The gravity dual of Lorentzian OPE blocks

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Based on 1912.04105

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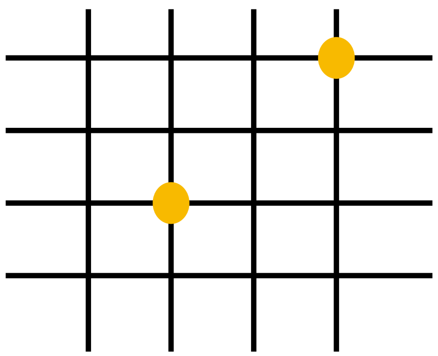
Conformal Field Theory

Conformal Field Theory (CFT) in Euclidean signature

plays a significant role in theoretical physics:

Ising model

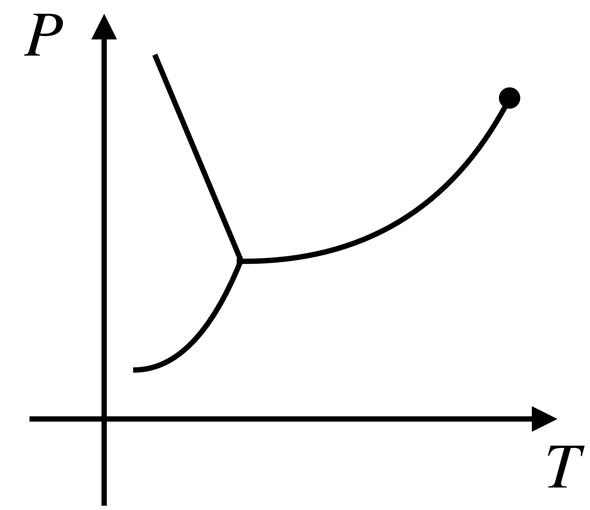
$$H = -J \sum_{<i,j>} s_i \cdot s_j$$



ϕ^4 theory

$$\mathscr{L} = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$$

vapor/liquid transition



All IR theories are described by same CFT!

RG flow and conformal symmetry

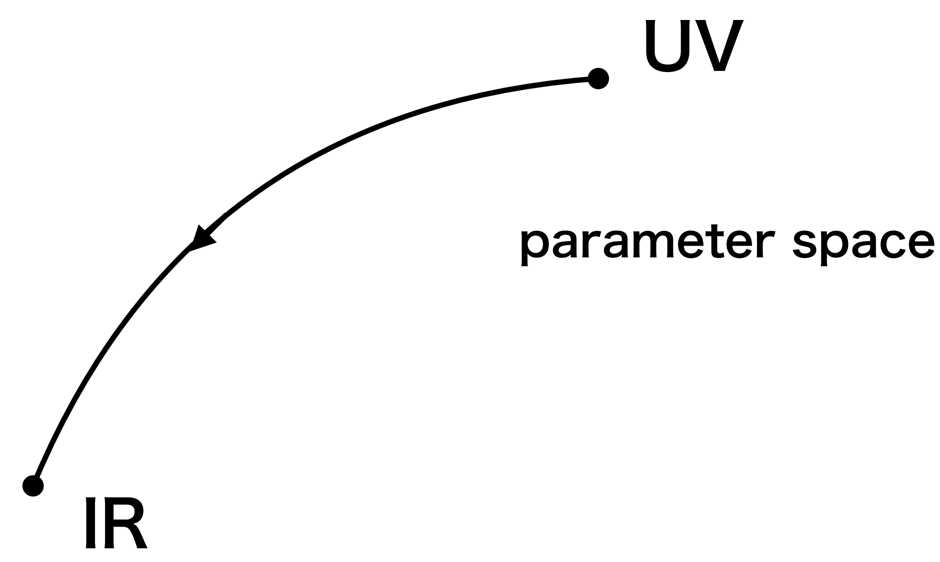
CFT can be defined on the fixed point of RG flow:

e.g

$$\mathscr{L} = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$$

$$\beta(g^i) \equiv \frac{dg^i}{d\log\mu} = 0 \quad \Rightarrow \quad g^i = g^i_*$$

RG scale



$\beta(g^i) = 0$ means the theory has scale invariance

conformal invariance

$$SO(d+1,1)$$

Power of conformal symmetry

Operators $O_{\Delta,\rho}(x)$ are characterized by

- Scaling dimension Δ
- Irreducible representation ρ of $SO(d)$ (or spin J)

2- and 3-point function are completely fixed by conformal symmetry:

$$\langle O_\Delta(x_1) O_{\Delta'}(x_2)\rangle = \delta_{\Delta\Delta'} \frac{1}{x_{12}^{\Delta}}$$

$$x_{ij} \equiv (x_i - x_j)^2$$

$$\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) O_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{x_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} x_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}} x_{31}^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}}}$$

C_{123} : OPE coefficient

Operator Product Expansion

One main feature of CFT is Operator Product Expansion (OPE)

$$O_i(x_1)O_j(x_2) = \sum_k C_{ijk}\, f_k(x_{12}, \partial_2) O_k(x_2)$$

Higher-point functions can be reduced to 2- or 3-point function

$$\langle O_1(x_1)O_2(x_2)\cdots O_n(x_n)\rangle = \sum_k C_{ijk}\, f_k(x_{12}, \partial_2)\langle O_k(x_2)\cdots O_n(x_n)\rangle = \cdots$$

Spectrum (Δ, ρ) and OPE coefficient C_{ijk} are only dynamical data that characterize a given CFT!

4-pt function and conformal bootstrap

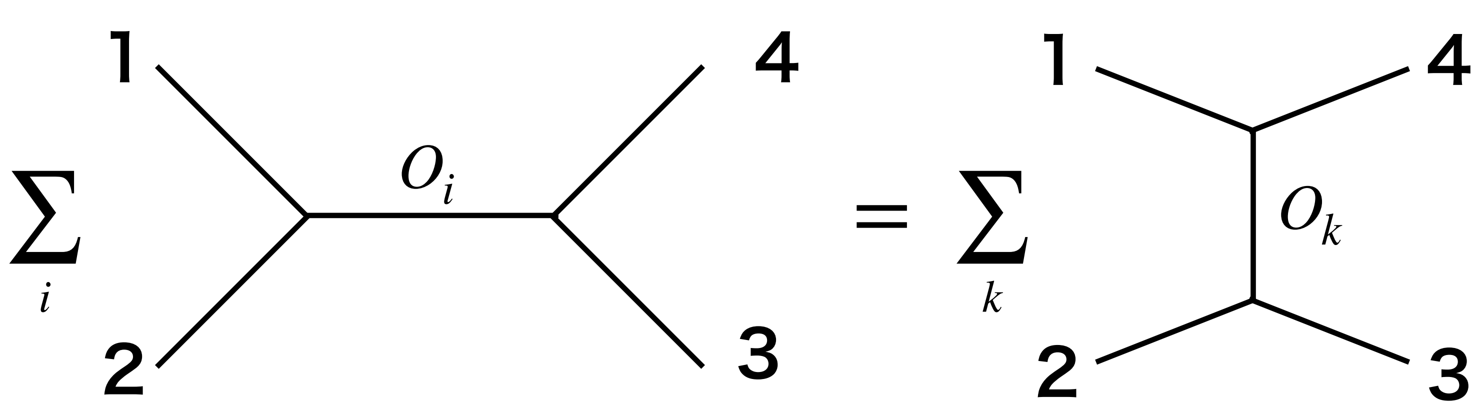
First nontrivial correlator is a 4-pt. function:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_i f_{12i}f_{34i}\langle O_k(x_2)O_k(x_4)\rangle$$

Must be equal!

On the other hand…

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_k f_{14k}f_{23k}\langle O_k(x_2)O_k(x_4)\rangle$$



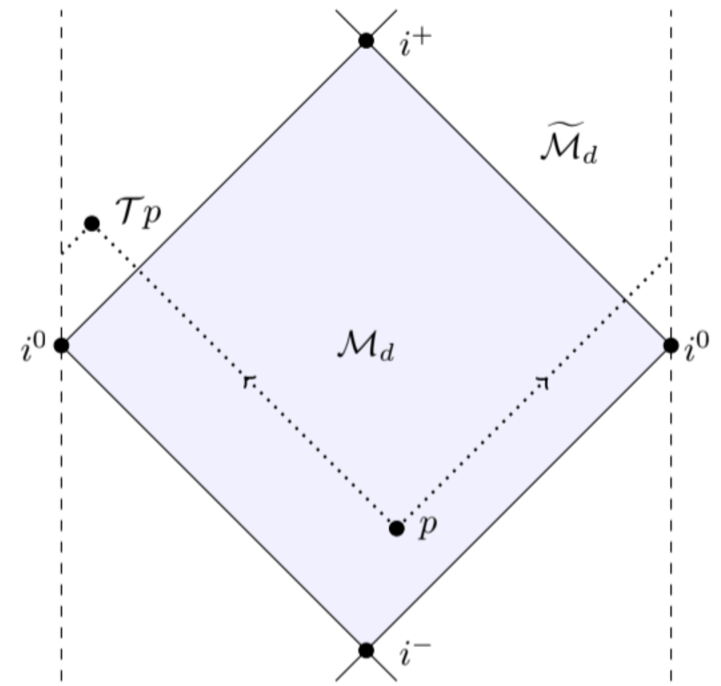
Bootstrap equation

[Rattazi-Rychkov-Tonni-Vichi 08,…]

Lorentzian CFT?

Unitary Lorentzian CFTs are related to reflection-positive Euclidean CFTs by Wick rotation:

[Luscher-Mack 75]



Question:

Do we care about Lorentzian CFT?

Can we calculate everything in Euclidean signature and just Wick rotate them?

Renewed interests on Lorentzian CFTs

There are many things deeply hidden in Euclidean signature…

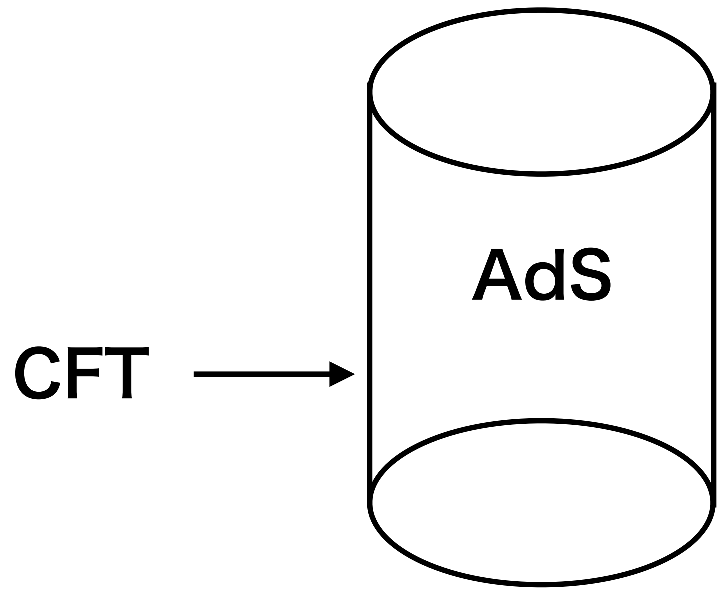
Lorentzian CFT gives us new perspective of kinematics and dynamics of CFT:

- Analytic Conformal Bootstrap [Fitzpatrick-Kaplan-Poland-Simmons-Duffin 12, …]
- Lorentzian inversion formula [Caron-Huot 17, Simmons-Duffin-Stanford-Witten 17]
- Causality constraint and ANEC [Hartman-Kundu-Tajdni 15]

Relation to AdS/CFT?

AdS/CFT correspondence

d-dimensional CFT = $d+1$-dimensional Quantum Gravity on Anti de Sitter Space



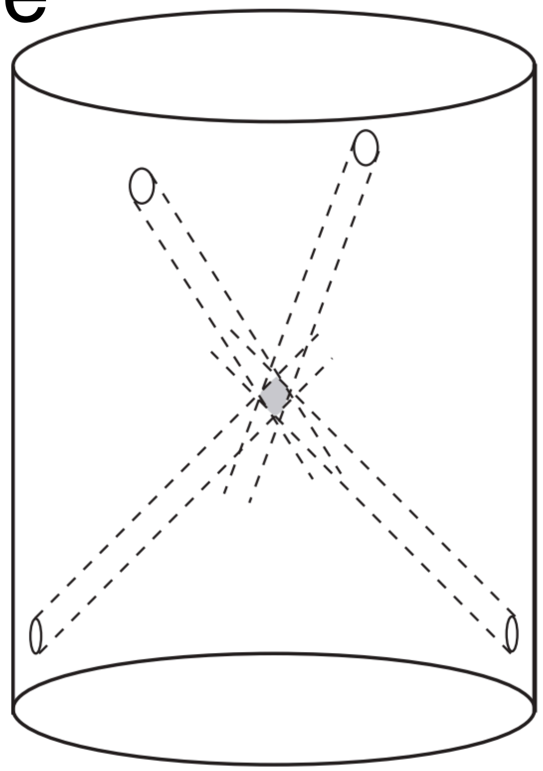
$$Z_{CFT} = Z_{AdS}$$

$$O_{CFT}(x) \leftrightarrow \phi_{AdS}(x, z)$$

Does Lorentzian CFT tell us a new insight to AdS/CFT?

Motivation

We would like to learn holographic perspective from Lorentzian CFT



Our analysis:

We revisit OPE structure of Lorentzian CFT and find its gravity dual

Outline of this talk

- Basics of Lorentzian CFT
- OPE blocks in momentum space
- The gravity dual of Lorentzian OPE block
  - Spacelike: HKLL field on AdS
  - Timelike: HKLL-type field on hyperboloid
- Summary and future problem

Basics of Lorentzian CFT

Dynamics

From reconstruction theorem, operator spectrum and OPE coefficient are given by Euclidean ones.

Kinematics

In Lorentzian signature, we have several correlators to compute

Time-ordered (Feynman) correlator, Wightman function, retated correlator, advanced correlator, …

Wightman function and Feynman correlator

Typical correlators are Wightman function and Feynman correlator.

Both can be computed from Euclidean correlators.

$E = \langle 0 | O_1(x_1) \cdots 0_n(x_n) | 0 \rangle_E$

Wightman function

$$W(x_1, \cdots x_n) = \langle 0 | O_1(x_1) \cdots O_n(x_n) | 0 \rangle_W = \lim_{\epsilon_i \to 0} E(x_1, \cdots x_n)$$

with $x_i^d = i x_i^0 + \epsilon_i$

Feynman correlator

$$F(x_1, \cdots x_n) = \langle 0 | O_1(x_1) \cdots O_n(x_n) | 0 \rangle_F = \lim_{\epsilon \to 0} E(x_1, \cdots x_n)$$

with $x_i^d = (i + \epsilon) x_i^0$

Correlators in momentum space

Difference of these correlators can be seen clearly in momentum space

Wightman 2-pt function

$$W_\Delta(p) \equiv \int d^d x e^{-ip\cdot x} W_\Delta(x) = \Theta\left(p^0\right)\Theta\left(-p^2\right)\frac{\pi^{d/2+1}}{2^{2\Delta-d-1}\Gamma(\Delta)\Gamma\left(\Delta-\frac{d-2}{2}\right)}\left(-p^2\right)^{\Delta-d/2}$$

→gives us the unitarity bound

Feynman 2-pt correlator

$$F_\Delta(p) \equiv \int d^d x e^{-ip\cdot x} F_\Delta(x) = -i\frac{\pi^{d/2}\Gamma\left(\frac{d}{2}-\Delta\right)}{2^{2\Delta-d}\Gamma(\Delta)}\left(p^2-i\epsilon\right)^{\Delta-d/2}$$

OPE revisited

Let us consider OPE of two scalar operators:

$$O_{\Delta_1}(x_1)O_{\Delta_2}(x_2) = \sum_{\Delta,J} C f_{\Delta,J}(x_{12}, \partial_2) O_{\Delta,J}(x_2)$$

$$= \sum_{\Delta,J} C B_{\Delta,J}(x_1, x_2)$$

$B_{\Delta,J}(x_1, x_2)$ is a bi-local operator so called OPE block which is kinematical and fixed by conformal symmetry

Fixing OPE block

To derive the form of OPE blocks, we may notice

- The sum of OPE is organized by irreducible representation of conformal group
- $B_{\Delta,J}$ is an eigenfunction of conformal casimir

$$(L_1 + L_2)^2 B_{\Delta,J}(x_1, x_2) = C_{\Delta,J} B_{\Delta,J}$$

It's useful to find the complete set consisting of the projectors onto each irreducible representation!

Projector in Euclidean spacetime

In Euclidean signature, such a projector can be found by using shadow formalism: [Ferrara-Gatto-Grillo-Parisi 72]

$$1 \sim \sum_{\Delta,J} \int d^d x \, |\tilde{O}_{\Delta,J}(x)\rangle\langle O_{\Delta,J}(x)|$$

$\tilde{O}_{\Delta,J}$: shadow operator with dimension $d - \Delta$ and spin J

which gives us the integral representation of OPE block

$$B_{\Delta,J} \sim \int d^d x \langle O_1(x_1) O_2(x_2) \tilde{O}_{\Delta,J}(x)\rangle O_{\Delta,J}(x)$$

Projector in Lorentzian signature?

Can we naively generalize Euclidean projector to Lorentzian?

$$1 \sim \sum_{\Delta,J} \int d^d x \, | \tilde{O}_{\Delta,J}(x) \rangle \langle O_{\Delta,J}(x) |$$

Ambiguity

- Is this projector complete even in Lorentzian?
- Are we allowed to integrate whole spacetime?
- What kind of correlator should we use?

Momentum space eigenstate

To avoid these ambiguities, it's better to move onto momentum space. Consider a state as follows:

$$|O_{\Delta,J}(p)\rangle \equiv \int d^d x e^{-ip\cdot x} O_{\Delta,J}(x^0 + i\epsilon, x^\mu, z)\,|0\rangle \qquad O(x,z) \equiv z^{\mu_1}\cdots z^{\mu_J} O_{\mu_1\cdots\mu_J}(x)$$

- spans orthogonal basis
- the norm is given by the Wightman 2-pt function

$$\langle O_{\Delta,J}\left(p_1, z_1\right) | O_{\Delta,J}\left(p_2, z_2\right)\rangle = (2\pi)^d \delta^{(d)}\left(p_1 + p_2\right) W_{[\Delta,J]}\left(p_2; z_1, z_2\right)$$

The projector in momentum space

Employing momentum eigenstates, we establish the projector:

$$\mathbf{1} = \sum_{\Delta,J} \int \frac{\mathrm{d}_{\mathrm{L}}^{d} p}{(2\pi)^{d}} \Theta\left(p^{0}\right) \Theta\left(-p^{2}\right) |\tilde{O}_{\Delta,J}(-p)\rangle\langle O_{\Delta,J}(p)|$$

The shadow operator in momentum space is defined by

$$\tilde{O}_{\Delta,J}(p) \equiv W_{[d-\Delta,J]}(p) O_{\Delta,J}(p)$$

One can easily confirm this projector is complete

$$\langle O(x) \underset{\substack{\uparrow \\ \mathbf{1}}}{} O(0)\rangle$$

OPE block in momentum space

Inserting the projector into OPE, we find that

$$B_{\Delta,J}(x_1,x_2)=\int d^dp\Theta(p^0)\Theta(-p^2)W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p)W_{[d-\Delta,J]}(-p)O_{\Delta,J}(p)$$

Partially Fourier transformed Wightman 3-pt function

$W_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p)$ can be computed from Euclidean 3-pt function $E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,-p_E)$:

$$E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,p_E)=\int d^dp_E e^{-ip_E\cdot x}E_{\Delta_1,\Delta_2,[\Delta,J]}(x_1,x_2,x)$$

Partially Fourier transformed 3-pt

From now on, we only consider scalar OPE block, for simplicity.

Relevant 3-pt function can be given by

$$E_{\Delta_1,\Delta_2,\Delta}(x_1,x_2,p_E) = \#(Q_{\Delta_1,\Delta_2,\Delta}(x_1,x_2,p_E) - \#E_\Delta(p_E)Q_{\Delta_1,\Delta_2,d-\Delta}(x_1,x_2,p_E))$$

where

$$Q_{\Delta_1,\Delta_2,\Delta}\left(x_1,x_2,-p_E\right) = -\frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}^+ + \Delta}{2}}}\left(\frac{p_E^2}{4x_{12}^2}\right)^{\frac{\Delta-h}{2}} \times \int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{12}^- + h}{2}-1}(1-u)^{\frac{h-\Delta_{12}^-}{2}-1} e^{ip_E\cdot\left(ux_1+(1-u)x_2\right)} I_{h-\Delta}\left(\sqrt{u(1-u)p_E^2x_{12}^2}\right)$$

- $\Delta_{12}^\pm = \Delta_1 \pm \Delta_2$
- $h = d/2$
- u: Schwinger/Feynman parameter

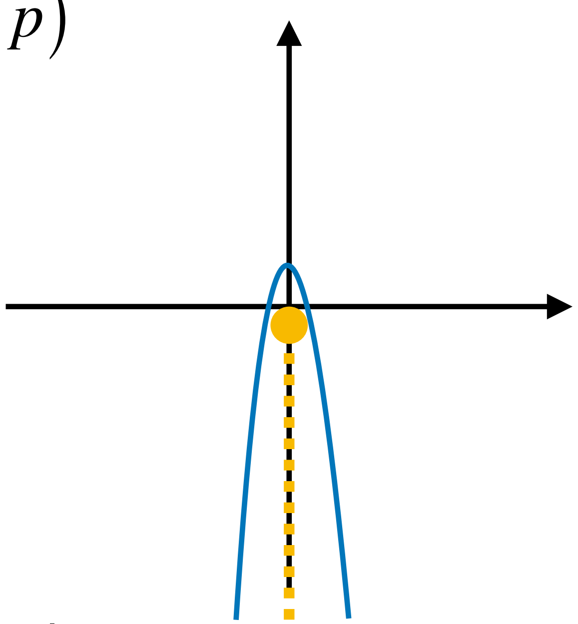
Wightman from Euclidean

To obtain $W_{\Delta_1,\Delta_2,\Delta}(x_1,x_2,p)$, we need analytic continuation

$x^d = ix^0 + \epsilon$:

$$W_{\Delta_1,\Delta_2,\Delta}\left(x_1,x_2,x\right) = \lim_{\epsilon_i\to 0}\int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^d}{2\pi} e^{-p^d\left(x^0-\mathrm{i}\epsilon_3\right)} E_{\Delta_1,\Delta_2,\Delta}\left(x_1,x_2,p\right)$$

Deforming the contour so that wrapping the cut,



$$W_{\Delta_1,\Delta_2,\Delta}\left(x_1,x_2,x\right) = -\mathrm{i}\lim_{\epsilon_i\to 0}\int \frac{\mathrm{d}^{d-1}\mathbf{p}}{(2\pi)^{d-1}} e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \int_{0}^{\infty} \frac{\mathrm{d}p^0}{2\pi} e^{-\mathrm{i}p^0\left(x^0-\mathrm{i}\epsilon_3\right)} \,\mathrm{Disc}\, E_{\Delta_1,\Delta_2,\Delta}\left(x_1,x_2,p\right)\Big|_{p^d\to ip^0}$$

Wightman 3-pt

Let's recall

$$E_{\Delta_1,\Delta_2,\Delta}(x_1, x_2, p_E) = \#(Q_{\Delta_1,\Delta_2,\Delta}(x_1, x_2, p_E) - \#E_\Delta(p_E)Q_{\Delta_1,\Delta_2,d-\Delta}(x_1, x_2, p_E))$$

We may notice that

- $\mathrm{Disc}Q_{\Delta_1,\Delta_2,\Delta} = 0$
- $\mathrm{Disc}E_\Delta(p) = -iW_\Delta(p)$

Then

$$W_{\Delta_1,\Delta_2,\Delta}(x_1, x_2, p) = -Q_{\Delta_1,\Delta_2,d-\Delta}(x_1, x_2, p)W_\Delta(p)$$

$$B_\Delta(x_1, x_2) = \int d^d p \Theta(p^0)\Theta(-p^2)W_{\Delta_1,\Delta_2,\Delta}(x_1, x_2, -p)W_{d-\Delta}(-p)O_\Delta(p)$$

OPE block in momentum space

Finally we find that

$$B_{\Delta,}\left(x_1, x_2\right) = -\int d^d p Q_{\Delta_1,\Delta_2,d-\Delta}\left(x_1, x_2, -p\right) O_\Delta(p)\Big|_{p^d \to \mathrm{i} p^0}$$

$$Q_{\Delta_1,\Delta_2,\Delta}\left(x_1, x_2, -p\right) = -\frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}^+ + \Delta}{2}}}\left(\frac{-p^2}{4x_{12}^2}\right)^{\frac{\Delta - h}{2}} \times \int_0^1 \mathrm{d}u u^{\frac{\Delta_{12}^- + h}{2}-1}(1-u)^{\frac{h-\Delta_{12}^-}{2}-1} e^{ip\cdot\left(ux_1 + (1-u)x_2\right)} J_{h-\Delta}\left(\sqrt{-u(1-u)p^2 x_{12}^2}\right)$$

which is not new, derived 70's ! [Dobrev-Mack-Petkova-Petkova-Todorov 77]

Can we express OPE block in position space and get new insight?

Two configuration in Lorentzian kinematics

There are two distinct case in Lorentzian OPE:

- Spacelike: $x_{12} > 0$
- Timelike: $x_{12} < 0$

Spacelike OPE blocks

We would like to somehow perform p-integral and obtain position space expression. To proceed, we employ

$$J_\nu(\sqrt{p_0^2 - \boldsymbol{p}^2}x) = \frac{1}{2^\nu \pi^h \Gamma(\nu - h + 1)} \left(\frac{\sqrt{p_0^2 - \boldsymbol{p}^2}}{x} \right)^\nu \int_{|y| \le x} \left[\mathrm{d}^d y \right]_\mathrm{E} \left(x^2 - |y|^2 \right)^{\nu - h} e^{\mathrm{i}\left(p_0 t + \mathrm{i}\boldsymbol{p} \cdot \boldsymbol{y}\right)}$$

which enable us to perform p-integral first.

(just the inverse Fourier transform!)

$$\int \left[\mathrm{D}^d p \right]_\mathrm{L} e^{\mathrm{i}\left(p_0 t + \mathrm{i}p \cdot y\right)} O_\Delta(p) = O_\Delta(-t, \mathrm{i}\boldsymbol{y})$$

Spacelike OPE blocks

We find that

$$B_\Delta\left(x_1,x_2\right)=\frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}}{2}}}\int_0^1 \mathrm{d}u u^{\frac{\Delta_{\overline{12}}}{2}-1}(1-u)^{-\frac{\Delta_{\overline{12}}}{2}-1}$$

$$\times\int_{t^2+y^2\le\eta(u)^2}\left[\mathrm{d}^d y\right]_{\mathrm{E}}\left(\frac{\eta(u)}{\eta(u)^2-t^2-y^2}\right)^{d-\Delta} O_\Delta(t(u)+t,x(u)+\mathrm{i}y)$$

- introducing $x(u)=ux_1+(1-u)x_2$ and $\eta(u)=\sqrt{u(1-u)x_{12}^2}$
- $(x(u),\eta(u))$ lives along the geodesic between x_1 and x_2 on Poincare AdS!

$$d^2s=\frac{d^2\eta+\eta_{\mu\nu}dx^\mu dx^\nu}{\eta^2}$$

HKLL field appearing

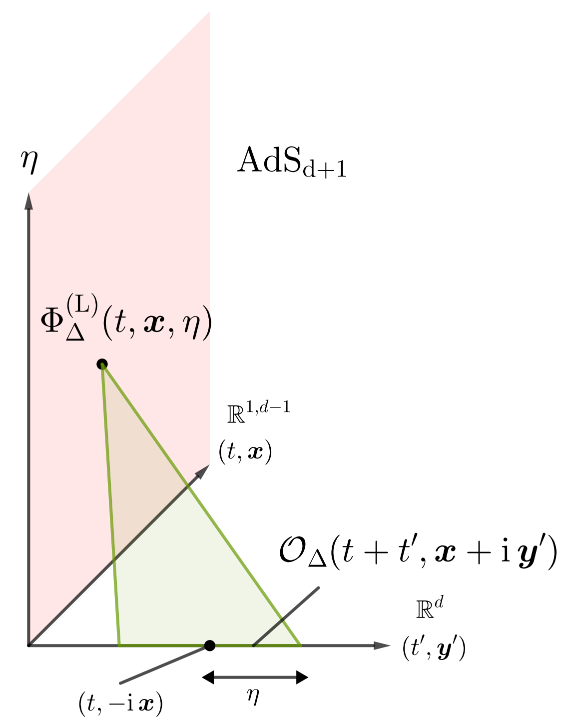
HKLL field is a solution of the free EOM on AdS;

$$(\Box_{AdS} - m^2)\Phi = 0$$

and can be expressed by a boundary CFT operator:

[Hamilton-Kabat-Lifschytz-Lowe 06, 07]

$$\Phi(x,\eta) = \int_{t'^2+y'^2\leq\eta^2} \mathrm{d}t'\mathrm{d}^{d-1}\boldsymbol{y}' \left(\frac{\eta}{\eta^2 - t'^2 - \boldsymbol{y}'^2}\right)^{d-\Delta} O_\Delta\left(t+t', \boldsymbol{x} + \mathrm{i}\boldsymbol{y}'\right)$$

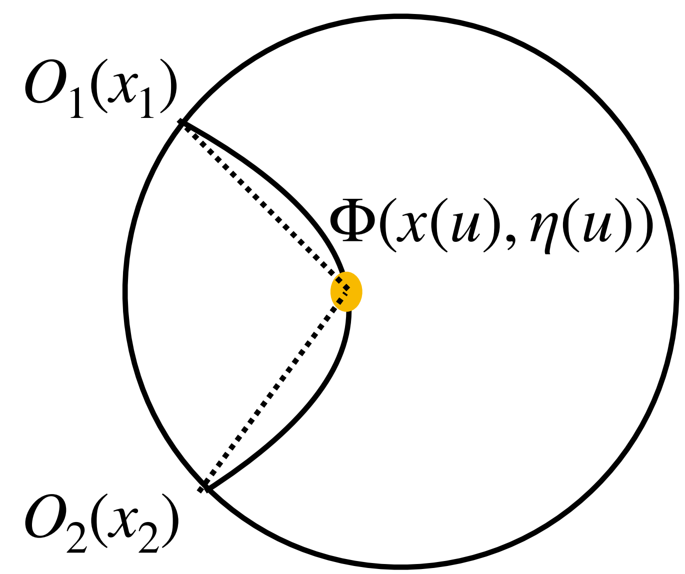


Gravity dual of spacelike OPE block

Now we can realize that spacelike OPE block has a natural gravity dual interpretation:

$$B_\Delta\left(x_1,x_2\right)=\frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}}{2}}}\int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{12}^{-}}{2}-1}(1-u)^{-\frac{\Delta_{12}^{-}}{2}-1}\Phi(x(u),\eta(u))$$

[Cunha-Guica 16]



Comments on higher-spin fields

We can repeatedly analyze spinning OPE block as well.

In particular, for conserved current we find that

$$B_\Delta\left(x_1, x_2\right) = \frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}}{2}}} \int_0^1 \mathrm{d}u\, u^{\frac{\Delta_{\bar{12}}}{2}-1}(1-u)^{-\frac{\Delta_{\bar{12}}}{2}-1} \Phi_{\mu_1 \cdots \mu_J}(x(u), \eta(u)) w^{\mu_1}(u) \cdots w^{\mu_J}(u)$$

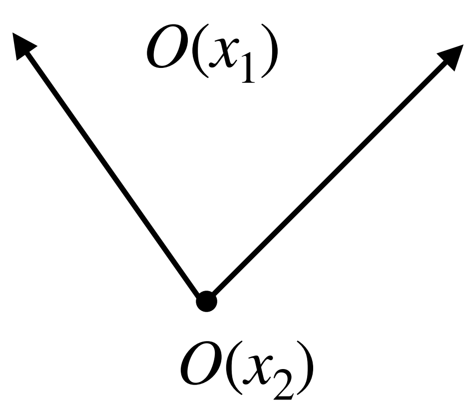
[Chen-Chen-NK-Nishioka 19]

where $\Phi_{\mu_1 \cdots \mu_J}$ is a bulk higher-spin gauge field and

generalization of HKLL field [Sakar-Xiao 14]

Timelike separation

To reach timelike separation, we need analytic continuation from spacelike case.



Replacing $x_{12}^2 \to e^{\pi i} |x_{12}^2|$

Timelike OPE block

Using similar techniques as in spacelike case, we end up with

$$B_\Delta\left(x_1,x_2\right)=\frac{1}{\left(x_{12}^2\right)^{\frac{\Delta_{12}^+}{2}}}\int_0^1 \mathrm{d}u u^{\frac{\Delta_{12}^-}{2}-1}(1-u)^{-\frac{\Delta_{12}^-}{2}-1}\Phi_\Delta^{(\mathrm{LT})}\left(x^\mu(u),\chi(u)\right)$$

Here parameters $x(u)=ux_1+(1-u)x_2$ and $\chi(u)=\sqrt{u(1-u)\,|\,x_{12}^2\,|}$ are not on AdS, but on the hyperboloid in $\mathbb{R}^{2,d}$;

$$d^2s=\frac{-d^2\chi+\eta_{\mu\nu}dx^\mu dx^\nu}{\chi^2}$$

which is an analytic continuation of de Sitter space time.

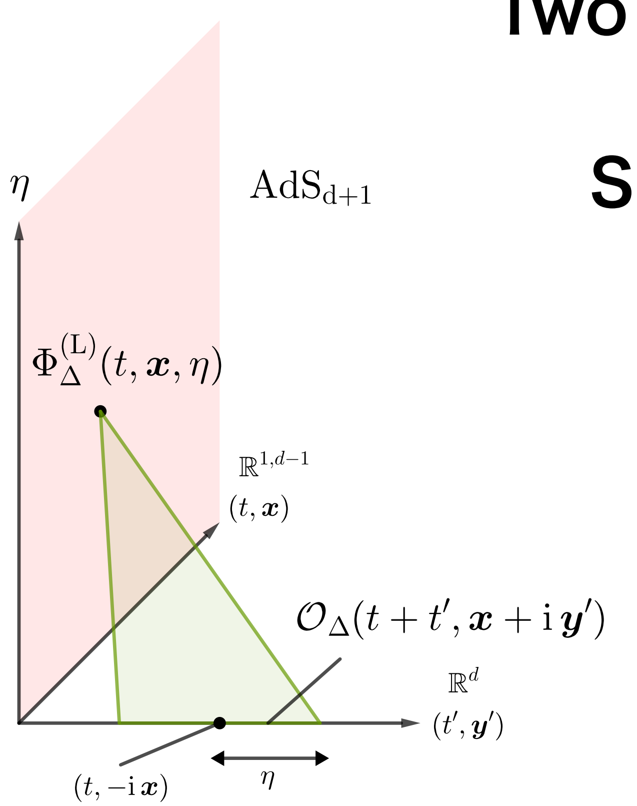
HKLL-like representation

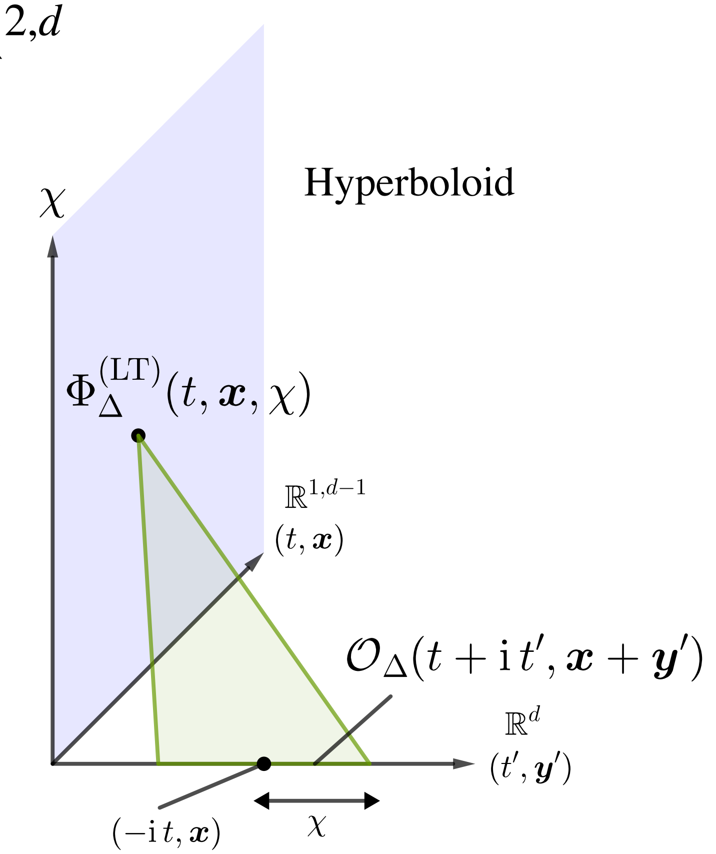
In timelike OPE block, we have defined HKLL-like representation of a scalar field on a hyperboloid:

$$\Phi_{\Delta}^{(\mathrm{LT})}(t,\boldsymbol{x},\chi) \equiv \int_{t^2+\mathbf{y}'^2 \leq \chi^2} \mathrm{d}^d y' \left(\frac{\chi}{\chi^2 - t'^2 - \boldsymbol{y}'^2} \right)^{d-\Delta} \mathcal{O}_{\Delta}\left(t + \mathrm{i}t', \boldsymbol{x} + \boldsymbol{y}'\right)$$

Two ways of extending $\mathbb{R}^{1,d-1}$ to $\mathbb{R}^{2,d}$

Spacelike **Timelike**





Summary

- Derived holographic representations of Lorentzian OPE blocks
- A new holographic dual of a bulk field on the hyperboloid found in the timelike case
- Can be generalized to higher-spin fields

Future problem

How to derive surface Witten representation of the timelike OPE block proposed by [Czech-Lamprou-McCandlish-Mosk-Sully 16, de Boer-Haehl-Heller-Myers 16]?

$$\mathcal{B}^{(\mathrm{L})}_{\Delta}(x_1, x_2) = \frac{1}{|x_{12}^2|^{\frac{\Delta^+_{12}}{2}}}$$