Classical Limit of Large N Gauge Theories with Conformal Symmetry

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15 October 2019 at Osaka University

based on the papers:

JHEP 1802 (2018) 019,arXiv:1710.07298 [hep-th],

arXiv:1907.05419 [hep-th].

Introduction

Why quantum gravity (QG) on AdS?

We often consider theory in “box”
in field theory and quantum mechanics
in order to avoid IR problems.
(e.g. to obtain discrete spectrum)

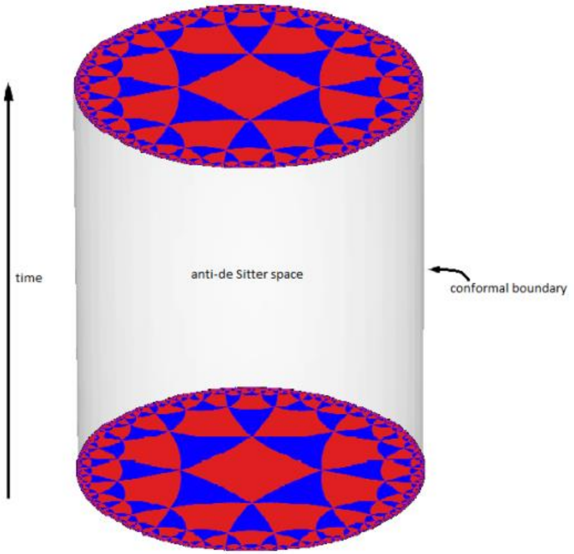
What is “simplest” QG in “box”?

Quantum gravity (QG) on AdS

What is "simplest" QG in "box"?

A. quantum gravity on (asymptotic) AdS

figure from Wikipedia



AdS/CFT (conjecture)

Maldacena

quantum gravity on (asymptotic) AdS

= conformal field theory

This conjecture comes from D-branes in string

Highly non-trivial and important!

Gravity Side:

Difficulties for quantization:

- Perturbatively non-renormalizable
- Un-boundedness of action
- Summations over different topologies

Dual CFT
(for QG with asymptotic AdS)

- Renormalizable,
- Positive definite action,
- No geometries (for finite N)

CFT could be a definition of QG.

However, **no** proof of AdS/CFT

(usual) formulation of AdS/CFT

GKPW relation
(for partition functions)

boundary condition in AdS

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Source terms in CFT

“equivalences of correlators“

Another formulation of AdS/CFT

In operator formalism,

equivalence between
Hilbert spaces and Hamiltonians
of gravity on AdS and CFT

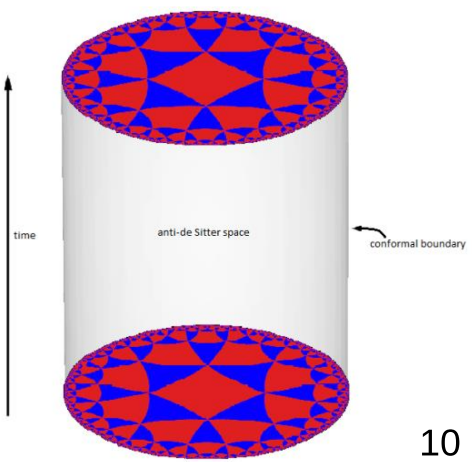
Another formulation of AdS/CFT

equivalence between
Hilbert spaces and Hamiltonians

$$\text{Gravity theory on global } AdS_{d+1}$$

$$\Updownarrow$$

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$



What we will determine explicitly:

Low energy spectrum of CFT_d
which is realized as
large N strong coupling gauge theory

- leading order in large N limit

- Not assuming SUSY, string, D-brane
- Not assuming dual gravity, AdS space
- Not assuming GKPW (nor BDHM)

We will study the spectrum under only 3 assumptions (natural for strong coupling large N CFT):

1. low energy spectrum is determined only by conserved symmetry currents

2. large N factorization, which was shown for perturbation theory

3. complete independence of spectrum except symmetry relations

From these assumptions we can determine low energy spectrum of the theory,

together with the conformal symmetry.

Here, low energy means $\mathcal{O}(N^0)(\ll N^2)$

To do so, we have NOT used any information on the possible gravity dual, or AdS.

We just considered the field theory.

Then, we can show explicitly:

Low energy spectrum of large N CFT_d

$\Updownarrow$ **equivalent!**

Spectrum of free gravity on AdS_{d+1}

The spectrum determine the theory itself.

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From CFT, we can

construct bulk local fields in AdS

derive the GKPW relation

How about 1/N corrections?
(i.e. interactions in bulk theory)

We will consider the classical limit of generic large N gauge theory with conformal symmetry.

We find that
the classical limit of it is Einstein gravity!

“a derivation of AdS/CFT”

Plan

1. **Introduction**
2. **Low energy spectrum of CFT**
3. **Deriving bulk local field and GKPW (will be skipped)**
4. **Classical limit of large N CFT**
5. **Conclusion**

Low energy spectrum of CFT

c.f. Balasubramanian-Kraus-Lawrence
Banks-Douglas-Horowitz-Martinec
Fitzpatrick-Kaplan

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$

unit radious

Conformal mapping from the complex plane to the cylinder (for $d = 2$)

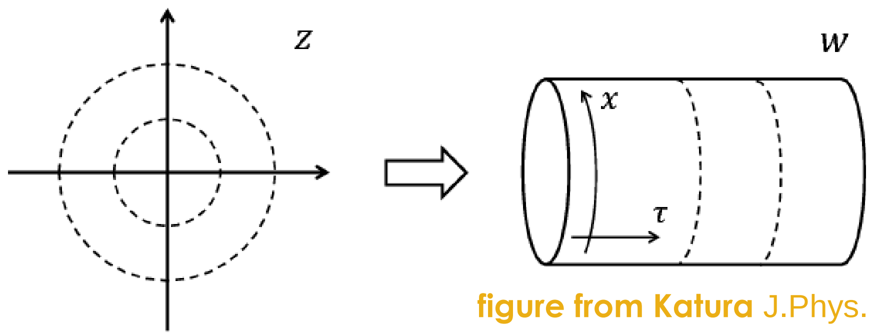

figure from Katura J.Phys. A45 (2012) 115003

CFT_d on $\mathbf{R}^d$ $\qquad$ CFT_d on $\mathbf{R} \times S^{d-1}$

$\hat{P}_\mu = \text{translations} \longrightarrow \hat{P}_\mu = ?$

$\hat{M}_{\mu\nu} = \text{rotations} \longrightarrow \hat{M}_{\mu\nu} = \text{rotations in } S^{d-1}$

$\hat{D} = \text{dilatation} \longrightarrow \hat{D} = \text{translation in } \mathbf{R} = \boxed{\hat{H}}$

$\hat{K}_\mu = \text{special conformal} \longrightarrow \hat{K}_\mu = ?$

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$

unit radious

Conformal algebra

$$\begin{cases} [\hat{D}, \hat{P}_\mu] = \hat{P}_\mu, \quad [\hat{D}, \hat{K}_\mu] = -\hat{K}_\mu, \\ [\hat{K}_\mu, \hat{P}_\nu] = 2\delta_{\mu\nu}\hat{D} - 2\hat{M}_{\mu\nu} \\ [\hat{D}, \hat{M}_{\mu\nu}] = 0, \ldots \end{cases}$$

Diagonalizing $\hat{H} = \hat{D}$ and "$\hat{M}_{\mu\nu}$",

$\hat{P}$, $\hat{K}$ are "creation" and "annihilation" operators

"Highest weight" represenation

Define primary state, $|\Delta\rangle$, s.t

$$\hat{K}_\mu|\Delta\rangle = 0, \quad \hat{D}|\Delta\rangle = \Delta|\Delta\rangle$$

Then, any state in CFT can be represented as

$$\hat{P}_{\mu_1}\hat{P}_{\mu_2}\cdots\hat{P}_{\mu_l}|\Delta\rangle$$

Primary field $\mathcal{O}_\Delta(x)$

$$\mathcal{O}_\Delta(x=0)|0\rangle = |\Delta\rangle = \hat{\mathcal{O}}_\Delta|0\rangle,$$

where $\hat{\mathcal{O}}_\Delta = \lim_{x\to 0}$ (regular part of $\mathcal{O}_\Delta(x)$)

ex. for $T_{\mu\nu}$ in CFT_2, $\hat{\mathcal{O}}_\Delta = L_{-2}$ or $\tilde{L}_{-2}$

Let us consider **large** N CFT_d

$$\hat{H}(\hat{P}_{\mu_1}\cdots\hat{P}_{\mu_l}|\Delta\rangle) = (\Delta + l)(\hat{P}_{\mu_1}\cdots\hat{P}_{\mu_l}|\Delta\rangle)$$

$\Delta \gg \mathcal{O}(N^0)$ for a generic state

because of the quatum corrections

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Only for symmetry currents,

energy is expected to be $\mathcal{O}(N^0)$

ex. for $T_{\mu\nu}$, $\Delta = d$

analogous to hydrodynamics

Large N factorization

'**t Hooft**

vanishing of connected n-pont func. for $n > 2$

i.e. $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \sum \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \cdots \langle \mathcal{O}_{l-1} \mathcal{O}_l \rangle$

$\Updownarrow$ by Wick theorem

$$[\hat{\mathcal{O}}_{\Delta_a}(x), \hat{\mathcal{O}}_{\Delta_b}(y)] = f(x-y),$$

$\Rightarrow$ $$[\hat{\mathcal{O}}_{\Delta_a}, (\hat{\mathcal{O}}_{\Delta_b})^\dagger] = \delta_{ab}$$

Polynomial

$\Rightarrow$ Low energy states: $R(\hat{P}^\mu, \hat{\mathcal{O}}_{\Delta_a})|0\rangle$

Complete independence

Furthermore, we assume

complete independene of states, $R(\hat{P}^{\mu}, \hat{\mathcal{O}}_{\Delta_a})|0\rangle$,

up to the relation $[\hat{\mathcal{O}}_{\Delta_a}, (\hat{\mathcal{O}}_{\Delta_b})^{\dagger}] = \delta_{ab}$

and symmetry relation, (ex. $\partial_{\mu} J^{\mu} = 0$)

because there is no specific scale for $N \to \infty$

and strong coupling.

Nothing special happens, randomeness or chaos.

Large N CFT spectrum

For scalar field, we conclude that large N CFT spectrum is the Fock space spanned by $\prod_{n,l,m}(\hat{a}^\dagger_{nlm})^{\mathcal{N}_{nlm}}|0\rangle$ where

$$\hat{a}^\dagger_{nlm} \equiv c_{nl}\, s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1}P_{\mu_2}\cdots P_{\mu_l}(P^2)^n\hat{\mathcal{O}}_\Delta$$

$$[\hat{a}_{nlm}, \hat{a}^\dagger_{n'l'm'}] = \delta_{n,n'}\delta_{l,l'}\delta_{m,m'}$$

where

- c_{nl} is the normalization constant
- $s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)}$ is a normalized rank l symmetric traceless constant tensor
- P^μ act on an operator such that $P^\mu\hat{\phi} = [\hat{P}^\mu, \hat{\phi}]$

Energy is given by $[\hat{H}, \hat{a}^\dagger_{nlm}] = \Delta + 2n + l$ and $n = 0, 1, 2, \cdots$ and $l = 0, 1, 2, \cdots$

Thus,
the low energy limit of the large N CFT is a free theory.

What is this free theory?

Thus,
the low energy limit of the large N CFT is
a free theory.

What is this free theory?

A. free theory on AdS space

Free scalar field in AdS_{d+1}

c.f. Breitenlohner-Freedman

The metric of global AdS_{d+1} $(l_{AdS} = 1)$ is

$$ds^2_{AdS} = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\Omega^2_{d-1}$$

where $0 \leq r < \infty$, $-\infty < t < \infty$ and

$d\Omega^2_{d-1}$ is the metric for round unit sphere S^{d-1}

$$= \frac{1}{\cos^2(\rho)}\left(-dt^2 + d\rho^2 + \sin^2(\rho)d\Omega^2_{d-1}\right)$$

where $r = \tan\rho$, $0 \leq \rho < \pi/2$

Boundary of AdS_{d+1} is located at $\rho = \pi/2$

Free scalar field in AdS_{d+1}

The action is

$$S_{scalar} = \int d^{d+1}x \sqrt{-\det(g)} \left(\frac{1}{2} g^{MN} \nabla_M \phi \nabla_N \phi + \frac{m^2}{2} \phi^2 \right)$$

The e.o.m. is

$$0 = -g^{MN} \nabla_M \nabla_N \phi + m^2 \phi^2.$$

We expand ϕ with spherical harmonics $Y_{lm}(\Omega)$,

$$\phi(t, \rho, \Omega) = \sum_{n,l,m} \left(a^\dagger_{nlm} e^{i\omega_{nl} t} + a_{nlm} e^{-i\omega_{nl} t} \right) \psi_{nlm}(\rho) Y_{lm}(\Omega)$$

Then, normalized solution for the e.o.m. is given as

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}} \sin^l(\rho) \cos^\Delta(\rho) \, {}_2F_1 \left(-n, \Delta + l + n, l + \frac{d}{2}, \sin^2(\rho) \right)$$

$$\omega_{nl} = \Delta + 2n + l$$

Gauss's hyper geometric function

where $\Delta = d/2 \pm \sqrt{m^2 + d^2/4}$

Free scalar field in AdS_{d+1}

Then, quantized field is

$$\hat{\phi}(t,\rho,\Omega) = \sum_{n,l,m} \left(\hat{a}^{\dagger}_{nlm} e^{i\omega_{nl}t} + \hat{a}_{nlm} e^{-i\omega_{nl}t} \right) \psi_{nlm}(\rho) Y_{lm}(\Omega)$$

$$\left\{ \begin{aligned} &\psi_{nlm}(\rho) = \frac{1}{N_{nl}} \sin^{l}(\rho) \cos^{\Delta}(\rho)\, {}_2F_1\left(-n, \Delta + l + n, l + \frac{d}{2}, \sin^2(\rho)\right) \\ &\omega_{nl} = \Delta + 2n + l \\ &N_{nl} = (-1)^n \sqrt{\frac{n!\Gamma(l+\frac{d}{2})^2 \Gamma(\Delta + n + 1 - \frac{d}{2})}{\Gamma(n + l + \frac{d}{2})\Gamma(\Delta + n + l)}} \end{aligned} \right.$$

where $\Delta = d/2 \pm \sqrt{m^2 + d^2/4}$

The commutation relation and Hamiltonian are

$$[\hat{a}_{nlm}, \hat{a}^{\dagger}_{n'l'm'}] = \delta_{n,n'}\delta_{l,l'}\delta_{m,m'} \qquad [\hat{H}, \hat{a}_{nlm}] = -\omega_{nl}$$

Free scalar field in AdS_{d+1}

In summary, quantized field is

$$\hat{\phi}(t,\rho,\Omega) = \sum_{n,l,m} \left(\hat{a}^\dagger_{nlm} e^{i\omega_{nl}t} + \hat{a}_{nlm} e^{-i\omega_{nl}t} \right) \psi_{nlm}(\rho) Y_{lm}(\Omega)$$

$$\begin{cases} \omega_{nl} = \Delta + 2n + l \\ \text{where } \Delta = d/2 \pm \sqrt{m^2 + d^2/4} \end{cases}$$

The commutation relation and Hamiltonian are

$$[\hat{a}_{nlm}, \hat{a}^\dagger_{n'l'm'}] = \delta_{n,n'}\delta_{l,l'}\delta_{m,m'} \qquad [\hat{H}, \hat{a}_{nlm}] = -\omega_{nl}$$

Therefore,

(low energy theory of)
large N CFT is equivalent to
a free theory on AdS

under the 3 assumptions.

(We considered the scalar.
We think that this is protected by the SUSY
or it has just accidentally low energy spectrum.)

Plan

1. **Introduction**
2. **Low energy spectrum of CFT**
3. **Deriving bulk local field and GKPW**
4. **Classical limit of large N CFT**
5. **Conclusion**

Deriving bulk local field and GKPW

Local field in bulk

Decompose local operator in bulk
to positive and negative frequency modes as

$$\hat{\phi}(t,\rho,\Omega) = \hat{\phi}^{+}(t,\rho,\Omega) + \hat{\phi}^{-}(t,\rho,\Omega)$$

Then using the map,

$$\hat{a}^{\dagger}_{nlm} \quad \longleftrightarrow \quad c_{nl}\, s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_{\Delta},$$

bulk local operator in CFT description is

$$\begin{aligned}\hat{\phi}^{+}(t=0,\rho,\Omega) &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) \hat{a}^{\dagger}_{nlm} \\ &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl}\, s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_{\Delta}\end{aligned}$$

Local field in bulk

In particular at $\rho = 0$, only $l = 0$ modes remain:

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}} \sin^l(\rho)\cos^\Delta(\rho)\, {}_2F_1\left(-n, \Delta + l + n, l + \frac{d}{2}, \sin^2(\rho)\right) \to \frac{1}{N_{n\,0}}$$

Then, bulk local operator at the origin in CFT description is

$$\begin{aligned}
\hat{\phi}^+(t=0,\rho,\Omega) &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl} s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1}P_{\mu_2}\cdots P_{\mu_l}(P^2)^n \hat{\mathcal{O}}_\Delta \\
&\to \sum_{n=0}^{\infty} \frac{1}{N_{n0}} c_{n0} (P^2)^n \hat{\mathcal{O}}_\Delta \\
&= \sqrt{\frac{\Gamma(\Delta)\Gamma(\Delta+1-\frac{d}{2})}{\Gamma(d/2)}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{n!\,\Gamma(\Delta+1-d/2+n)} (P^2)^n \hat{\mathcal{O}}_\Delta
\end{aligned}$$

which agrees with the previous results

Miyaji-Numasawa-Shiba-Takayanagi-Watanabe
Verlinde
Nakayam-Ooguri

Bulk local field near boundary

Below, we will show

$$\lim_{\rho\to\pi/2} \frac{\hat{\phi}^{+}(t=0,\rho,\Omega)}{\cos^{\Delta}(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\left(e^{P_\mu x^\mu}\hat{\mathcal{O}}_\Delta\right)|_{x^2=1}.$$

First, for the wave function,

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}}\sin^{l}(\rho)\cos^{\Delta}(\rho)\,{}_2F_1\left(-n,\Delta+l+n,l+\frac{d}{2},\sin^2(\rho)\right)$$

at the boundary is evaluated using

$$\frac{c_{nl}}{N_{nl}}\,{}_2F_1\left(-n,\Delta+l+n,l+\frac{d}{2},1\right) = \frac{2^{-2n-l}}{n!}\frac{1}{\Gamma(n+l+d/2)}\sqrt{\frac{\Gamma(d/2)\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)}}.$$

Bulk local field near boundary

Expansion formula of plain wave by spherical harmonics:

$$e^{ik_\mu x^\mu} = (d-2)!! \sum_{l=0}^{\infty} i^l\, j_l^d(kr) \sum_m Y_{lm}^*(\Omega_k) Y_{lm}(\Omega)$$

$$= \sum_{l=0}^{\infty} i^l \sqrt{\frac{\pi}{2}} (kr)^l \sum_{n=0}^{\infty} 2^{-2n-l} \frac{\Gamma(d/2)(ikr)^{2n}}{n!\Gamma(n+l+d/2)} \sum_m Y_{lm}^*(\Omega_k) Y_{lm}(\Omega),$$

where $\left\{ \begin{array}{l} j_l^d \text{ is hyper spherical Bessel function,} \\ r = \sqrt{x^\mu x_\mu},\ k = \sqrt{k^\mu k_\mu}, \\ \Omega \text{ and } \Omega_k \text{ are the angular variables for } x^\mu \text{ and } k^\mu, \end{array} \right.$

Applying these to (with $r=1$ and $k_\mu = -iP_\mu$,)

$$\hat{\phi}^+(t=0,\rho,\Omega) = \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl} s_{(l,m)}^{\mu_1\mu_2\ldots\mu_l} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta,$$

we find $\displaystyle \lim_{\rho\to\pi/2} \frac{\hat{\phi}^+(t=0,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}} \left(e^{P_\mu x^\mu} \hat{\mathcal{O}}_\Delta\right)|_{x^2=1}.$

Bulk local field near boundary

$$\lim_{\rho\to\pi/2}\frac{\hat{\phi}^+(t=0,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}e^{P_\mu x^\mu}\hat{\mathcal{O}}_\Delta,$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\hat{\mathcal{O}}^+_\Delta(x)|_{x^2=1}.$$

Operator on cylinder $\mathbf{R}\times S^{d-1}$ is given by $\mathcal{O}^{cy}_\Delta(\tau,\Omega)=\mathcal{O}_\Delta(x)e^{\Delta\tau}$

where $\tau=\ln(x^2)/2$

from the operator $\mathcal{O}_\Delta(x)$ which is radially quantized on $\mathbf{R}^d$.

Thus, bulk operator at bounadry is CFT field:

$$\lim_{\rho\to\pi/2}\frac{\hat{\phi}(t,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\mathcal{O}^{cy}_\Delta(t,\Omega),$$

BDHM

Bulk local field near boundary

Thus, bulk operator at bounadry is CFT field:

$$\lim_{\rho\to\pi/2}\frac{\hat{\phi}(t,\rho,\Omega)}{\cos^{\Delta}(\rho)}=\sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\mathcal{O}^{cy}_{\Delta}(t,\Omega),$$

BDHM

GKPW relation

GKPW relation is essentially obtained from this BDHM

With a background "non-normalizable" mode

$\delta\phi = (\cos(\rho))^{\Delta^-}\bar{\phi} + \cdots$ with $\Delta^- = d - \Delta$,

$$\delta S = -\int_{\text{boundary}} d^d x \left((\cos(\rho))^{1-d} \delta\phi \, \frac{\partial}{\partial_\rho} \phi \right) \sim \int_{\text{boundary}} d^d x \, (\bar{\phi} \, \mathcal{O}^{cy}_{\Delta}) \, ,$$

This is a GKPW relation

c.f. HKLL

We can show that

for conserved current and
energy momentum tensor,
low energy theory of
large N CFT is equivalent to
free theory limit of gauge field
and graviton on AdS, respectively

under the 3 assumptions.

c.f. Ishibashi-Wald

Plan

1. Introduction
2. Low energy spectrum of CFT
3. Deriving bulk local field and GKPW
4. Classical limit of large N CFT
5. Conclusion

Classical limit of large N CFT (including 1/N corrections)

Classical limit as large N limit

In gravity side, action would be

$$S_{grav} = \frac{1}{2(l_p)^{d-1}} \int d^{d+1}x \sqrt{-\det g} \left(-\frac{d(d+1)}{2l_{AdS}^2} + R + \alpha_1 D^2 R + \alpha_2 l_{AdS}^2 R^2 + \cdots \right),$$

Expansion around AdS solution (vacuum),

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + h_{\mu\nu}$$

$$\begin{aligned} S =& N^2 \int d^{d+1}x \sqrt{-\det(g^{AdS})} \\ & \times \left(-f_0(h)h^2 + f_1(h)(Dh)^2 + \alpha_1 f_2(h)(DDh)^2 + \alpha_2 f_3(h)(Dh)^4 + \cdots \right), \end{aligned}$$

perturbation use $\tilde{h}_{\mu\nu} \sim N h_{\mu\nu}$

Algebra of Energy momentum tensor

Virasoro algebra of 2d CFT

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conserved charge = modes of EM tensor

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P,K,M,D alg. of EMT

for CFT (d>2)

Mode expansion

For scalar,

$$\mathcal{O}_\Delta(x) = \sum_{l=0}^{\infty} \sum_{m=0}^{m_{max}(l)} Y_{lm}(\Omega) \sum_{\omega} |x|^{\omega-\Delta} \mathcal{O}_{\Delta\omega l m},$$

In the previous notation,

$$\mathcal{O}_{\Delta\omega l m} = 2^{-(2n+l)} \frac{1}{n!} \frac{\Gamma(\frac{d}{2})}{\Gamma(n+\frac{d}{2}+l)} s_{(l,m)}^{\mu_1\mu_2\ldots\mu_l} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta,$$

$$\text{for } \omega \geq \Delta + l$$

(singular part of) OPE $\leftrightarrow$ commutator

Equal time commutator is given by

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)]_{|x|=|y|=1}$$

$$= \lim_{\epsilon \to 0} \left(\mathcal{O}_1(x)|_{|x|=1+\epsilon} \mathcal{O}_2(y)|_{|y|=1} - \mathcal{O}_1(x)|_{|x|=1} \mathcal{O}_2(y)|_{|y|=1+\epsilon} \right) .$$

Important formula:

$$\frac{1}{(x-y)^{2\Delta}} = \frac{1}{(r_>)^{2\Delta}} \sum_{s=0}^{\infty} \left(\frac{r_<}{r_>} \right)^s \sum_{n=0}^{[\frac{1}{2}s]} (d^\Delta)_s^{s-2n} \sum_m Y^*_{(s-2n)m}(\Omega) Y_{(s-2n)m}(\Omega').$$

$$(d^\Delta)_s^{s-2n} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} \frac{\Gamma(\Delta+s-n)}{\Gamma(\Delta)} \frac{\Gamma(\Delta+1-\frac{d}{2}+n)}{\Gamma(\Delta+1-\frac{d}{2})} \frac{1}{n!} \frac{\Gamma(\frac{d}{2})}{\Gamma(s-n+\frac{d}{2})}$$

(singular part of) OPE $\leftrightarrow$ commutator

Equal time commutator is given by

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)]_{|x|=|y|=1}$$

$$= \lim_{\epsilon \to 0} \left(\mathcal{O}_1(x)|_{|x|=1+\epsilon} \mathcal{O}_2(y)|_{|y|=1} - \mathcal{O}_1(x)|_{|x|=1} \mathcal{O}_2(y)|_{|y|=1+\epsilon} \right) .$$

We can show

$$[\mathcal{O}_{\Delta\omega l m}, \mathcal{O}_{\Delta\omega' l' m'}]$$

$$= \int d\Omega Y_{lm}(\Omega) \int d\Omega' Y'_{l'm'}(\Omega')$$

$$\left(\mathrm{P}_+[|x|^{\Delta-\omega}|y|^{\Delta-\omega'} \mathcal{O}_1(x)\mathcal{O}_2(y)] - \mathrm{P}_-[|x|^{\Delta-\omega}|y|^{\Delta-\omega'} \mathcal{O}_1(x)\mathcal{O}_2(y)] \right)$$

Algebra of EMT

OPE of energy momentum tensor is

$$T^{\mu_1\nu_1}(x)T^{\mu_2\nu_2}(0) =$$

$$C_T\left(\frac{I^{\mu_1}{}_{\alpha}(x)I^{\nu_l}{}_{\beta}(x)\left(\frac{1}{2}(\delta_{\alpha\mu_2}\delta_{\beta\nu_2}+\delta_{\alpha\nu_2}\delta_{\beta\mu_2})+\frac{1}{d}\delta_{\alpha\beta}\delta_{\mu_2\nu_2}\right)}{x^{2d}}\right)$$

$$+s_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(x,\partial)T^{\mu_3\nu_3}(0)+\cdots\ ,$$

Osborn-Petkou

where $C_T = \mathcal{O}(N^2)$

and $s_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(x,\partial)$ is uniquely determined

if we require causality

Camanho-Edelstein-Maldacena-Zhiboedov

Algebra of EMT

Thus, algebra of energy momentum tensor is fixed with one parameter N as

$$[L_i, L_j] = N^2 \omega_{ij} \mathbf{1} + f_{ij}^k L_k$$

where L_i is a mode of EMT

This is a Lie algebra

(We neglect the muti-trace operators here, for simplicity. But, such operatros can be taken into account.)

Different normalizations of L

1. Creation operators

$$L_i^F = \frac{1}{N} L_i$$

$$[L_i^F, L_j^F] = \omega_{ij} \mathbf{1} + \frac{1}{N} f_{ij}^k L_k^F .$$

2. Classical limit

$$L_i^{cl} = \frac{1}{N^2} L_i$$

$$N^2 [L_i^{cl}, L_j^{cl}] = \omega_{ij} \mathbf{1} + f_{ij}^k L_k^{cl} .$$

Classical limit of the CFT

$$N^2[L_i^{cl}, L_j^{cl}] = \omega_{ij}\mathbf{1} + f_{ij}^k L_k^{cl}.$$

In a large N limit, this is regarded as a Poisson bracket

$$-i\{L_i^{cl}, L_j^{cl}\}_P = \omega_{ij}\mathbf{1} + f_{ij}^k L_k^{cl},$$

$$\text{where } N^2 \sim 1/\hbar$$

The classical state is given by deformation of coherent state of harmonic oscillators

Classical gravity in Asymptotic AdS

$$S_{grav} = \frac{1}{2(l_p)^{d-1}} \int_{AdS_{d+1}} d^{d+1}x \sqrt{-\det g}\,(R - 2\Lambda)$$
$$+S_{GH} + S_{ct},$$

For the free approximation (linearized gravity), the classical Hamiltonian and phase space are shown to be equivalent to the classical limit of the CFT

Coordinates of the phase space are the creation op. L_i^+

Boundary stress tensor (Brown-York tensor)

$$T^{\mu\nu}_{bndy}(x) = \frac{2}{\sqrt{-\det g^{(0)}_{\mu\nu}}} \frac{\delta S_{grav}}{\delta g^{(0)}_{\mu\nu}(x)},$$

where $g^{(0)}_{\mu\nu}(x)$ is the boundary metric

This is defined on the boundary and is conserved and traceless.

The conformal generators are defined from this, and then, this is transformed as the primary field.

Boundary stress tensor (Brown-York tensor)

$$T^{\mu\nu}_{bndy}(x) = \frac{2}{\sqrt{-\det g^{(0)}_{\mu\nu}}} \frac{\delta S_{grav}}{\delta g^{(0)}_{\mu\nu}(x)},$$

Thus, the boundary stress tensor also can be expanded to the modes: L_i^{bndy},

and the algebra of L_i^{bndy} defined by the Poisson bracket is uniquely determined with a constant $N^2 \sim G_N$.

Here, the coordinates of the phase space are L_i^+

Thus, the classical limit of the CFT is the classical Einstein gravity on asymptotic AdS_{d+1} because Hamiltonian and Poisson bracket are same.

Conclusion

- **Spectrum of large N CFT is identical to spectrum of free gravitational theory in AdS under some assumptions which are expected to be valid.**

- **Thus, two theories are equivalent for the low energy region under the assumptions.**

- **Using this equivalence, the bulk local field is constructed and the GKPW relation is derived.**

Conclusion (continued)

- **Classical limit of the generic large N gauge theory with conformal symmetry is the classical Einstein gravity on asymptotic AdS because the Hamiltonian and the Poisson bracket are same.**

Fin.