Introduction to Resurgence & Complex saddle point analysis

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Purpose of This talk

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To emphasize importance of complex saddle point analysis

in which we mind

- Saddle points which are not on original (path) integral contour
- Global steepest descents (Lefschetz thimbles)

I will mainly explain applications to resurgence.

Contents (most=review)

1. Expectations on weak coupling perturbative series in QFT
2. What is resurgence?

$\geqq$40min.

3. Application to a 3d SUSY CS theory

[The simplest example in Fujimori-M.H.-Kamata-Misumi-Sakai '18]

4. Summary & Outlook

1. Expectations on weak coupling perturbative series in QFT

- Perturbative series in typical QFT
- Borel resummation
- Borel summability in QFT?

Perturbative expansion in QFT

- Typically non-convergent [Dyson '52]
- Naïve sum of all-orders → divergent

General questions

- What does perturbative series actually know?
- Is there a way to obtain exact answer from perturbative expansion?
- If yes, how?

More precise question

Perturbative series around saddle points:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell + \sum_{I\in \text{saddles}} e^{-S_I(g)} \sum_{\ell=0}^{\infty} c_\ell^{(I)} g^\ell \ ??$$

Can we get the exact result by using the coefficients?

= What is a correct way to resum the perturbative series?

(∼continuum definition of QFT?)

A standard resummation

Borel transformation:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell} \Rightarrow \boxed{\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}}$$

Borel resummation (along θ):

$$\boxed{S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t)}$$

(usually, θ=arg(g)=0)

Why Borel resummation may be nice

(Let us take θ=arg(g))

$$S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{BO}(t) \qquad \mathcal{BO}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

① Reproduce original perturbative series:

$$S_\theta \mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} \int_0^{e^{i\theta}\infty} dt\ t^{a+\ell-1} e^{-\frac{t}{g}} = \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell}$$

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② Finite for any g if

1. Borel trans. is convergent
2. Its analytic continuation does not have singularities along the contour
3. The integration is finite

"Borel summable (along θ)"

related to exact result?

Expectations in typical QFT

['t Hooft '79]

Non-Borel summable due to singularities along R_+

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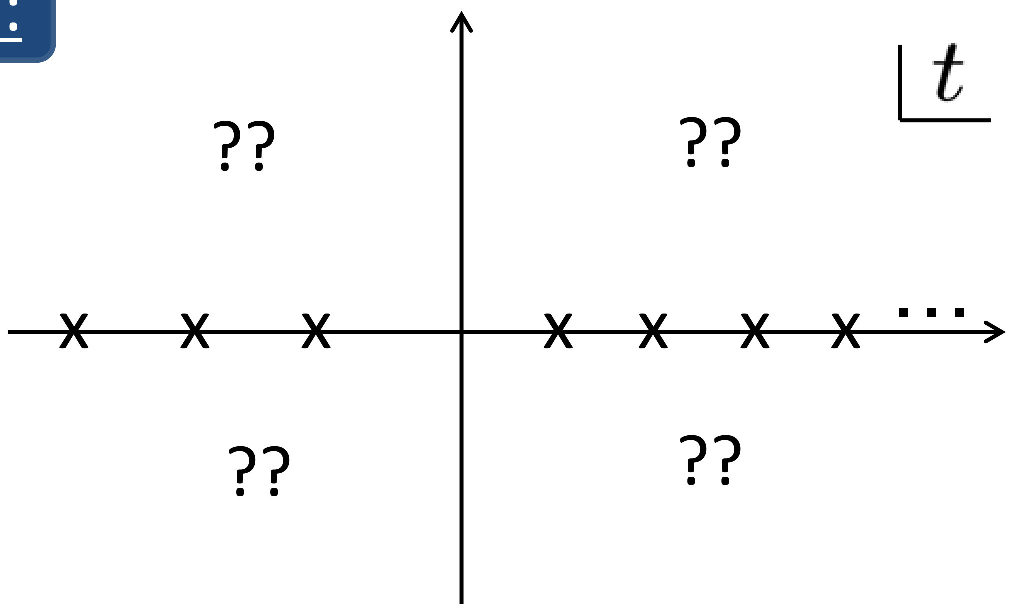
Non-Borel summable due to singularities along R_+

Borel plane:



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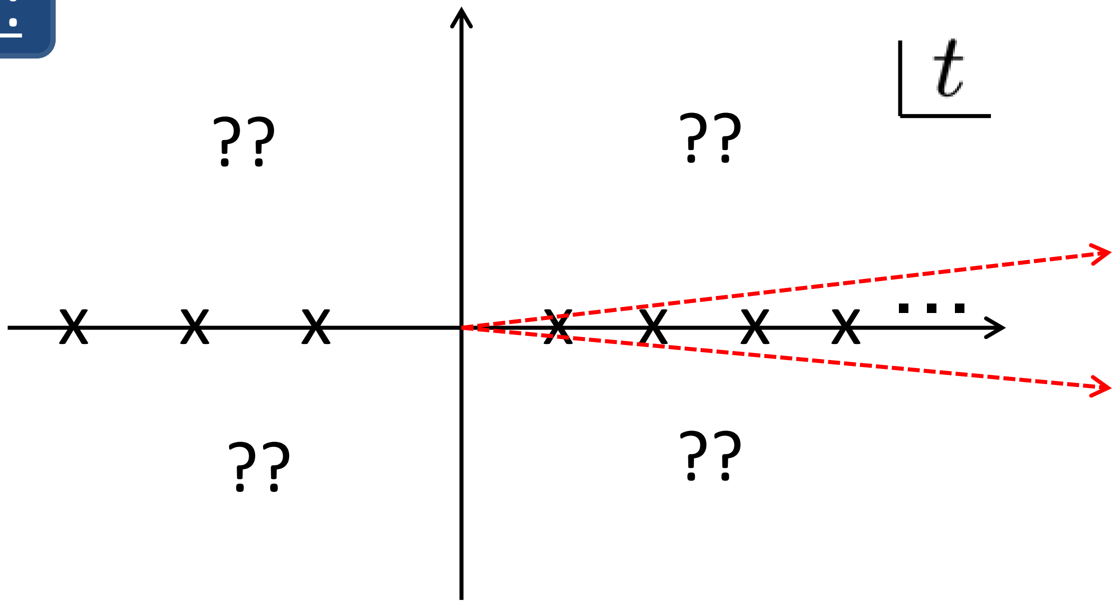

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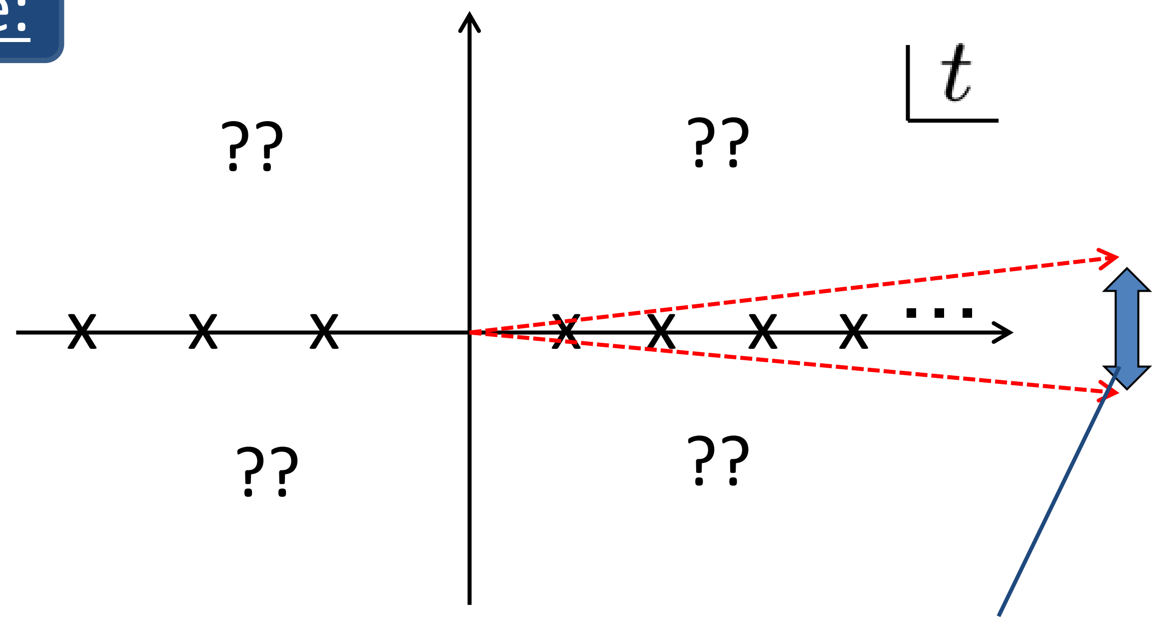

Integral depends on a way to avoid singularities

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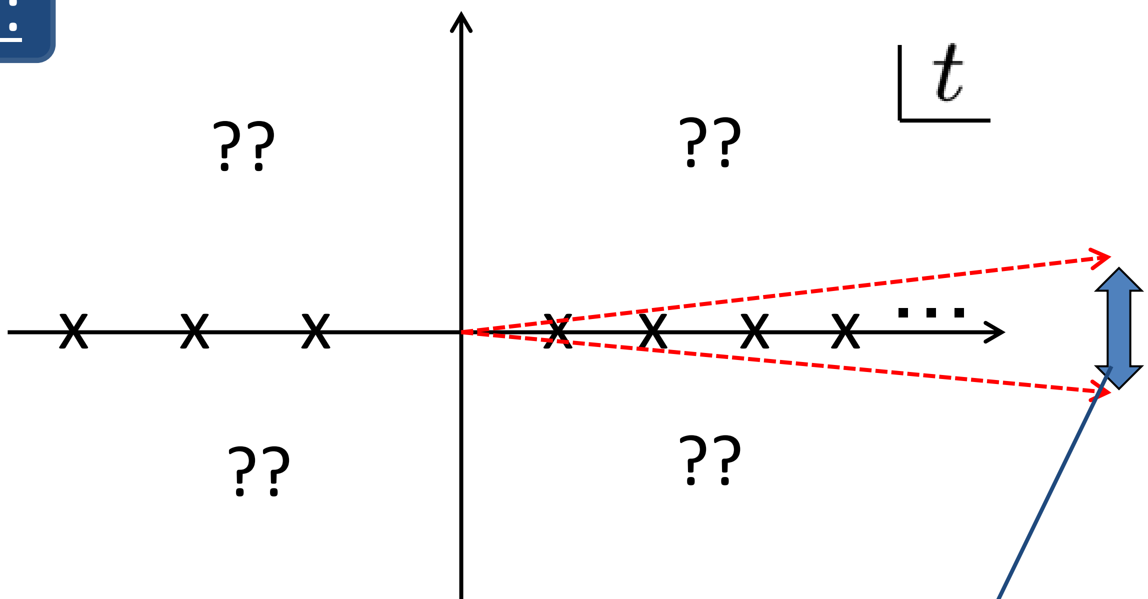

Integral depends on a way to avoid singularities

$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{BO}(t) \Rightarrow (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Non-perturbative effect?

Interpretation of Borel singularities

[Lipatov '77]

$$Z(g) = \int D\Phi e^{-\frac{1}{g}S[\Phi]} \simeq \sum_{\ell} c_{\ell} g^{\ell}$$

Large order coefficient:

$$c_{\ell} = \frac{1}{2\pi i}\oint \frac{dg}{g^{\ell+1}} Z(g) = \frac{1}{2\pi i}\oint dg \int D\phi e^{-\frac{1}{g}S[\phi] - (\ell+1)\ln g} \qquad (\ell \to \infty)$$

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$$\simeq e^{-\frac{1}{g_*}S[\phi_*]-(\ell+1)\ln g_*} \qquad \left(\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_*} = 0,\ -\frac{1}{g_*^2}S[\phi_*] + \frac{\ell+1}{g_*} = 0 \right)$$

Interpretation of Borel singularities

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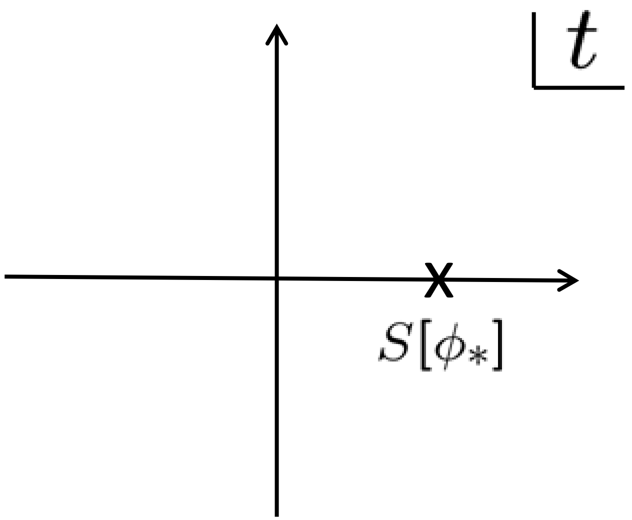
$$\Rightarrow \quad \mathcal{B}Z(t) \simeq \sum_\ell (S[\phi_*])^{-\ell} = \frac{1}{1-\frac{t}{S[\phi_*]}}$$



Nontrivial saddle point gives
Borel singularities

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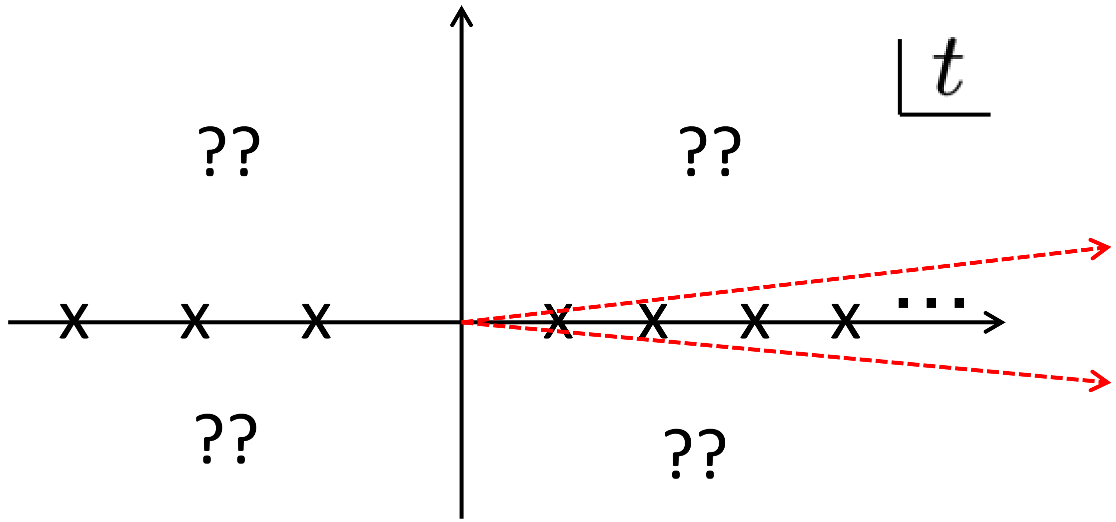
Basic idea of Resurgence

Suppose perturbation around trivial saddle is non-Borel summable:



$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \Rightarrow \text{(Residue)} \sim e^{-\frac{\sharp}{g}}$$

Ambiguity!

But this ambiguity is cancelled by ambiguities of perturbations around different saddles.

Toy model

[Cherman-Dorigoni-Unsal ’14, Cherman-Koroteev-Unsal ’14]

0d Sine-Gordon model:

$$Z(g) = \frac{1}{\sqrt{g}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \; e^{-\frac{1}{2g}\sin^2 x} = \frac{\pi}{\sqrt{g}} e^{-\frac{1}{4g}} I_0\left(\frac{1}{4g}\right)$$

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Saddle point:

$$0 = \frac{d}{dx}\sin^2 x\Big|_{x=x_*} = \sin(2x_*) \quad \Rightarrow \quad x_* = 0,\ \pm\frac{\pi}{2}$$

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$$0 = \frac{d}{dx}\sin^2 x\Big|_{x=x_*} = \sin(2x_*) \Rightarrow x_* = 0, \; \pm\frac{\pi}{2}$$

"Action":

$$\left(S(x) = \frac{1}{2g}\sin^2 x\right)$$

$$S(x=0) = 0$$ trivial

$$S\left(x = \pm\frac{\pi}{2}\right) = \frac{1}{2g}$$ Non-perturbative

Expansion around the saddle pts:

$$Z(g) \sim \underbrace{\sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell}_{x_* = 0} + \underbrace{e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell}_{x_* = \pm\frac{\pi}{2}} \;??$$

Expansion around the saddle pts:

$$Z(g) \sim \underbrace{\sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell}_{x_* = 0} + e^{-\frac{1}{2g}} \underbrace{\sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell \;??}_{x_* = \pm\frac{\pi}{2}}$$

Trivial saddle:

$$Z(g)|_{x_*=0} = \sqrt{2\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1/2)^2 2^\ell}{\Gamma(\ell+1)\Gamma(1/2)^2} g^\ell \equiv \Phi_0(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_0(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(0)}}{\ell!} t^\ell = \sqrt{2\pi}\; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t\right)$$

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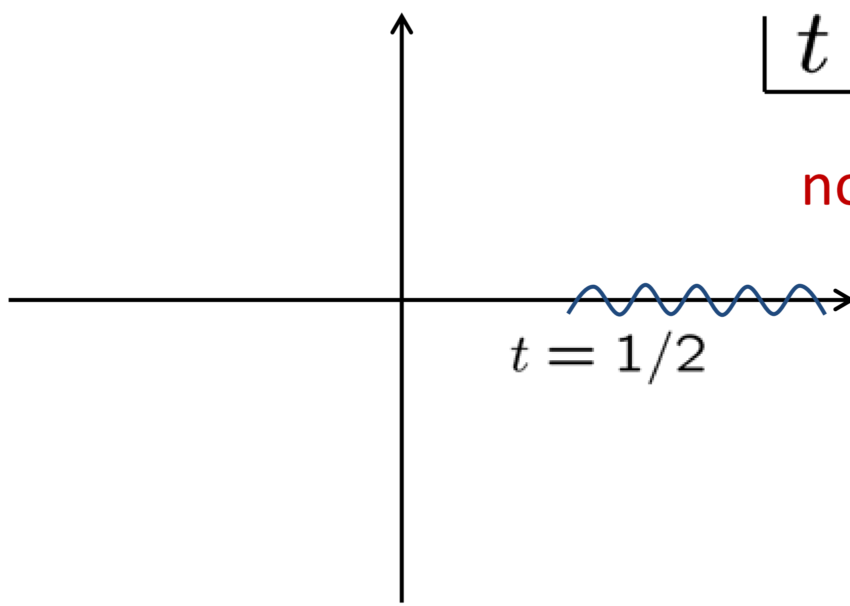
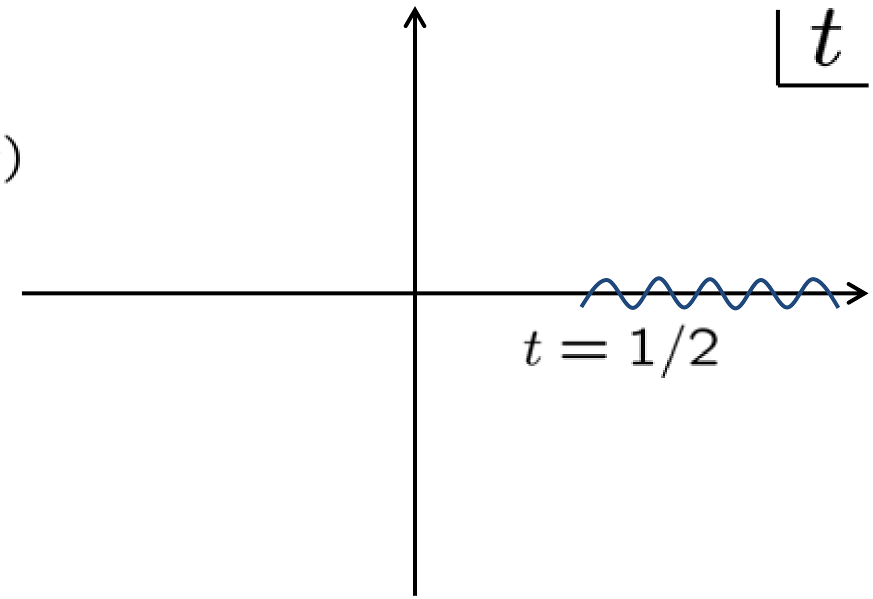


non-Borel summable!

$$\begin{cases} S_\theta \Phi_0(g) = \dfrac{1}{g} \displaystyle\int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\Phi_0(t) \\ \mathcal{B}\Phi_0(t) = \sqrt{2\pi}\ {}_2F_1\left(\dfrac{1}{2}, \dfrac{1}{2}, 1; 2t\right) \\ g = |g| e^{i\theta} \end{cases}$$



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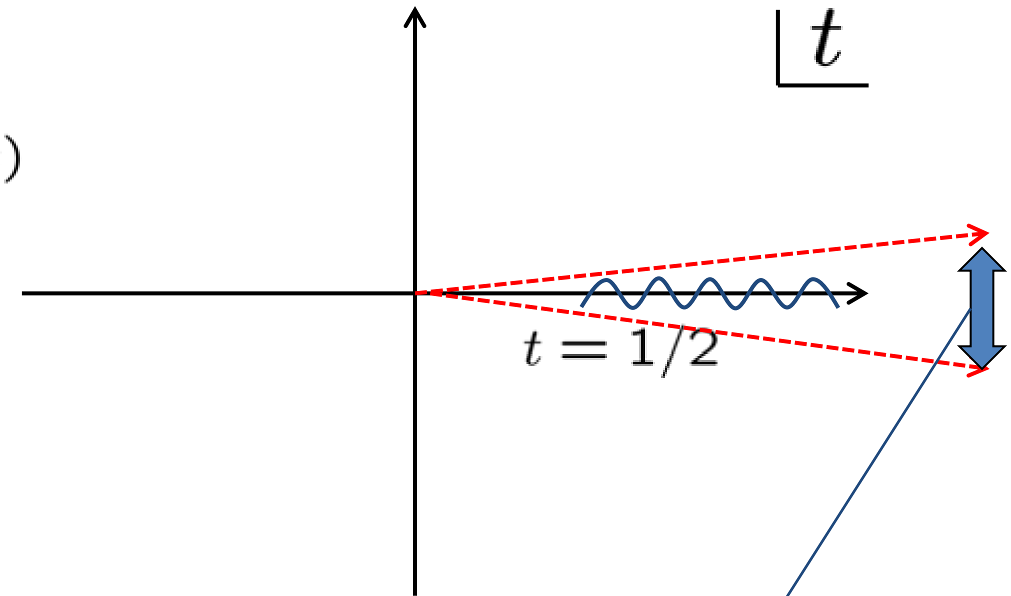
\end{cases}
$$



Ambiguity:

$$
\left(S_{0+} - S_{0-}\right)\Phi_0(g) = e^{-\frac{1}{2g}} \times \frac{2i\sqrt{2\pi}}{g}\int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right) \neq 0
$$

Related to contribution from $x_* = \pm\pi/2$?

Expansion around nontrivial saddle

$$\begin{cases} e^{-S(x)} = e^{-\frac{1}{2|g|}e^{-i\theta}x^2+\cdots} & x_* = 0 \qquad \left(g = |g|e^{i\theta}\right) \\ e^{-S(x)} = e^{-\frac{1}{2g}} \times e^{\frac{1}{2|g|}e^{-i\theta}(x-\pm\frac{\pi}{2})^2+\cdots} & x_* = \pm\frac{\pi}{2} \end{cases}$$

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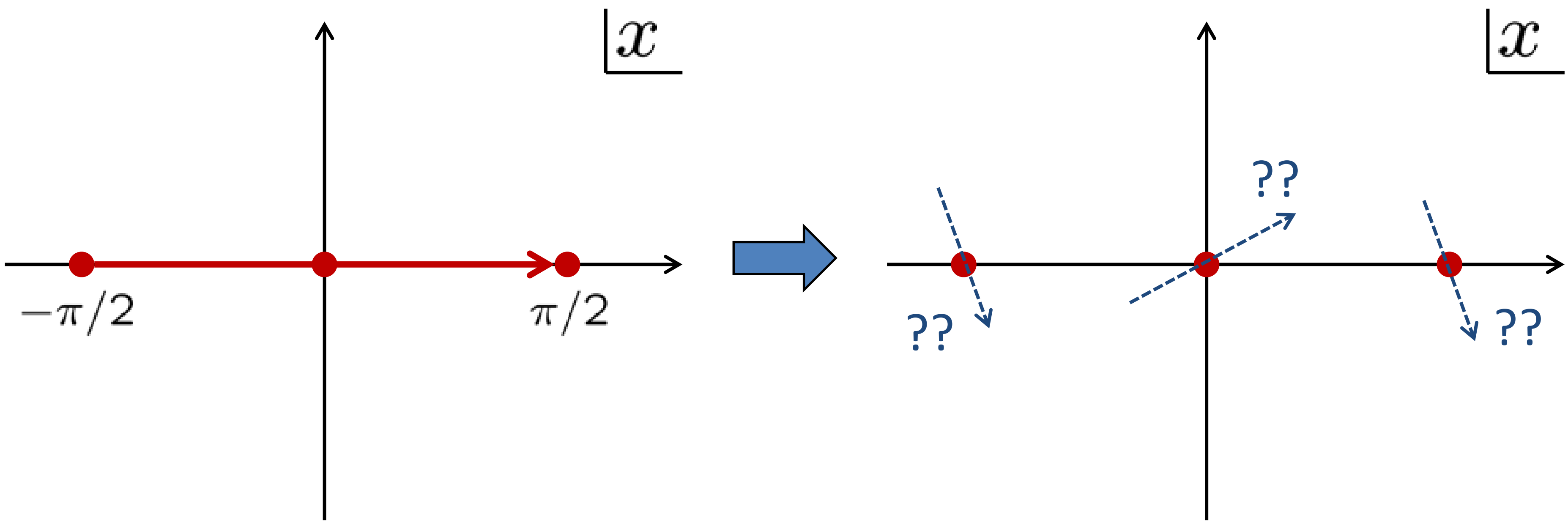
To pick up saddles, change the integral contour to steepest descent s.t.

1. passes the saddles w/ appropriate angle
2. Keep Im(S(x)) to avoid oscillation
3. Keep final result



Appropriate contour = Lefschetz thimble

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., $\exists$ unique Lefschetz thimble J_I :

$$\frac{dz(t)}{dt} = \overline{\frac{\partial S(z)}{\partial z}}, \quad \text{with } z(t \to -\infty) = z_I$$

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Properties:

$a)$ $\mathrm{Im}S(z)|_{J_I}=\mathrm{Im}S(z_I)$ $\left(\frac{d}{dt}\mathrm{Im}S \propto \frac{d}{dt}(S-\bar{S})=\frac{dz}{dt}\frac{\partial S}{\partial z}-\frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}}=0\right)$

$b)$ $\mathrm{Re}S(z)|_{J_I}\geq \mathrm{Re}S(z_I)$ $\left(\frac{d}{dt}\mathrm{Re}S \propto \frac{dz}{dt}\frac{\partial S}{\partial z}+\frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}}=2\frac{\partial S}{\partial z}\overline{\frac{\partial S}{\partial z}}\geq 0\right)$

$c)$ Decomposition of cycle:

$$C=\sum_{I\in \text{saddle}} n_I J_I \qquad (n_I\in \mathbf{Z})$$

may jump as changing parameters

Dual thimble = steepest ascent

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., $\exists$ unique dual thimble K_I :

$$\frac{dz(t)}{dt} = -\overline{\frac{\partial S(z)}{\partial z}}, \text{ with } z(t \to -\infty) = z_I$$

Properties:

a) $\mathrm{Im}S(z)|_{K_I} = \mathrm{Im}S(z_I)$

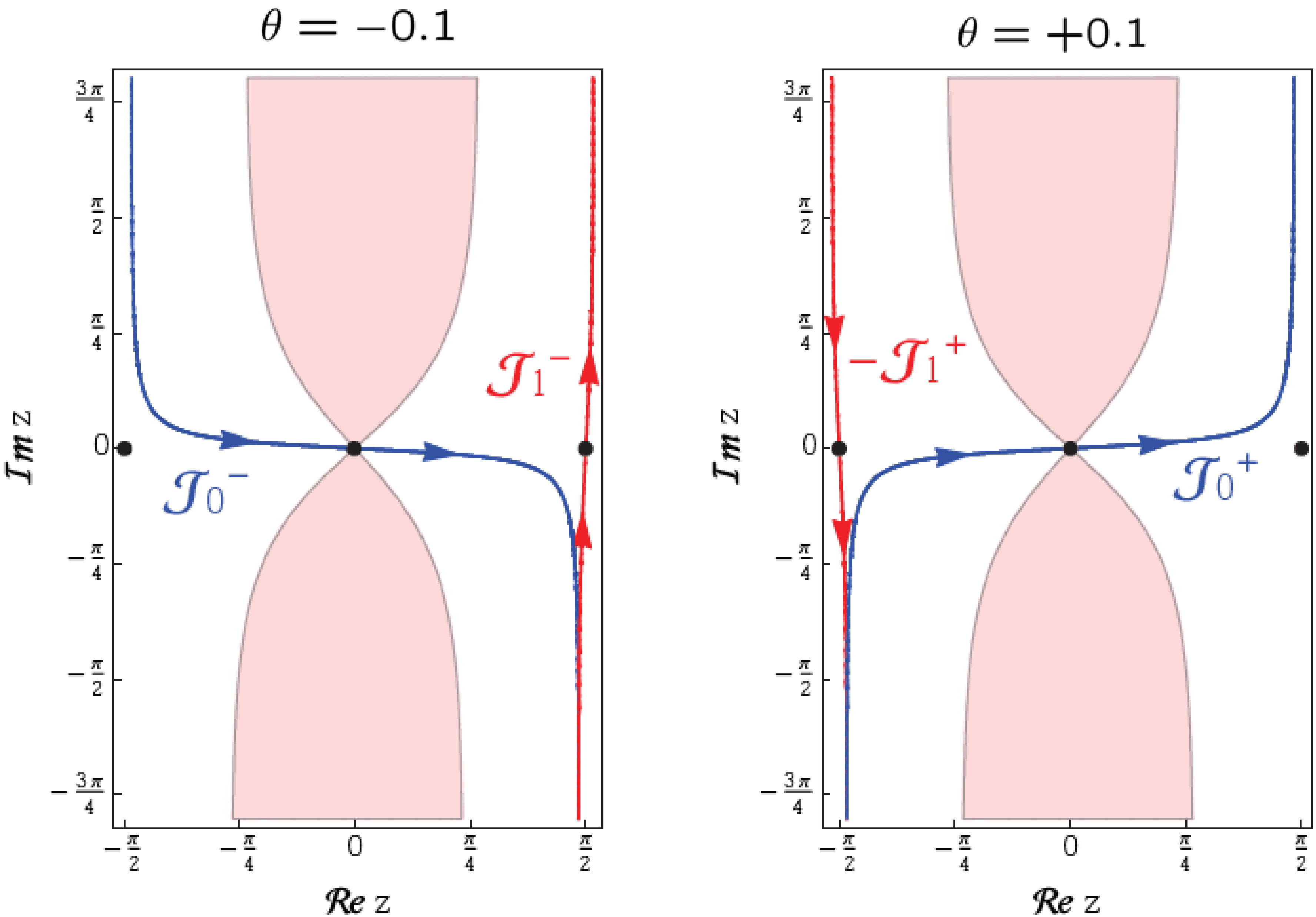
b) $\mathrm{Re}S(z)|_{K_I} \leq \mathrm{Re}S(z_I)$

c) Decomposition of cycle:

$$C = \sum_{I \in \text{saddle}} n_I J_I, \qquad n_I = \text{intersection } \sharp \text{ of } (C, J_I)$$

Back to the toy model

[Fig.1 in Cherman-Dorigoni-Unsal '14]



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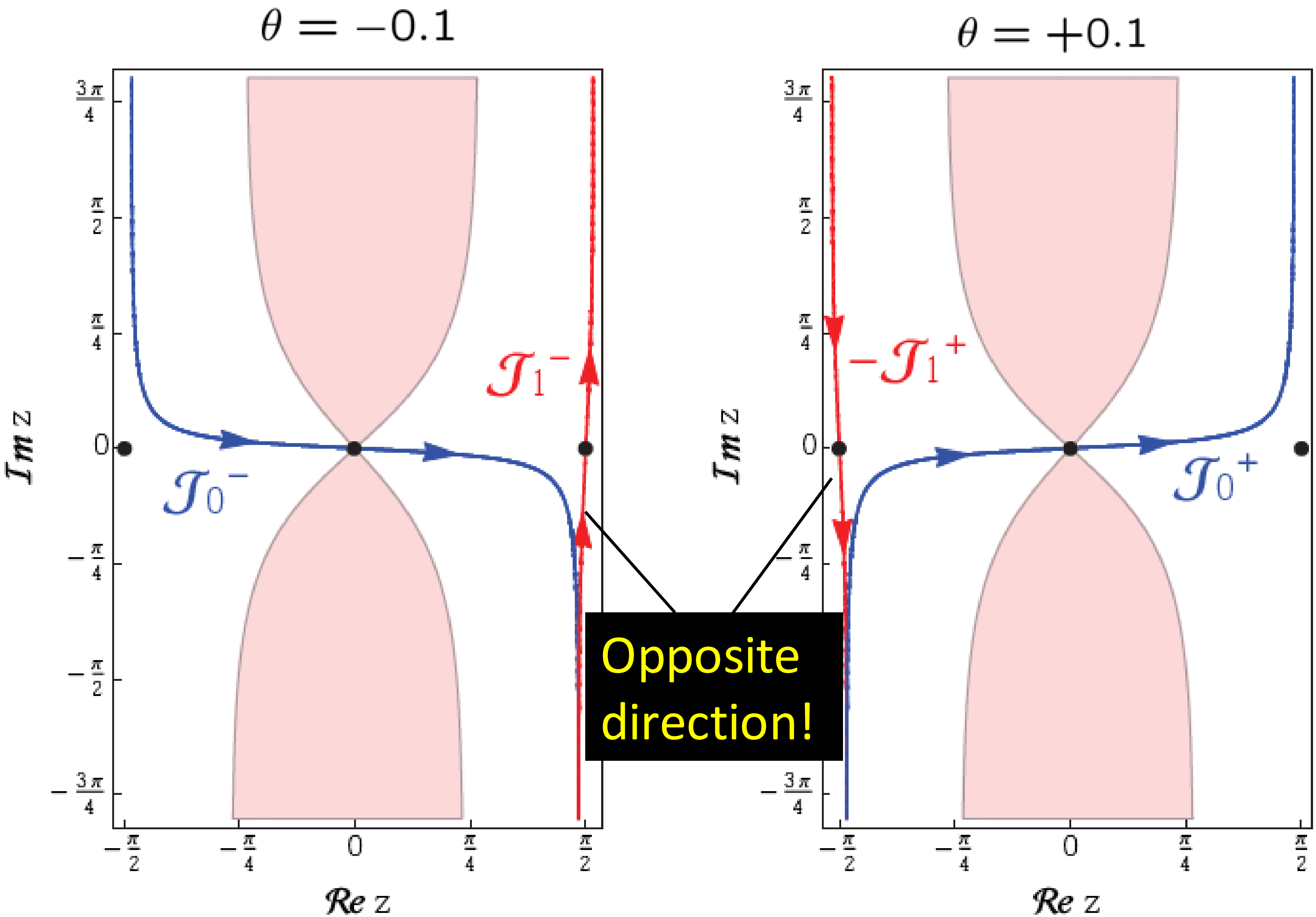
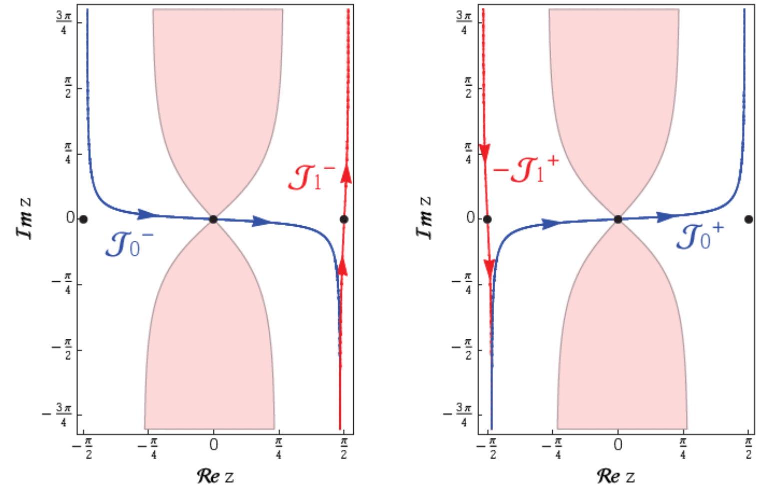


Contribution from nontrivial saddle

- Either x=+π/2 or -π/2 contributes
- Contours smoothly change
  in the ranges 0<θ<π and -π<θ<0
- Contours through nontrivial saddles
  are opposite between θ<0 & θ>0



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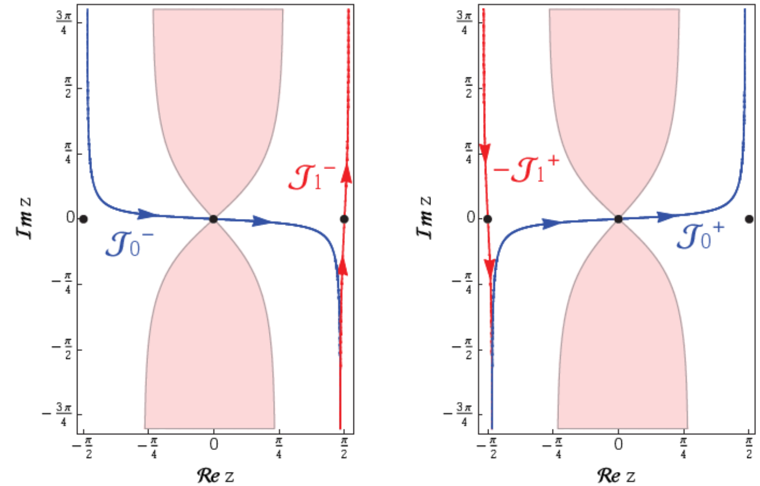

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$$Z(g)|_{x_*=\pm\frac{\pi}{2}} = \begin{cases} +e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta < 0) \\ -e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta > 0) \end{cases}$$

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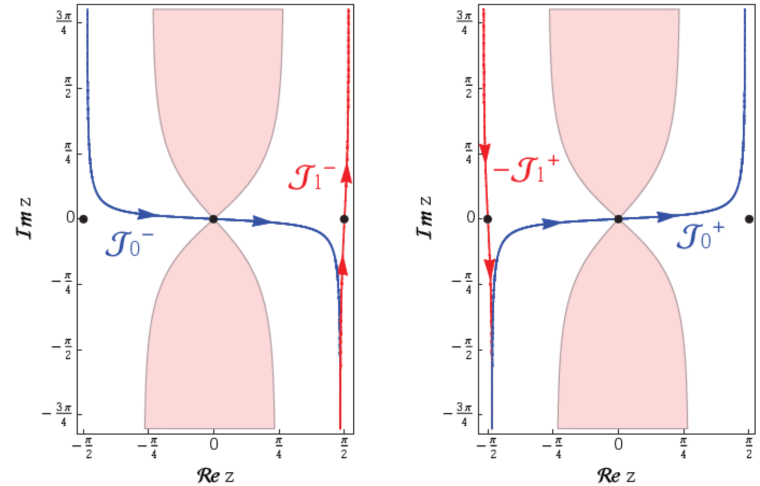

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∃ Jump at θ=0!! ("Stokes phenomenon")

Expansion around nontrivial saddle is also ambiguous at θ=0

Expansion around nontrivial saddle

$$\pm e^{-\frac{1}{2g}}\sqrt{2\pi}\sum_{\ell=0}^{\infty}\frac{(-2)^{\ell}\Gamma(\ell+1/2)^2}{\Gamma(\ell+1)\Gamma(1/2)^2}g^{\ell}\equiv\pm e^{-\frac{1}{2g}}\Phi_1(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_1(t)=\sum_{\ell=0}^{\infty}\frac{c_\ell^{(1)}}{\ell!}t^{\ell}=\sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

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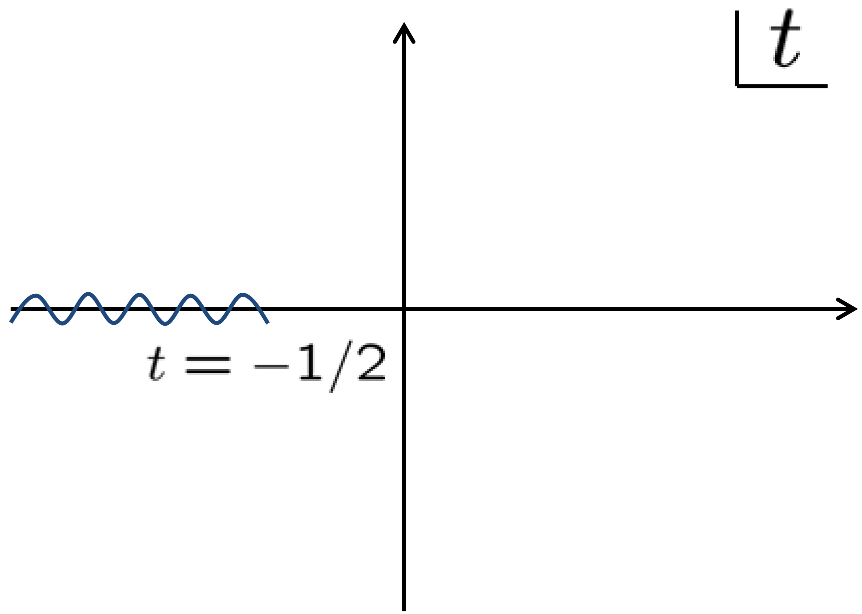
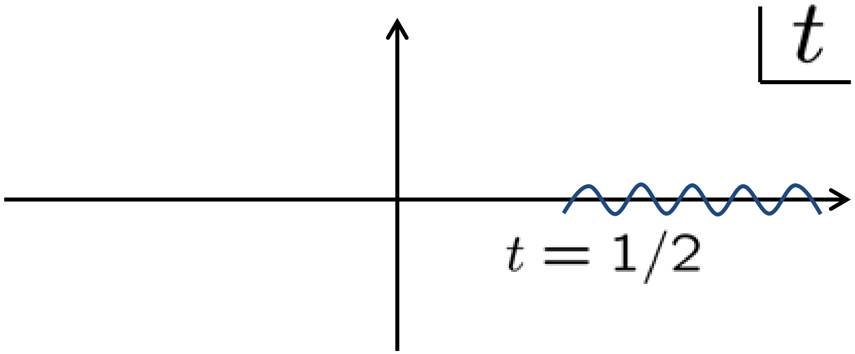


Borel trans. itself is OK but ∃ ambiguity at θ=0 because of Stokes phenomena

Comparison of ambiguities (at θ=0)

Trivial saddle



By the branch cut, ambiguity:

$$\left(S_{0+} - S_{0-}\right) \Phi_0(g)$$
$$= e^{-\frac{1}{2g}} \frac{2i\sqrt{2\pi}}{g} \int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$

Nontrivial saddle

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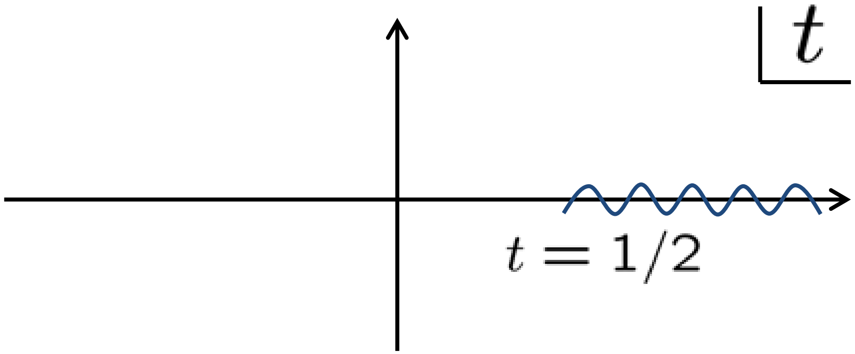

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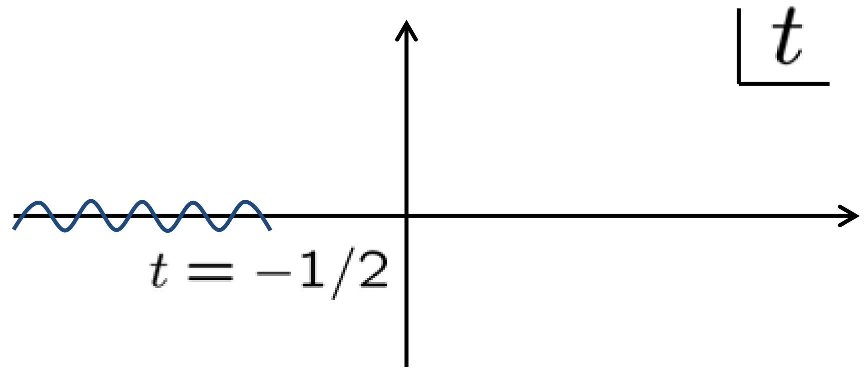
$$= e^{-\frac{1}{2g}} \frac{2i\sqrt{2\pi}}{g} \int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$

Nontrivial saddle

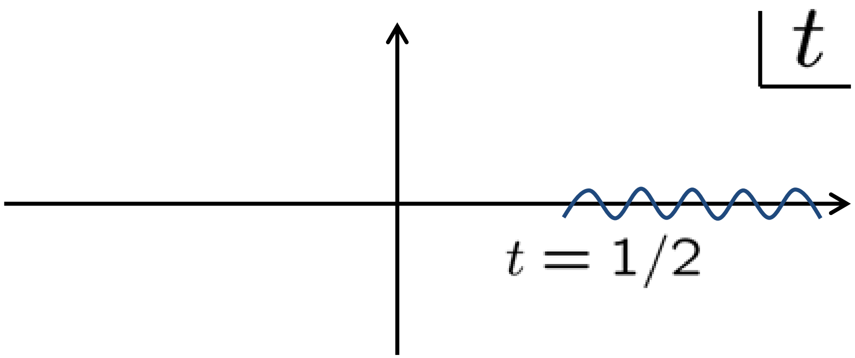


By the Stokes phenomena,

$$Z(g)|_{x_* = \pm\frac{\pi}{2}} = \begin{cases} +ie^{-\frac{1}{2g}} S_\theta \Phi_1(g) & (\theta < 0) \\ -ie^{-\frac{1}{2g}} S_\theta \Phi_1(g) & (\theta > 0) \end{cases}$$

Comparison of ambiguities (at θ=0)

Trivial saddle



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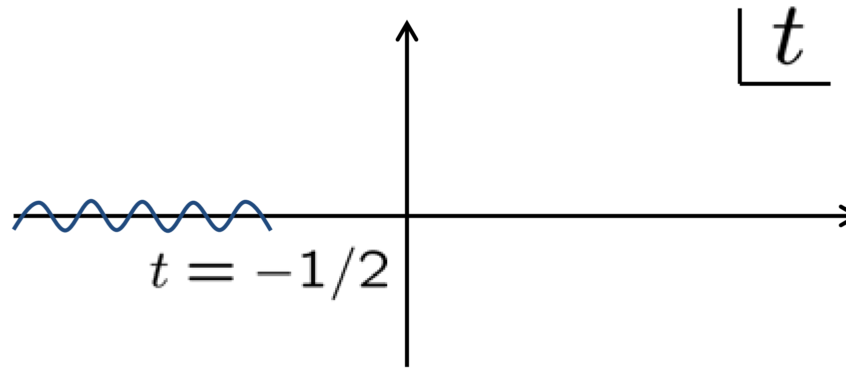

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<u>Ambiguity:</u>

$$-2ie^{-\frac{1}{2g}} S_0 \Phi_1(g) = -\frac{2i\sqrt{2\pi}}{g} e^{-\frac{1}{2g}} \int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -2t\right)$$
$$= -\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$

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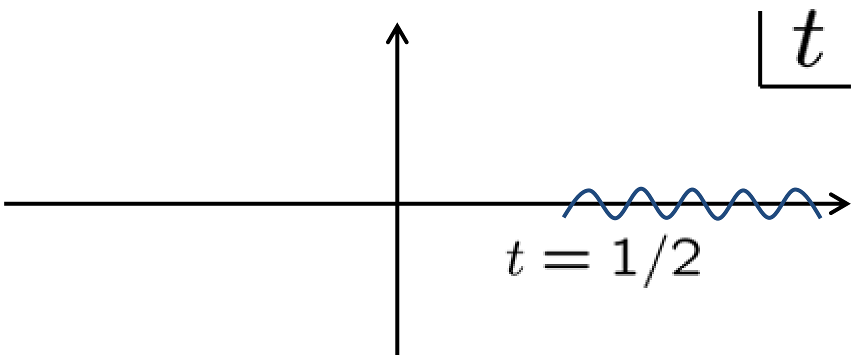

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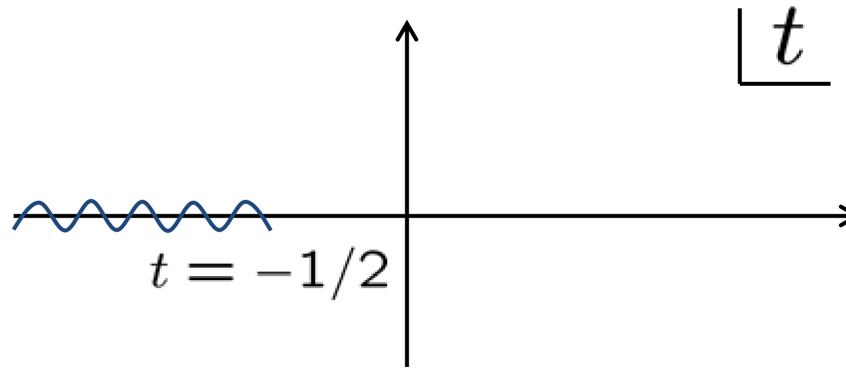

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$$\boxed{= -\left(S_{0+} - S_{0-}\right)\Phi_0(g)}$$

Resurgence

(Ambiguity from trivial saddle point)

||

−(Ambiguity from nontrivial saddle point)

Resummation from a saddle point may be ambiguous but the ambiguity is cancelled by other saddles

In the toy model, resurgence gives the exact result:

$$Z(g \in \mathbf{R}_{\geq 0}) = \lim_{\theta \to 0_{\pm}} \left[S_\theta \Phi_0(g) \mp i e^{-\frac{1}{2g}} S_\theta \Phi_1(g) \right] = \mathrm{Re} S_0 \Phi_0(g)$$

Samples of successful examples

Quantum mechanics

- Quartic/Periodic potential [Zinn Justin-Jentschura '04]
- CP^N [Fujimori-Kamata-Misumi-Nitta-Sakai '17]
- Slightly broken SUSY [Dunne-Unsal , Kozcaz-Sulejmanpasic-Tanizaki-Unsal '16]

2d QFT

- CP^N /O(N) sigma model [Dunne-Unsal '12, Misumi-Nitta-Sakai, etc..] [Dunne-Unsal '15]
- Principal chiral model [Cherman-Dorigoni-Unsal '15]
- Pure Yang-Mills [Ahmed-Dunne '17, Okuyama-Sakai '18]

3d QFT

- Pure Chern-Simons [Gukov-Marino-Putrov '16]
- N=2 SUSY Chern-Simons matter theories

[M.H. '16, Gukov-Pei-Putrov-Vafa '17, Fujimori-M.H.-Kamata-Misumi-Sakai '18]

Remark: more than weak coupling expansion in QFT

We could apply resurgence to other types of expansions.

For example,

- 1/N expansion
- strong coupling expansion
- Weak coupling expansion in gravity (string)
- high/low temperature expansion
- ε-expansion
- Derivative expansion in effective theory

etc...

Contents

1. Expectations on weak coupling perturbative series in QFT

2. What is resurgence?

3. Application to a 3d SUSY CS theory

[The simplest example in Fujimori-M.H.-Kamata-Misumi-Sakai ’18]

4. Summary & Outlook

Let us consider

$\mathcal{N}=3\ U(1)_k$ CS theory w/ charge-1 hyper & real mass m

Let us consider

$\mathcal{N}=3\ U(1)_k$ CS theory w/ charge-1 hyper & real mass m

Sphere partition function:

$$Z=\int_{-\infty}^{\infty} d\sigma\ \frac{e^{\frac{i}{g}\sigma^2}}{2\cosh\frac{\sigma-m}{2}}, \qquad g=\frac{4\pi}{k}$$

(σ: Coulomb branch parameter
=const. configuration of the adjoint scalar "A_4")

Now we are interested in small-g (large level) expansion of this object.

Borel trans. hidden in localization formula

$$(k > 0)$$

$$Z = \int_{-\infty}^{\infty} d\sigma \, \frac{e^{\frac{i}{g}\sigma^2}}{2\cosh\frac{\sigma - m}{2}} = \int_0^{\infty} d\sigma \, \sum_{\pm} \frac{e^{\frac{i}{g}\sigma^2}}{2\cosh\frac{\sigma \pm m}{2}}$$

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Changing the variable as $\sigma = \sqrt{it}$,

$$\begin{cases} Z(g) = \displaystyle\int_0^{-i\infty} dt \ e^{-\frac{t}{g}} \mathcal{B}Z(t) \\ \mathcal{B}Z(t) = \displaystyle\frac{i}{4\sqrt{it}} \sum_{\pm} \frac{1}{\cosh\frac{\sqrt{it} \pm m}{2}} \end{cases}$$

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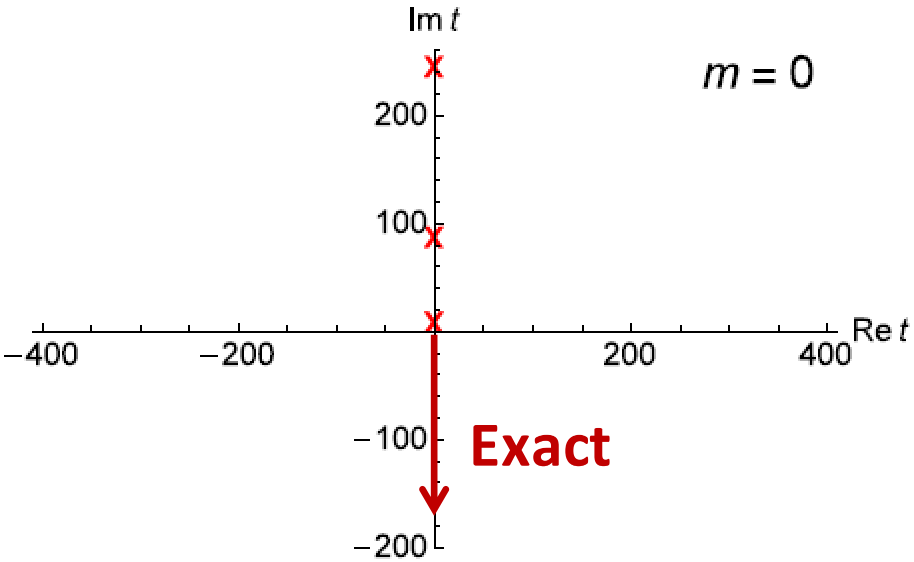
This is equivalent to the Borel resummation formula
w/ different integral contour! $S_\theta I(g) = \int_0^{e^{i\theta}\infty} dt \; e^{-\frac{t}{g}} \; \mathcal{B}I(t)$

[M.H. '16]

Analytic property of Borel trans.

$$t_{\text{pole}} = -i\,[m + (2n+1)\pi i]^2$$



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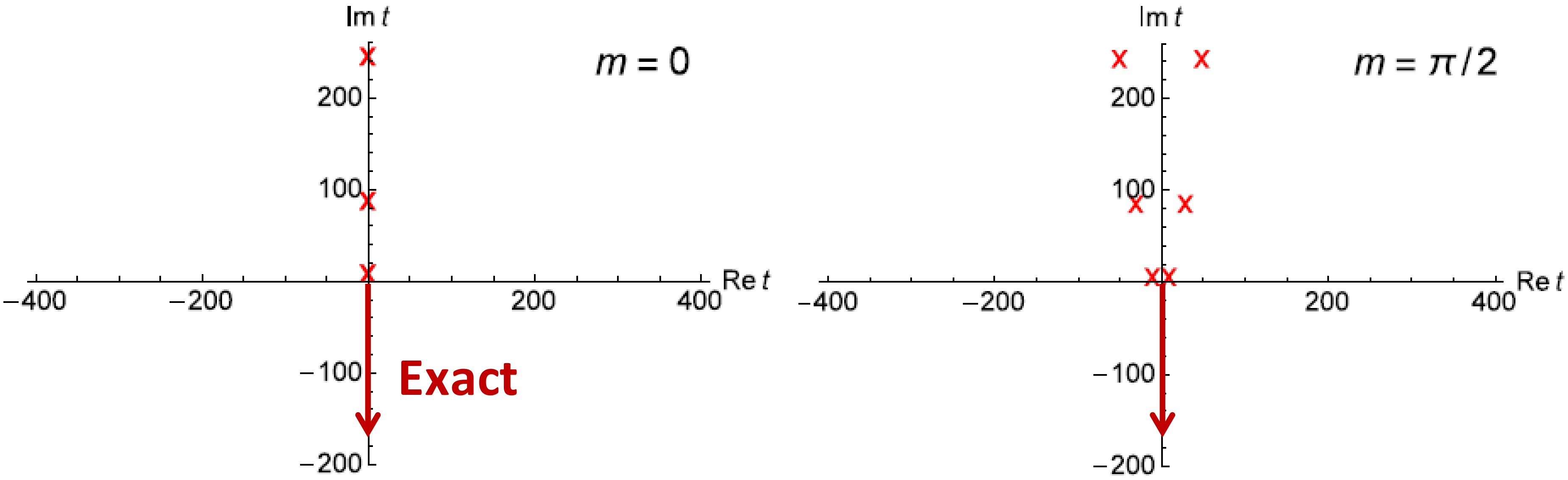

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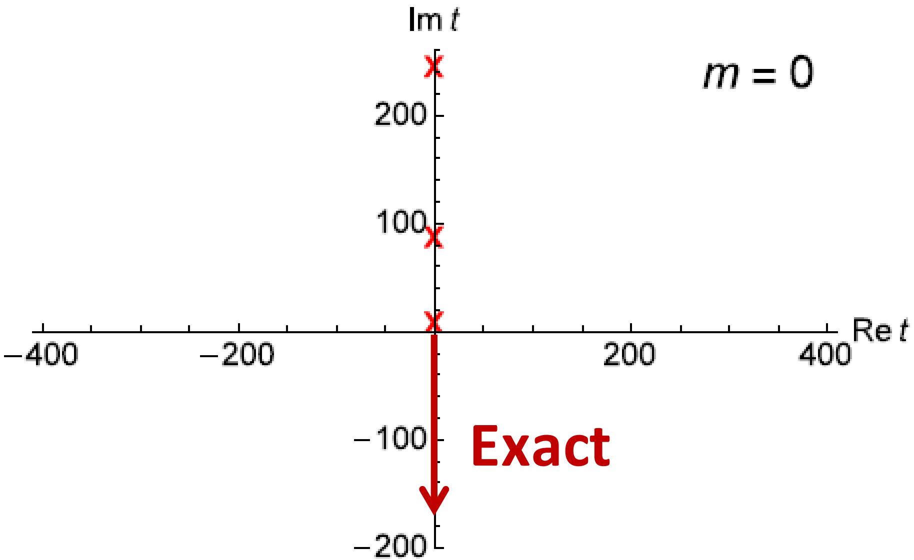

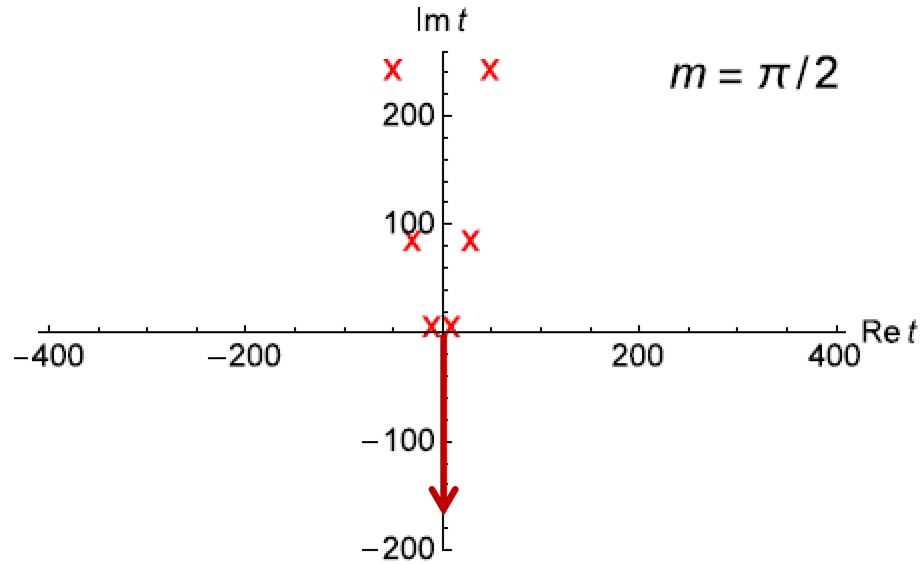

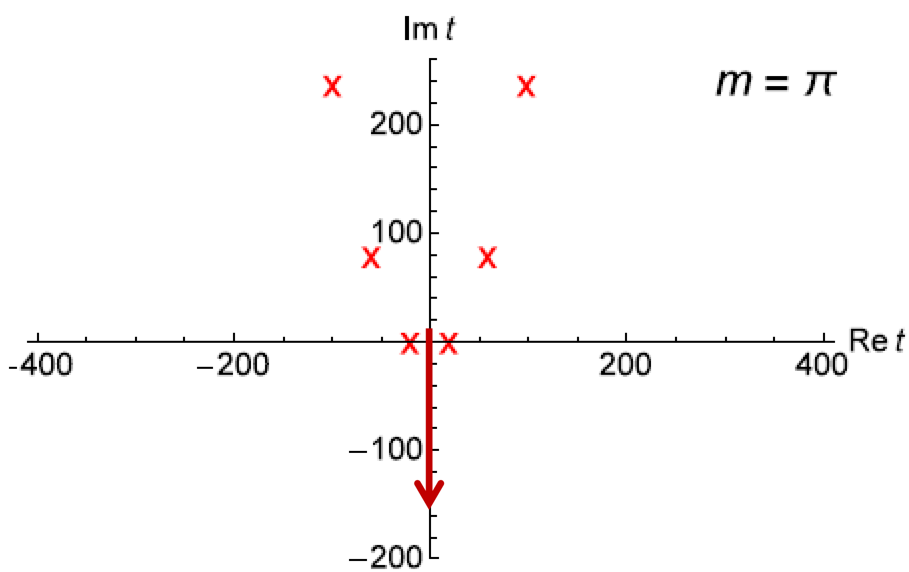

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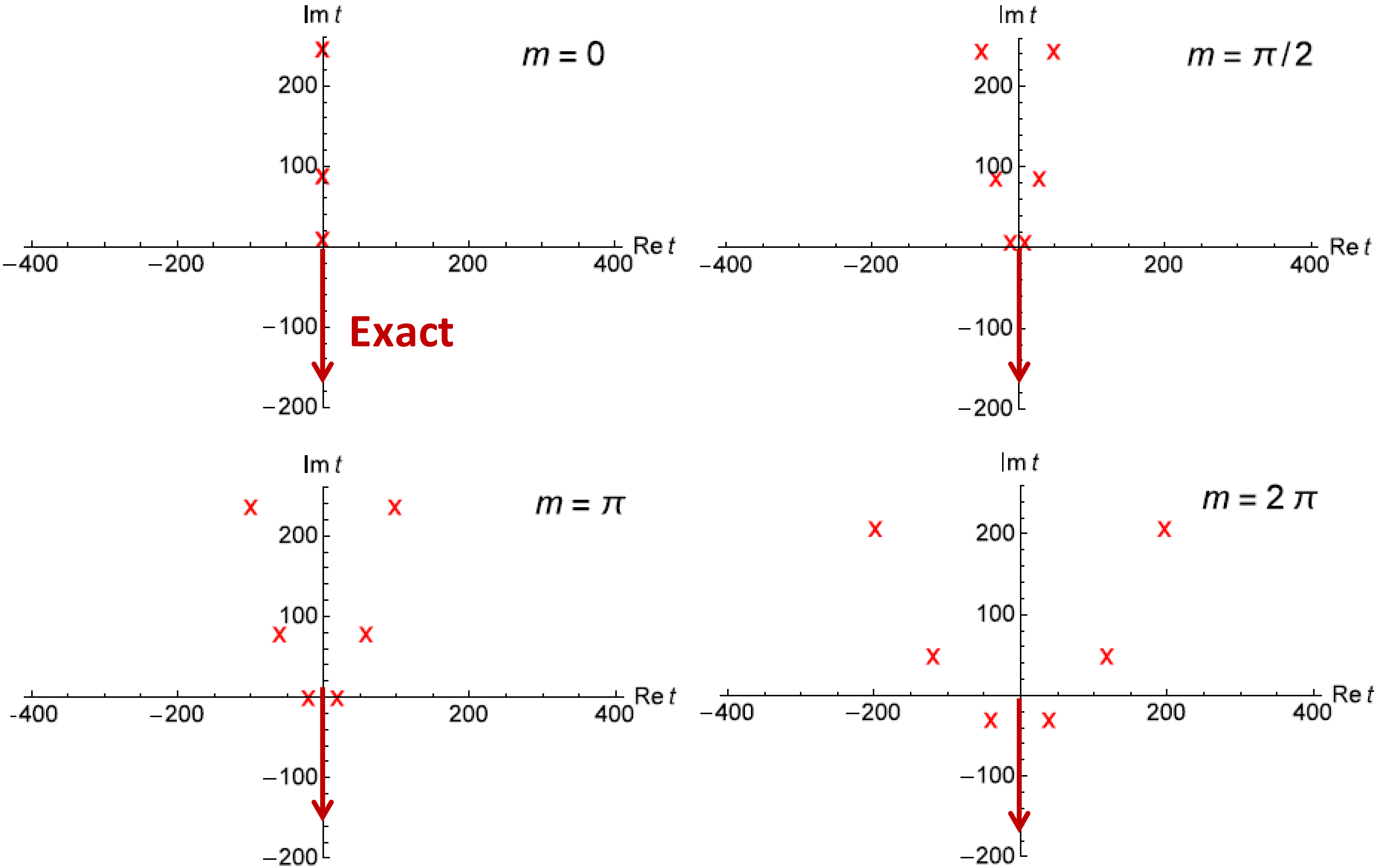

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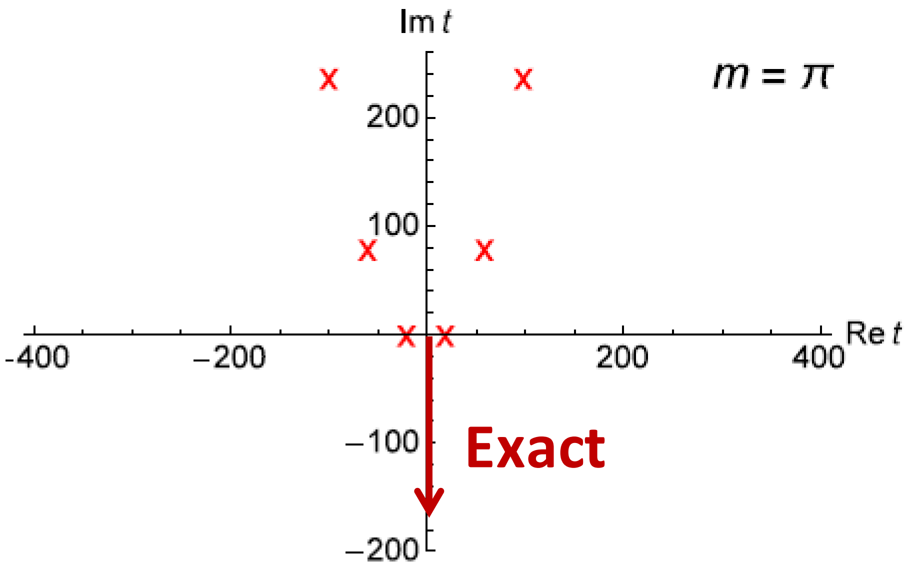

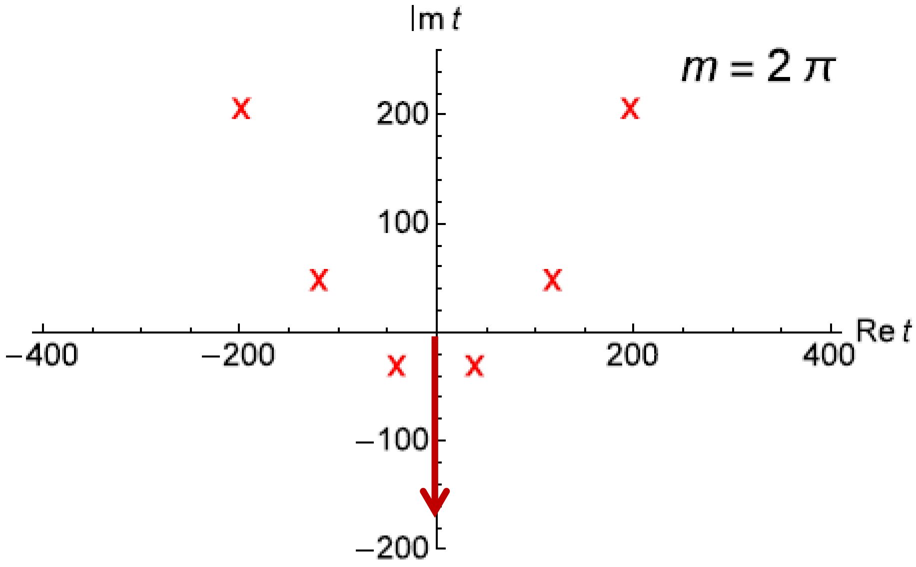

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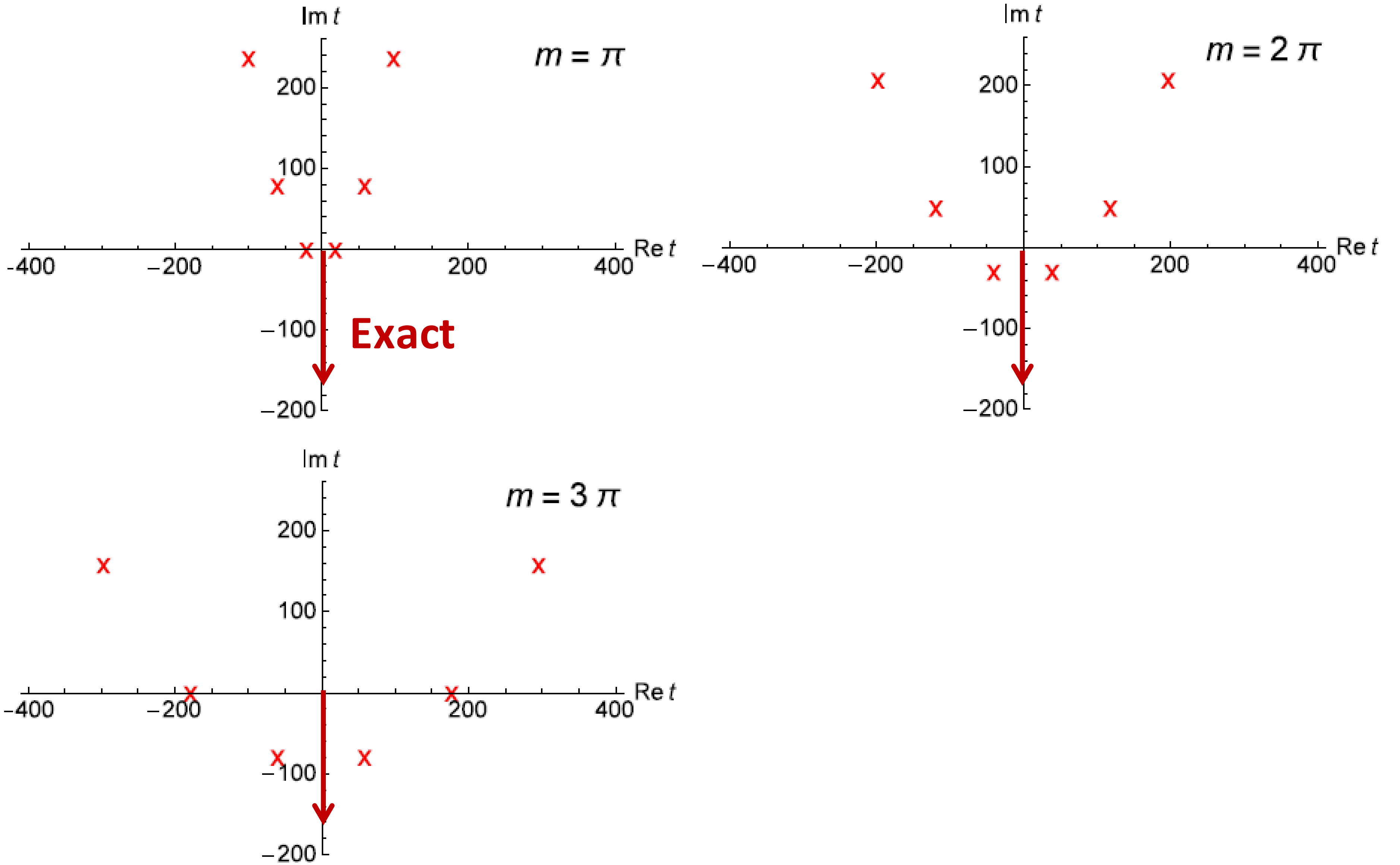

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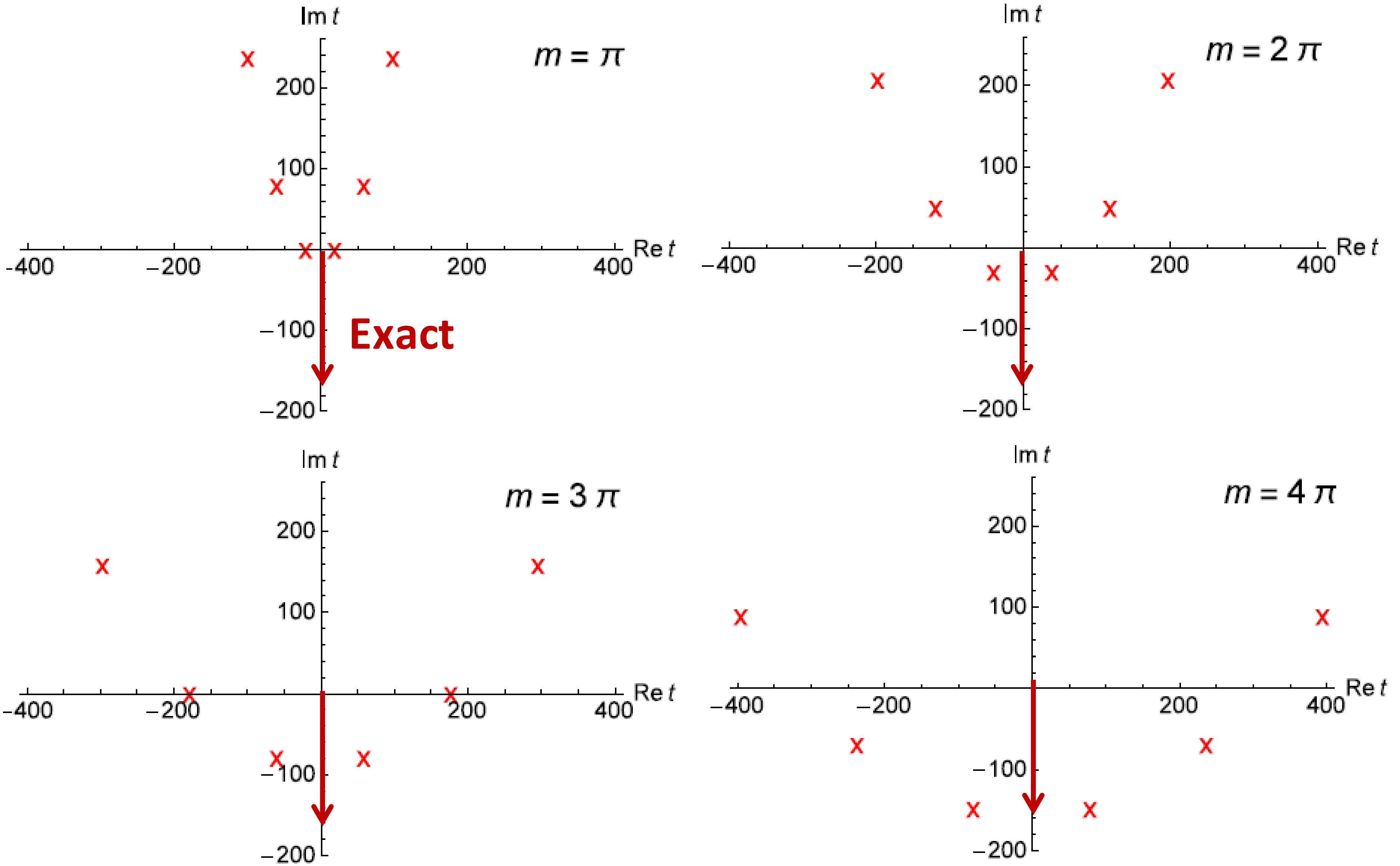

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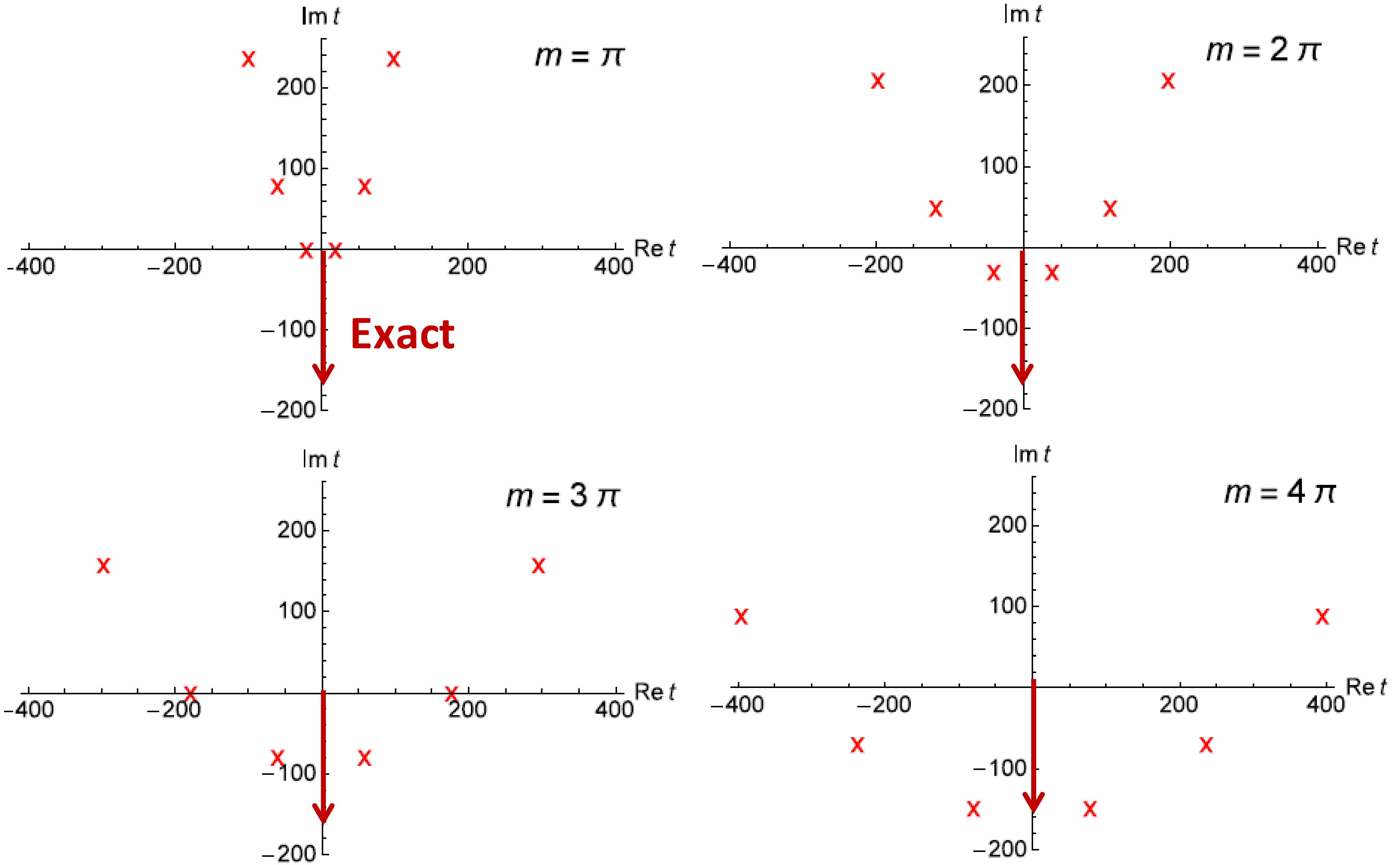
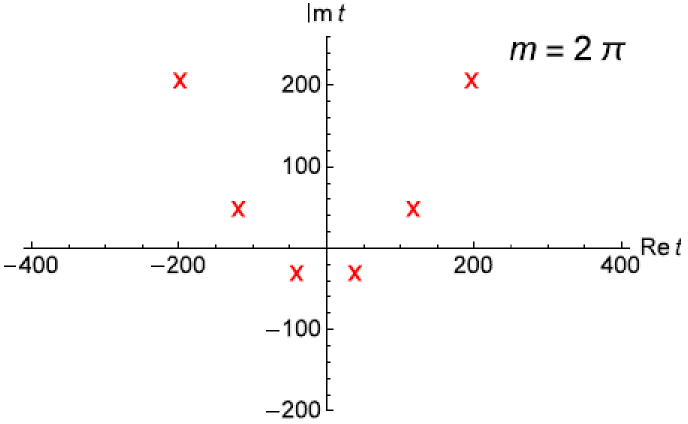


This is repeated infinitely many times...

Trans-series expression

In terms of Borel resum. along R+,

$$Z = \int_0^\infty dt\ e^{-\frac{t}{g}} \mathcal{B}Z(t) + \sum_{\text{poles}\in\text{4th quadrant}} \mathrm{Res}_{t=t_{\text{pole}}} \left[e^{-\frac{t}{g}} \mathcal{B}Z(t) \right]$$



Trans-series expression

In terms of Borel resum. along R$^+$,

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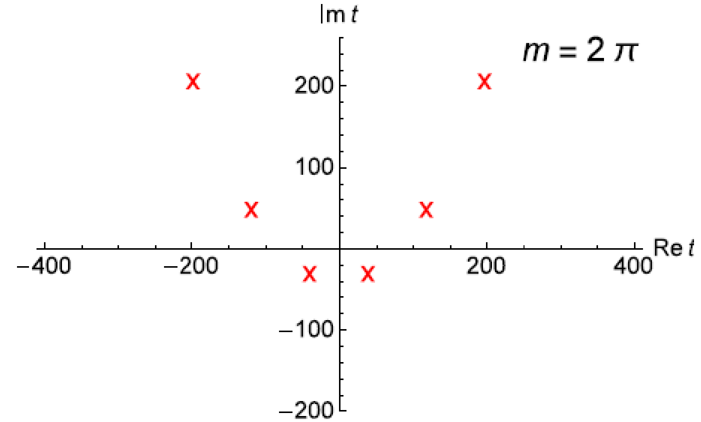

Decompose this into "perturbative part" & "non-pert. part":

$$Z = Z_{\text{pt}} + \sum_{n=1}^{\infty} Z_{\text{np}}^{(n)} ,$$

Perturbative part:

$$\boxed{Z_{\text{pt}} = \int_0^\infty dt\ e^{-\frac{t}{g}} \mathcal{B}Z(t),}$$

$$\left(\frac{\sqrt{ig}}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{\infty} \frac{E_{2(q+a)} \Gamma(q+1/2)}{2^{2(q+a)} \Gamma(2q+1) \Gamma(2a+1)} m^{2a} (ig)^q \right)$$

Non-pert. part:

$$\boxed{Z_{\text{np}}^{(n)} = \theta\left(m - (2n-1)\pi\right) 2\pi (-1)^{n-1}\, e^{\frac{i}{g}[m+(2n-1)\pi i]^2}}$$

Trans-series expression (Cont'd)

$$Z = Z_{\mathrm{pt}} + \sum_{n=1}^{\infty} Z_{\mathrm{np}}^{(n)} \,,$$

$$Z_{\mathrm{pt}} = \int_0^{\infty} dt\ e^{-\frac{t}{g}} \mathcal{B}Z(t), \qquad Z_{\mathrm{np}}^{(n)} = \theta\left(m - (2n-1)\pi\right) 2\pi(-1)^{n-1}\, e^{\frac{i}{g}[m+(2n-1)\pi i]^2}$$

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For m=(2n-1)π, perturbative part is ambiguous:

$$Z_{\rm pt}(g, m = (2n-1)\pi + 0_+) - Z_{\rm pt}(g, m = (2n-1)\pi + 0_-)$$

$$= -\mathrm{Res}_{t=2(2n-1)^2\pi^2}\left[e^{-\frac{t}{g}}\mathcal{B}Z(t)\right]$$

Trans-series expression (Cont'd)

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Ambiguity in non-pert. part:

$$Z_{\rm np}^{(\ell)}(m=(2n-1)\pi + 0_+) - Z_{\rm np}^{(\ell)}(m=(2n-1)\pi + 0_-) = \begin{cases} 0 & \text{for } \ell \neq n \\ +\mathrm{Res}\left[e^{-\frac{t}{g}} \mathcal{B}Z(t)\right] & \text{for } \ell = n \end{cases} .$$

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$$Z = Z_{\rm pt} + \sum_{n=1}^{\infty} Z_{\rm np}^{(n)} ,$$

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Canceled! → Unambiguous answer

Lefschetz thimble analysis

$$Z = \int_{-\infty}^{\infty} d\sigma \; e^{-S[\sigma]}, \qquad S[\sigma] \, = \, -\frac{i}{g}\sigma^2 - \log \frac{1}{2\cosh \frac{\sigma - m}{2}}$$

Lefschetz thimble analysis

$$Z = \int_{-\infty}^{\infty} d\sigma \; e^{-S[\sigma]}, \qquad S[\sigma] \; = \; -\frac{i}{g}\sigma^2 - \log \frac{1}{2\cosh\frac{\sigma - m}{2}}$$

Critical point:

$$\left.\frac{\partial S[z]}{\partial z}\right|_{z=z^c} \; = \; -\frac{2i}{g} z^c + \frac{1}{2}\tanh\frac{z^c - m}{2} \; = \; 0\,.$$

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Flow equation:

$$\begin{cases} \left.\dfrac{dz}{ds}\right|_{\mathcal{J}_I} = \overline{\dfrac{\partial S[z]}{\partial z}} = +\dfrac{2i}{g}\bar{z} + \dfrac{1}{2}\tanh\dfrac{\bar{z} - m}{2}\,, \\ \lim\limits_{s\to-\infty} z(s) = z_I^c, \end{cases}$$

We solve these numerically but let us first understand weak coupling behavior analytically

Analytic argument for weak coupling

Critical point:

$$-\frac{2i}{g} z^c + \frac{1}{2} \tanh \frac{z^c - m}{2} = 0 \quad \xrightarrow{g \to 0} \quad z^c \cosh \frac{z^c - m}{2} = 0$$

$$\lim_{g \to 0} z^c(g, m) = 0, \ \underline{m + (2\ell + 1)\pi i}$$

pole of integrand

$$S[0] \simeq 0, \ S[m+(2\ell+1)\pi i] \simeq -\frac{i}{g}[m+(2\ell+1)\pi i]^2$$

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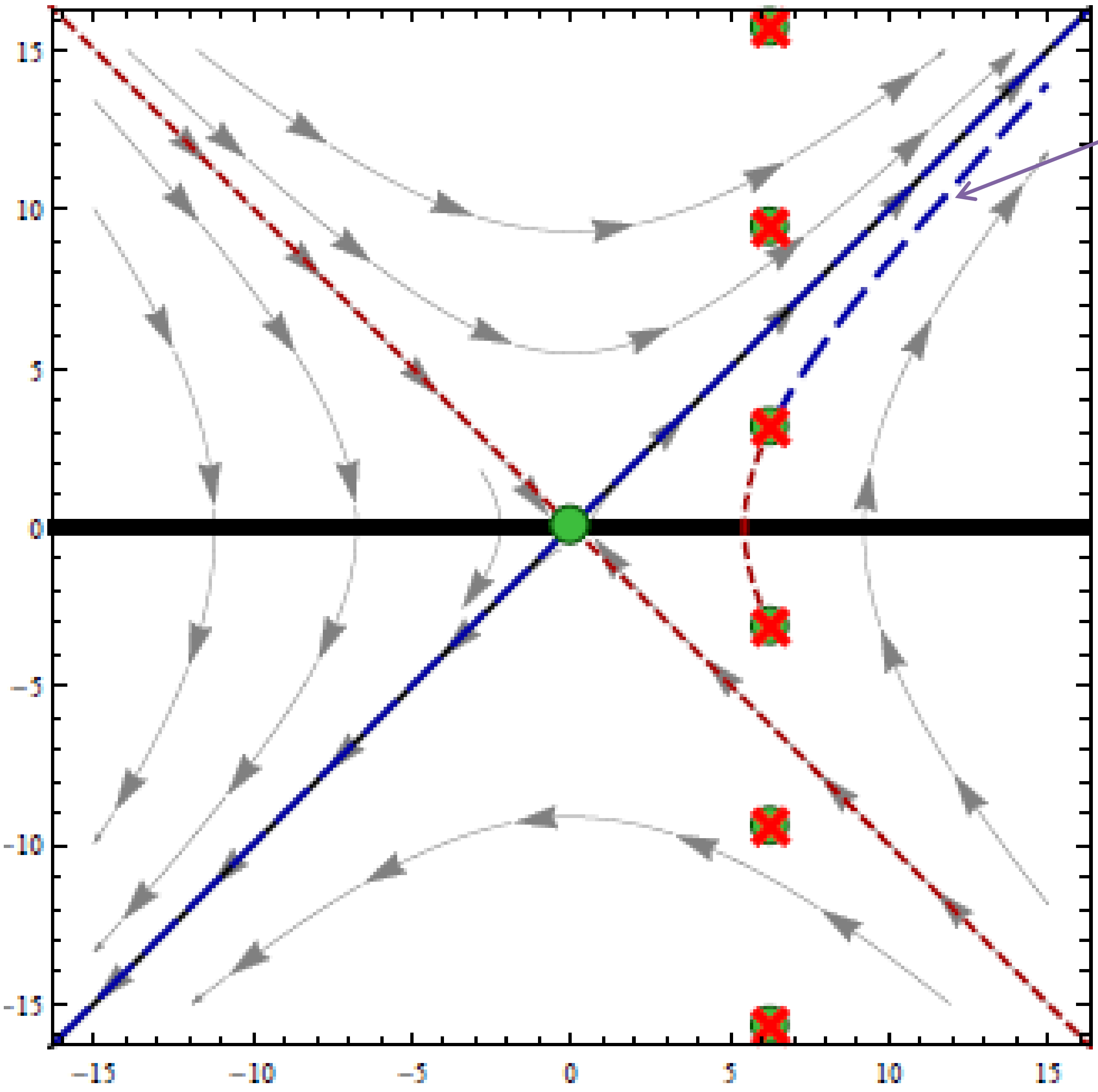
Lefschetz thimble associated w/ zc=0 :

$$\frac{dz}{ds} = +\frac{2i}{g}\bar{z} + \frac{1}{2}\tanh\frac{\bar{z} - m}{2} \quad \xrightarrow{g \to 0} \quad \frac{dz}{ds} = +\frac{2i}{g}\bar{z}$$

$$\lim_{g \to 0} z_{\mathrm{pt}}(g, m; s) = \epsilon \exp\left(\frac{2}{g}s + \frac{\pi i}{4}\right)$$ (∼contour for Fresnel integral)

Numerical result for g=0.1 & m=2π



Residue integral

● : critical pt.

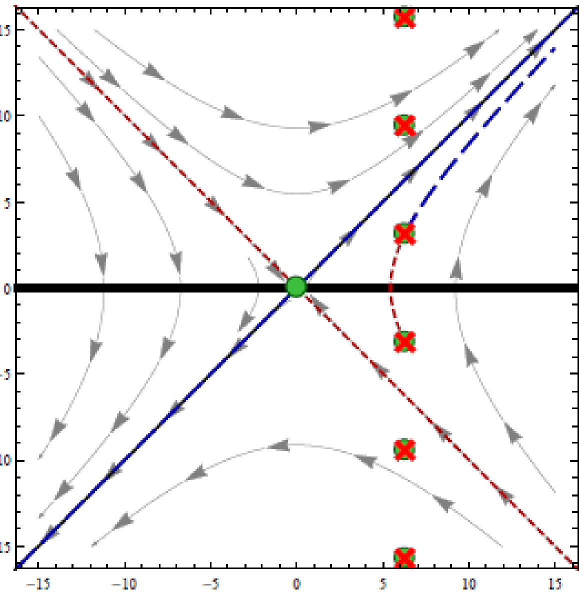
x : pole

line: steepest desc.

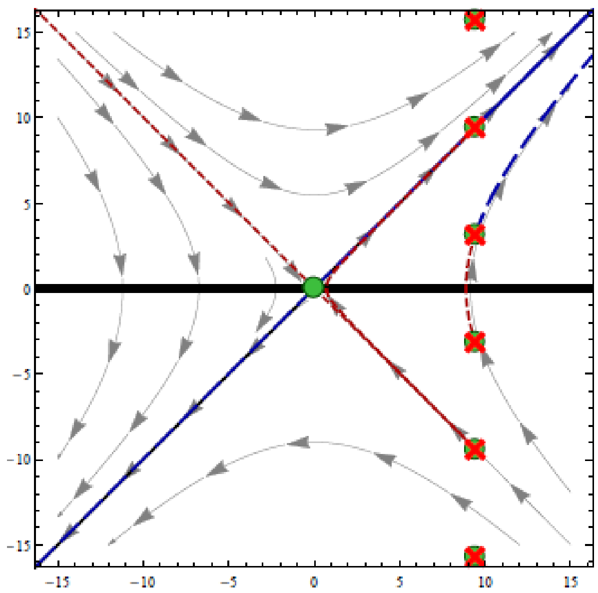
line: steepest asc.

Numerical result for g=0.1 & various m

m=2π



m=3π



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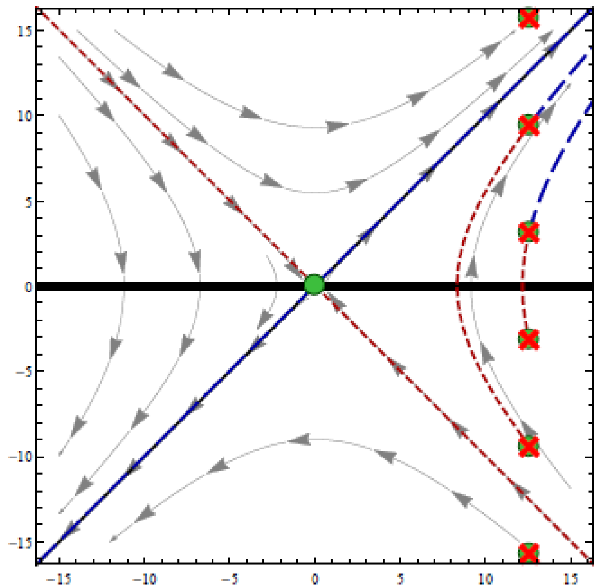

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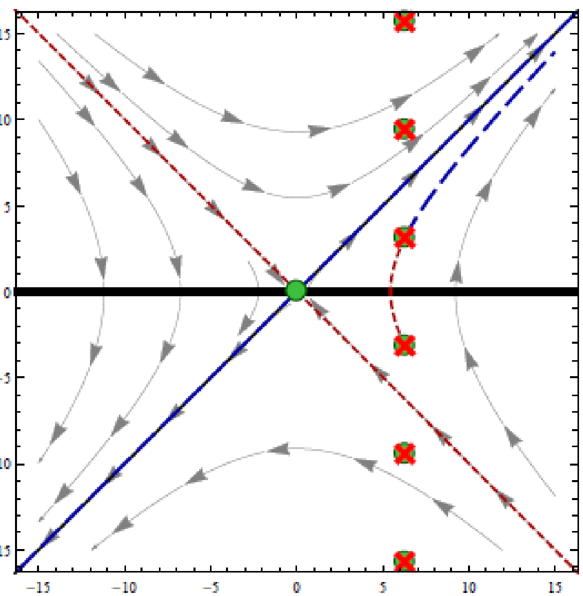
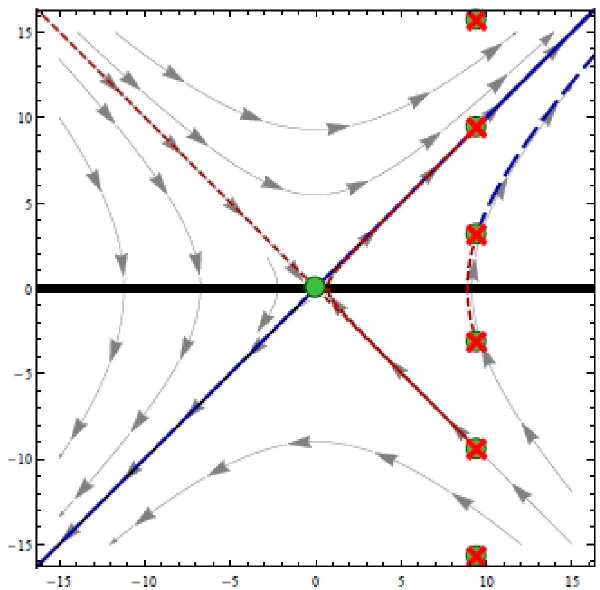


m=3π



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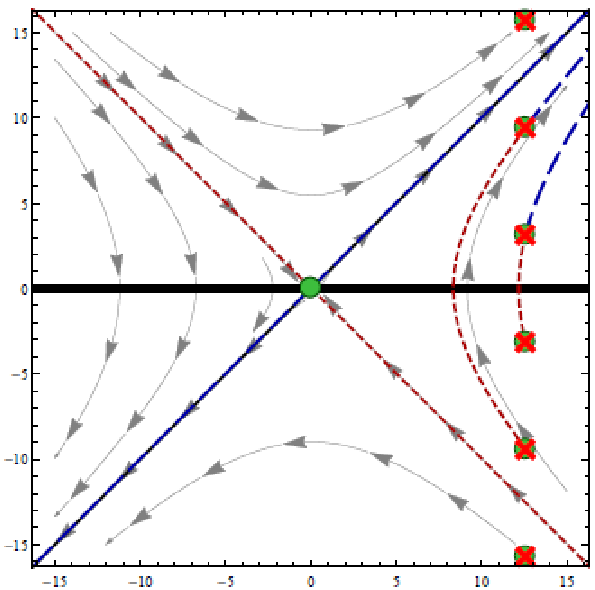
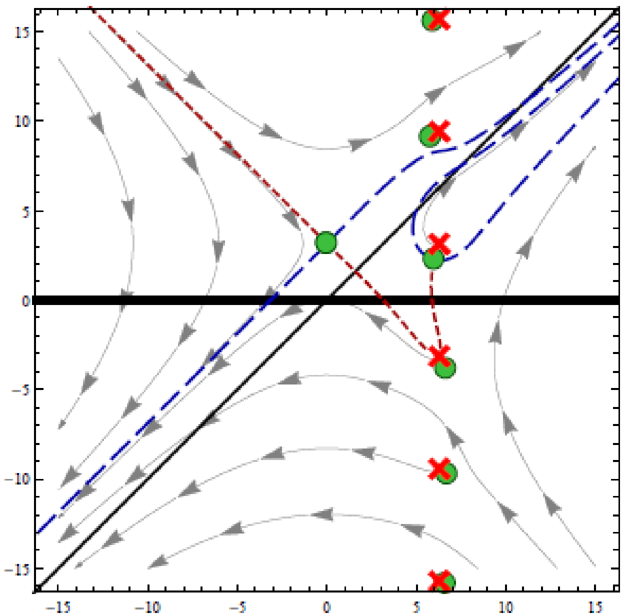


Stokes phenomena!

Numerical result for g=4π & various m

m=2π



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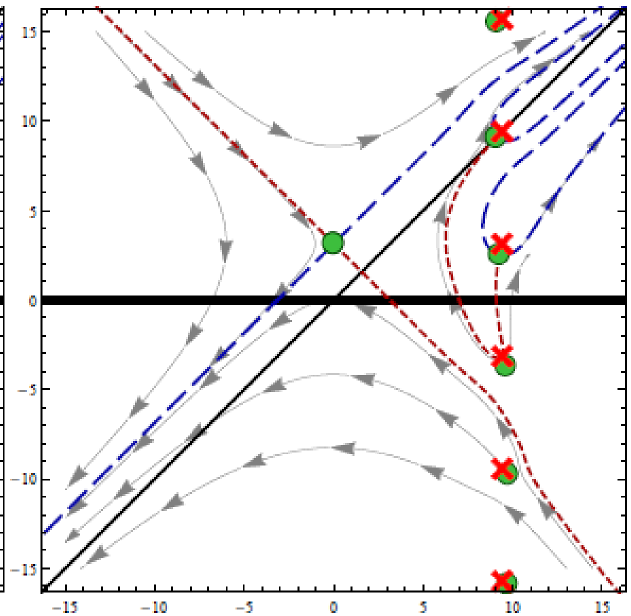
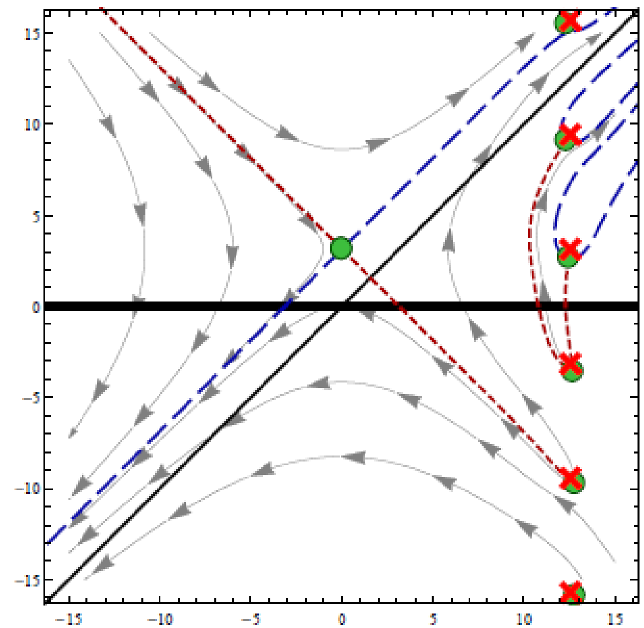


m=4π



Thimble decomposition & Resurgent trans-series

Let us label the critical points by

$$\lim_{g\to 0} z^{\mathrm{c}}_{\mathrm{pt}}(g,m) = 0, \quad \lim_{g\to 0} z^{\mathrm{c}}_{\ell}(g,m) = m + (2\ell+1)\pi i.$$

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Thimble decomposition:

$$\begin{cases} Z(g,m) = \int_{\mathcal{J}_{\mathrm{pt}}} dz e^{-S[z]} + \sum_{\ell=0}^{\infty} \theta(m - \tilde{m}_{\ell}(g)) \int_{\mathcal{J}_{\ell}} dz e^{-S[z]}, \\ \lim_{g\to 0} \tilde{m}_{\ell}(g) = (2\ell+1)\pi. \end{cases}$$

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Small-g expansion

$$\boxed{Z(g,m)=Z_{\mathrm{pt}}(g,m)+\sum_{\ell=0}^{\infty}\theta\left(m-(2\ell+1)\pi\right)\mathrm{Res}_{z=z^*_\ell}\left[e^{-S[z]}\right],}$$

Resurgent trans-series

Interpretation of Borel singularities

[M.H. '17]

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Complexified SUSY Solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q(\text{fields}) = 0$

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Proposal:

If there are n_B bosonic & n_F fermionic solutions with action $S=S_c/g$, then

$$\boxed{(\text{Borel trans.}) \supset \prod_{\text{solutions}} \frac{1}{(t - S_c)^{n_B - n_F}}}$$

Summary & Outlook

Summary of Complex saddle point analysis

$$\int_C dx\ e^{\frac{i}{g}S(x)} \qquad |g| \ll 1$$

Steps of saddle point method:

1. Find all the saddle points

2. Deform the contour to steepest descents
   without changing the final result:

$$C = \sum_{I \in \text{saddle}} n_I J_I \qquad (n_I \in \mathbf{Z})$$

Notes: (a) Saddle points may not be on the original contour
(b) It may be impossible to pass all the saddles
—We get contributions from the ones which we can pass

3. Compute perturbative series around contributing saddles

Various applications of Lefschetz thimbles

- Resurgence
- Sign problem in Monte Carlo simulation

  [Cristoforetti-Di Renzo, Fujii-Honda-Kato-Kikukawa-Komatsu-Sano, etc…]
- Real time path integral

  [Tanizaki-Koike, Cherman-Unsal, Alexandru-Basar-Bedaque-Vartak-Warrington, etc…]
- Complex saddle point analysis in general relativity

  [Feldrugge-Lehners-Turok, Brown-Cole-Shiu-Cottrell]
- Analytic continuation of Chern-Simons & Knot

  [Witten]
- Thimble integral in d-dim. theory
  = (d+1)-dim. SUSY theory

  [Witten, Fukushima-Tanizaki, Nekrasov]
- SUSY breaking [M.H., work in progress]

Thanks!