Seminar@Osaka Univ.

Non-perturbative Baryogenesis and the Resurgence

Tomohiro Matsuda(松田智裕)
/ Saitama Institute of Technology(埼玉工業大学)
And
Seishi Enomoto（榎本　成志）
/ Sun Yat-sen University（中山大学）

“Baryogenesis from the Berry phase”
Phys.Rev. D99 (2019) no.3, 036005 / 1811.06197
“Asymmetric preheating”
Int.J.Mod.Phys. A33 (2018) no.25 / 1850146
And a new paper in preparation

This work combines three topics in physics

Baryogenesis

- CP violation
- Many models (Baryogenesis Zoo)

Non-perturbative Particle production

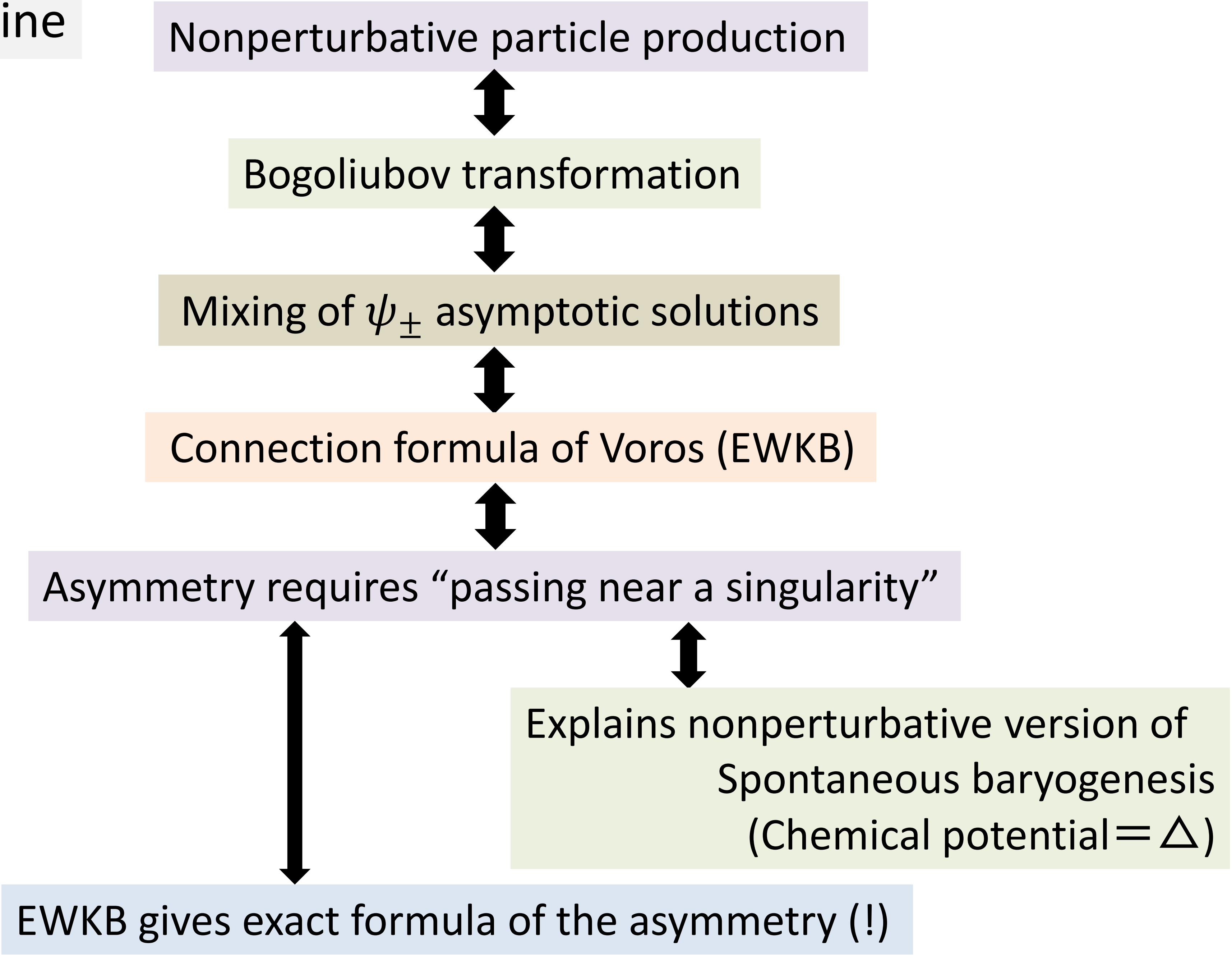
- Bogoliubov transformation
- Time-dependent background
- Preheating of the Universe

Resurgence

- Borel summation
- Includes Exact WKB analysis
- Many applications

Normally, each topic requires lengthy introduction.
We are trying to make a “bird’s eye view” introduction

Outline



Nonperturbative particle production

Bogoliubov transformation

Mixing of $\psi_\pm$ asymptotic solutions

Connection formula of Voros (EWKB)

Asymmetry requires “passing near a singularity”

Explains nonperturbative version of
Spontaneous baryogenesis
(Chemical potential＝△)

EWKB gives exact formula of the asymmetry (!)

First

Baryogenesis

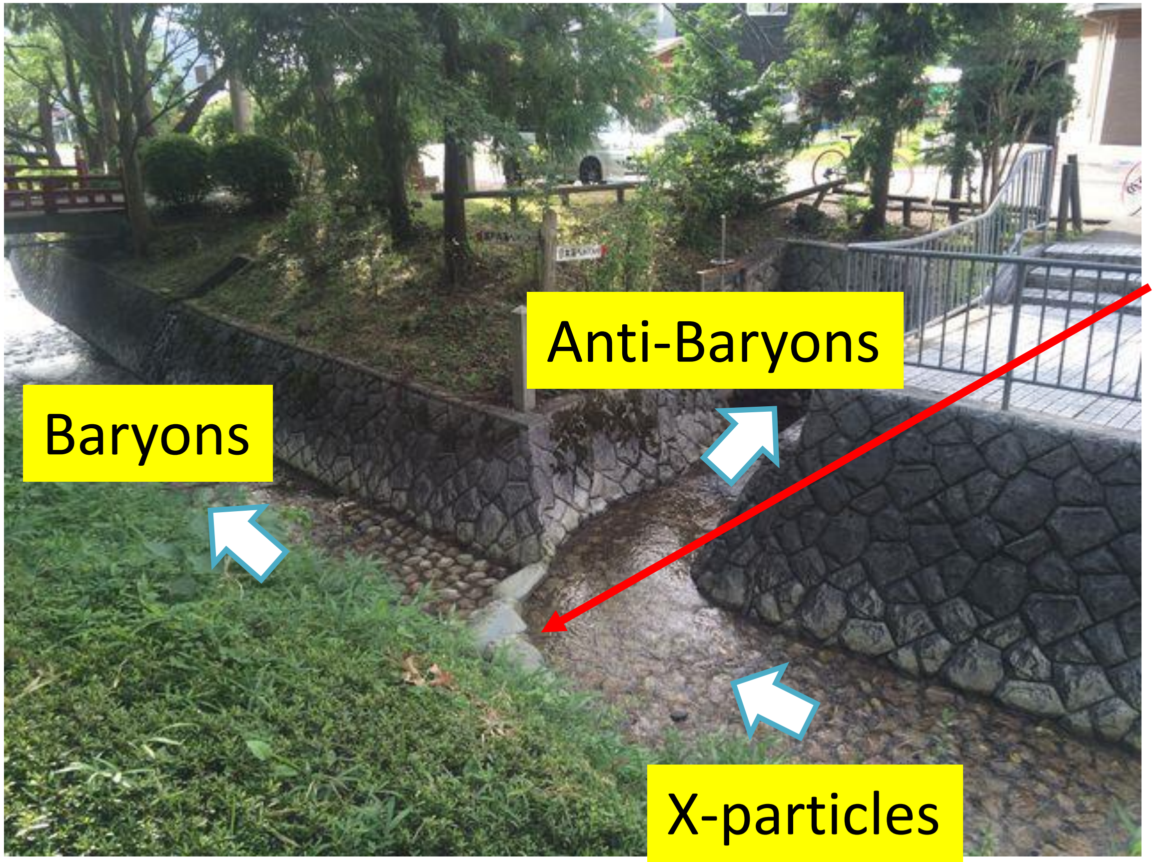
1. Our Universe is not symmetric
   “$n_{matter} \neq n_{antimatter}$”
2. Particles are generated after inflation
   (Reheating)

↓

We need a mechanism
Works after inflation

“Baryogenesis”

Baryogenesis from (heavy) X-decay



GUT Baryogenesis / Leptogenesis

$$|g_B| \neq |g_{\bar{B}}|$$

Difference in the Branching Ratio

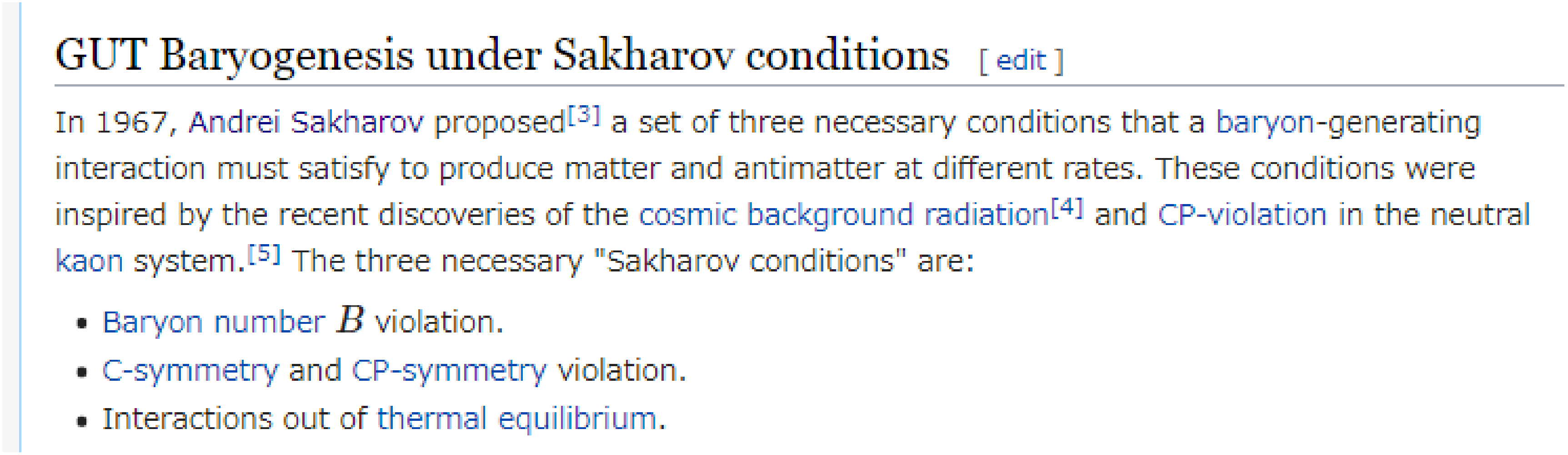
idea

If there is an asymmetry in particle production, there MUST be something at the branch(=interaction).

However, $|g_B| \neq |g_{\bar{B}}|$ cannot generate the asymmetry in the thermal equilibrium

The required conditions are not trivial

“Baryogenesis” from Wikipedia



GUT Baryogenesis under Sakharov conditions [edit]

In 1967, Andrei Sakharov proposed[3] a set of three necessary conditions that a baryon-generating interaction must satisfy to produce matter and antimatter at different rates. These conditions were inspired by the recent discoveries of the cosmic background radiation[4] and CP-violation in the neutral kaon system.[5] The three necessary "Sakharov conditions" are:

- Baryon number B violation.
- C-symmetry and CP-symmetry violation.
- Interactions out of thermal equilibrium.

But, this “minimal” condition was considered for perturbative particle production

Since we are thinking about non-perturbative production, we have to reconsider actual conditions

t-dependent background violates CPT!

Second

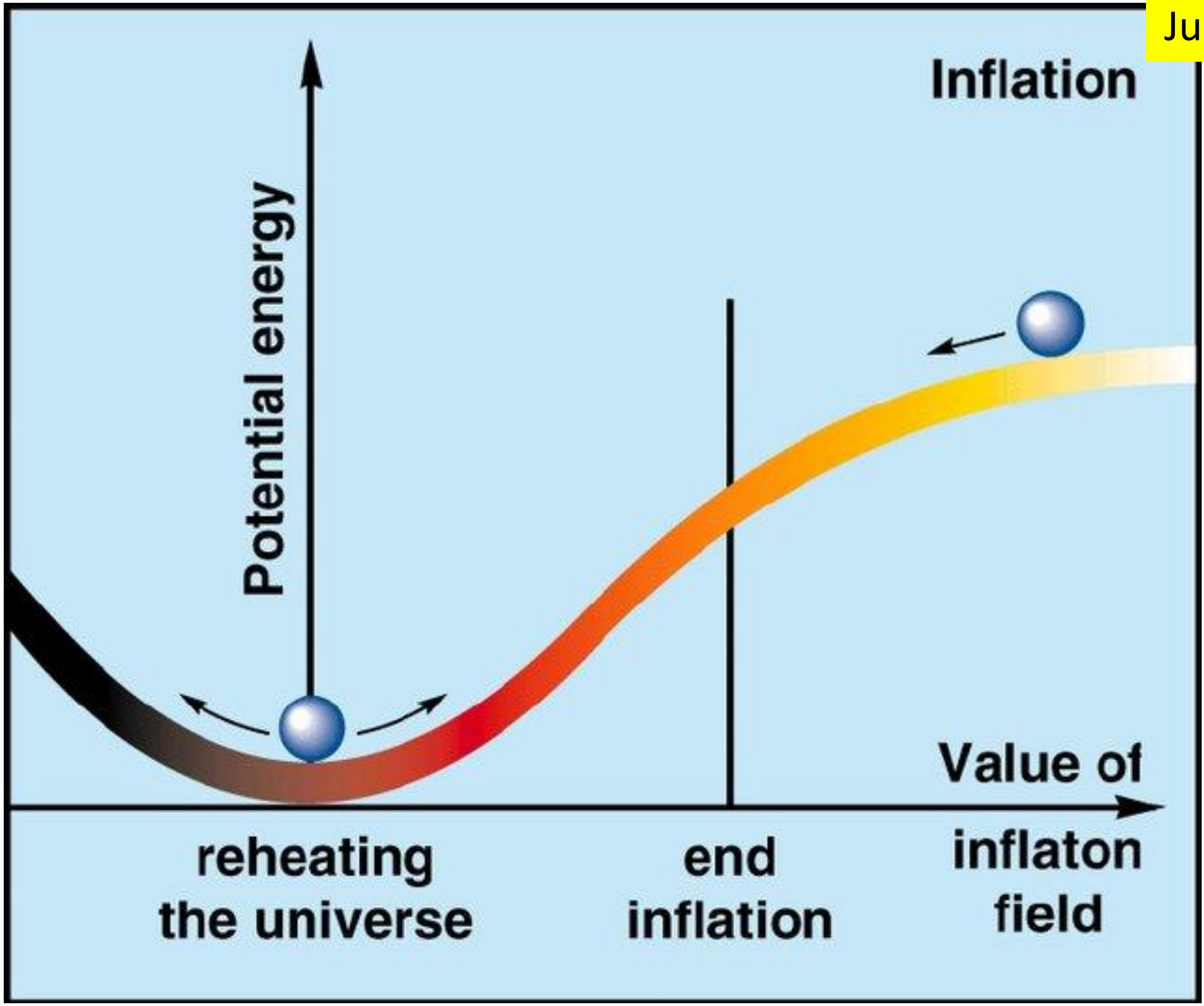
Non-perturbative Particle production

For particle physicists and cosmologists
the most familiar scenario would be
“preheating”

1. **Towards the theory of reheating after inflation**
Lev Kofman (Inst. Astron., Honolulu), Andrei D. Linde (Stanford U., Phys. Dept.), Alexei A. Starobinsky (Landau Inst.). Apr 1997. 40 pp.
Published in **Phys.Rev. D56 (1997) 3258-3295**
IFA-97-28, SU-ITP-97-18
DOI: 10.1103/PhysRevD.56.3258
e-Print: **hep-ph/9704452** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service
レコードの詳細 - Cited by 1313 records 1000+

2. **Reheating after inflation**
Lev Kofman (Hawaii U.), Andrei D. Linde (Stanford U., Phys. Dept.), Alexei A. Starobinsky (Kyoto U., Yukawa Inst., Kyoto & Landau Inst.). May 1994. 9 pp.
Published in **Phys.Rev.Lett. 73 (1994) 3195-3198**
UH-IFA-94-35, SU-ITP-94-13, YITP-U-94-15
DOI: 10.1103/PhysRevLett.73.3195
e-Print: **hep-th/9405187** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
KEK scanned document; ADS Abstract Service
レコードの詳細 - Cited by 1299 records 1000+

Juan Garcia-Bellido '99



Particle production from oscillating background

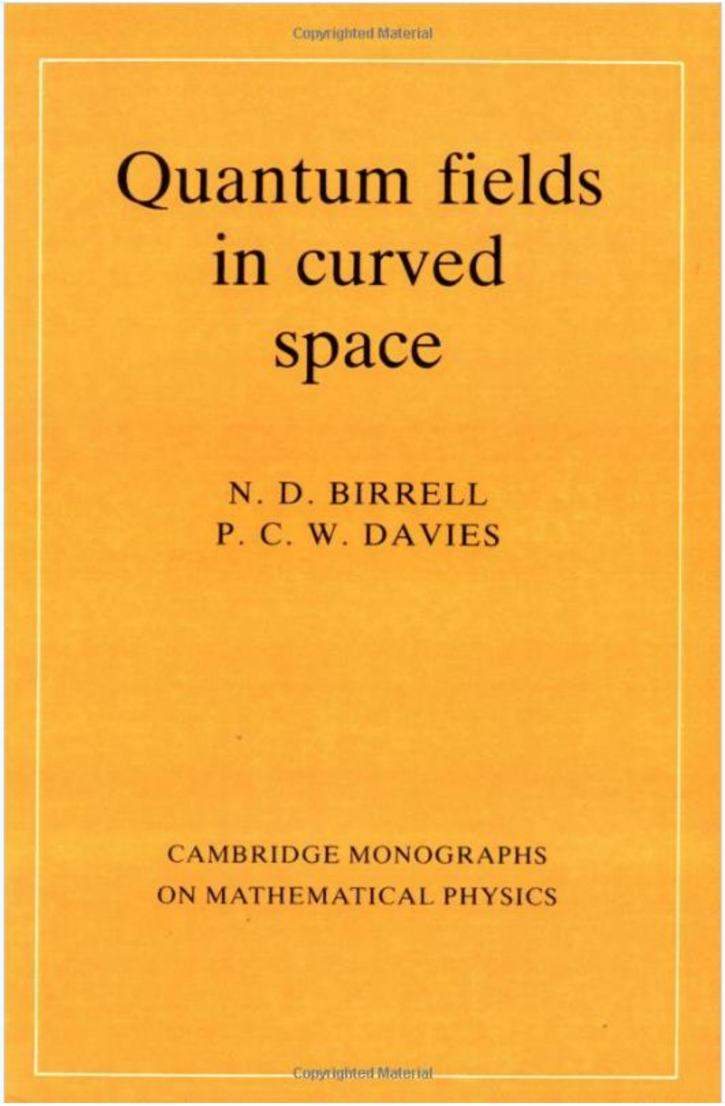
The inflaton starts to oscillate after inflation

The oscillation causes reheating of the Universe

The process can be non-perturbative = preheating

“Preheating” uses particle production with a time-dependent (homogeneous) background

The basic idea uses "$m(t)$"


Copyrighted Material

Quantum fields
in curved
space

N. D. BIRRELL
P. C. W. DAVIES

CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

Copyrighted Material

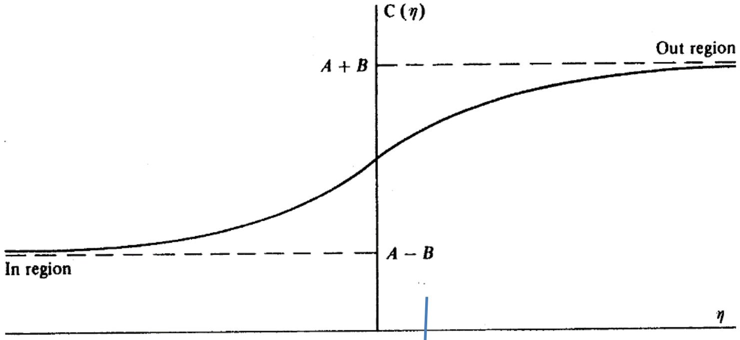


Fig. 7. The conformal scale factor $C(\eta) = A + B \tanh \rho\eta$ represents an asymptotically static universe that undergoes a period of smooth expansion.

Suppose that

$$C(\eta) = A + B \tanh \rho\eta, \quad A, B, \rho \text{ constants,} \tag{3.84}$$

then in the far past and future the spacetime becomes Minkowskian since

$$C(\eta) \to A \pm B, \quad \eta \to \pm\infty$$

(see fig. 7). We consider the production of massive, minimally coupled scalar particles in this spacetime; an investigation first carried out by Bernard & Duncan (1977). Note that in two dimensions minimal and conformal coupling are equivalent (see (3.27)).

Since $C(\eta)$ is not a function of x (the spatial coordinate) spatial translation invariance is still a symmetry in this spacetime, so we can separate the variables in the scalar mode functions appearing in (3.30):

$$u_k(\eta, x) = (2\pi)^{-\frac{1}{2}} e^{ikx} \chi_k(\eta). \tag{3.85}$$

Substituting (3.85) in place of ϕ into the scalar field equation (3.26), with $\xi = 0$ and the metric given by (3.83), one obtains an ordinary differential equation for $\chi_k(\eta)$:

$$\boxed{\frac{d^2}{d\eta^2}\chi_k(\eta) + (k^2 + C(\eta)m^2)\chi_k(\eta) = 0.} \tag{3.86}$$

This equation can be solved in terms of hypergeometric functions. The normalized modes which behave like the positive frequency Minkowski

Why particles are generated when the mass is time-dependent?

Bogoliubov transformation

At the end of your “Quantum Mechanics” class, your teacher may have started to refer to condensed matter physics, and you my have seen…

$$H = \sum a_k^\dagger \left(\frac{k^2}{2m} - E_F \right) a_k + \frac{1}{2L} \sum V_q a_{k-q}^\dagger a_{k'+q}^\dagger a_{k'} a_k$$

can be reduced to

$$H = \sum \begin{pmatrix} b_q & b_{-q}^\dagger \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \begin{pmatrix} b_q^\dagger \\ b_{-q} \end{pmatrix}$$

Because of the off-diagonal elements, the creation/annihilation operators has to be redefined to diagonalize H.
This redefinition is called the Bogoliubov transformation
（Mixing between creation/annihilation operators)

Mixing between creation/annihilation operators

Why?

Particle appears from the vacuum

Answer

$$\boxed{a_{old}|0\rangle = 0}$$

$$a_{new}|0\rangle = \left(\alpha a_{old} + \beta^* a_{old}^{\dagger}\right)|0\rangle$$
$$= \beta^* \, a^{\dagger}|0\rangle$$

In terms of the "new" particle, the "old" vacuum is filled with
$n = |\beta|^2$

Indeed, the same thing will happen when the mass is time-dependent

Particle production with $m(t)$

a_k for negative $e^{-i(\omega t)}$
$a_k^\dagger$ for positive $e^{+i(\omega t)}$

A (free) scalar field can be decomposed as

$$\chi_{in} = \int dk \left[a_k\, e^{-i(\omega t - kx)} + a_k^\dagger\, e^{i(\omega t - kx)}\right]$$

Since the mass is time-dependent, after a time interval the time-dependent function may be

$$\chi_{end} = \int dk \left[f_k(t) a_k\, e^{i(kx)} + f_k^*(t) a_k^\dagger\, e^{-i(kx)}\right]$$

If the positive/negative solutions are mixed in f_k(t) as

$$f_k(\mathrm{t}) \to \alpha_k\; e^{-i\omega t} + \beta_k\; e^{+i\omega t}$$

this gives the Bogoliubov transformation

$$\widehat{a_k} = \alpha_k\; a_k + {\beta^*}_k\; b^\dagger_{-k}$$

This mixes the definition of a_k and $a_k^\dagger$!

Key! Mixing between ±solutions is the source of particle production which can be caused by $m(t)$

There are many Models which can be solved exactly.

1. From the textbook of Birrell and Davies

$$\left(\frac{d^2}{d\eta^2} + k^2 + m^2(A + B\tanh\rho\eta\right)\chi_k(\eta) = 0$$

Obviously, the asymptotic solutions are

$t = -\infty$ $\quad u_k^{in}(\eta) = \frac{e^{\pm i\omega_{in}\eta}}{\sqrt{4\pi\omega_{in}}}$ ⇒ $u_k^{out}(\eta) = \frac{e^{\pm i\omega_{out}\eta}}{\sqrt{4\pi\omega_{out}}}$ $\quad t = +\infty$

$\omega_{in} = \sqrt{k^2 + m^2(A - B)}$ $\qquad \omega_{out} = \sqrt{k^2 + m^2(A + B)}$

There is a hypergeometric function that connects these solutions.
The “linear transformation property” of the function gives

$$u_k^{in}(\eta) = \alpha_k u_k^{out} + \beta_k (u_k^{out})^*$$

Mixed during the evolution

$$\alpha_k(\beta_k) = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1 - \frac{i\omega_{in}}{\rho}\right)\Gamma\left(-\frac{i\omega_{out}}{\rho}\right)}{\Gamma\left(\mp i\frac{\omega_{in} \pm \omega_{out}}{2\rho}\right)\Gamma\left(1 \mp i\frac{\omega_{in} \pm \omega_{out}}{2\rho}\right)}$$

This problem is very familiar for physicists, because…

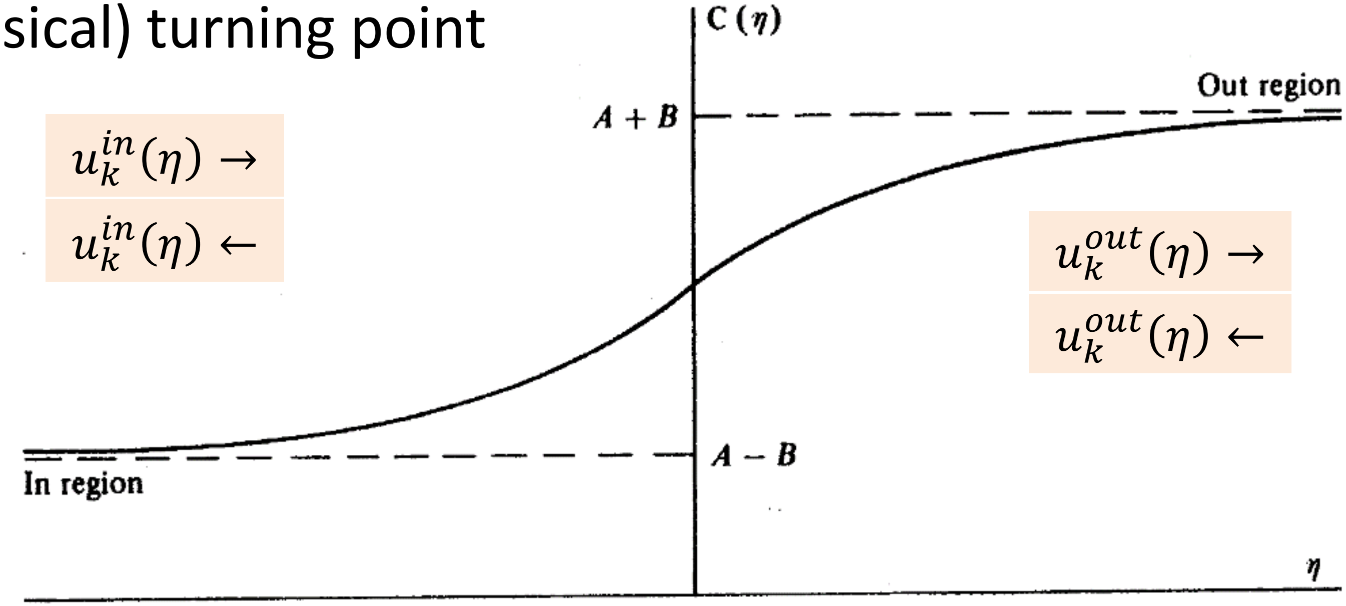
“mixing of the asym. solutions” ⇔ “scattering” in QM

$$\left(\frac{\hbar^2}{2m}\nabla^2 - (\mathrm{V(x)} - \mathrm{E})\right)\Psi(x) = 0 \quad \Leftrightarrow \quad \left(\frac{d^2}{d\eta^2} + k^2 + m^2(A + B\tanh\rho\eta)\right)\chi_k(\eta) = 0$$

V(x) in Schrödinger

$m(t)$ in Field theory

*A very shallow potential
No (classical) turning point



$u_k^{in}(\eta) \rightarrow$

$u_k^{in}(\eta) \leftarrow$

$u_k^{out}(\eta) \rightarrow$

$u_k^{out}(\eta) \leftarrow$

This might be misleading because ;
free particle with $m(t) \simeq$ Scattering problem of QM
but particle with interaction $\gg$ Scattering problem of QM

*Interaction raises the rank

2. “preheating” scenario

Replaced by classical $\phi(t) \Leftrightarrow m(t)$
Not introducing a genuine interaction

Introduce an interaction (for real scalar fields)

$\sim g^2\,\phi^2\chi^2$ for an oscillating field(inflaton) $\phi(t) \sim \Phi_n \sin(m_\phi t)$

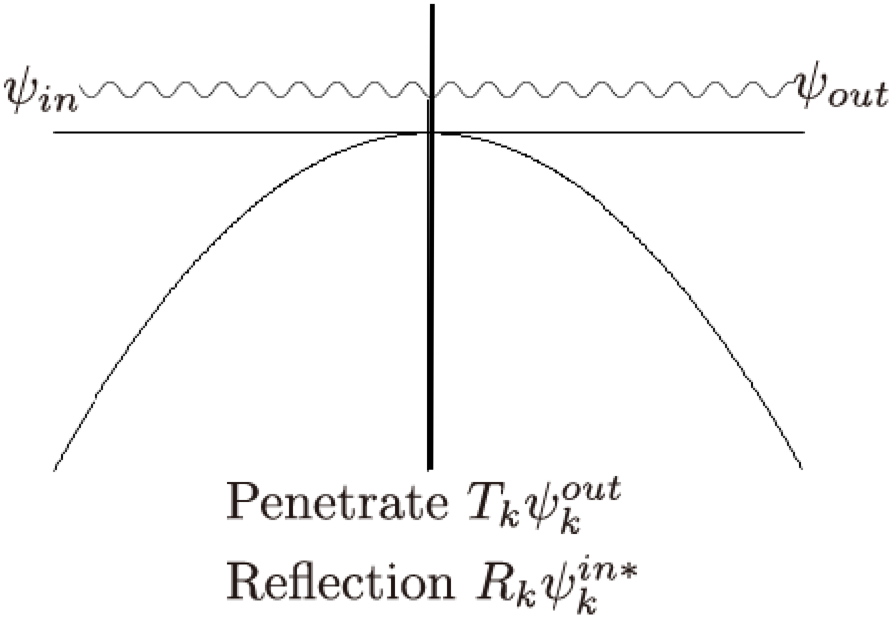
gives $m_\chi^2(t) \sim m_0^2 + g^2\Phi_n^2\sin^2(m_\phi t)$

Scattering by a decaying sinusoidal potential

Near the origin $(t \sim 0\,;\phi \sim 0)$ this can be approximated as

$$m_\chi^2(t) \sim m_0^2 + g^2\Phi_n^2 m_\phi^2 t^2$$

This corresponds to a scattering problem at a negative parabolic potential



Solved using
a parabolic cylinder function

$$R = -ie^{i\varphi}\left(1+e^{\pi\kappa^2}\right)^{\frac{1}{2}}$$

$$T = e^{i\varphi}\left(1+e^{-\pi\kappa^2}\right)^{\frac{1}{2}}$$

$$\kappa \equiv \frac{k^2}{g\Phi_n m_\phi}, \qquad \varphi\text{: } phase$$

Note

"Particle production by an oscillating inflaton"

$$\left(\frac{d^2}{d\eta^2}+k^2+m^2(t)\right)\chi_k(\eta)=0$$

$$m_\chi^2(t)\sim m_0^2+g^2\Phi_n^2\sin^2(m_\phi t)$$

gives the Mathieu Equation, which represents
"QM with degenerated vacua"
The solution requires "Trans-series expansion"
i.e, "summation of instantons" for Eigenstate problem
or "summation of scattering from many bumps" for scattering

One can assume particle decay at large Φ_n(where χ is heavy),
and it "resets" the condition before the next event
If not, one cannot ignore "Trans-series" (or a "resonance")

I will be back to this topic briefly after introducing EWKB

If we are very lucky, we can find the exact solution.
Else, we have to calculate it using approximations.

Question (a simple example that requires approximation)

Preheating requires $\sim g^2 \, \phi^2 \chi^2$ for particle production

Then, do you think "preheating" is a special scenario that works only when the inflaton has the explicit interaction?

Our answer is NO!

"Beauty is more attractive: particle production and moduli trapping with higher dimensional interaction"
Seishi Enomoto (KMI, Nagoya & Warsaw U.), Satoshi Iida (Nagoya U.), Nobuhiro Maekawa (KMI, Nagoya & Nagoya U.), Tomohiro Matsuda (Saitama Inst. Tech.).
JHEP 1401 (2014) 141 / arXiv:1310.4751

Non-renormalizable terms $\propto M_p^{-n}$ can be used for preheating

Preheating is very generic!

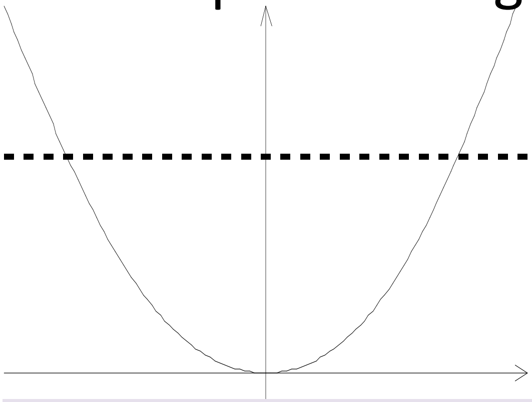
In that paper we have calculated (both numerical / analytical) the particle production with $m(t)^2 \propto t^n, (n > 2)$ from $\frac{\phi^n}{M_p^{n-2}}\chi^2$

Our approximation was based on WKB and the steepest descent for the integration (on complex t)

Here we omit the calculation because its quite lengthy

After a while we came to know that on the Schrödinger side eigenstate problem for anharmonic potential is a hot topic in the light of the Resurgence

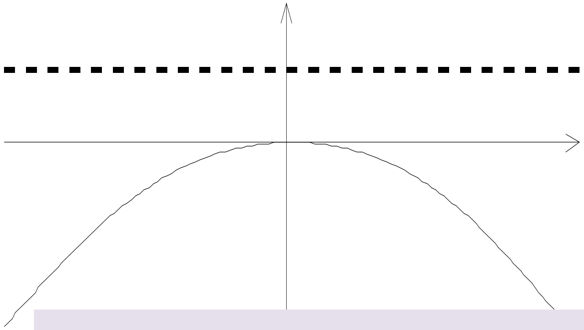
$V(x) \sim x^n, (n \neq 2)$



Eigenstate problem of anharmonic V

⇔

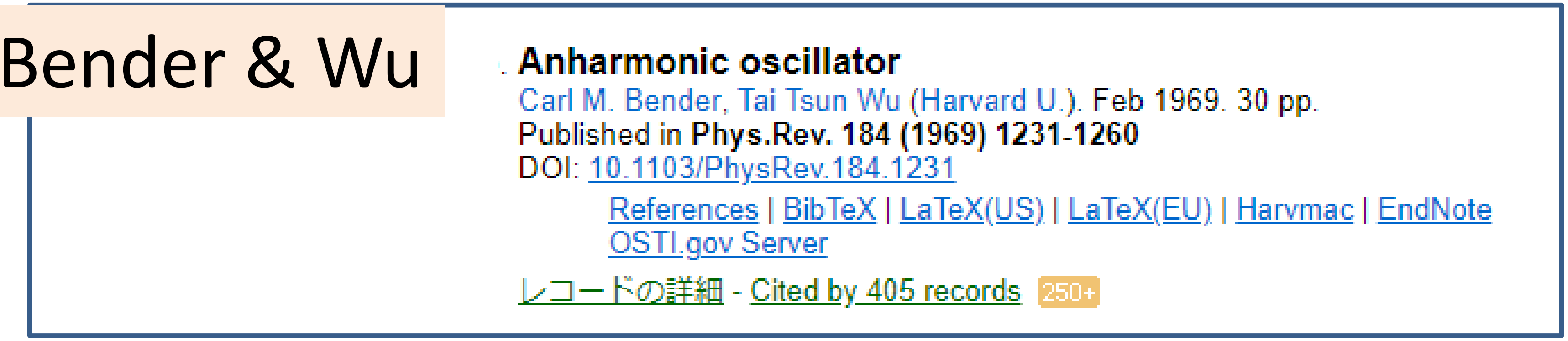
Obvious relation



Scattering problem with nonrenormalizable int.

The study of
Anharmonic oscillator and the Resurgence was started by

Bender & Wu



Anharmonic oscillator

Carl M. Bender, Tai Tsun Wu (Harvard U.). Feb 1969. 30 pp.

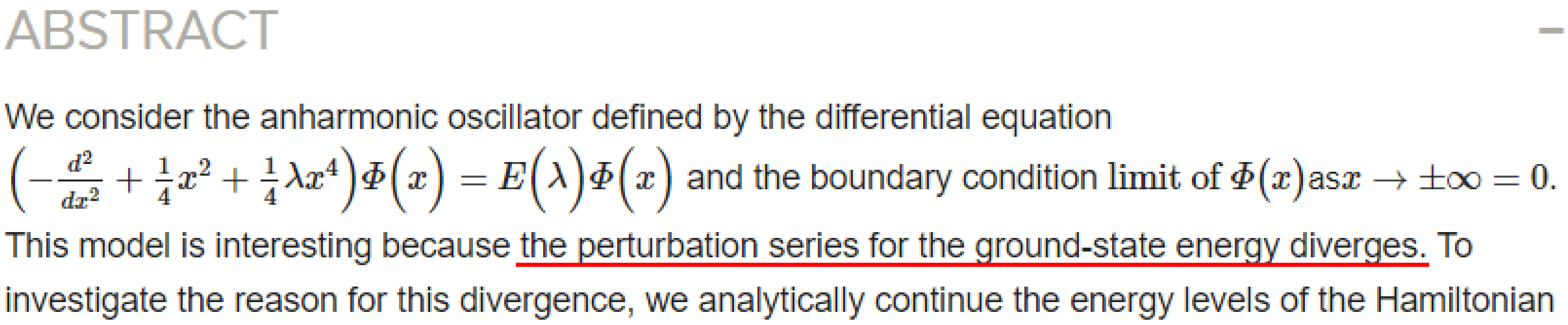
Published in **Phys.Rev. 184 (1969) 1231-1260**

DOI: 10.1103/PhysRev.184.1231

References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote

OSTI.gov Server

レコードの詳細 - Cited by 405 records 250+



ABSTRACT

We consider the anharmonic oscillator defined by the differential equation $\left(-\frac{d^2}{dx^2}+\frac{1}{4}x^2+\frac{1}{4}\lambda x^4\right)\Phi\left(x\right)=E\left(\lambda\right)\Phi\left(x\right)$ and the boundary condition $\lim$ of $\Phi\left(x\right)\mathrm{as} x\to\pm\infty=0.$ This model is interesting because the perturbation series for the ground-state energy diverges. To investigate the reason for this divergence, we analytically continue the energy levels of the Hamiltonian

Perturbation ($O(\lambda^n)$ expansion) based on WKB gives a divergent series, but it could be cured by the Borel summation

Strong impact!

Since the Resurgence is useful for solving various eigenstate problems,
it is (obviously) useful for scattering problems
and for solving the Bogoliubov transformation
in the non-pertubative particle production

⇩

We searched previous works, which refers to the relation between non-perturbative particle production and the Resurgence

We found
No work

*Some papers refers to "Scattering in QM ⇔ Resurgence"

Since the equation becomes higher if we introduce interaction
(multiple elements => higher derivative)
“resurgence for higher derivative ” must be important

Note! Baryogenesis requires
B-violating interaction

We searched previous works, which refers to the relation between higher-order differential equations and the resurgence

We found
“Exact WKB analysis”

So, the study of non-perturbative Baryogenesis
in the light of the Resurgence
is a new frontier for physicists,
but mathematicians already know
how to solve the problems

Our tool

Exact WKB analysis / Resurgence

Our target

Baryogenesis
from non-perturbative particle production

Third

Resurgence

What is the Resurgence?

Unification of perturbation theory and non-perturbative physics

"non-perturbative effect" appears in a divergent (perturbative) series expansion

You might wonder
All the non-perturbative effects are supposed to appear in perturbation?

Large N
Instanton
etc

At this moment the resurgence is defined for specific models.
So, (for now) we cannot say the same is true for the "real" QCD.

How to deal with the divergent perturbative series?

Borel (re)summation

What is the Borel summation?

Inverse Laplace transformation
is called "Borel transformation"
IF it is applied to a divergent power series

$f \to f_B$

Laplace transformation
of the Borel transformed function
is called "Borel summation"

$f_B \to \hat{f}$

f is a divergent power series

$\hat{f}$ is integral of a function with singularities

Given by Different formula!

One can see the origin of the "divergence" form the "singularities"

A very simple example

$$\left(-\frac{d}{dz}+1\right)\psi(z)=\frac{1}{z}$$ has a power series solution $$f=\sum_{n=0}^{\infty}\frac{(-1)^n n!}{z^{n+1}}$$

Divergent because of $n!$

If one defines the Borel transformation(Inverse Laplace)

$$f(z)=e^{\zeta_0 z}\sum_{n=0}^{\infty}\frac{f_n}{z^{n+\alpha}} \Rightarrow f_B(\zeta)=\sum_{n=0}^{\infty}\frac{f_n}{\Gamma(n+\alpha)}(\zeta+\zeta_0)^{n+\alpha-1}$$

one finds

$$f_B=\sum_{n=0}^{\infty}(-1)^n\zeta^n=\frac{1}{\zeta+1}$$

Converges, but a singularity appears

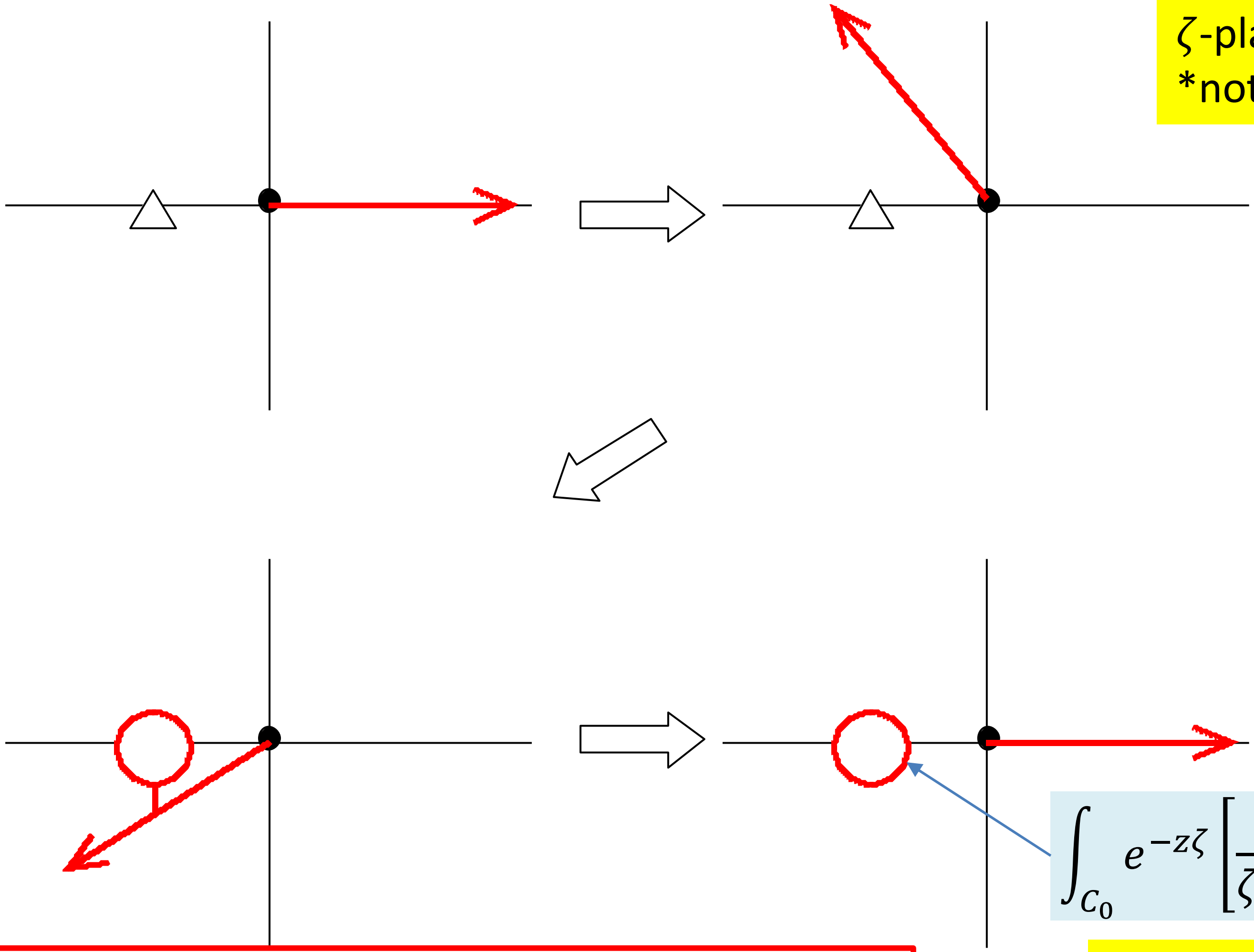
The Borel summation(Laplace) of f_B is

$$\hat{f}=\int_0^{\infty}e^{-z\zeta}\left[\frac{1}{\zeta+1}\right]d\zeta$$

NOTE!
This is not the “singularity of the equation”!
This is the “singularity of the Borel summation”

Convergent, but the path can be rotated by $z\to ze^{i\theta}$.
Then the path may hit the singularity at $\zeta=-1$

ζ-plane
*not for (complex) z



$$\int_{C_0} e^{-z\zeta}\left[\frac{1}{\zeta+1}\right]d\zeta = 2\pi i e^{z}$$

"Stokes phenomenon"
gives exponential factor

The integration path can be rotated by $z \to ze^{i\theta}$.
Then the path may hit the singularity at $\zeta = -1$
After $\theta = 0 \Rightarrow \theta = 2\pi$ one will find $+2\pi i e^{z}$

The basic Idea of the Resurgence is very simple.
Borel sum of PT => Stokes phenomenon => Explains nPT?

One can apply this simple idea to the familiar WKB expansion

Exact WKB analysis (strategy)

WKB expansion is usually a divergent series expansion and gives $f^{\pm}$

⇩

Can be cured by the Borel summation

⇩

The global structure of $\hat{f}^{\pm}$ gives connection formula of $f^{\pm}$ (Stokes phenomenon)

⇩

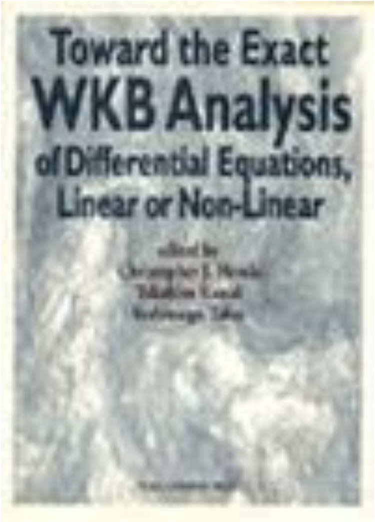
Exact result of the Bogoliubov transformation
can be obtained from the stokes curves
without knowing the exact solution

The most useful textbooks of EWKB (Up to 2^{nd} order)

岩波
オンデマンドブックス

特異摂動の
代数解析学

河合隆裕
竹井義次 著



Toward the Exact
WKB Analysis
of Differential Equations,
Linear or Non-Linear

Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear
Kawai and Takei

In our introduction we have to omit many “crucial” proofs of the method. Please refer to these textbooks.

Exact WKB analysis

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x)\right)\psi(x,\eta)=0$$

with

Assume Exp form

$$\psi(x,\eta)=e^{\int^x S\,dx}$$

Large η

$$-\left(S^2+\frac{\partial S}{\partial x}\right)+\eta^2 Q=0$$

This is not $O(x^n)$!!
$O(\eta^{-n})$ with x-dependent coefficients

$$S=S_{-1}(x)\eta+S_0(x)+S_1\eta^{-1}+\cdots$$

$S_{-1}^2=Q$ ⇔ Solution with ± sign

$$2S_{-1}S_j=-\left(\sum_{\substack{k+l=j-1\\k,l\geq 0}} S_k S_l+\frac{dS_{j-1}}{dx}\right),(j\geq 0)$$

S_j Diverges at turning points (Q=0)

Result (f)

$$\psi_\pm=\frac{1}{\sqrt{S_{odd}}}e^{\pm\int_{x_0}^{x} S_{odd}dx}$$

$$S_{odd}=\sum_{j\geq 0} S_{2j-1}\eta^{1-2j}$$

You might think this is strange.
Although we are solving equation of x,
Borel summation is used for η (!!)

For the lowest part we define

$$s(x) \equiv \int_{x_0}^{x} S_{-1}\, dx$$

and expand terms of $S_j, (j \geq 1)$

η −independent

$$\psi_+(x,\eta) = e^{s(x)\eta} e^{\int_{x_0}^{x} S_0 dx} \underbrace{\left[\exp\left(\sum_{j\geq 1} \int_{x_0}^{x} S_j \eta^{-j} dx\right)\right]}$$

Divergent power series

$$1 + \left[\int_{x_0}^{x} S_1\, dx\right]\eta^{-1} + \left(\left[\int_{x_0}^{x} S_2\, dx\right] + \frac{1}{2}\left(\int_{x_0}^{x} S_1\, dx\right)^2\right)\eta^{-2} + \cdots$$

One can use the "Borel transformation" and the "Borel summation"

$$f = e^{s\eta} \sum_{n=0}^{\infty} f_n\, \eta^{-\alpha-n} \Rightarrow f_B(y) = \frac{1}{\Gamma(\alpha+n)} \sum_{n=0}^{\infty} f_n\, (y+s)^{\alpha+n-1}$$

$$\hat{f} = \int_{-s}^{\infty} e^{-\eta y} f_B(y) dy$$

$z \to \eta, \zeta_0 \to s, \zeta \to y$
"x-dependent coefficients f_n"

$$L[t^n] = \frac{n!}{s^{n+1}}$$

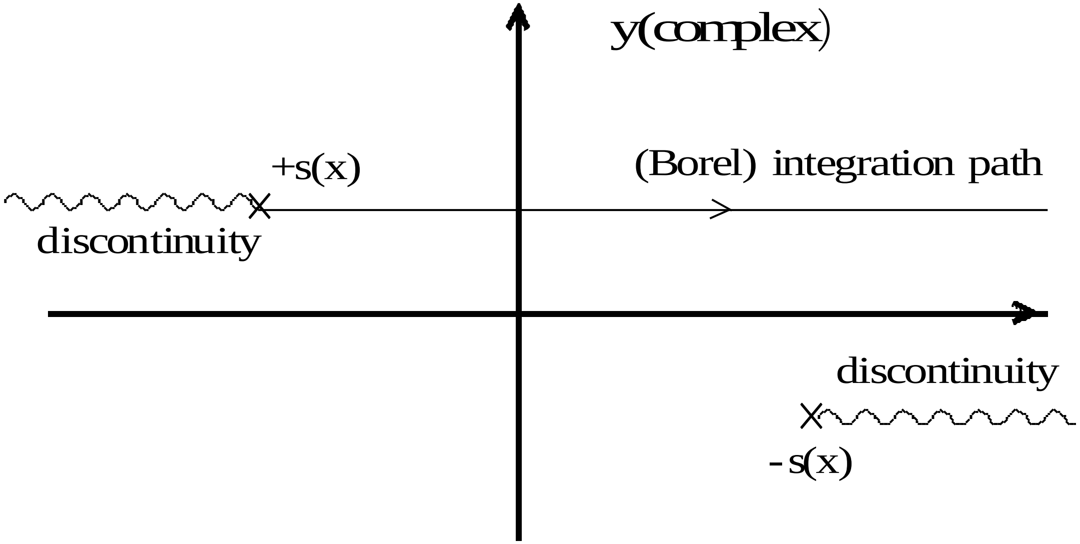
$\Gamma(k) \Leftrightarrow k!$

What is the Stokes phenomena in EWKB?

The starting point of the Borel integral is

$$s(x) = \int_{x_0}^{x} S_{-1}(x')dx'$$

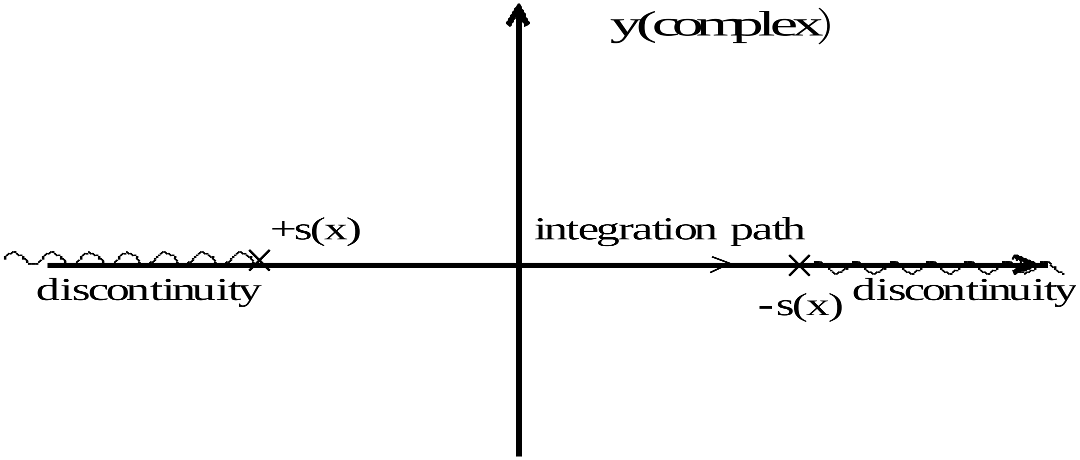
The path of integration



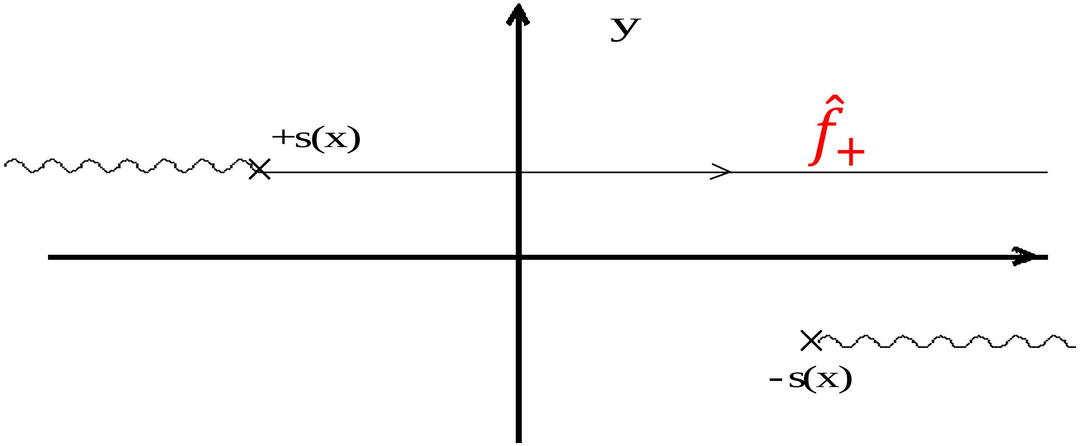
At the Turning point

1. $\pm s$ degenerates
2. WKB diverges

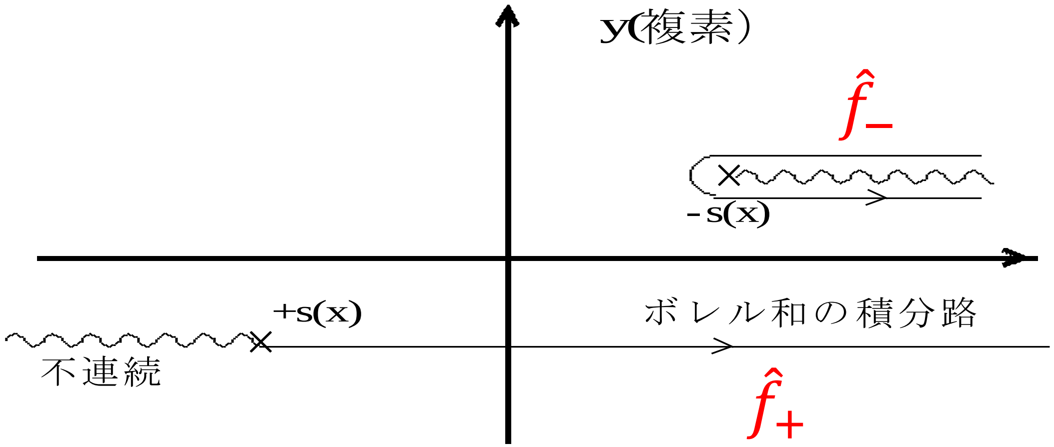
Stokes phenomena around the turning point is the key



Before analytic continuation
（y is real, Im s=0 ）
The integration path steps on
the singularity of f_B



Singularity can be avoided
by analytic continuation
（ Im s>0）

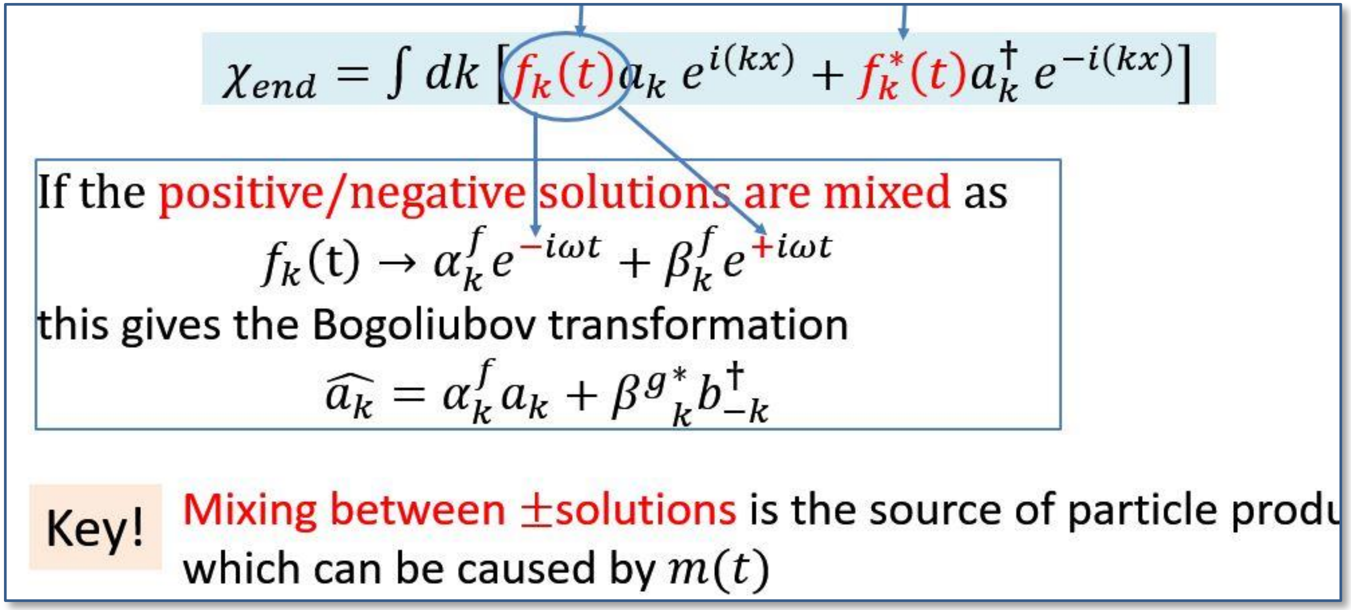


Deformation from Im s>0
to Im s<0 gives an additional
contribution
Stokes phenomena generates
$\hat{f}_-$ from $\hat{f}_+$
(after careful calculation)

Previously we said...

“Mixing between $f^{\pm}$” ⇔ “Bogoliubov” ⇔ “NP particle production”

Slide #11



$$\chi_{end} = \int dk\, [f_k(t) a_k\, e^{i(kx)} + f_k^*(t) a_k^\dagger\, e^{-i(kx)}]$$

If the positive/negative solutions are mixed as

$$f_k(\mathrm{t}) \to \alpha_k^f e^{-i\omega t} + \beta_k^f e^{+i\omega t}$$

this gives the Bogoliubov transformation

$$\widehat{a_k} = \alpha_k^f a_k + \beta^{g}{}^{*}_{k} b^{\dagger}_{-k}$$

Key! Mixing between ±solutions is the source of particle produ which can be caused by $m(t)$

Now we can add to these relations...

“Stokes phenomenon” ⇔ “Mixing between $f^{\pm}$”

⇔ “Bogoliubov” ⇔“NP particle production”

Of course, the connection formula of the exact solution considers the stokes phenomenon.
Not a new thing. Very common.

The simplest example $Q(x) = x$

Airy type〜Near TP

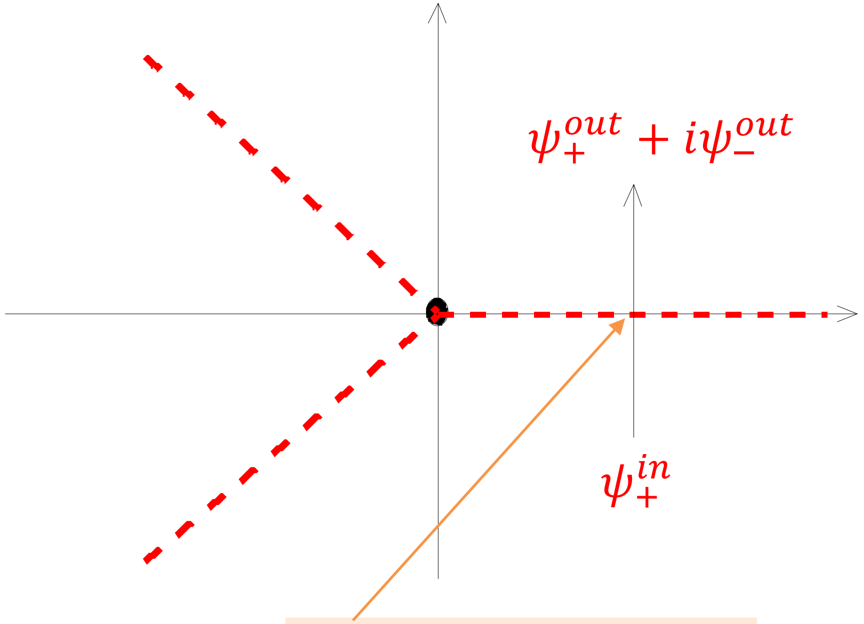
$$\left(-\frac{d^2}{dx^2} + \eta^2 x\right)\psi(x,\eta) = 0$$

$$S_{-1} = \pm\sqrt{x} \quad \Leftrightarrow \quad \int_0^x S_j dx' = -\frac{2}{3j} c_j x^{-\frac{3j}{2}}$$

Stokes curve is defined by $Im\ x^{3/2} = 0$
（So you can see three lines from $x = 0$）

This may be called
“anti-stokes line”
We follow WKB papers

The origin is the turning point$(Q(x) = 0)$



$Im\ s$ changes the sign

If $Re\left[\int_{x_0}^{x} \sqrt{Q_0(x')}dx'\right] > 0$

then $\begin{cases} \psi_+^{in} = \psi_+^{out} \pm i\psi_-^{out} \\ \psi_-^{in} = \psi_-^{out} \end{cases}$

*Stokes of Airy

Applying this idea widely, one can find

2. Connection formula of Voros

Passing across the stokes line

$$Re\left[\int_{x_0}^{x}\sqrt{Q_0(x')}dx'\right] < 0 \quad => \quad \begin{cases} \psi_+^{in} = \psi_+^{out} \\ \psi_-^{in} = \psi_-^{out} \pm i\psi_+^{out} \end{cases}$$

$$Re\left[\int_{x_0}^{x}\sqrt{Q_0(x')}dx'\right] > 0 \quad => \quad \begin{cases} \psi_+^{in} = \psi_+^{out} \pm i\psi_-^{out} \\ \psi_-^{in} = \psi_-^{out} \end{cases}$$

＋ for an “anti-clockwise” motion around the turning point

“in” and “out” are the solution in the former and the latter area

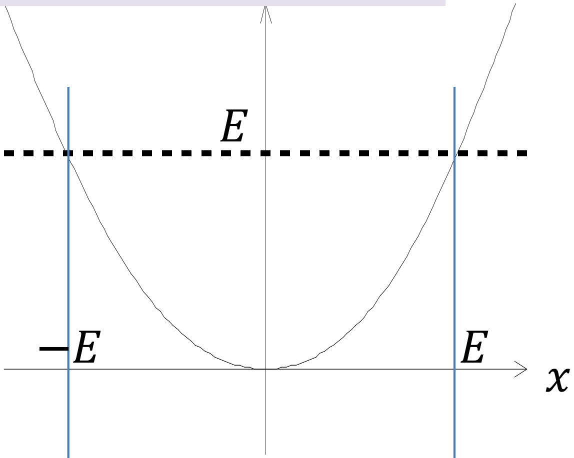
＊If the base point cannot be shared by “in” and “out”,
one has to replace it

Stokes lines

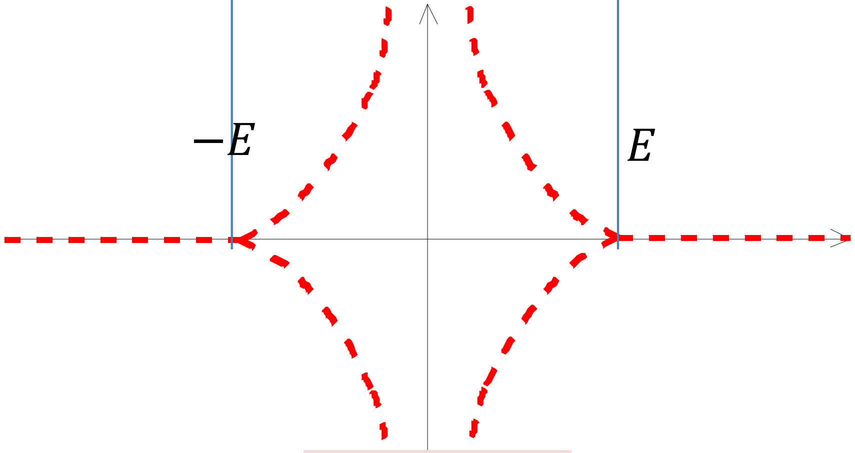
Eigenstate problem

$-Q(x) = E - x^2$

Potential: $V(x) = x^2$



analytic cont. $x \to z$

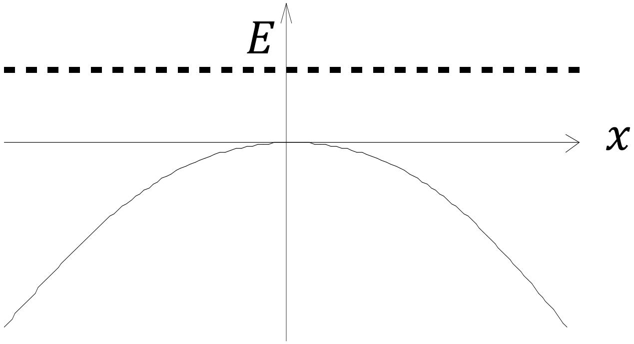


Stokes lines

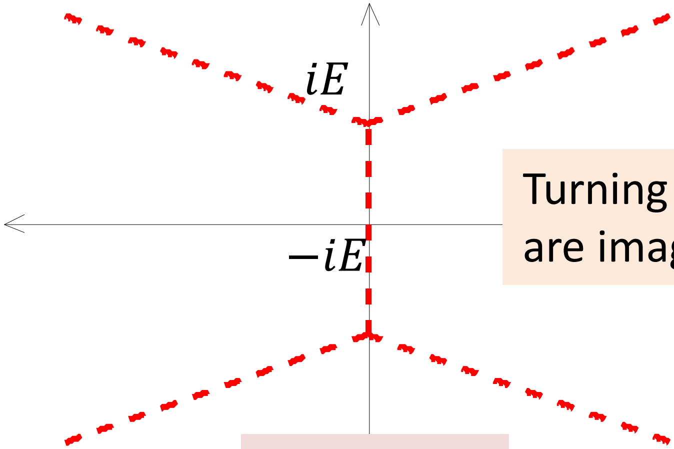
Scattering problem

$-Q(x) = E + x^2$

Potential : $V(x) = -x^2$



analytic cont. $x \to z$

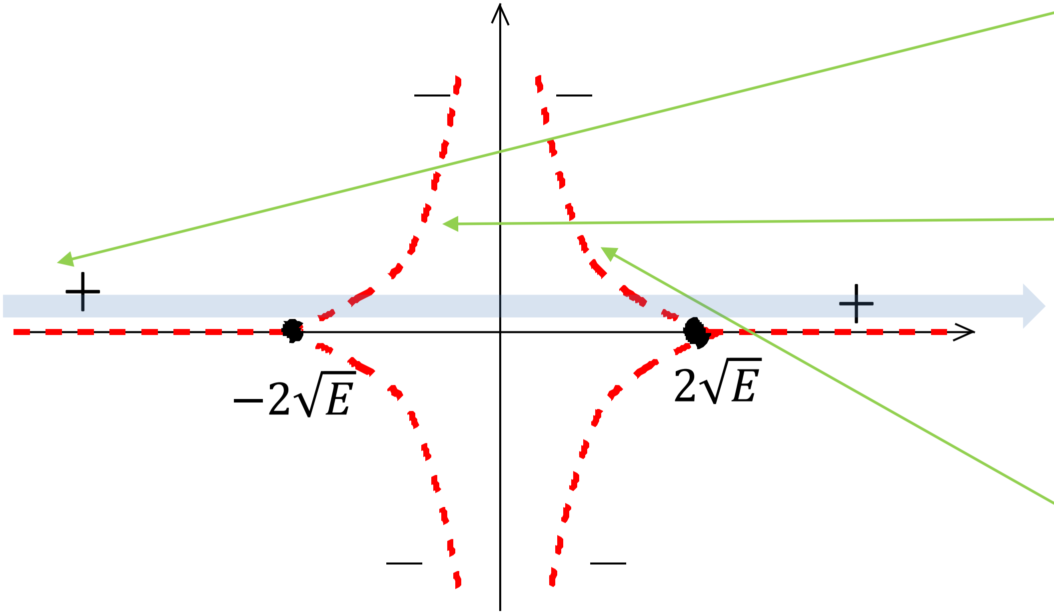


Turning points are imaginary

Stokes lines

Eigenstate problem for a harmonic oscillator

$$Q(x) = \frac{1}{4}x^2 - E$$



Boundary condition is
$\psi(\pm\infty) \to 0$
ψ_- => from the left

The first stokes line（−）gives
$\psi_- \to \psi_- - i\psi_+$
$(-2\sqrt{E}$ is the shared base point)

The second line（−）gives
$\psi_- - i\psi_+ \to \psi_- - iU\psi_+$
$(-2\sqrt{E} \Rightarrow 2\sqrt{E})$ replaced

$$U = 1 + \exp\left[-2\int_{-2\sqrt{E}}^{2\sqrt{E}} S_{odd}\,dx\right] = 1 + \mathrm{e}^{-2\pi i\eta E}$$

$\psi(\pm\infty) \to 0 \Leftrightarrow$ U=0
$\Leftrightarrow E = \eta^{-1}\left(N + \frac{1}{2}\right)$

Rather trivial: Identical result can be found from the conventional WKB approximation
Perhaps you have seen it in QM class

Extension to “Anharmonic” oscillator

Not “exact WKB analysis”

Bender and Wu : “Anharmonic Oscillator”
Phys.Rev. 184 (1969) 1231-1260

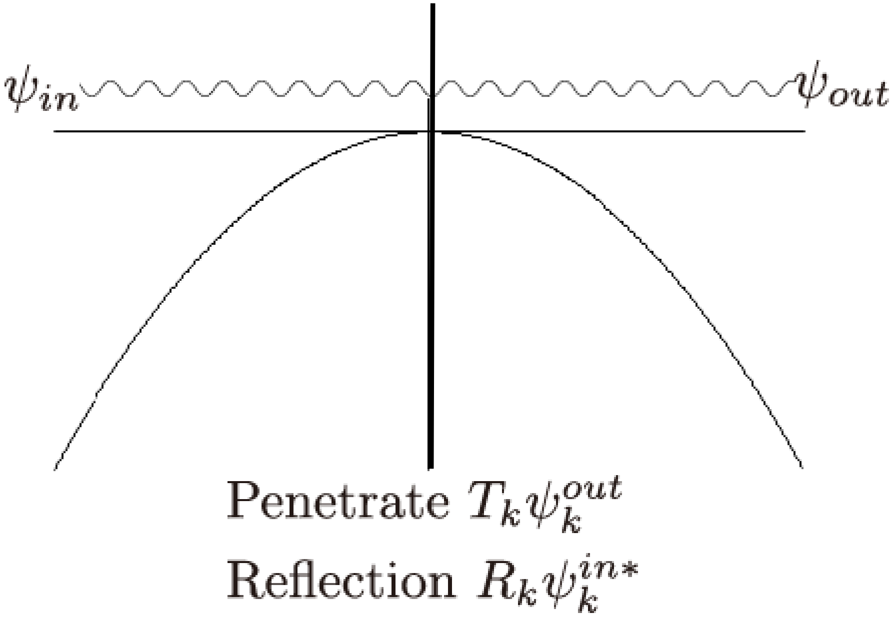
“Exact WKB analysis : Connection formula of Voros”

Aoki, T., T. Kawai and Y. Takei,
“The Bender-Wu analysis and the Voros theory”,
ICM-90 Satellite Conference Proceedings, Special Functions, Springer-Verlag, 1991, pp. 1-29.
Kawai, T. and Y. Takei,
“Secular equations through the exact WKB analysis”,
Proc., "Algebraic Analysis of Singular Peturbations".
など

EWKB for the Scattering problem

Equation of motion

$$[\partial_t^2 + k^2 + g^2(\mu^2 + v^2 t^2)]u_k = 0$$



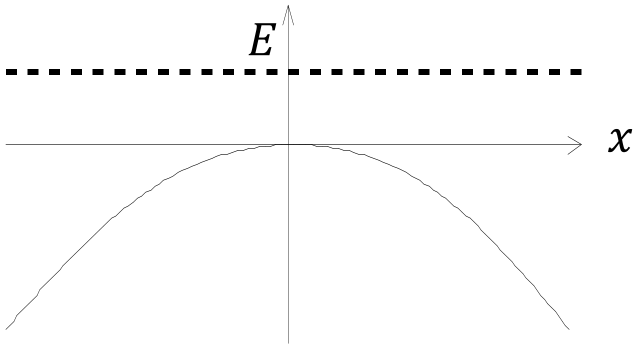
$t \rightarrow x$ gives Schrödinger Eq.

Parabolic Cylinder function gives the exact solution but…

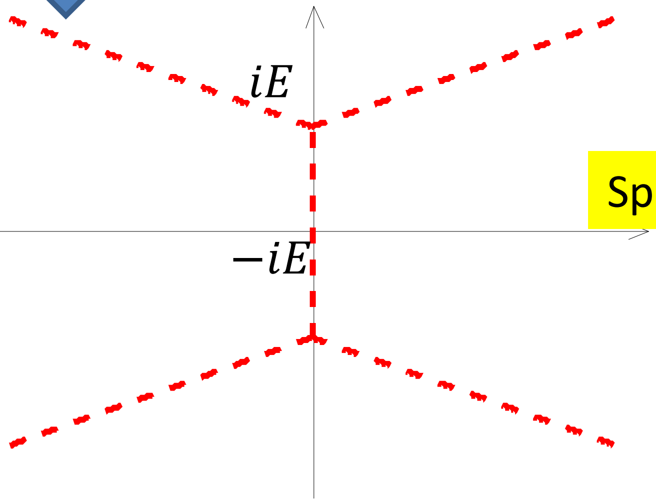
One can choose exact WKB to get the connection formula

$-Q(x) = E + x^2$

$V(x) = -x^2$



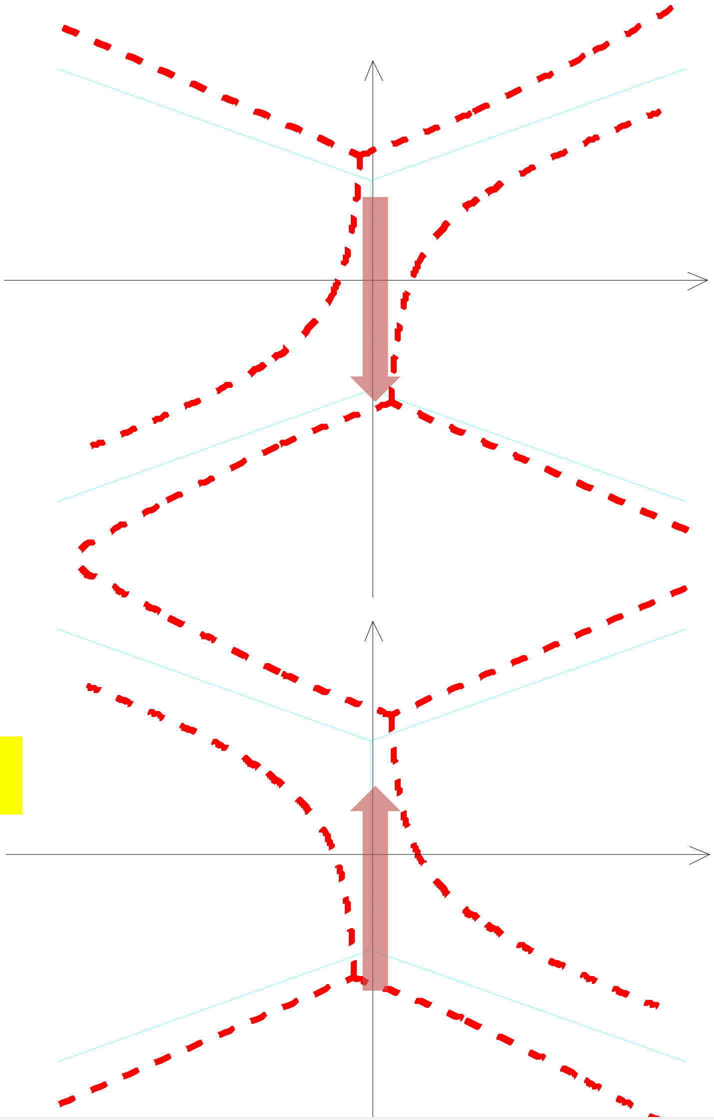
real to complex($x \to z$)



Split by $\eta \pm i\epsilon$

Depends on the sign of $\eta \pm i\epsilon$ but the final results are identical(as expected from the monodromy)

Delabaere, Dillinger, Pham('97)



Exact WKB analysis is obviously useful for calculating nPT particle production

Outline

Nonperturbative particle production

Bogoliubov transformation

Mixing of $\psi_{\pm}$ asymptotic solutions

Connection formula of Voros (EWKB)

We are Here!
Interaction is the key

Asymmetry requires a “singularity”

Explains nonperturbative version of
Spontaneous baryogenesis
(Chemical potential＝△)

EWKB gives exact formula of the asymmetry (!)

Previous study on “interaction” and the nP particle production

Introducing interaction in Preheating

“Quenching preheating by light fields”
O.Czerwińska, S,Enomoto, Z.Lalak
Phys.Rev. D96 (2017) 023510

"Influence of interactions on particle production induced by time-varying mass terms"
Seishi Enomoto, Olga Fuksińska, Zygmunt Lalak
JHEP 1503, 113 (2015)

These works consider
WKB Approximation (not exact WKB)
+ Perturbation

To understand thermalization of the Universe,
interaction is very important.
But, this approach is not enough to understand the asymmetry.

Previous study on “asymmetry” and the nPT particle production

``Baryogenesis during reheating in natural inflation and comments on spontaneous baryogenesis,''
A. Dolgov, K. Freese, R. Rangarajan and M. Srednicki,
Phys. Rev. D56, 6155 (1997) [hep-ph/9610405].

Spontaneous baryogenesis using
non-perturbative particle production

Their claim: Rotation in B-violating interaction => chemical potential
Because of the chemical potential, the asymmetry is generated.

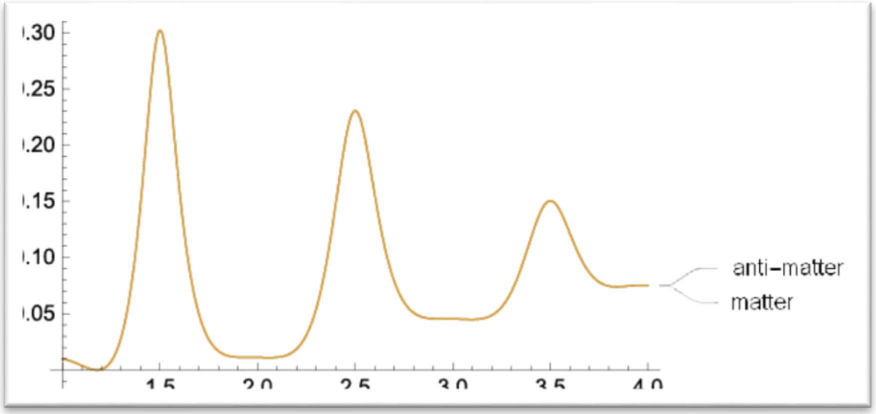
This paper includes very important idea for
solving asymmetry problem in non-perturbative particle production.
BUT
“Spontaneous baryogenesis” is based on the chemical potential
and therefore Quite misleading ---why?

For more details, see
“Baryogenesis from the Berry phase”
Phys.Rev. D99 (2019) no.3, 036005
1811.06197

Chemical potential in non-perturbative particle production

A complex scalar χ (free) + $m_\chi(t)$ + chemical potential

$$L_{chem} = -\frac{\partial\varphi}{M_*} J^\mu, \quad J^\mu = -i(\chi\partial^\mu\chi^* - \chi^*\partial^\mu\chi)$$



No asymmetry!

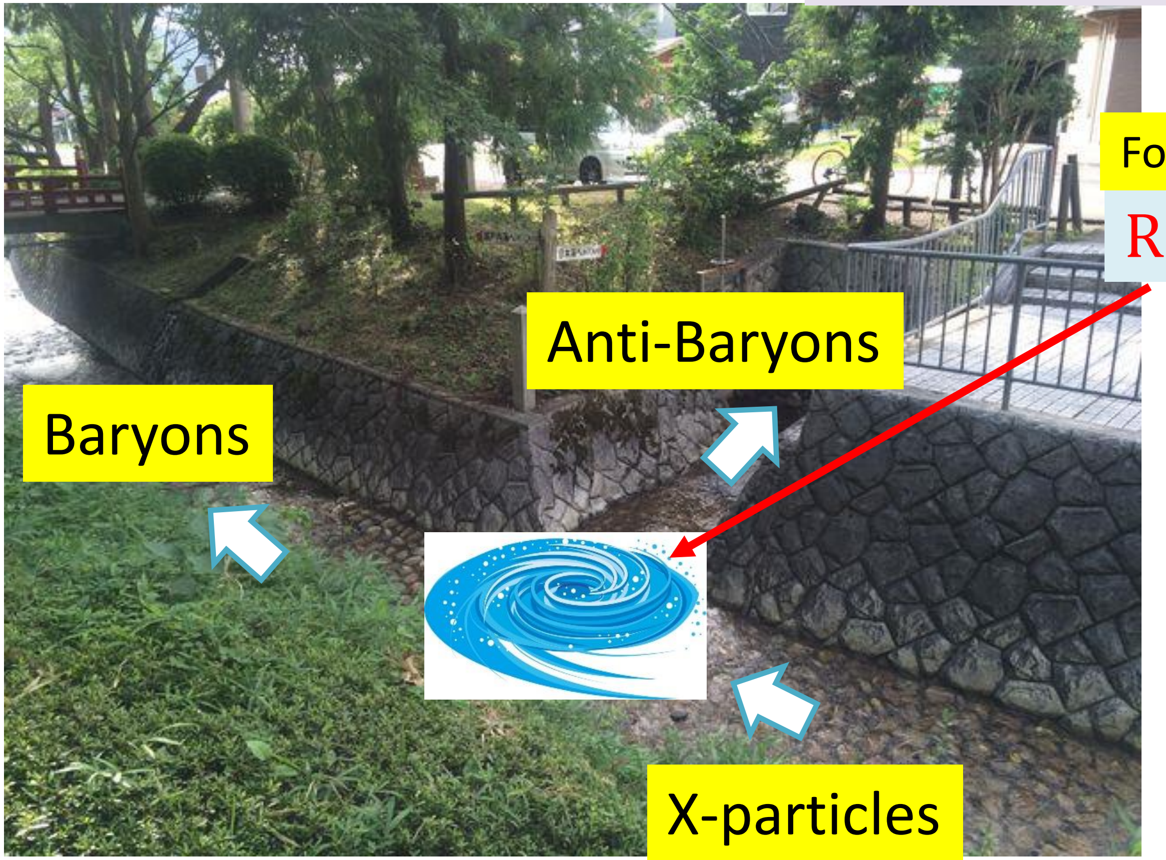
Indeed, calculating the Hamiltonian one can easily find that the chemical potential goes away.

Strictly speaking, L_{chem} is not a chemical potential

Redefining

Asymmetry in spontaneous baryogenesis

A. Dolgov, K. Freese, R. Rangarajan and M. Srednicki



Forget the “Chemical potential”

Rotation of B-violating Int

⇕ Different

GUT baryogenesis
Interference

idea

If there is an asymmetry in particle production, there MUST be something at the branch(=interaction).

Our focus is now clear !

How to understand “rotation in the interaction”

Normally, the non-perturbative particle production uses $m(t)$
In the equation of motion of matter-antimatter system(2×2),
$m(t)$ appears in the diagonal element.

Since “interaction” appears in the off-diagonal,
this is Motion in the off-diagonal element

Example

Scalar field with a CP violation $\sim \left[G(t)\chi^2 + G(t)^*{\chi^*}^2\right]$

Expand $\chi = \int \frac{d^3p}{2\omega(2\pi)^3}\,[a_k e^{-i\omega t} + b^\dagger_{-k} e^{+\omega t}]$ at $t = \pm\infty$

$\chi = \int \frac{d^3p}{2\omega(2\pi)^3}\,[a_k f_k(t) + b^\dagger_{-k} g_k^*]$ during evolution

One will find

$$\begin{bmatrix} \partial_t^2 + k^2 + m^2 & G(t)^* \\ G(t) & \partial_t^2 + k^2 + m^2 \end{bmatrix} \begin{pmatrix} f_k \\ g_k \end{pmatrix} = 0$$

Differential equations of f_k and g_k are now 4th order.

*Can be reduced to 2nd order using conventional perturbative expansion

This is what Dolgov, Freese et. al. considered in their paper

Let us see their strategy!

Perturbation

0th

$$\begin{bmatrix}\partial_t^2+k^2+m^2 & 0\\ 0 & \partial_t^2+k^2+m^2\end{bmatrix}\begin{pmatrix}f_0\\ g_0\end{pmatrix}=0$$

⇒

Asymptotic form

$$\begin{pmatrix}f_0\\ g_0\end{pmatrix}=\begin{pmatrix}e^{-i\omega t}\\ e^{+i\omega t}\end{pmatrix}$$

1st

$$\begin{bmatrix}\partial_t^2+k^2+m^2 & 0\\ 0 & \partial_t^2+k^2+m^2\end{bmatrix}\begin{pmatrix}f_1\\ g_1\end{pmatrix}+\begin{bmatrix}0 & G(t)^*\\ G(t) & 0\end{bmatrix}\begin{pmatrix}f_0\\ g_0\end{pmatrix}=0$$

⇓

$$\begin{bmatrix}\partial_t^2+k^2+m^2 & 0\\ 0 & \partial_t^2+k^2+m^2\end{bmatrix}\begin{pmatrix}f_1\\ g_1\end{pmatrix}=\begin{pmatrix}G(t)^*e^{-i\omega t}\\ G(t)e^{+i\omega t}\end{pmatrix}$$

⇓ Fourier

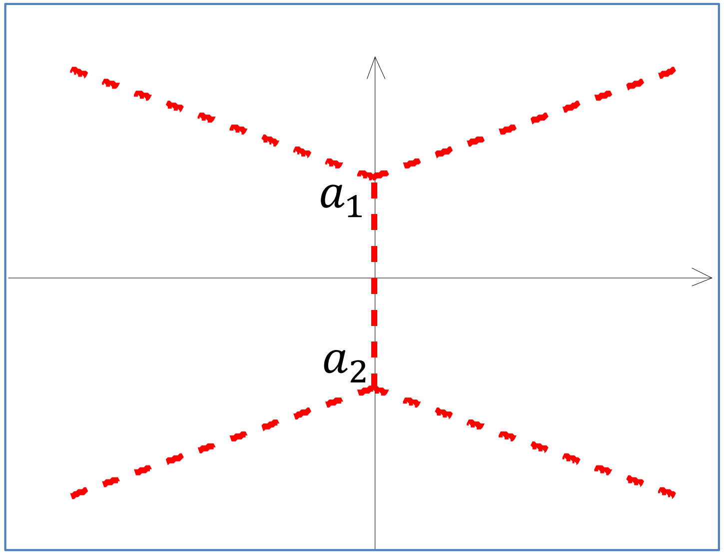
$$\begin{pmatrix}f_1\\ g_1\end{pmatrix}=\begin{pmatrix}\int\frac{d\omega'}{2\pi}\frac{\tilde{G}(\omega'-\omega)^*}{\omega'^2-\omega^2}e^{-i\omega' t}\\ \int\frac{d\omega'}{2\pi}\frac{\tilde{G}(\omega'-\omega)}{\omega'^2-\omega^2}e^{+i\omega' t}\end{pmatrix},\qquad \begin{aligned}\tilde{G}(\omega)&\equiv\int dt\;\,G(t)e^{i\omega t}\\ \tilde{G}(\omega)^*&\equiv\int dt\;\,G(t)^*e^{i\omega t}\end{aligned}$$

Pole at $\omega'=-\omega$ gives $e^{+i\omega t}$ (mixing) in f_k ⇔ particle production

$\left|\frac{\tilde{G}(-2\omega)^*}{2\omega}\right|\neq\left|\frac{\tilde{G}(-2\omega)}{2\omega}\right|$ gives the asymmetry

Easy to find an example
BUT the "origin" is unclear

What is the origin of the asymmetry in EWKB?



Sample:
Stokes line of Scattering on an inverse
quadratic potential
Analytic continuation of the time (t=>z)

$Q(z) = \Pi(z - a_i)$,
a_i are the turning points

$CP: a_i \rightarrow a_i^*$

CP flips the imaginary axis

IF [Turning points + stokes lines] only, CP cannot generate asymmetry (seems obvious, not proved)

Particle production is indistinguishable between matter/antimatter.

In this case, Asymmetry is impossible

Where is the way out?

What distinguishes matter/antimatter in the exact WKB?

Hint

A “Singularity” appears when interaction vanishes

Rotational interaction of SPB ⇔ Rotation around Singularity

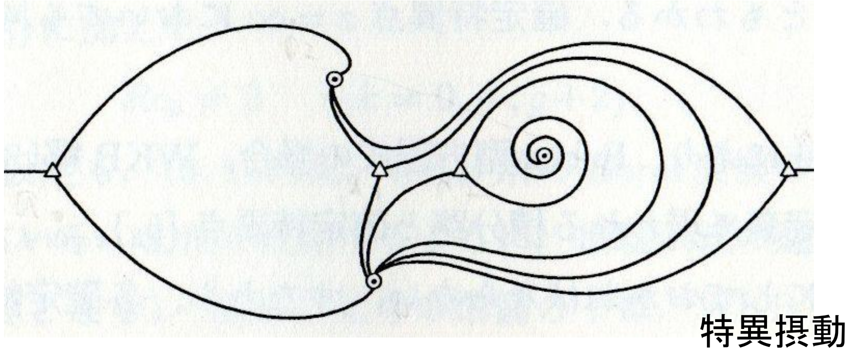
More naively,

Singularity = the origin of the asymmetry (!?)

EWKB for Fuchsian type differential equation(sample)

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x)\right)\psi = 0, \qquad Q(x) = \frac{(x^2-9)\left(x^2-\frac{1}{9}\right)}{(x^3-e^{i\pi/8})^2}$$

For Math
Connection formula of Voros is a Powerful tool for calculating monodromy around singularities



One can draw Stokes lines
◎=Regular singularity (Denominator)
△=Turning point (Numerator)

特異摂動の代数解析学 （河合・竹井）より

CP ⇔ Flip of the Imaginary Axis

Turning point (trivial monodromy)

Singularity (nontrivial)

△

|| $n_B = \bar{n}_B$

△

◎ anticlockwise

∦ $n_B \neq \bar{n}_B$

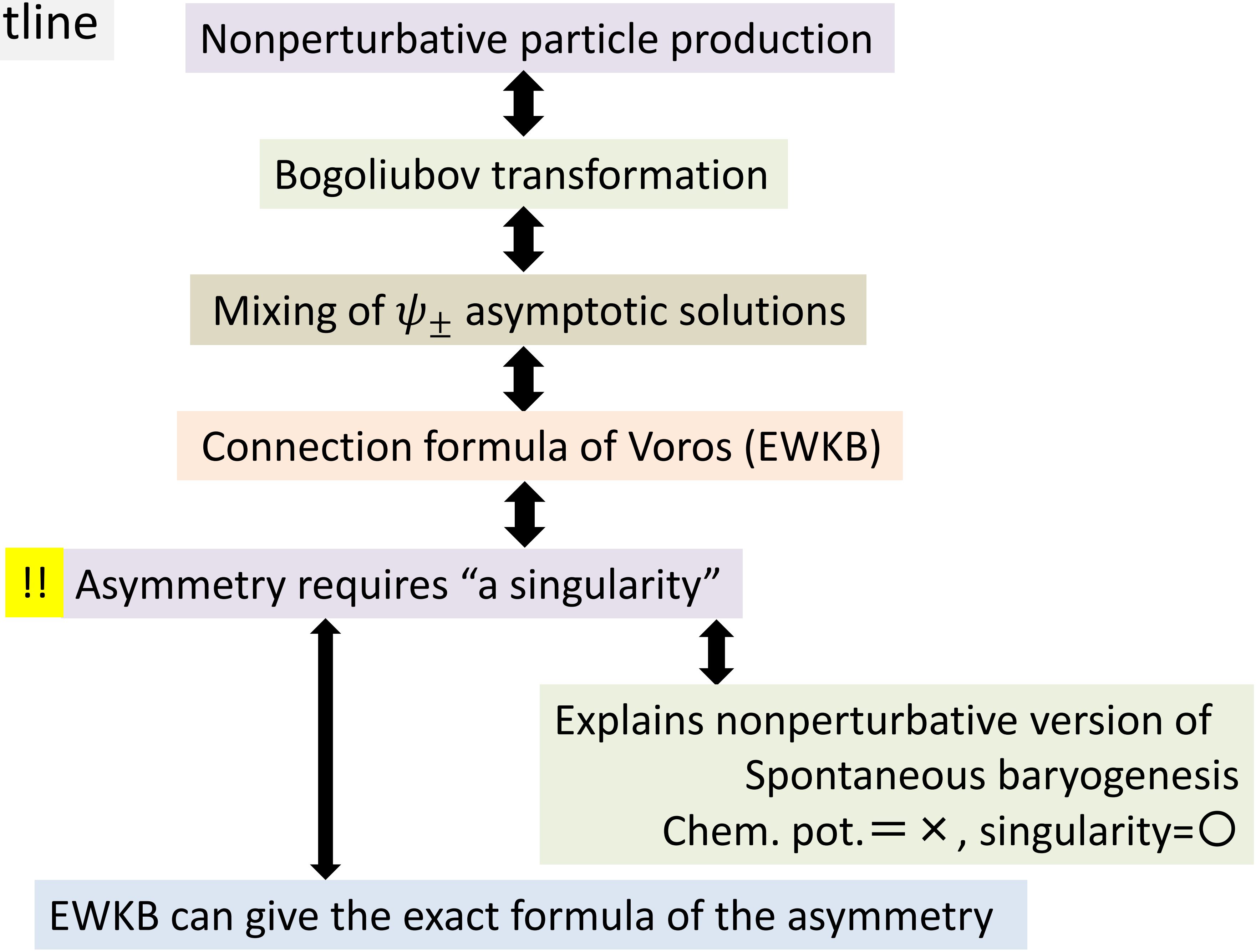
◎ clockwise

The Voros’s connection formula can be extended to include “stokes line ending at a singularity”

Then, the connection factor is determined by the characteristic exponent (特性指数）

The asymmetry (the baryon number) could be determined by the characteristic exponent
(Is our expectation true?)

Outline



Nonperturbative particle production

Bogoliubov transformation

Mixing of $\psi_{\pm}$ asymptotic solutions

Connection formula of Voros (EWKB)

!! Asymmetry requires “a singularity”

Explains nonperturbative version of
Spontaneous baryogenesis
Chem. pot.＝×, singularity=○

EWKB can give the exact formula of the asymmetry

To introduce B(L)-violating interaction we consider a toy model with the Majorana fermion, since the order of the differential equation is two (lower than the scalar particle).

To avoid confusions please remember

$$(\nu_L)^c = \nu_R^c$$

For the Majorana fermions and
the helicity is defined in the massless limit

Baryogenesis (realistic) requires dirty calculation
We are avoiding this complexities in this seminar.

See also
“Particle production with left-right neutrino oscillations”
SE and TM, PhysRevD.93.063504 arXiv:1602.07454

The simplest example of non-perturbative baryogenesis

Majorana Fermion(2 × 2)

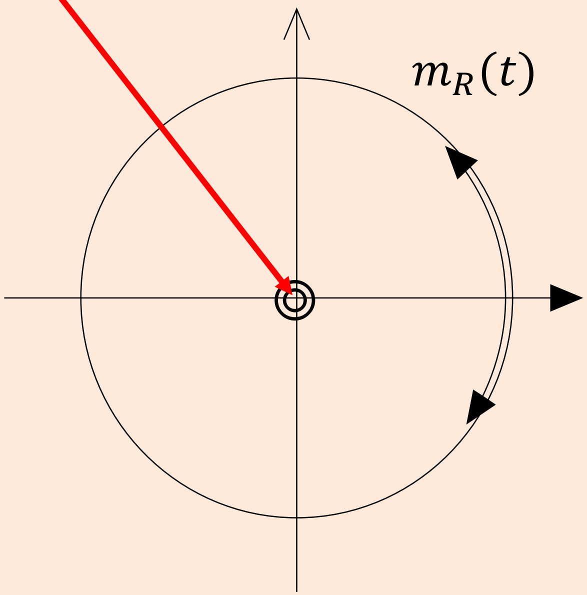
For $\Psi_R^t \equiv (\psi_R, \psi_L^c)$,

$$\mathcal{L}_m = \bar{\Psi}_R \left(\begin{array}{cc} 0 & m_R \\ m_R^* & 0 \end{array} \right) \Psi_R.$$

The Off-diagonal element is the Majorana mass

Singularity

Rotational osc.



$$m_R(t) = M_R e^{i\theta(t)}$$

If you expand,

Discriminating

$$(\psi_R)_\alpha = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=\pm} (e^s_{\boldsymbol{k}})_\alpha$$

$$\times \left[u^s_k(x^0) a^s_{\boldsymbol{k}} + s v^s_{\boldsymbol{k}}(x^0) a^{s\dagger}_{-\boldsymbol{k}} \right],$$

The EOM becomes

$$(i\partial_t + s|\boldsymbol{k}|)u_{\boldsymbol{k}}^s = s\boxed{m_R^* v_{\boldsymbol{k}}^s},$$
$$(i\partial_t + s|\boldsymbol{k}|)v_{\boldsymbol{k}}^{s*} = -s\boxed{m_R^* u_{\boldsymbol{k}}^{s*}}.$$

Off-diagonal

This equation can be converted into
a famous “Landau-Zener” 2-state transition model

Setting $\Psi^t \equiv (v_k^s, u_k^s)$, we find

$$i\frac{d}{dt}\Psi = H\Psi$$

$$H = \begin{pmatrix} -s|k| & s\,M_R\boxed{e^{-i\theta(t)}} \\ s\,M_R\boxed{e^{i\theta(t)}} & s|k| \end{pmatrix}$$

$$\theta(t) = \theta_0(t)\cos m_\theta\, t$$

Rotational Oscillation

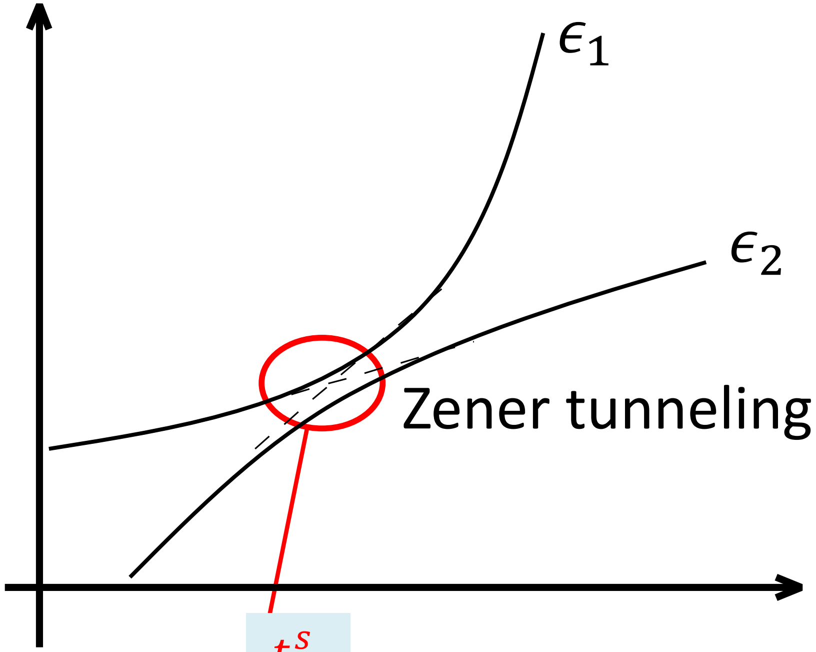
Using $\psi^R \equiv e^{i\theta}\psi$, "$e^{i\theta}$" in the off-diagonal can be removed

$$i\frac{d}{dt}\psi^R = \hat{H}^R\psi^R,$$

$$\hat{H}^R = \begin{pmatrix} \overset{\epsilon_1}{-s|\boldsymbol{k}| + \frac{\theta_0 m_\theta}{2}\sin m_\theta t} & sM_R \\ sM_R & \overset{\epsilon_2}{s|\boldsymbol{k}| - \frac{\theta_0 m_\theta}{2}\sin m_\theta t} \end{pmatrix}.$$

This 2-state model gives “Landau-Zener tunneling”

*approximation at the crossing



t_k^s

Landau-Zener tunneling gives transition rate at the crossing

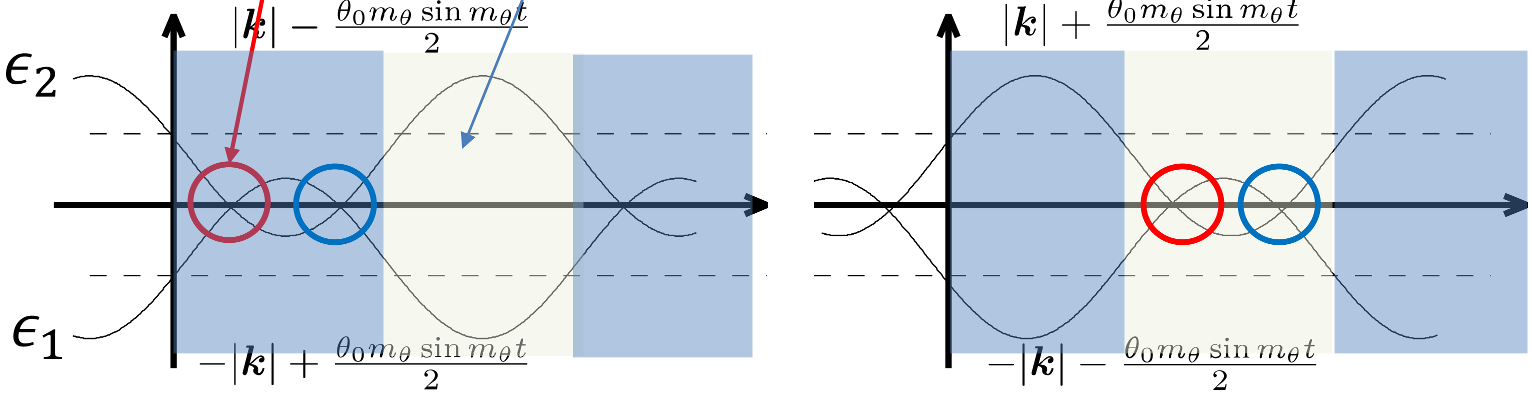
Bogoliubov trans.
(number densities)

$$\beta_k^s \simeq P_k^s \simeq e^{-\pi\, p_k^s}$$

$$p_k^s \equiv \frac{2M_R^2}{|\theta_0^s m_\theta^2 \cos m_\theta t_k^s|}$$

*Helicity ($s = \pm 1$)
flips the sign

“Return path” gives nothing



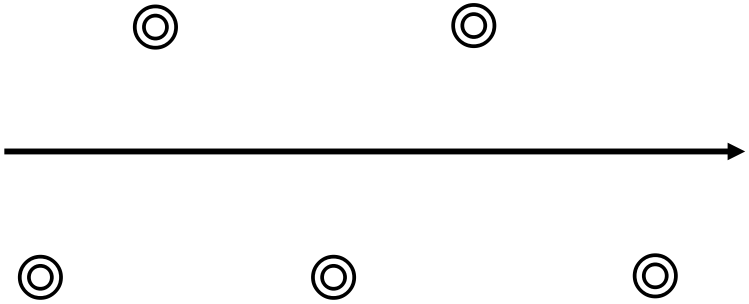
This picture shows “particle production is not simultaneous”
After the 2nd half, total asymmetry remains if the oscillation damps

Where is the "singularity"?

Off-diagonal element vanishes

$m(t) = M_R e^{i\theta(t)}$ for $\theta(t) = A\cos t$

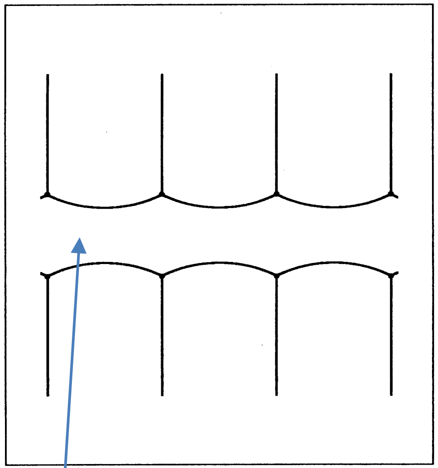
$$
\begin{aligned}
&m(t) = 0 \\
&\quad \mathrm{i}\theta \Rightarrow -\infty \\
&\quad e^{it} + e^{-it} \Rightarrow i\infty \\
&\quad t_{-} = \left(\frac{\pi}{2} - i\infty\right) + 2n\pi \\
&\quad t_{+} = \left(-\frac{\pi}{2} + i\infty\right) + 2n\pi
\end{aligned}
$$



Not "Symmetric" with respect to the flip of Im axis

Unfortunately…the characteristic exponent of these singularities are trivial
We are just seeing the effect of alternate singularity $t_{\pm}$

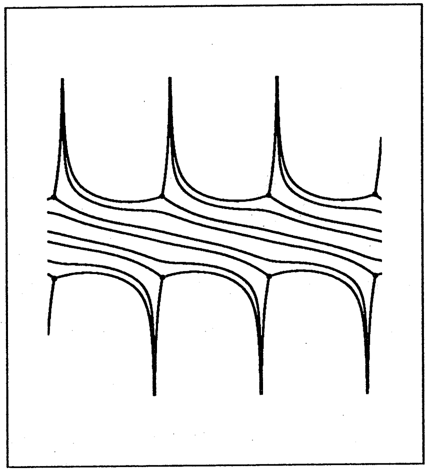
What if the particles do not decay?
--- Landau-Zener is not a good approximation



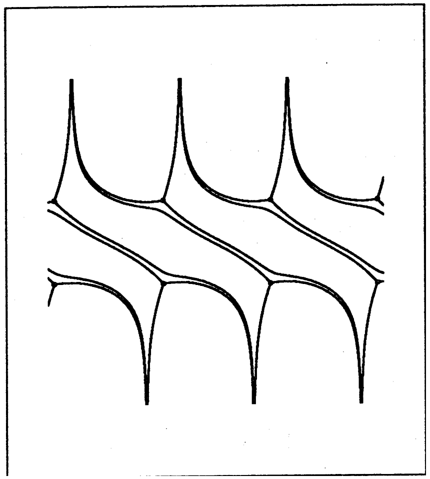
$E > V(t) = \cos t$

“scattering by a shallow bumps”

$\eta \to \eta + i\epsilon$



$\epsilon \ll 1$



$\epsilon \simeq 1$

Infinitely many stokes lines are degenerated
(Scattering from infinitely many bumps)

“Exact WKB analysis for Schorödinger equations with periodic potentials” T.Koike
数理解析研究所講究録 (1999), 1088: 22-38

cannot see the line because of the boundary in PC
(This is a numerical calculation)

What is the crucial difference from the perturbative approach (A. Dolgov, K. Freese, R. Rangarajan and M. Srednicki)?

Previously we said "for the perturbative expansion"

$\left|\frac{\tilde{G}(-2\omega)^*}{2\omega}\right| \neq \left|\frac{\tilde{G}(-2\omega)}{2\omega}\right|$ gives the asymmetry

This looks like an interference between terms

A simple (perturbative) calculation shows that

$$G(t) = a + i\, b\, t$$

cannot generate asymmetry (i.e, no interference)

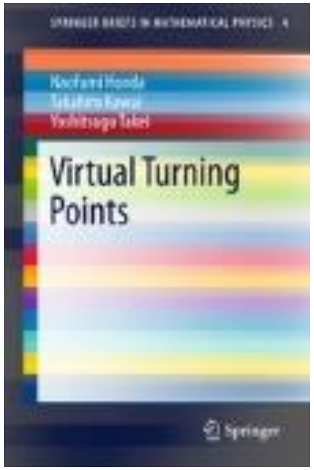
But, in the light of the EWKB, the position of the singularity is important.

$$G(t) = 0 \quad \Leftrightarrow \quad \hat{t} = \frac{ia}{b}$$

Since the evolution path is on the real axis, $\hat{t} \neq \widehat{t_*}$ has to introduce asymmetry

*This work is Still in progress

Beyond the 2nd order equations


Virtual Turning Points

Springer

Virtual Turning Points
Authors: **Honda**, Naofumi, **Kawai**, Takahiro, **Takei**, Yoshitsugu

The discovery of a virtual turning point truly is a breakthrough in WKB analysis of higher order differential equations. As M.V. Fedoryuk once lamented, global asymptotic analysis of higher order differential equations had been thought to be impossible to construct. In 1982, however, H.L. Berk, W.M. Nevins, and K.V. Roberts published a remarkable paper indicating that the traditional Stokes geometry cannot globally describe the Stokes phenomena of solutions of higher order equations; a new Stokes curve is necessary.

Traditional Stokes line is NOT ENOUGH to analyze Higher differential equation
Breakthrough is the discovery of a new stokes line and a new “Turning Point”

We have no time for the discussion.

Conclusion and discussions

Our work started with "Preheating with higher dimensional operator".

This interaction is important, since

1. $O\left(M_p^{-n}\right)$ gravitational int. naturally violates Global symmetry
2. Singlet inflaton may not have renormalizable int. with SM

Then we came to know

1. Baryogenesis by preheating is not well understood (Except for a "decay of a heavy particle" scenario)
2. Spontaneous Baryogenesis (with chemical potential) has a (non-trivial) problem in its setup
3. Resurgence is widely used for solving eigenstate problems, but people are not using it for preheating
4. Asymmetry requires "asymmetric singularity"

Required condition for baryogenesis

The origin of the asymmetry can be revealed using EWKB.

Phenomenological arguments(toward thermalization after inflation) requires (dirty) numerical calculation

運動方程式は

$$(i\partial_t + s|\boldsymbol{k}|)u^s_{\boldsymbol{k}} = s m^*_R v^s_{\boldsymbol{k}},$$

$$(i\partial_t + s|\boldsymbol{k}|)v^{s*}_{\boldsymbol{k}} = -s m^*_R u^{s*}_{\boldsymbol{k}}.$$

2準位状態を $\Psi^t \equiv (v^s_k, u^s_k)$ とすると

$$i\frac{d}{dt}\Psi = H\Psi$$

$$H = \begin{pmatrix} -s|k| & s\, m^*_R(t) \\ s\, m_R(t) & s|k| \end{pmatrix}$$

2階の常微分方程式へ

$$\left[\partial_t^2 - \frac{\dot{m}^*_R}{m^*_R}\partial_t + |\boldsymbol{k}|^2 + is|\boldsymbol{k}|\frac{\dot{m}^*_R}{m^*_R}\right]u^s_{\boldsymbol{k}} = -|m_R|^2 u^s_{\boldsymbol{k}}.$$

$u_{\boldsymbol{k}}^s = e^{\int \frac{\dot{m}_R^*}{2m_R^*} dt} U_{\boldsymbol{k}}^s,$ という定番の置き換えで

特徴は$m_R^* = 0$ のPole

$$\ddot{U}_{\boldsymbol{k}}^s + \underbrace{\left[\frac{d}{dt}\left(\frac{\dot{m}_R^*}{m_R^*} \right) - \frac{1}{4}\left(\frac{\dot{m}_R^*}{m_R^*} \right)^2 + |\boldsymbol{k}|^2 + |m_R|^2 + is|\boldsymbol{k}| \frac{\dot{m}_R^*}{m_R^*} \right]}_{\boxed{-\mathrm{Q(t)}}} U = 0$$