)/(2pt functions) captures the EWCS @ large-c

\[- \frac{\partial}{\partial n} G_{\sigma_n} (u, v) \bigg|_{n \to 1} \]

\[ G_{\sigma_n} \sim e^{-\frac{c}{12} (n - \frac{1}{n}) \sigma_{min}} \]

\[ = \frac{c}{6} \sigma_{min} = \frac{1}{4G_N} \sigma_{min} = E_W \]

! For the sector with (twist#)=1, the above \( \sigma_n \) will be dominant @ large-c
Extension to the BTZ blackholes

BTZ $t=0$

- Assume the heavy state as an optimized purification
- Reduce to heavy-light Virasoro conformal blocks
Our result

CFT quantities which describe EWCS/EoP

(1) EoP in holographic code model $\rightarrow$ “bulk” twist op.

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[Hirai, KT, Yokoya ‘18]
Future directions

Optimization within the framework of QFT
What is V in QFT? TTbar?

Extension to the higher dimension
HKLL map for defects in the bulk?

Relation to the Kinematic space
e.g. Czech–Lamprou–McCandlish–Mosk–Sully ’16

Thank you!
backup
\[ \phi_n \text{ should be reconstructed from } \sigma_n \]

\[
\langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle = \langle 0 | (K \sigma_n)(K \bar{\sigma}_n) | 0 \rangle
\]

| \Psi_{opt.} \rangle = V_f^\dagger \cdots V_1^\dagger | 0 \rangle

\[
\equiv V^\dagger | 0 \rangle
\]

(isometry: \( V^\dagger V = 1 \))
\( \phi_n \) should be reconstructed from \( \sigma_n \)

\[
\langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle = \langle 0 | (K\sigma_n)(K\bar{\sigma}_n) | 0 \rangle
\]

\[
| \Psi_{opt.} \rangle = V^\dagger | 0 \rangle
\]

(isometry: \( V^\dagger V = 1 \))

\[
V \phi_n(z)V^\dagger \equiv (K\sigma_n)(z)
\]

\(~ \text{Bulk reconstruction} \)

[Hamilton, Kabat, Lifschytz, Lowe '06]