Towards Entanglement of Purification for Conformal Field Theories

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[1803.10539] with Hayato Hirai, Tsuyoshi Yokoya

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18/07/17 seminar @ Osaka U.

What objects in QFT can capture

quantum entanglement

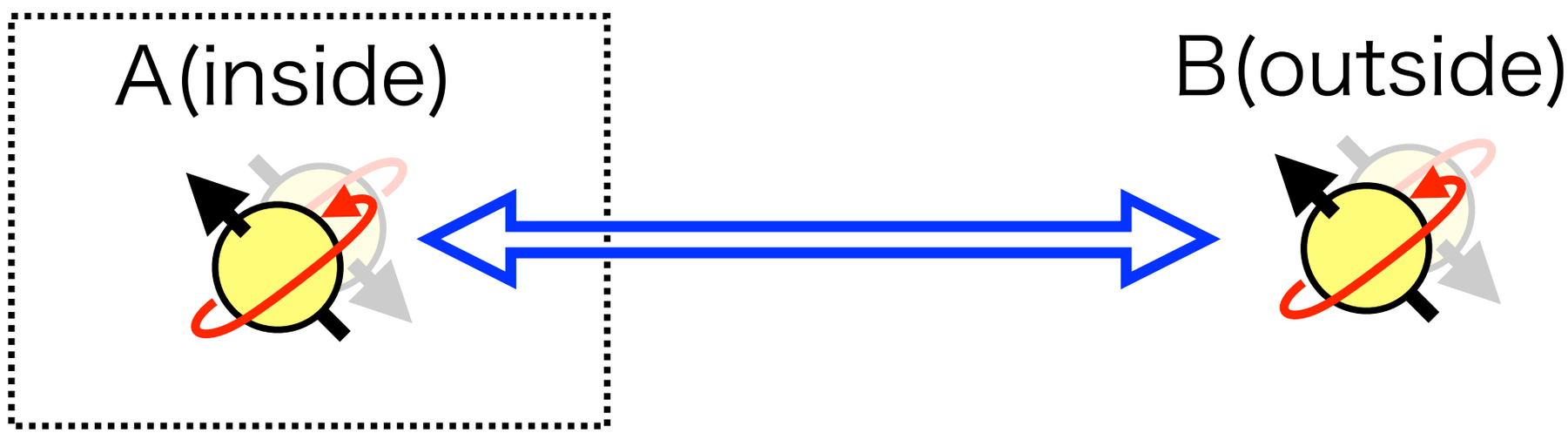
finer than the entanglement entropy?

Today, I will show an explicit (calculable) example!

What was entanglement entropy (EE)?

EE = # of EPR pair-like correlation

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{in}|\uparrow\rangle_{out} + |\downarrow\rangle_{in}|\downarrow\rangle_{out})$$



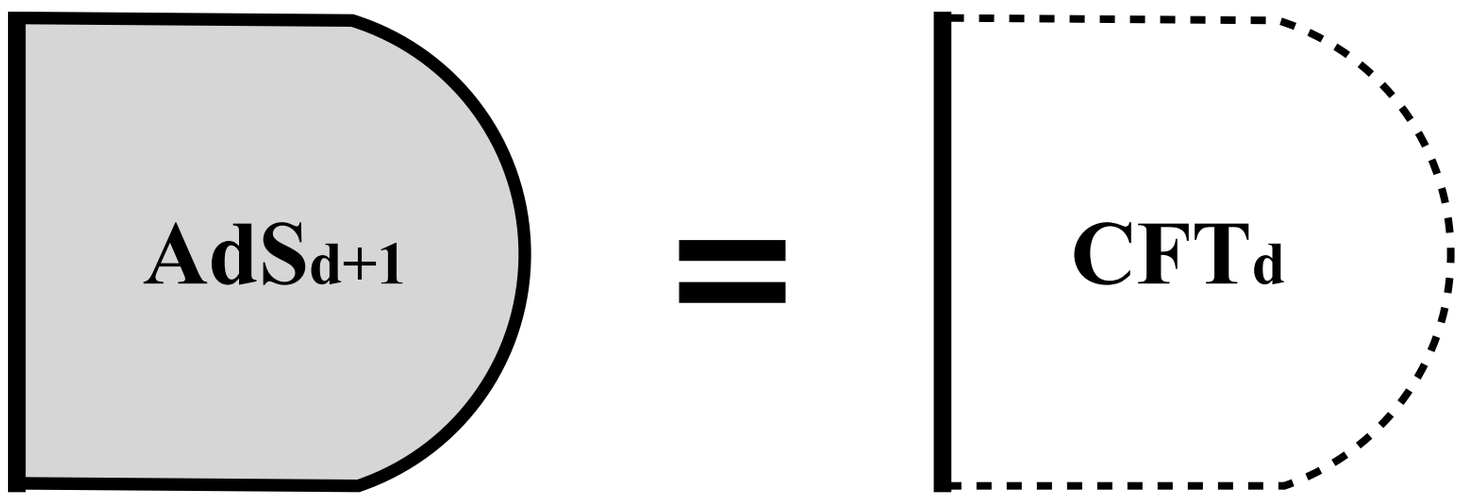
$$S_{\mathrm{EE}} = -\mathrm{Tr}_{\mathcal{H}_{in}}\left(\hat{\rho}_{\mathrm{in}} \log \hat{\rho}_{\mathrm{in}}\right) = \log 2$$

where $\hat{\rho}_{in} = \mathrm{Tr}_{\mathcal{H}_{out}} |\Psi\rangle\langle\Psi|$

How can we learn about gravity from QFT?

The AdS/CFT correspondence

[Maldacena '97]



[Heemskerk-Penedones-Polchinski-Sully '09]

How about **metric** in gravity side?

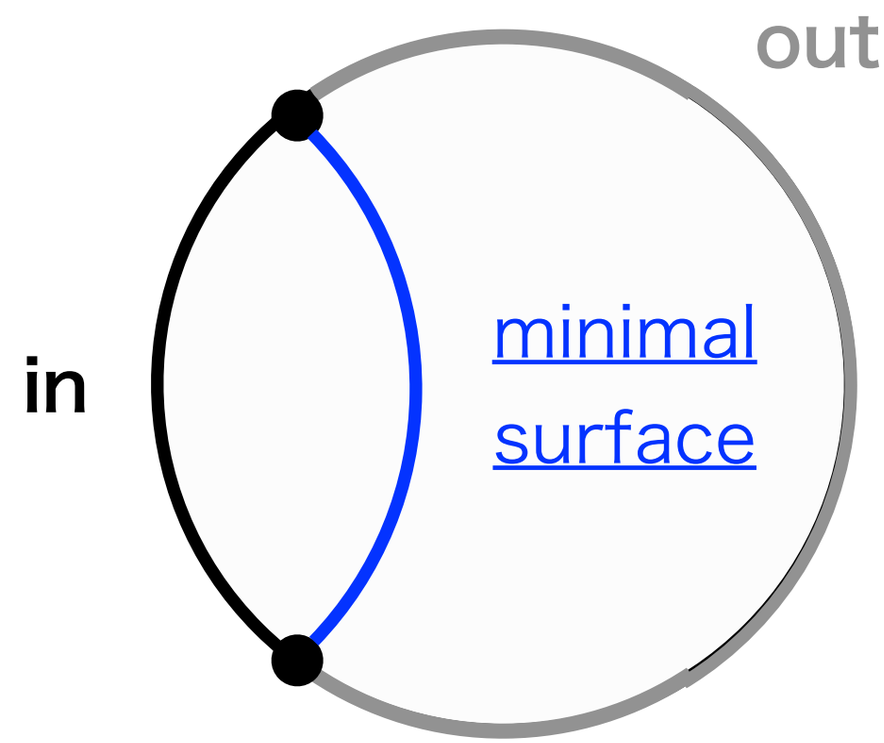
Quantum entanglement plays a key role!

How can we learn about metric from QFT?

Entanglement entropy (EE) ↔ Area of minimal surface

$$S_{\rm in} = \frac{\text{Area}}{4G_N}$$

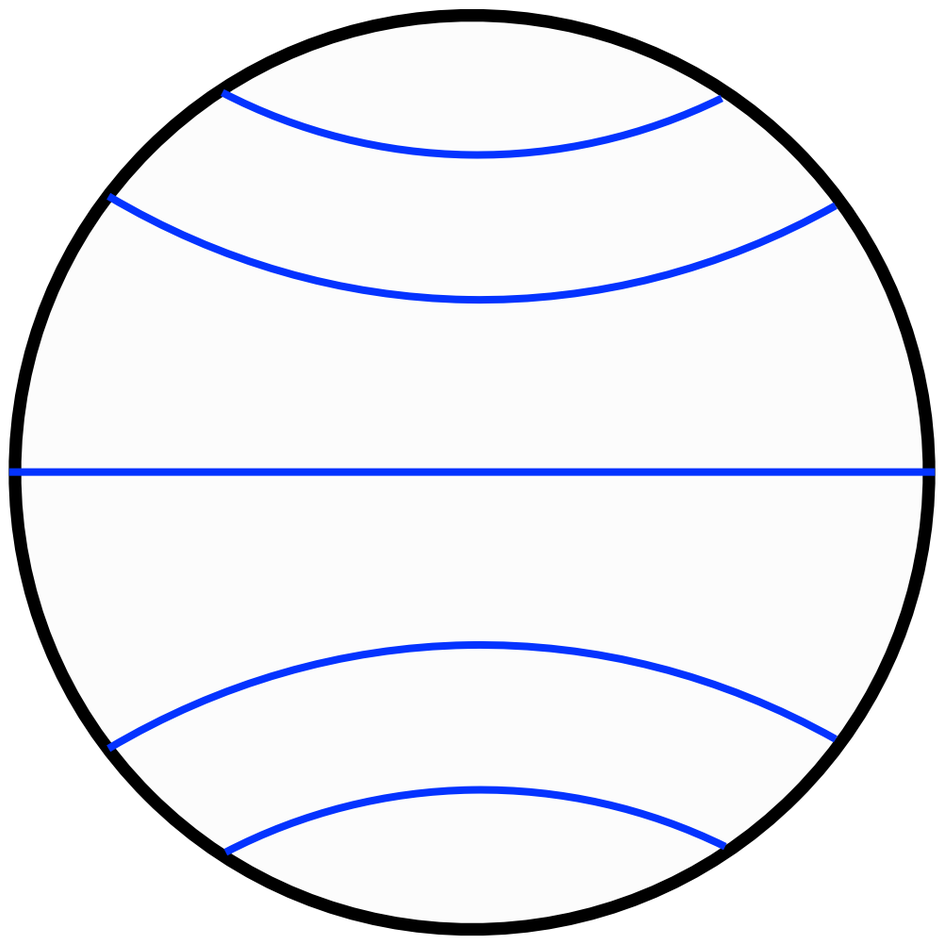
[Ryu-Takayanagi '06]



- Evaluate EE in QFT → Explore metric
- To what extent？ (Is EE enough?)

Q. Is EE enough ?

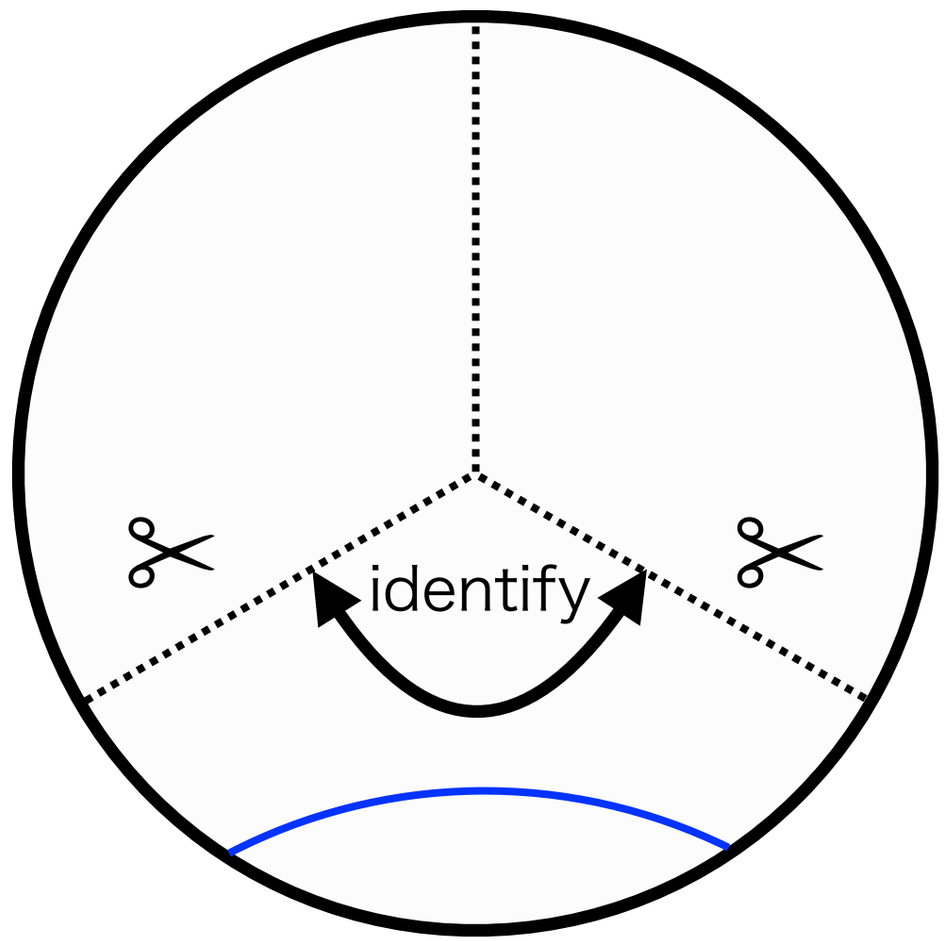
A. For time slice of AdS3, Yes!



minimal geodesics can pass through the center

Q. Is EE enough ?

A. In general, No…



a example: cut & glue the slice (orbifold)

minimal geodesics can NOT pass through the center

How to probe deep inside the bulk ?

an example:

Bulk: Entanglement Wedge Cross Section (EWCS)

conjecture

[Umemoto-Takayanagi '17],
[Nguyen-Devakul-Halbasch-Zaletel-Swingle '17]
c.f. intensive lecture by Takayanagi-san last year

Boundary: Entanglement of Purification (EoP)

I will explain these definition later

Our result

CFT quantities which describe EWCS/EoP

(1) EoP in holographic code model → “bulk” twist op.

(2) CFT calculation

(@ large-c, reduce to CB with internal twist op.s)

[Hirai, KT, Yokoya ‘18]

Plan

(1) EWCS & EoP (Review)

(2) EoP in holographic code model → “bulk” twist op.

(3) CFT calculation

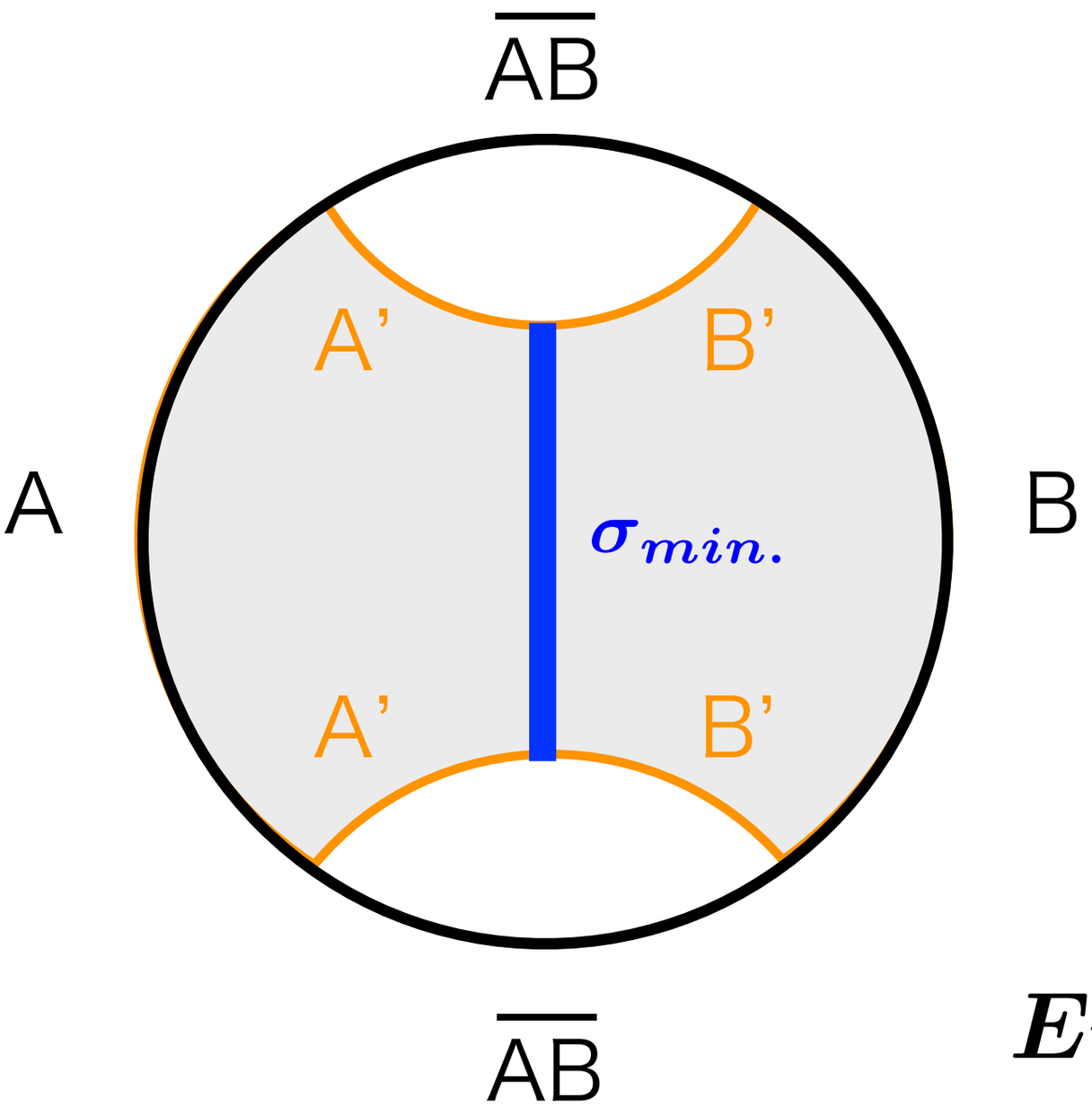
(@ large-c, reduce to CB with internal twist op.s)

[Hirai, KT, Yokoya ‘18]

(4) Discussion

Entanglement Wedge Cross Section

[Umemoto-Takayanagi '17], [Nguyen-Devakul-Halbasch-Zaletel-Swingle '17]



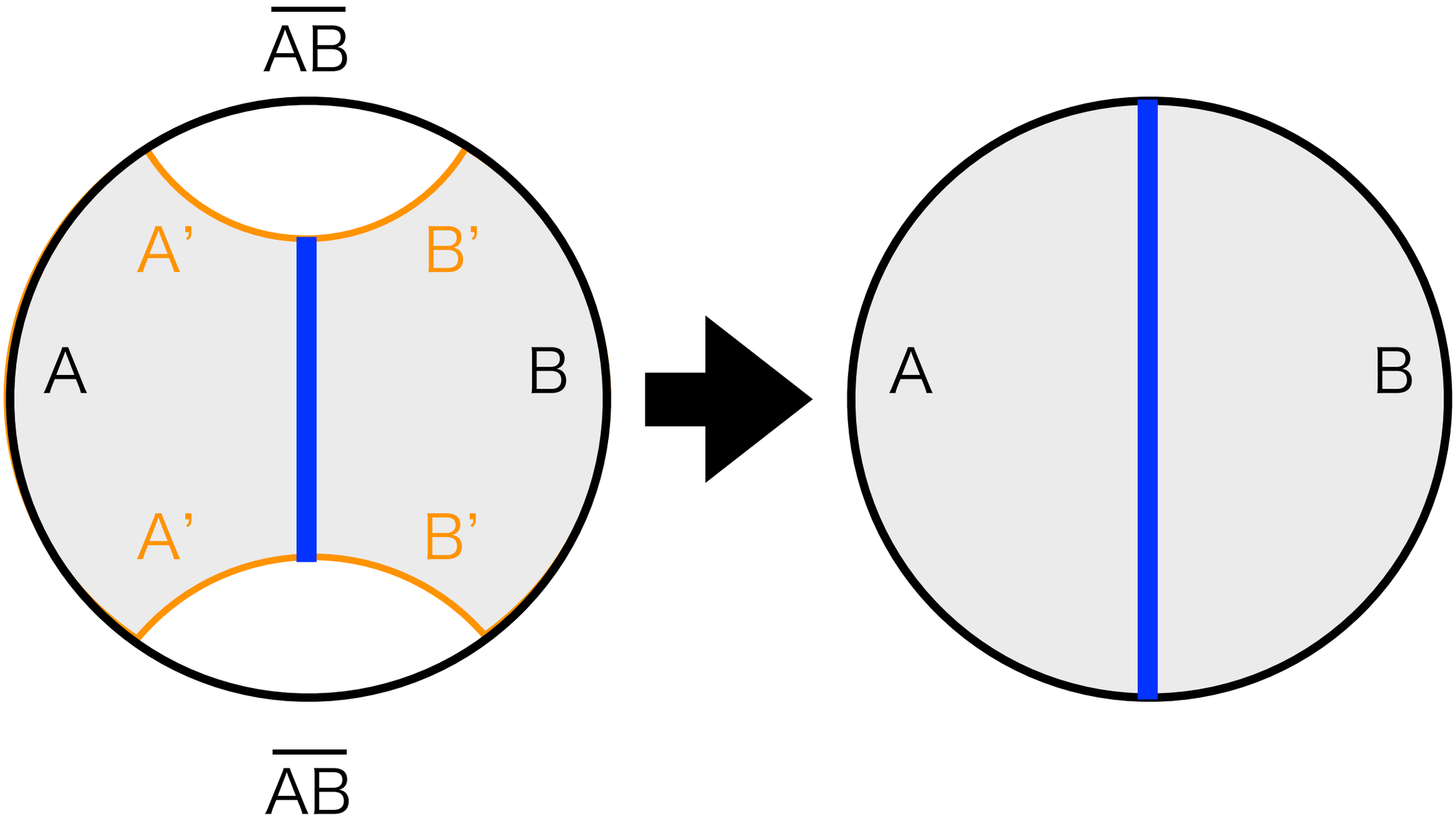
1. Divide AB and $\overline{AB}$
2. Write minimal surfaces

EW of AuB

Region surrounded by AuBu(min. surfaces)

$$E_W(A:B) = \frac{1}{4G_N}\sigma_{min.}$$

EWCS : a generalization of min. surface



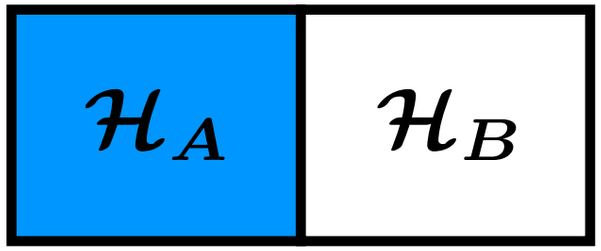
Take the size of $\overline{AB}$ →0: original RT formula

Entanglement of Purification(EoP)

a correlation measure for two subregions

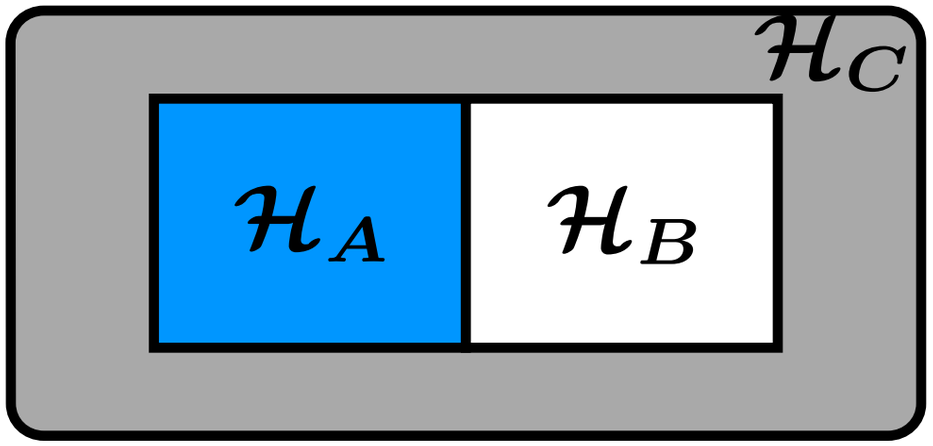
[Terhal-Horodecki-Leung-DiVincenzo '02]

$\hat{\rho}_{AB}$

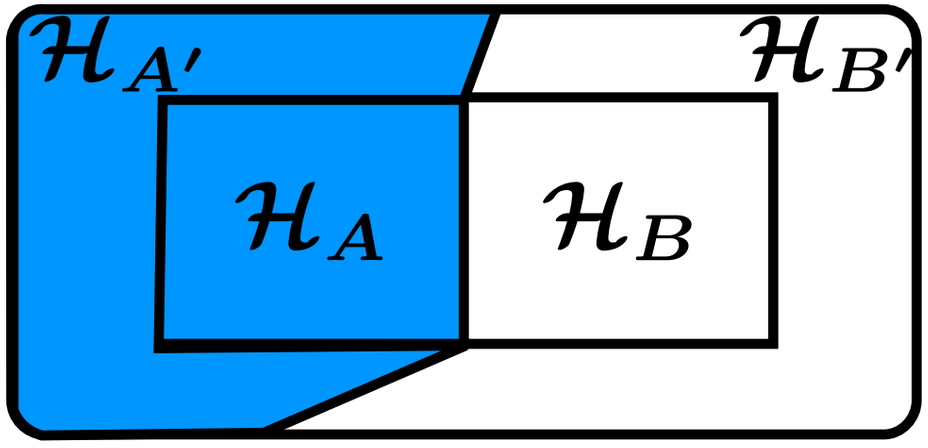


$|\psi\rangle_{ABC}$

s.t. $\rho_{AB} = \mathrm{Tr}_C\, |\psi\rangle\langle\psi|$



$$E_P(A:B) = \min_{|\psi\rangle_{ABA'B'}} S(\rho_{AA'})$$



Entanglement of Purification(EoP)

Minimal EE for all possible purification

[Terhal-Horodecki-Leung-DiVincenzo '02]

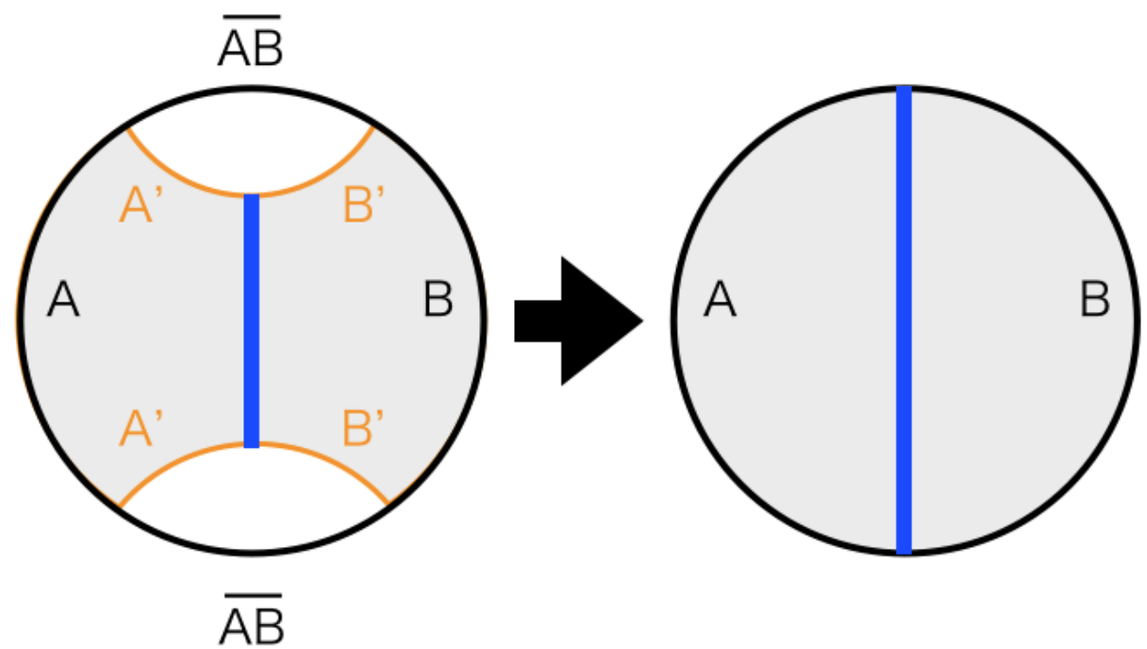
For a pure state: $|\psi\rangle_{AB}$

$$E_P(A:B) = S_A (= S_B)$$

A generalization of EE in this sense

Summary so far

EWCS:



EoP:

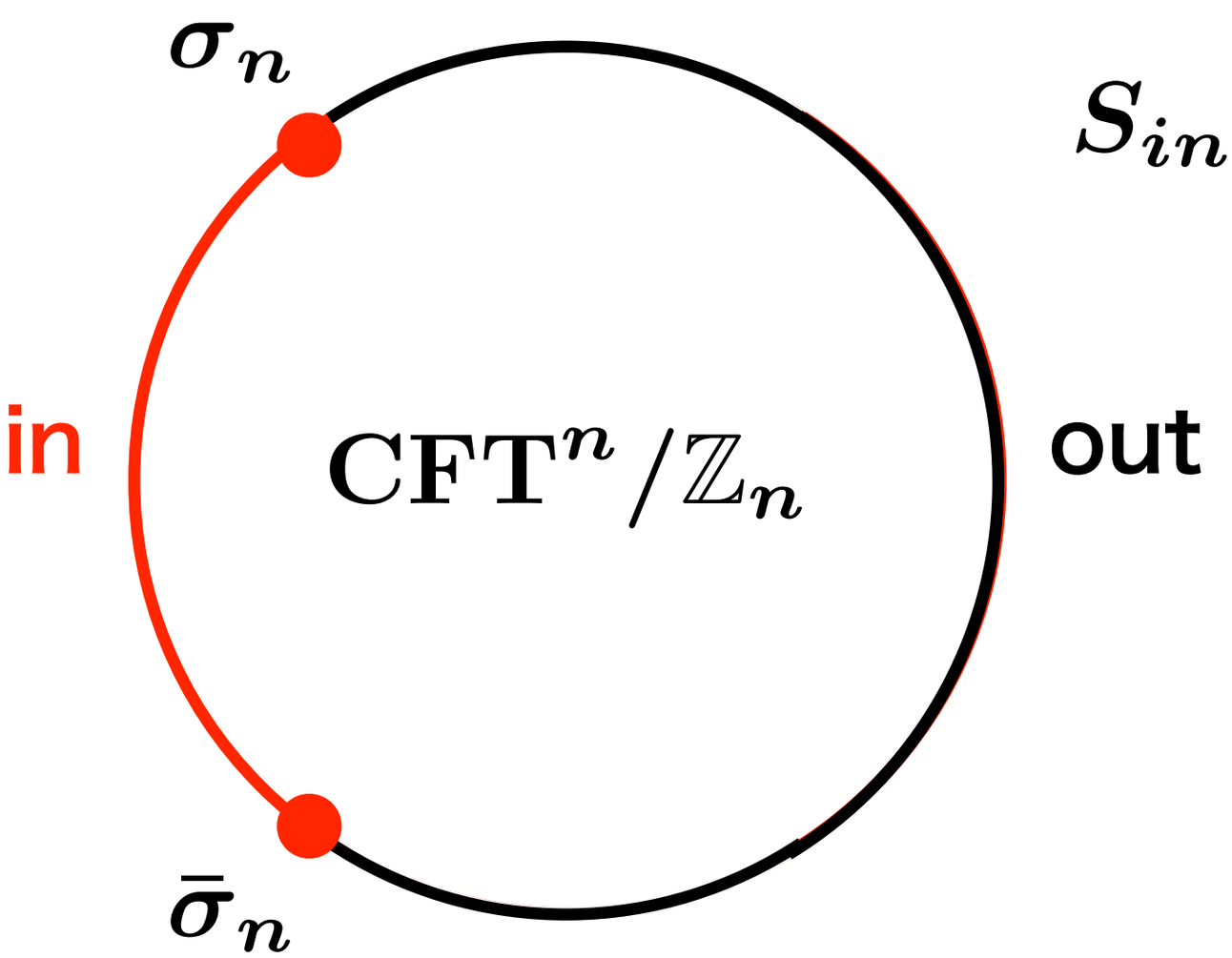
- a correlation measure for A vs B
- pure state → reduce to EE

(2) EoP in holographic code model

Replica method → “bulk” twist op.s

[Hirai, KT, Yokoya ‘18]

EE from twist op.s correlation function (Replica method)



$$S_{in} = -\frac{\partial}{\partial n}\langle \sigma_n \bar{\sigma}_n \rangle \Big|_{n\to 1}$$

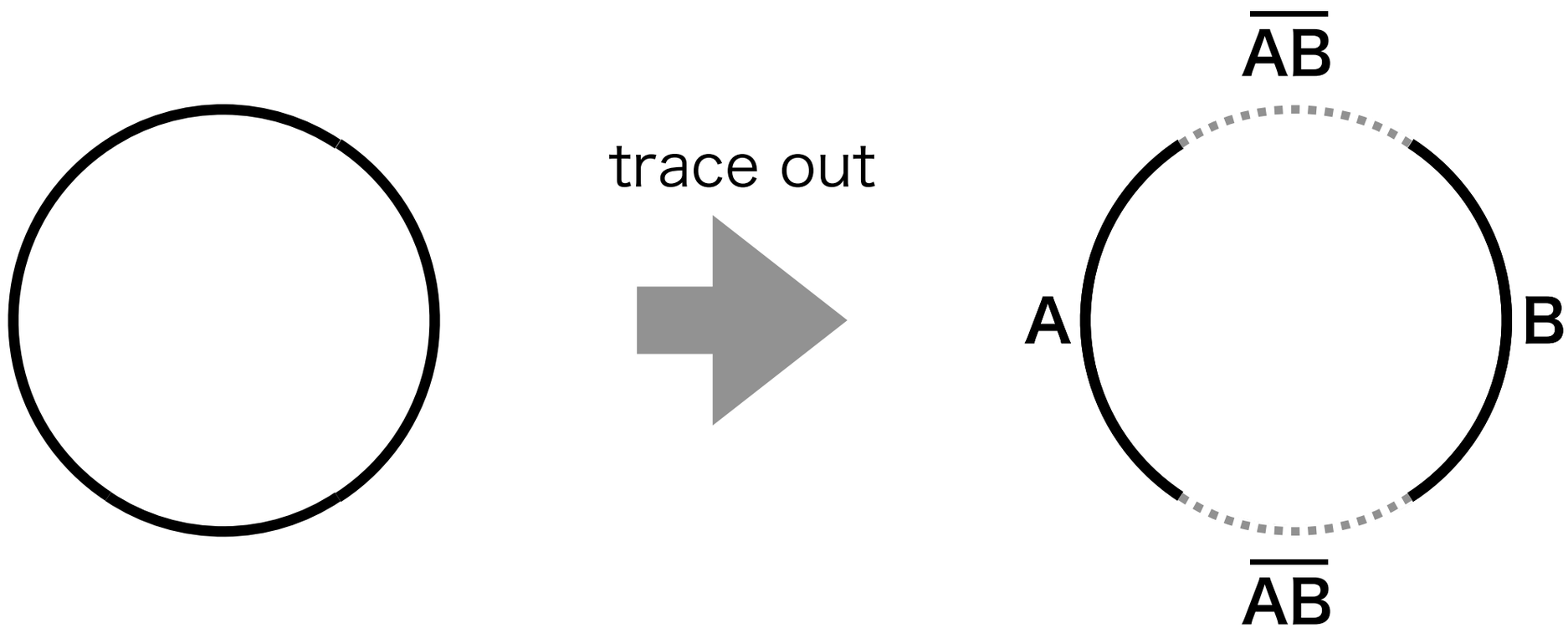
[Calabrese, Cardy '04]

! Scaling dim. of twist op.

→ from Conformal WT id.

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

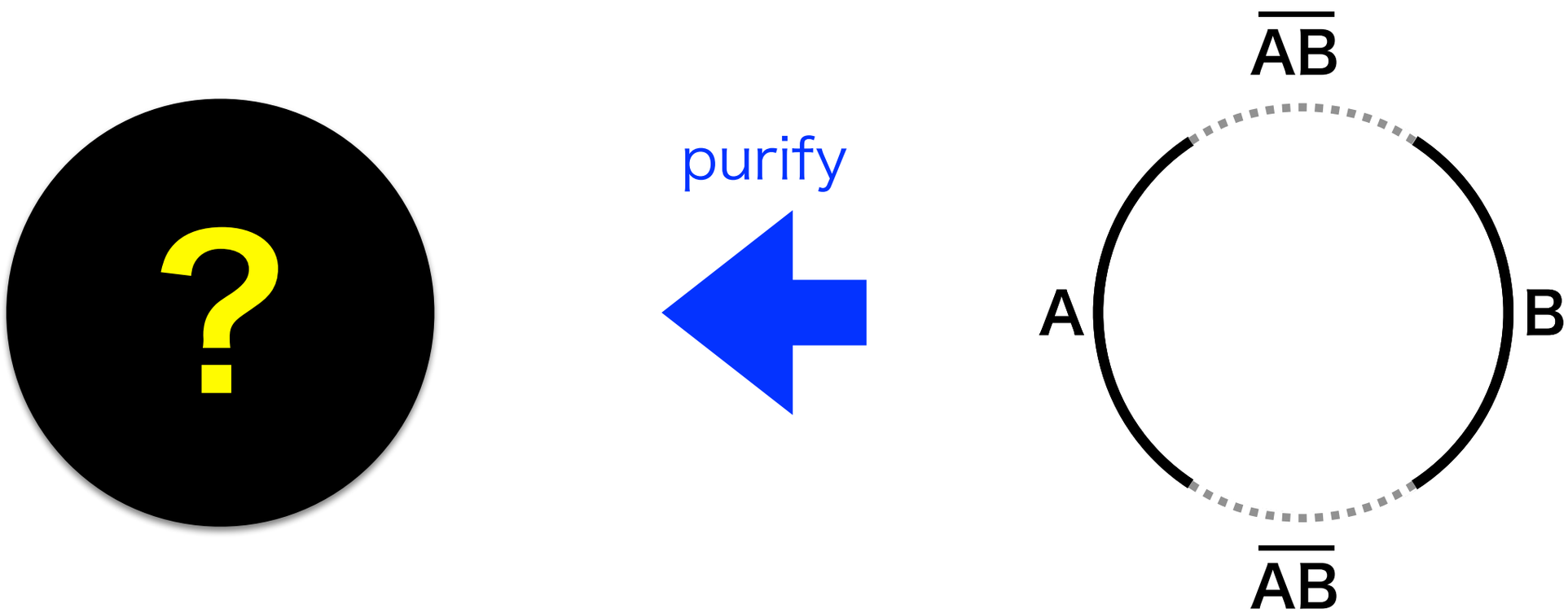
Let us consider EoP for mixed state



$|0\rangle$

$$\rho_{AB} = \mathrm{Tr}_{\overline{AB}} |0\rangle\langle 0|$$

What is a optimized purification ?



Can ask for help to the Tensor Network Model

→ We used Holographic code model

[Pastawski, Yoshida, Harlow, Preskill '15]

Punch line of the following argument

Within the holographic code model,

Twist op. for optimized solution

= “bulk dual of the CFT twist op.”

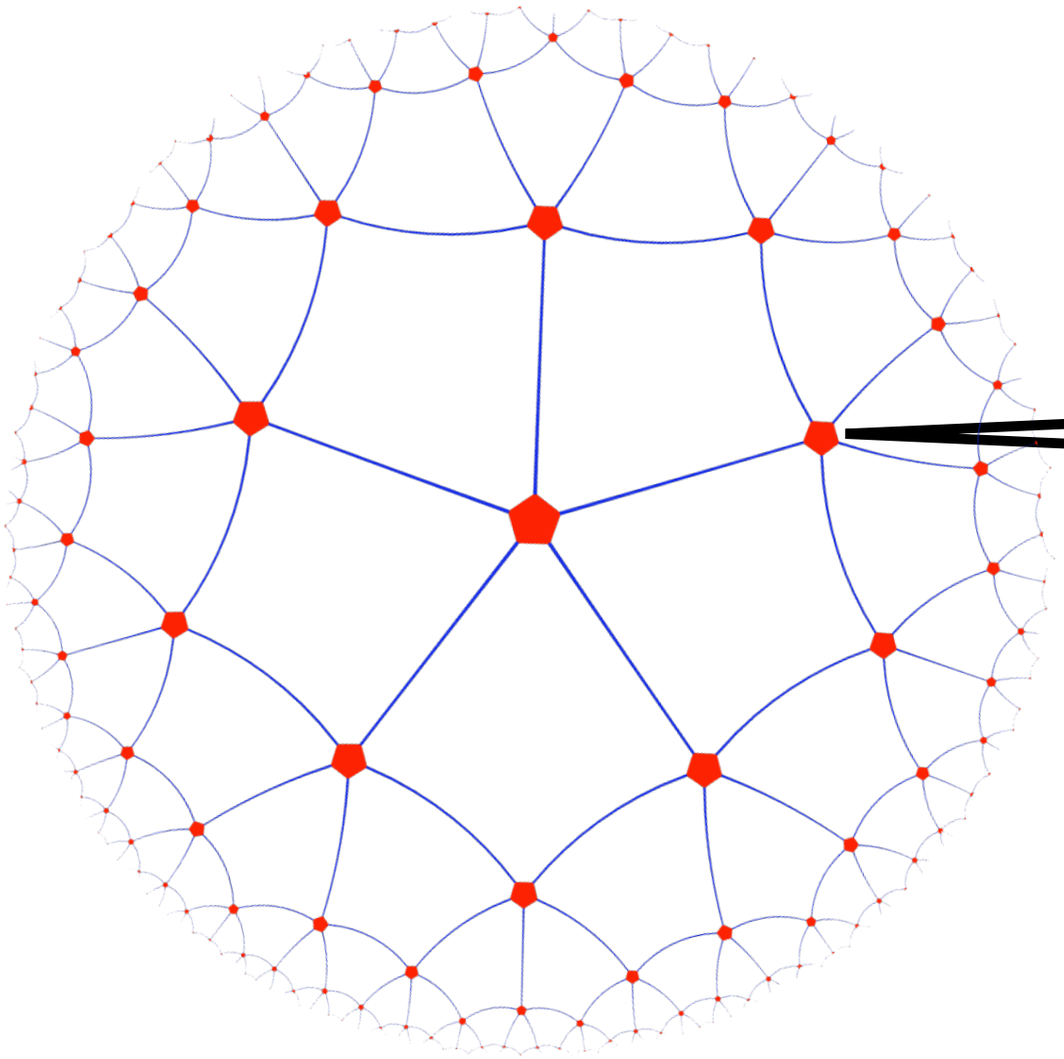
$$\phi_n \left|\Psi_{opt.}\right\rangle \simeq \left(K\sigma_n\right)\left|0\right\rangle$$

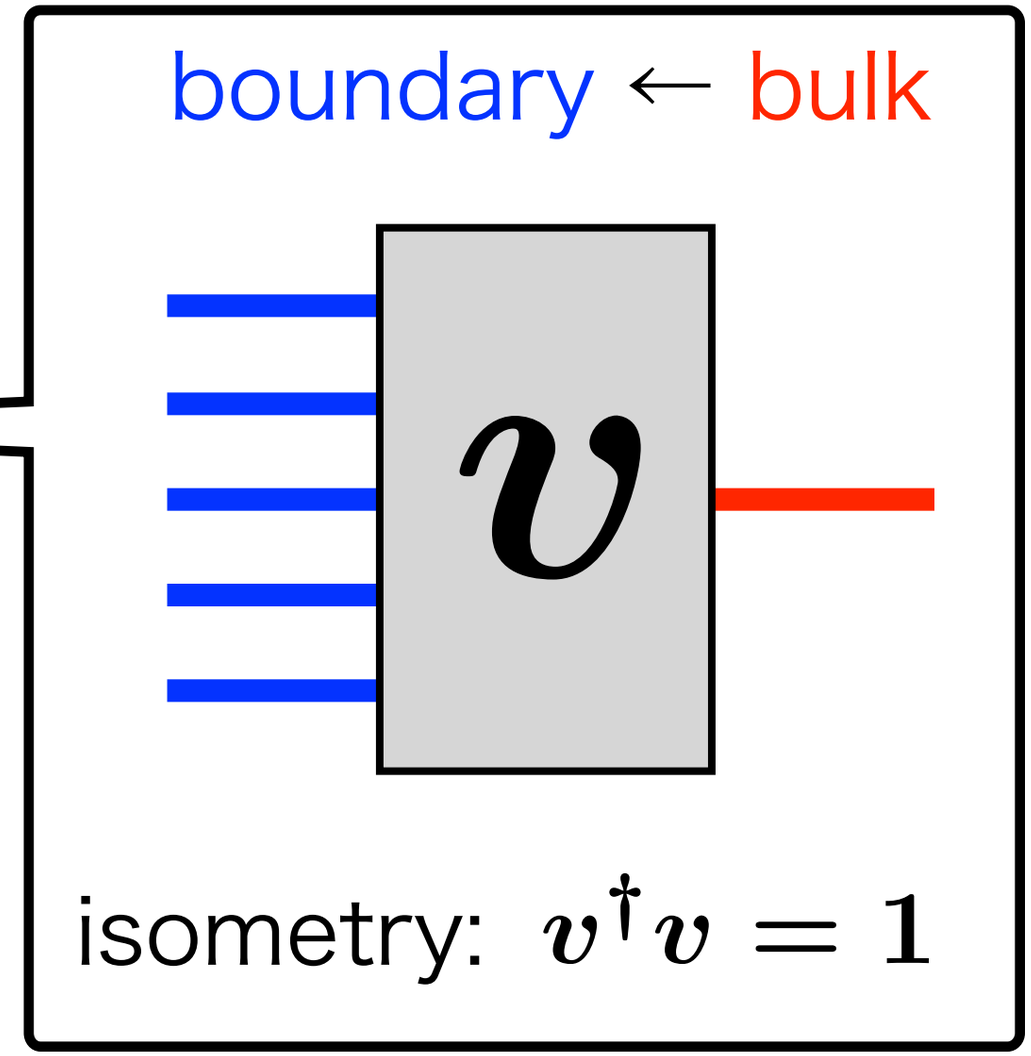
(i.e. CFT twist op. smeared by HKLL kernel $\mathbf{K}$)

[Hamilton, Kabat, Lifschytz, Lowe '06]

Holographic code model : duality as isometry

[Pastawski, Yoshida, Harlow, Preskill '15]

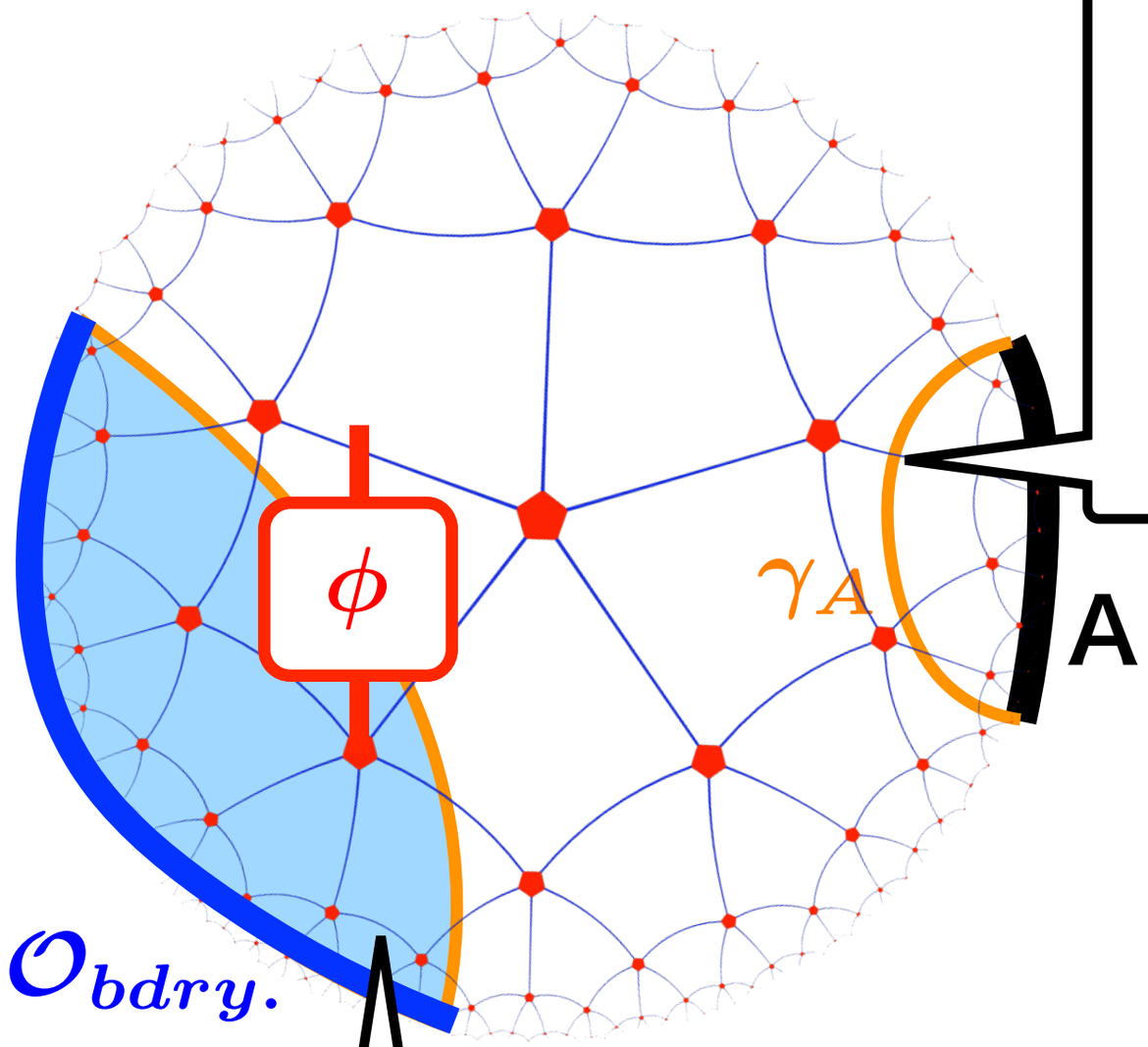




$$\left(\prod v\right) : \underline{|0\rangle_{bulk}} \mapsto \underline{|0\rangle_{bdry.}} \equiv |0\rangle$$

Holographic code model : duality as isometry

[Pastawski, Yoshida, Harlow, Preskill '15]



① <u>Ryu-Takayanagi formula</u>

$$S_A = \text{const.} \times \text{Area}(\gamma_A) + S_{\text{bulk}}$$

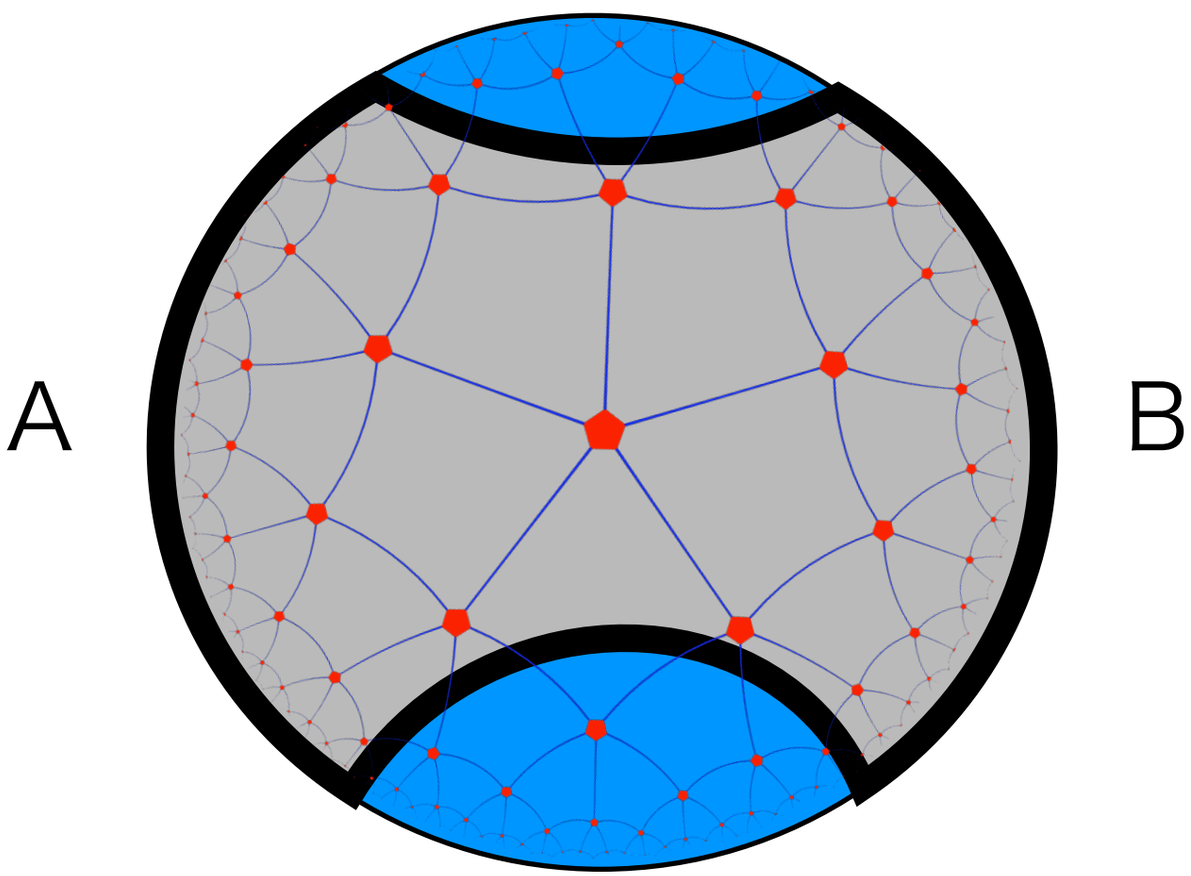
② <u>HKLL bulk reconstruction</u>

[Hamilton, Kabat, Lifschytz, Lowe '06]

$$\phi_{bulk.} |0\rangle_{bulk.} \mapsto \mathcal{O}_{bdry.} |0\rangle_{bdry.}$$

Optimized purification within the holographic code model

$$\rho_{AB} = \mathrm{Tr}_{\overline{AB}}(|0\rangle\langle 0|) \xrightarrow{\text{purify}} |\Psi_{opt.}\rangle \simeq V^\dagger |0\rangle$$



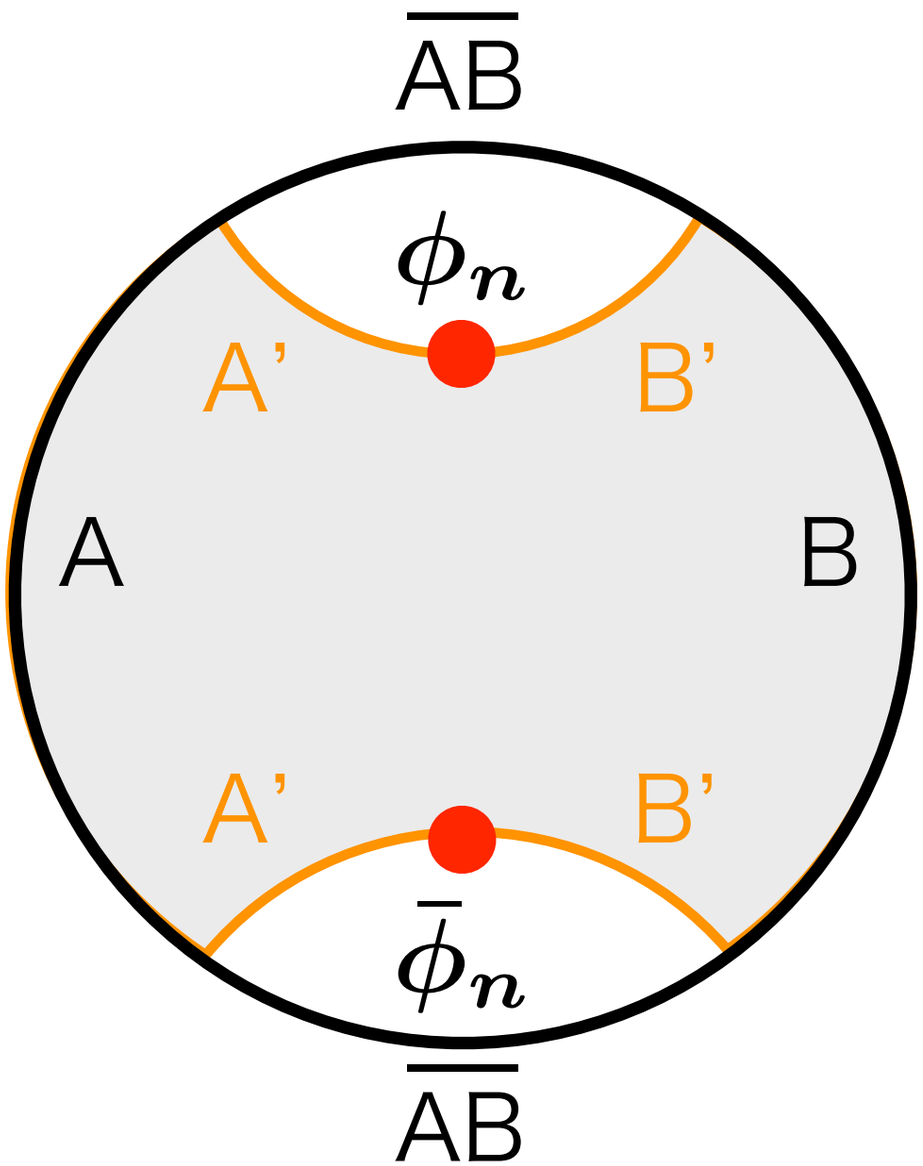
Let us assume

"Duality between
new boundary and bulk"

c.f. Miyaji-Takayanagi '15

Then, new boundary
→ C=A'B'

Replica method can apply to EoP



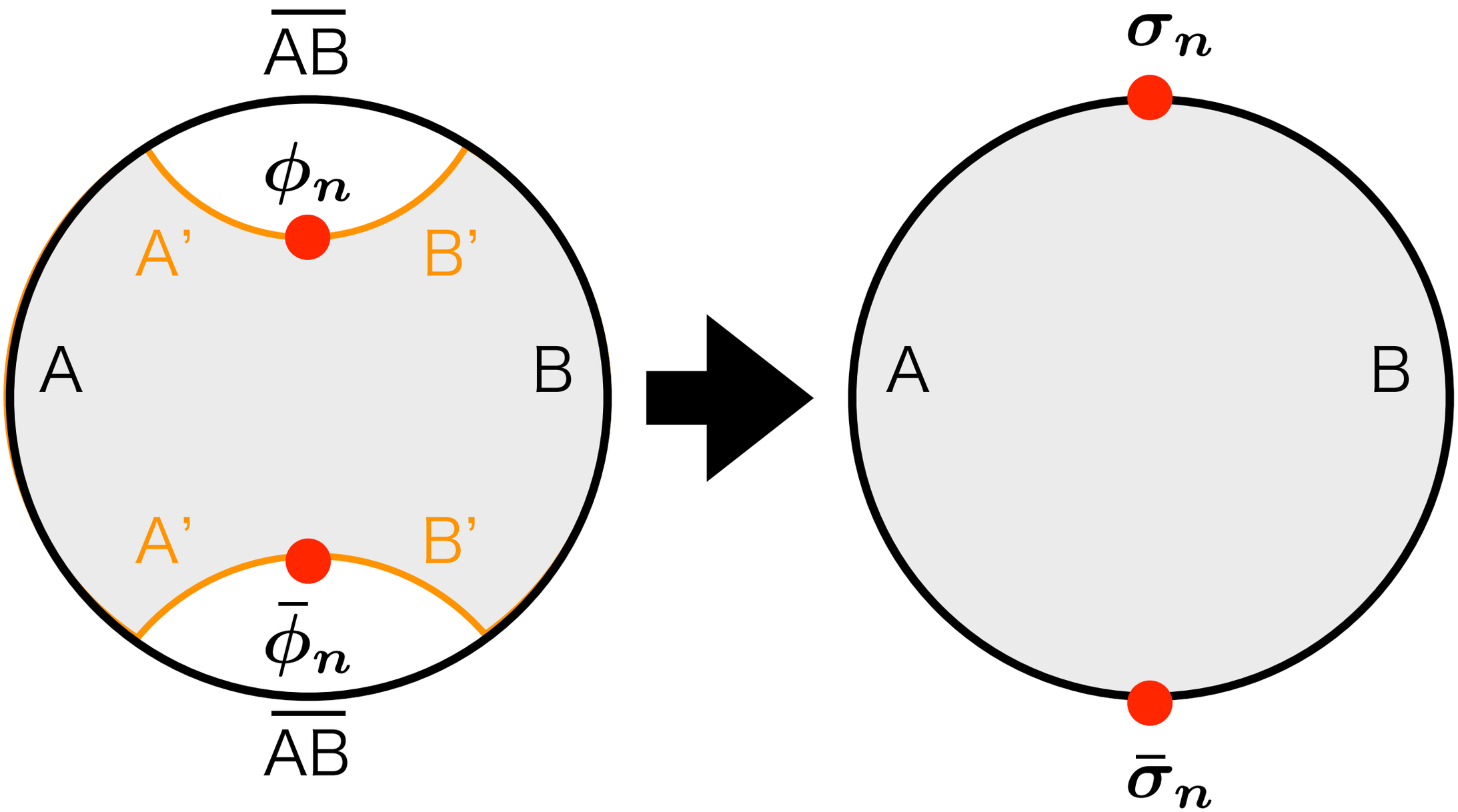
$$S_{AA'}|_{min.}$$

$$= -\frac{\partial}{\partial n} \langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle \Big|_{n \to 1}$$

ϕ_n :(so far) unknown twist op.

$$\rho_{AB} = \mathrm{Tr}_C(|0\rangle\langle 0|) \xrightarrow{\text{purify}} |\Psi_{opt.}\rangle \simeq V^\dagger |0\rangle$$

Recap: for pure states, EoP → EE



Take the size of $\overline{AB}$ →0: original RT formula

“Bulk twist op.” → twist op. in CFT

<u>Boundary condition</u>

$$\phi_n \xrightarrow{r \to \infty} \sigma_n$$

$$m_{\phi_n}^2 R_{AdS}^2 = \Delta_n(\Delta_n - 2)$$

! Within framework of the Holographic code model,
ϕ_n is just the HKLL map of σ_n

$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

(3) From CFT correlation function:

the channel including internal twist op.

[Hirai, KT, Yokoya ‘18]

<u>Set Up:</u> $\mathbf{CFT}^n/\mathbb{Z}_n$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}} W_{\mathcal{O}}^{12,34}(x_i)$$

Consider a channel including twist op. $\sigma_n \in \mathcal{O}$

(at large-c limit, the twist op. will dominate)

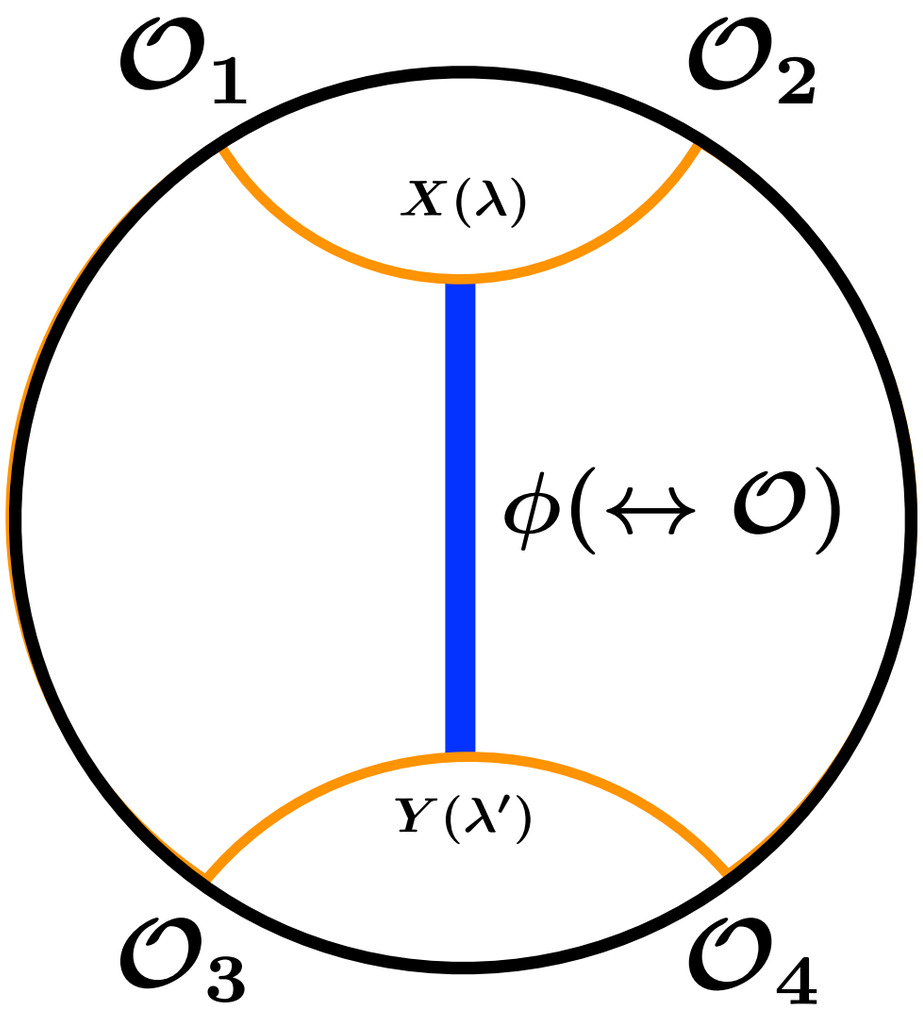
$$W_{\mathcal{O}}^{12,34}(x_i) = \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \times G_{\mathcal{O}}(u,v)$$

u,v: cross ratio

We will consider **<u>sum of $G_{\mathcal{O}}$</u>** rather than $\langle O_1 O_2 O_3 O_4 \rangle$

Conformal Block from AdS geodesics

[Hijano-Kraus-Perlmutter-Snively '15]



Conformal Block=

$$\int d\lambda \int d\lambda' G_{bb}^{\Delta,0}(X(\lambda), Y(\lambda'))$$

$$\sim e^{-\Delta \sigma_{min}}$$

$$1 \ll \Delta \simeq m R_{AdS}$$

At the semiclassical limit, only σ_{min} contribute!

(Now we are treating so-called "light operators" ($1 \ll \Delta_i, \Delta \ll c$))

(a 4pt function)/(2pt functions) captures the EWCS @ large-c

$$-\frac{\partial}{\partial n} G_{\sigma_n}(u,v)\Big|_{n\to 1}$$

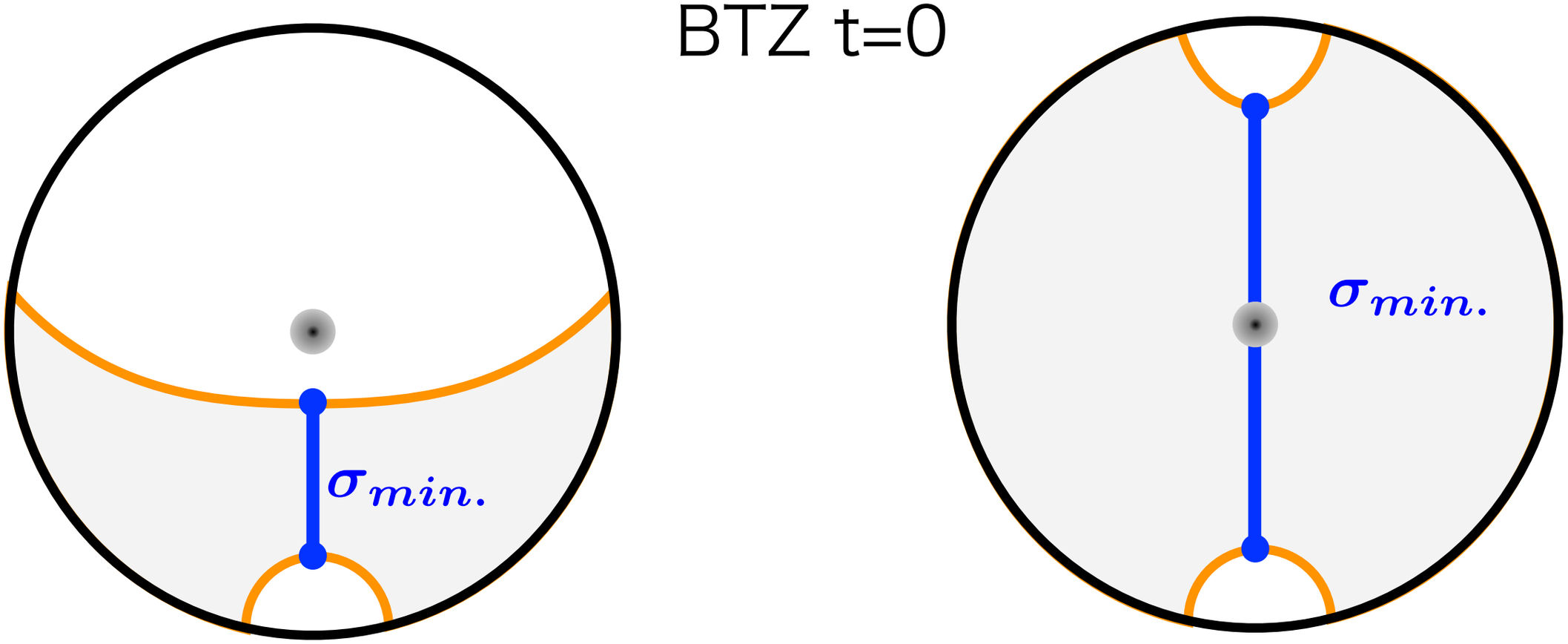
$$G_{\sigma_n} \sim e^{-\frac{c}{12}\left(n-\frac{1}{n}\right)\sigma_{min}}$$

$$= \frac{c}{6}\sigma_{min} = \frac{1}{4G_N}\sigma_{min} = E_W$$

! For the sector with (twist#)=1,
the above **σ_n will be dominant @ large-c**

Extension to the BTZ blackholes

BTZ t=0



- Assume the heavy state as an optimized purification
- Reduce to heavy-light Virasoro conformal blocks

Our result

CFT quantities which describe EWCS/EoP

(1) EoP in holographic code model → “bulk” twist op.

(2) CFT calculation

(@ large-c, reduce to CB with internal twist op.s)

[Hirai, KT, Yokoya ‘18]

Future directions

Optimization within the framework of QFT

What is V in QFT? TTbar?

Extension to the higher dimension

HKLL map for defects in the bulk?

Relation to the Kinematic space

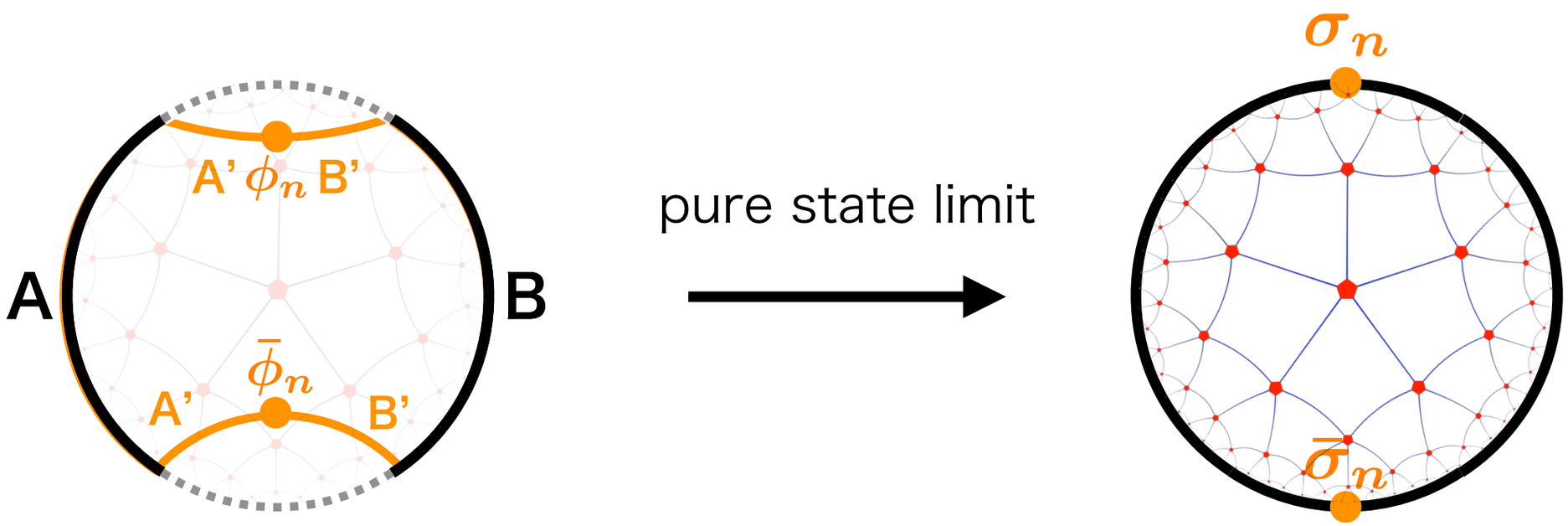
e.g. Czech-Lamprou-McCandlish-Mosk-Sully '16

Thank you !

backup

ϕ_n should be reconstructed from σ_n

$$\langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle = \langle 0 | (K\sigma_n)(K\bar{\sigma}_n) | 0 \rangle$$



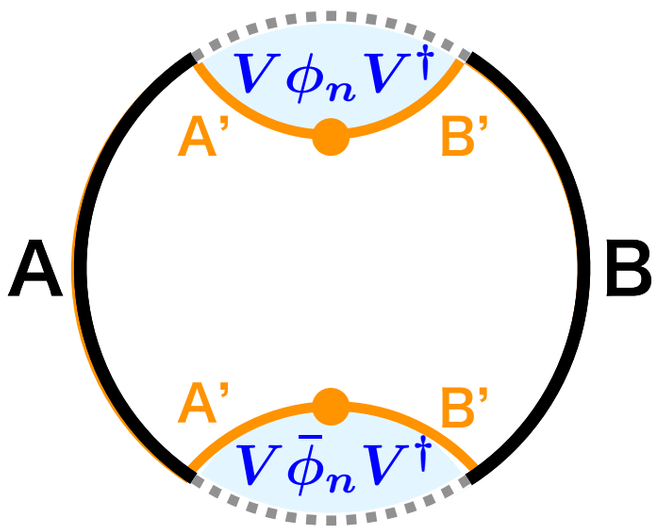
$$|\Psi_{opt.}\rangle = V_f^\dagger \cdots V_1^\dagger |0\rangle \qquad\qquad |0\rangle$$

$$\equiv V^\dagger |0\rangle$$

(isometry: $V^\dagger V = 1$)

ϕ_n should be reconstructed from σ_n

$$\langle \Psi_{opt.} | \phi_n \bar{\phi}_n | \Psi_{opt.} \rangle = \langle 0 | (K\sigma_n)(K\bar{\sigma}_n) | 0 \rangle$$



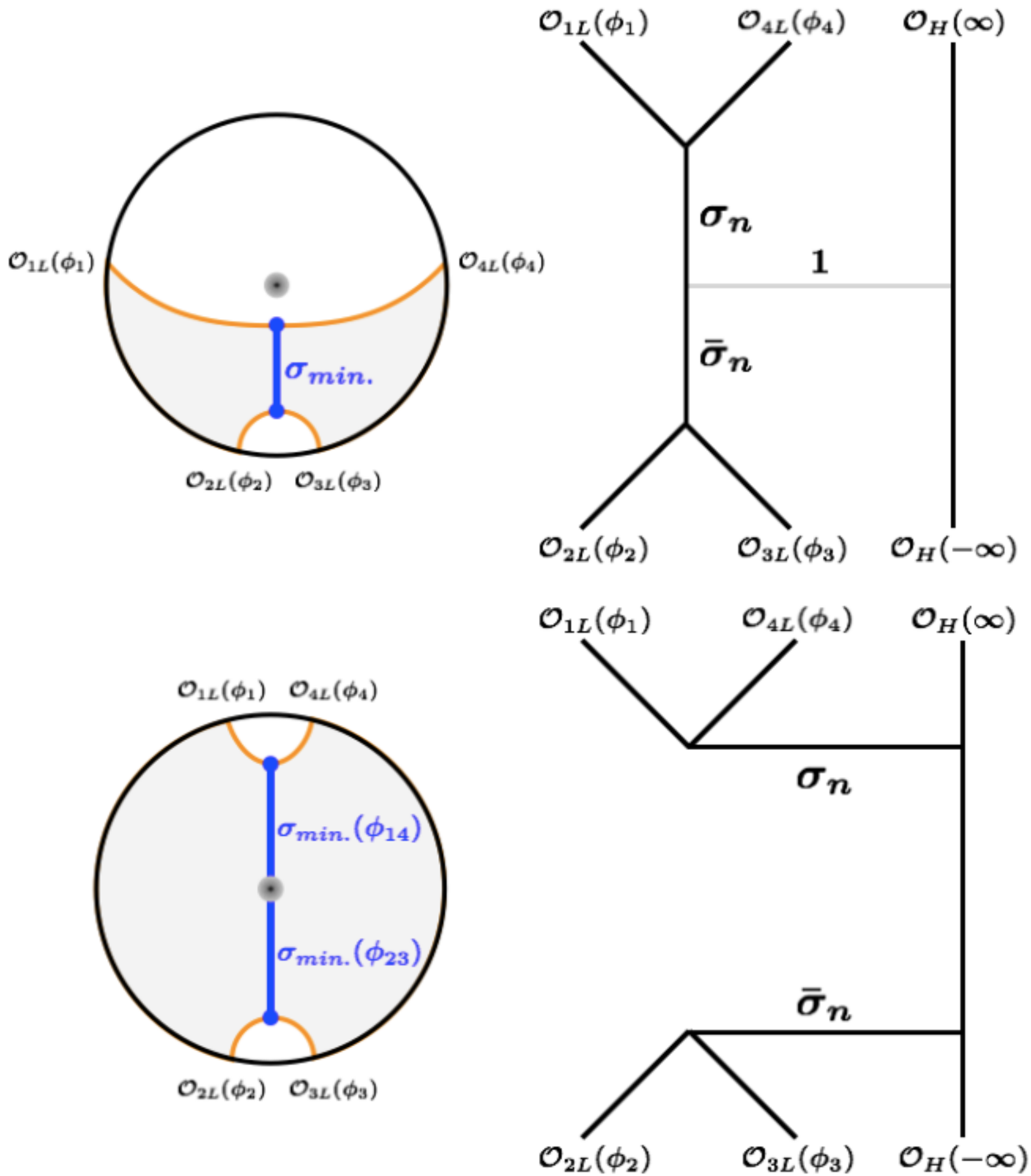
$$V\phi_n(z)V^\dagger \equiv (K\sigma_n)(z)$$

$\sim$ Bulk reconstruction

[Hamilton, Kabat, Lifschytz, Lowe '06]

$$|\Psi_{opt.}\rangle = V^\dagger |0\rangle$$

(isometry: $V^\dagger V = 1$)


$\mathcal{O}_{1L}(\phi_1)$
$\mathcal{O}_{4L}(\phi_4)$
$\sigma_{min.}$
$\mathcal{O}_{2L}(\phi_2)$
$\mathcal{O}_{3L}(\phi_3)$
$\mathcal{O}_{1L}(\phi_1)$
$\mathcal{O}_{4L}(\phi_4)$
$\mathcal{O}_H(\infty)$
σ_n
1
$\bar{\sigma}_n$
$\mathcal{O}_{2L}(\phi_2)$
$\mathcal{O}_{3L}(\phi_3)$
$\mathcal{O}_H(-\infty)$
$\mathcal{O}_{1L}(\phi_1)$
$\mathcal{O}_{4L}(\phi_4)$
$\sigma_{min.}(\phi_{14})$
$\sigma_{min.}(\phi_{23})$
$\mathcal{O}_{2L}(\phi_2)$
$\mathcal{O}_{3L}(\phi_3)$
$\mathcal{O}_{1L}(\phi_1)$
$\mathcal{O}_{4L}(\phi_4)$
$\mathcal{O}_H(\infty)$
σ_n
$\bar{\sigma}_n$
$\mathcal{O}_{2L}(\phi_2)$
$\mathcal{O}_{3L}(\phi_3)$
$\mathcal{O}_H(-\infty)$