Sphaleron from gradient flow

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Based on
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Osaka Univ. seminar (16th, Jun. 2020)

Introduction

Standard Model (SM)

After the Higgs boson's discovery, the Standard Model has been established.

https://www.theguardian.com/science/blog/2012/jul/04/higgs-boson-discovered-live-coverage-cern

Mysteries

But there are still many mysteries unanswered by the SM.

Baryon asymmetry

Dark matter

Neutrino mass

Gauge hierarchy

etc.

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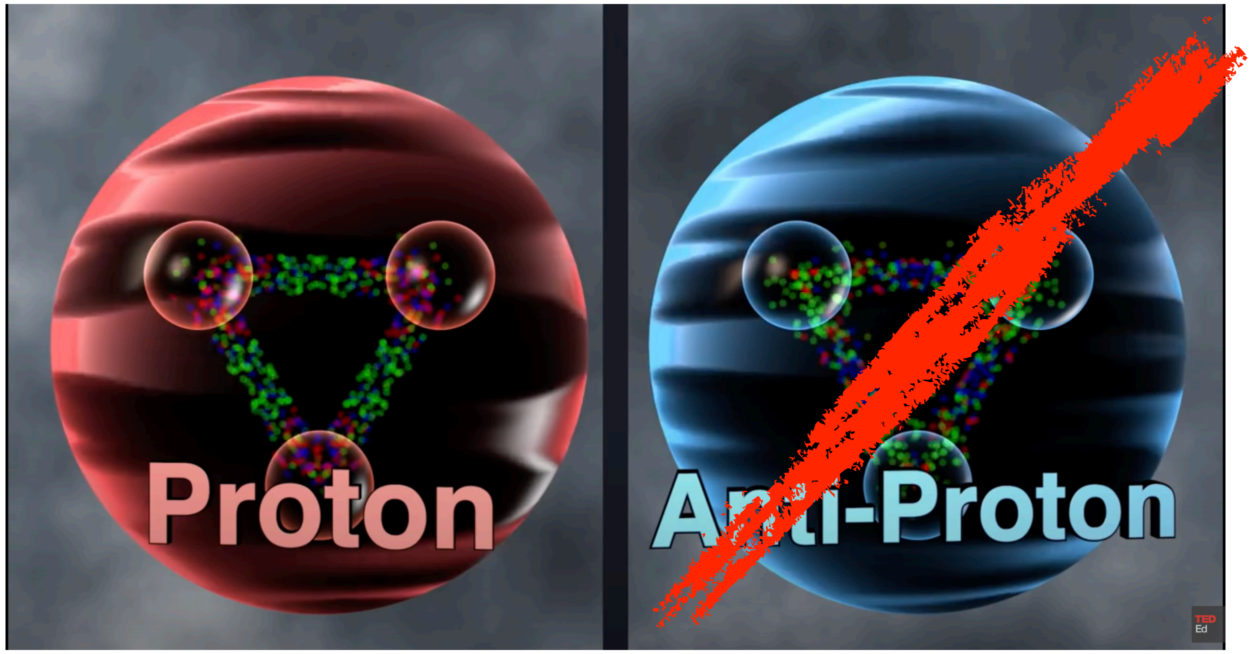
Our universe is (slightly) baryon asymmetric:

$$\boxed{Y_B \equiv \frac{n_B - n_{\bar{B}}}{s} \sim 10^{-10}}$$

s : entropy density

n_B ($n_{\bar{B}}$) : (anti-)baryon # density



[From www.youtube.com/watch?v=CtR5EkvLNfg]

Baryon # and Chern-Simons

- **Baryon asymmetry can be produced by a topology change of the gauge fields.**

Chiral anomaly: $$\partial_\mu j_B^\mu = \frac{g^2}{16\pi^2} \mathrm{tr}\, W_{\mu\nu} W^{\mu\nu} + \frac{g'^2}{32\pi^2} Y_{\mu\nu} Y^{\mu\nu}$$

integrating over spacetime $\longrightarrow$ $$\boxed{\Delta B = 3 \Delta N_{CS}}$$

$$B(t) \equiv \int d^3x\, j_B^0$$

where N_{CS} is a topological quantity called as the Chern-Simons #:

$$\boxed{N_{CS}(t) = \frac{-1}{16\pi^2} \int d^3x\, \mathrm{tr} \left[WF - \frac{2}{3} W^3 \right]}$$

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(Pontryagin index)

If one takes $W = U^\dagger dU$, $\longrightarrow$ $$N_{CS} = \frac{2}{3} \int d^3x\, \mathrm{tr}\, (U^\dagger dU)^3 \in \mathbb{Z}$$

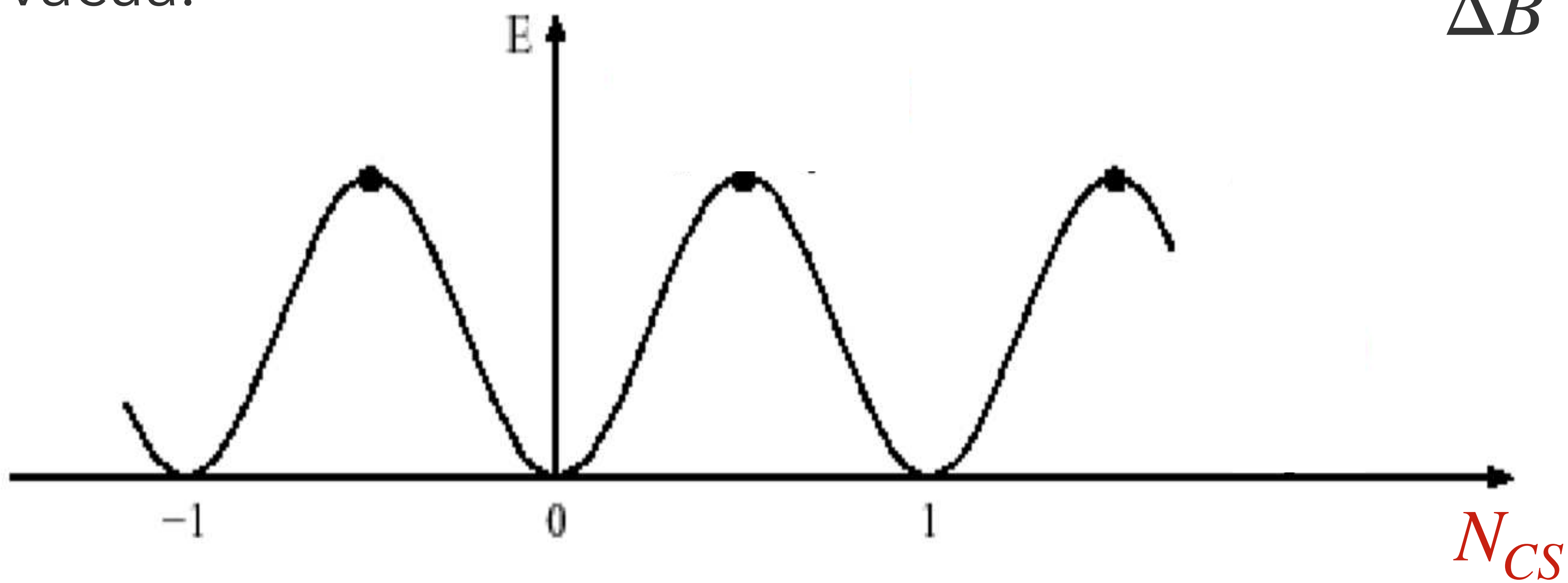
(vacuum configuration)

degenerated vacua are labeled by $N_{CS} = 0, 1, \cdots$

Sphaleron process

- To change N_{CS} , we need jump an energy barrier between the vacua.

$$\Delta B = 3\Delta N_{CS}$$



There are two processes to change N_{CS}

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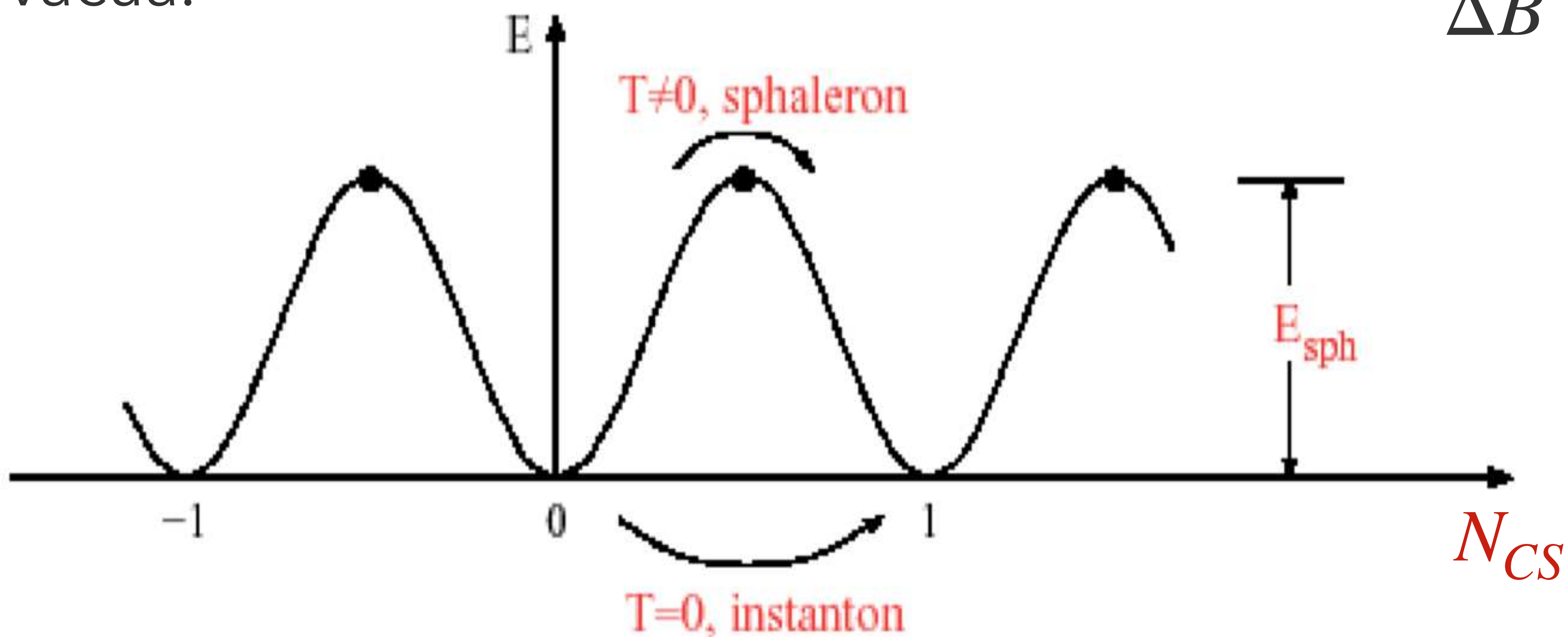


There are two processes to change N_{CS}

- Quantum tunneling (instanton effect) $\sim e^{-\frac{8\pi^2}{g^2}}$: tiny
- **Sphaleron process** (thermal jump) $\sim e^{-\frac{E_{sph.}}{T}}$

$E_{sph.}$ is the energy of **the sphaleron solution.**

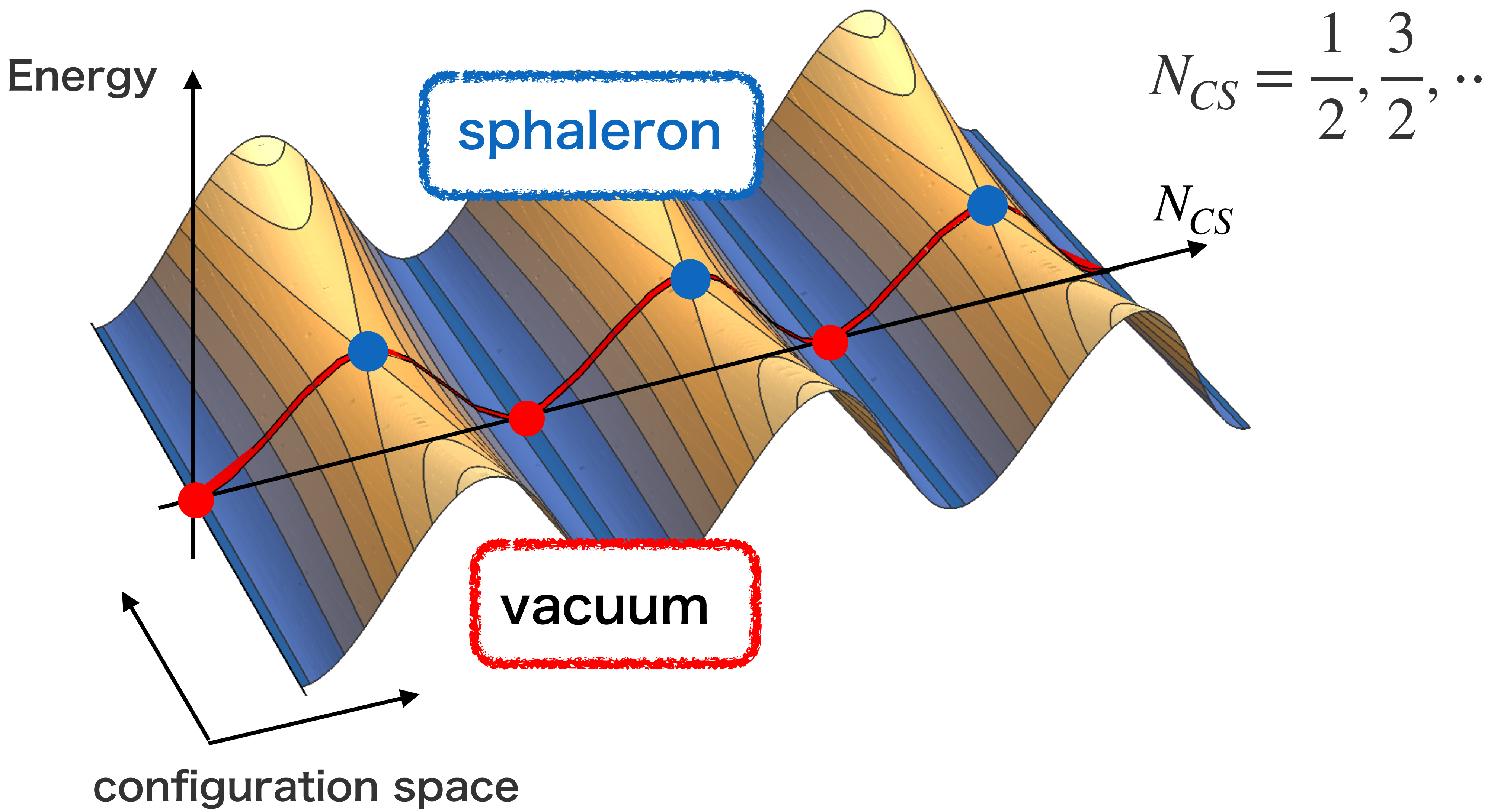
Sphaleron

[Manton '83]

[Klinkhamer-Manton '84]

σφαλερος (sphaleros) ``ready to fall''

- saddle point solution of classical EOMs
- maximum point on least-energy path connecting two vacua



Motivation

- For predictions of baryogenesis, it is important to obtain the sphaleron solution accurately.

- However, the conventional method is technically difficult (explained later) except for simple models (e.g., $g_Y = 0$).

- In usual, people obtain the sphaleron energy for $g_Y = 0$ first, and then treat g_Y perturbatively (not solve full EOMs).

Our work

- **We propose a simple method to obtain the sphaleron using gradient flow.**
- **It can be applied to various models other than SM!**

Plan of talk

- Introduction (9p.)
- Our method (12p.)
- Result for $SU(2)$-Higgs model (9p.)
- Summary

Our method

Conventional method (Min & Max procedure)

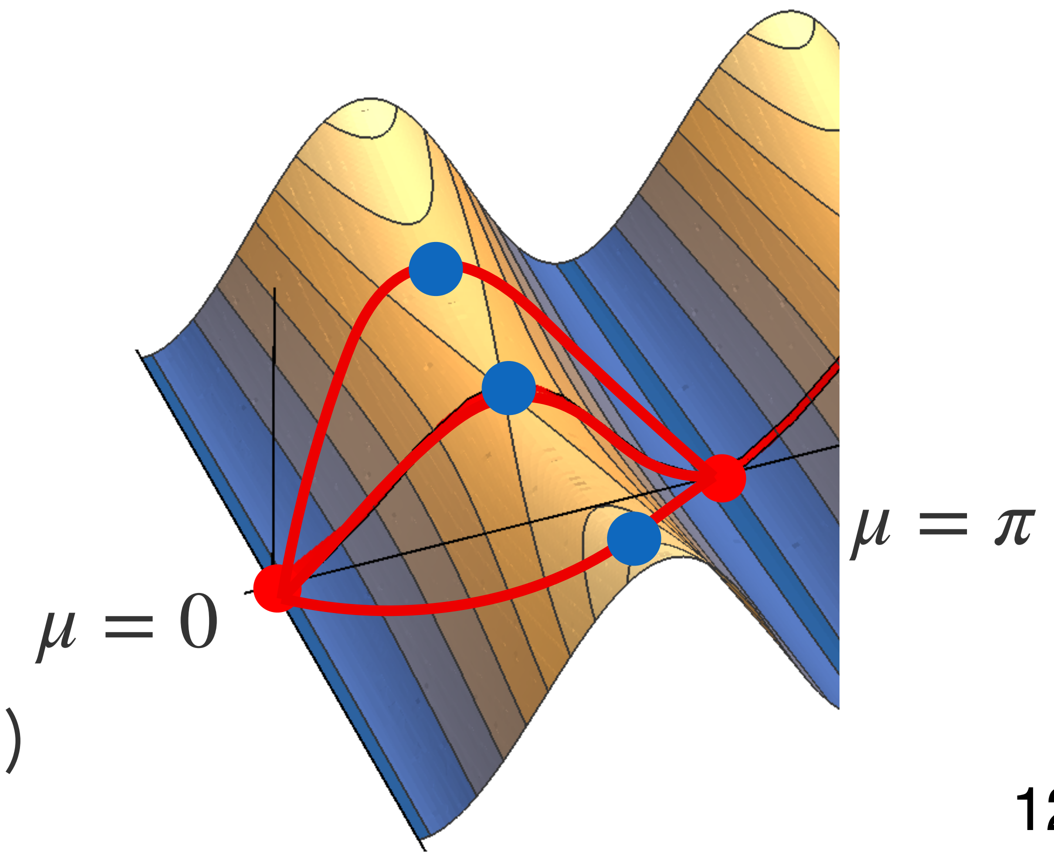
[Manton, 1983]

- Consider a family of paths (**red lines**) connecting two vacua.
- Each path is parametrized with a parameter $\mu \in [0,\pi]$.
- Find the maximum-energy point on each path (**blue dots**).
- The sphaleron is the minimum point among the maximum points.

$$E_{sph.} = \min_{\text{path}} \max_{0\leq\mu\leq\pi} E(\mu)$$

Obviously, it is not an easy task!

(In the SM, this works only when $g_Y \to 0$.)



Gradient flow (relaxation method)

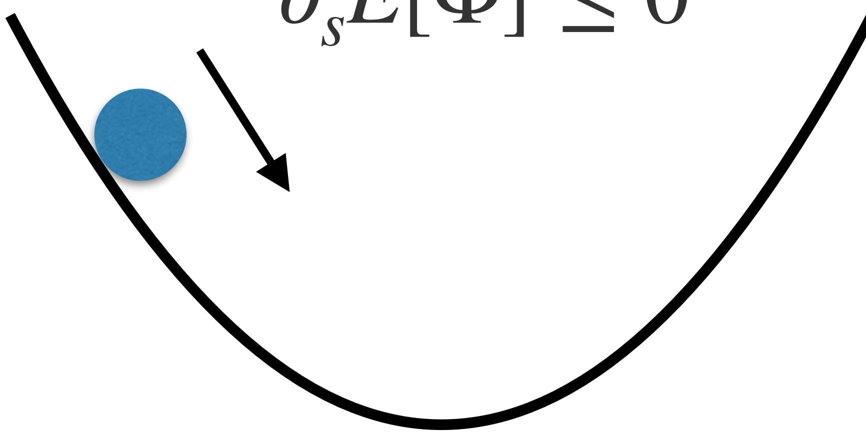
(cf.):
[Luscher '14]
[Luscher-Weisz '11]

- Useful method to find a (locally) minimum-energy configuration
- Introduce a fictitious time s in addition to D-dim. coordinates x.
- Evolve a field configuration following the flow equation:

$$\boxed{\partial_s \Phi_A(x,s) = -\frac{\delta E[\Phi]}{\delta \Phi_A(x,s)}}$$

$$\Phi_A = \{\underbrace{\phi}_{\text{Higgs}}, \underbrace{W^a_\mu}_{\text{gauge}}, \cdots\}$$

$$\partial_s E[\Phi] \leq 0$$



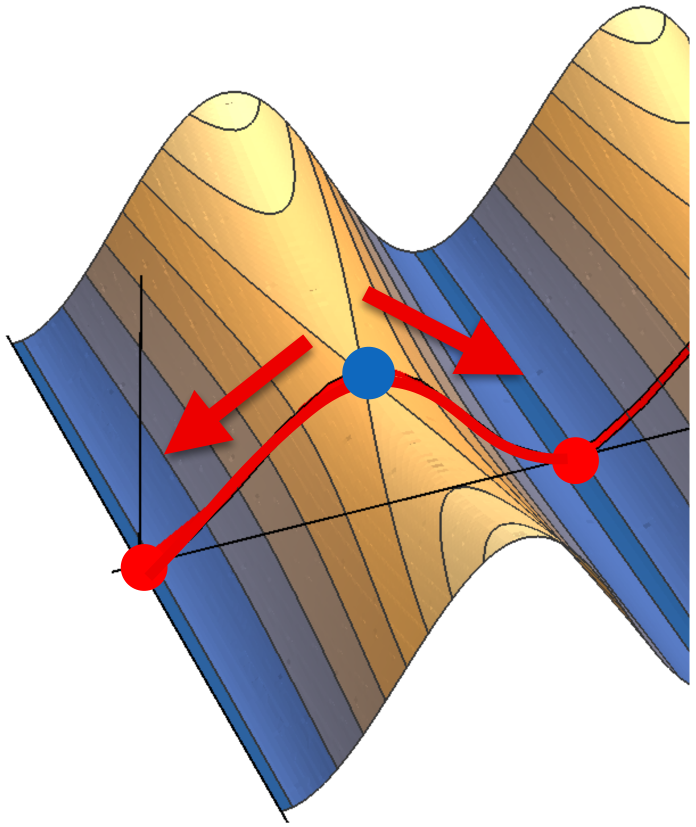
- If the flow converges, the configuration is **a solution of EOM**: $\delta E / \delta \Phi_A = 0$.
  (locally minimum-energy configuration)

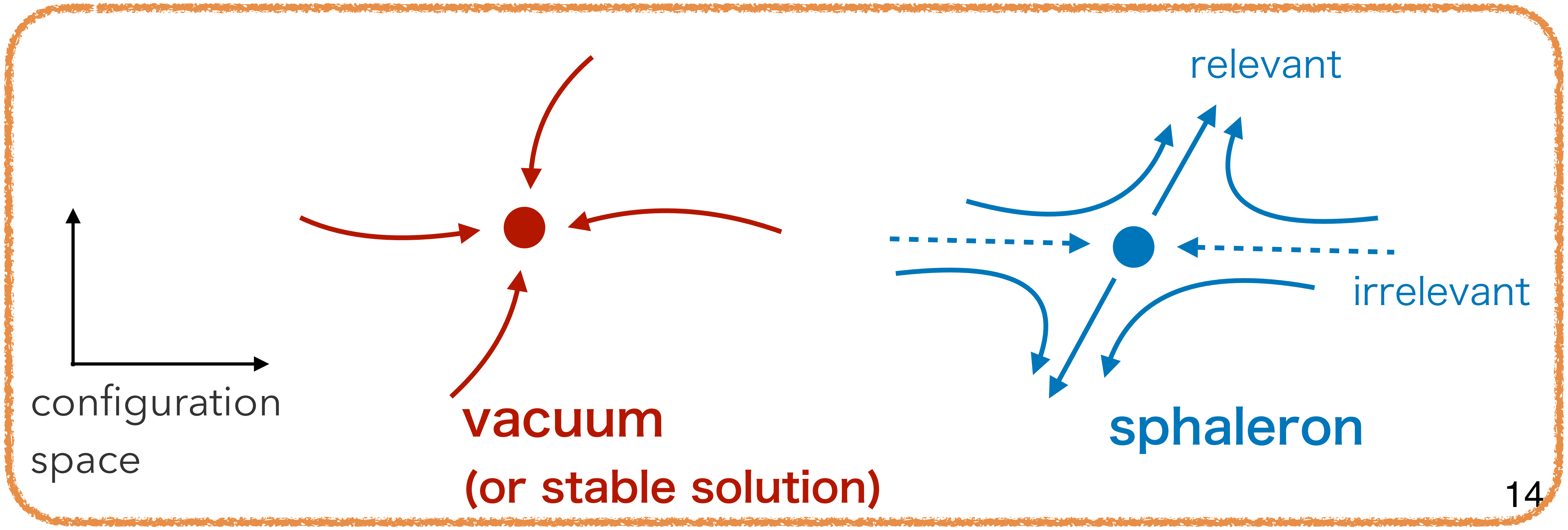
- Note: In quantum field theory, gradient flow provides an interesting property (finiteness of correlation functions). But we do not consider such a quantum aspect but **classical field theories.**

Gradient flow for sphaleron

- Although the sphaleron is a solution of EOM, the gradient flow does not converge to it because it is an unstable solution.

- In other words, the sphaleron is a fixed point with a single relevant direction.





Mathematical reason (skippable)

- Introduce the quadratic curvature $\mathscr{M}_{AB}$ of the sphaleron

$$\mathscr{M}_{AB} \equiv \left. \frac{\delta^2 E[\Phi]}{\delta\Phi_A \delta\Phi_B} \right|_{\Phi=\Phi_{sph}}$$

Sphaleron config.

- Let $|\chi^{(n)}\rangle$ be eigenfunctions of $\mathscr{M}$:

$$\mathscr{M}\,|\chi^{(n)}\rangle = \lambda_n\,|\chi^{(n)}\rangle \qquad n = 0,1,\cdots$$

For the sphaleron, the lowest eigenvalue is negative: $\lambda_0 < 0$

$\longrightarrow$ A perturbation $\propto \chi^{(0)}(x)$ is an unstable direction.

(the others $\lambda_{n\geq 1} > 0$, and $\chi^{(n\geq 1)}$ are stable directions)

Mathematical reason (skippable) (con't)

- Expand a configuration around the sphaleron as

$$\Phi_A(x,s) = \Phi_A^{sph.}(x) + a_n(s)\,\chi_A^{(n)}(x)$$

- Substituting into the flow eq: $\partial_s \Phi_A(x,s) = -\delta E/\delta\Phi_A$,

$$\dot{a}_n(s)\chi_A^{(n)}(x) = -\cancel{\frac{\delta E[\Phi]}{\delta\Phi_A(x,s)}}\Bigg|_{\Phi_{sph.}} - a_n(s)\mathcal{M}_{AB}\chi_B^{(n)} + \mathcal{O}(a_n^2)$$

$$\simeq -\lambda_n a_n(s)\chi_A^{(n)}$$

$$\longrightarrow \quad \dot{a}_n(s) \simeq -\lambda_n a_n(s) \qquad \text{(not sum for } n\text{)}$$

- $a_{n\geq 1}$ exponentially decay, but a_0 exponentially growths.

$\longrightarrow$ **Cannot converge to the sphaleron!**

Modify the flow

[Chigusa-Moroi-Shoji '19]

- By adding a "lifting" term to the flow eq, we can lift up the unstable direction!

$$\boxed{\partial_s \Phi_A(x,s) = -\frac{\delta E[\Phi]}{\delta \Phi_A(x,s)} + C(s)\,\mathcal{G}_A(x,s)}$$

$$C(s) \equiv \beta \int d^3x \sum_A \frac{\delta E[\Phi]}{\delta \Phi_A(x,s)} \mathcal{G}_A(x,s)^\dagger$$

$\beta > 1$ (const.)

where $\mathcal{G}_A(x,s)$ is proportional to the unstable direction

$$\mathcal{G}_A \propto \chi_A^{(0)}(x)$$

and normalized as $\int d^3x\, |\mathcal{G}|^2 = 1.$

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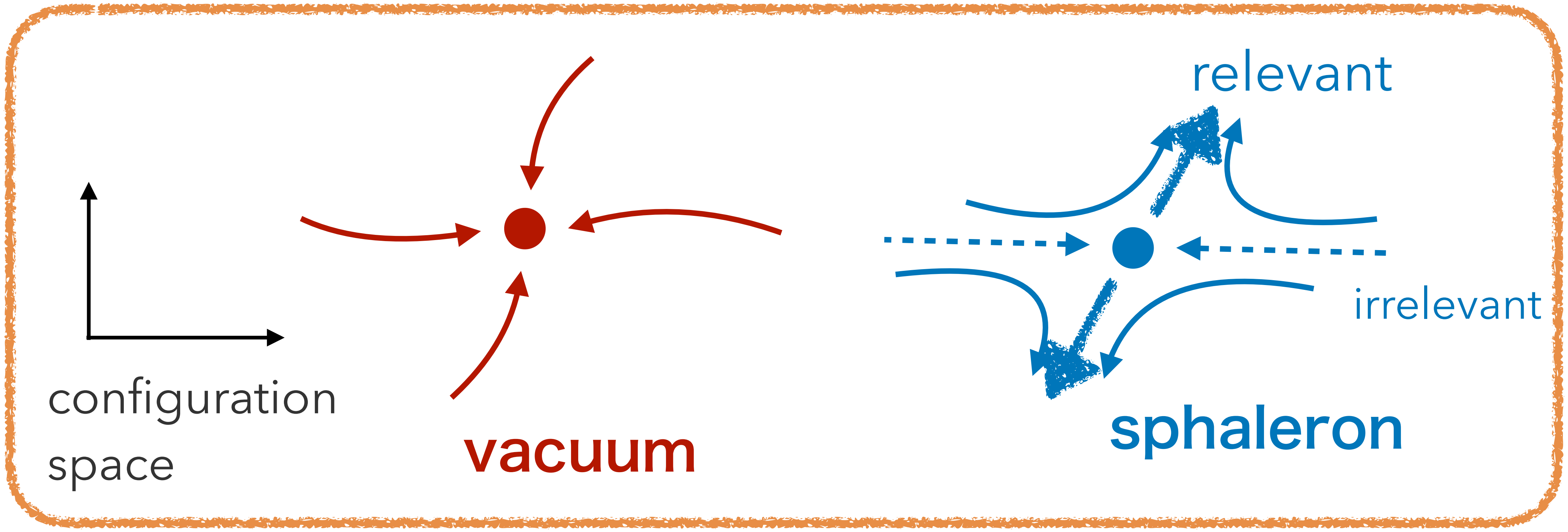
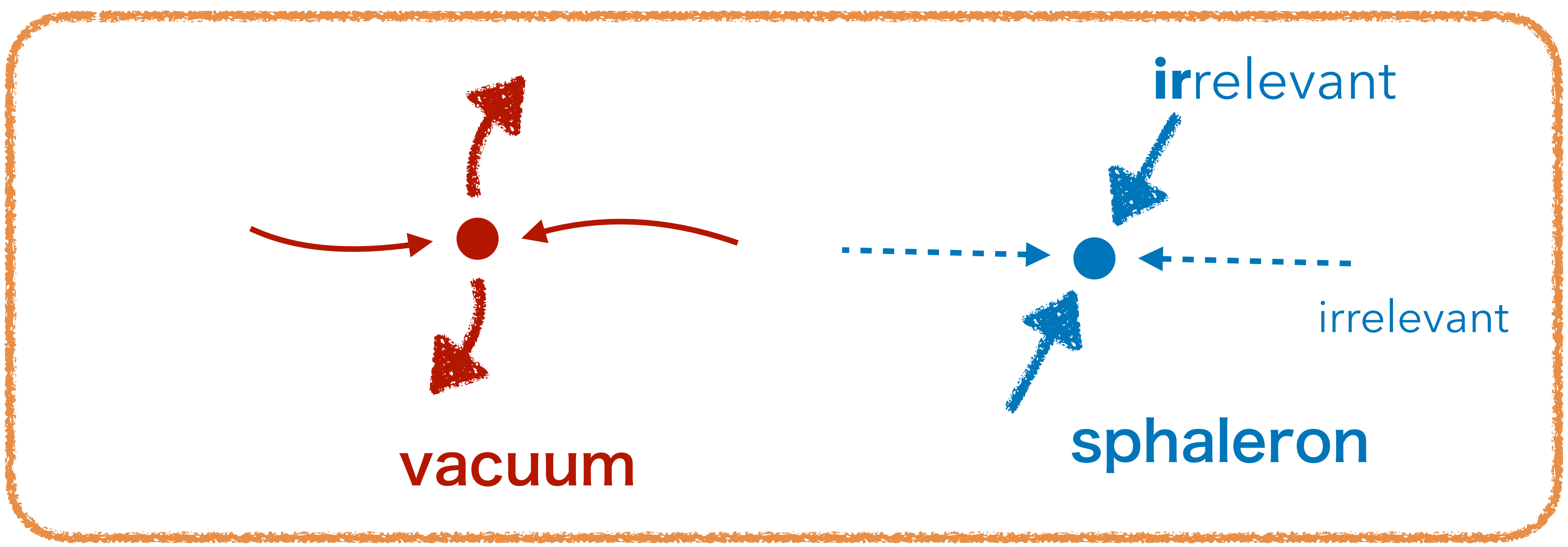
Picture of the modified flow



Lifting term



``Proof'' for convergence (skippable) [Chigusa-Moroi-Shoji '19]

- Again, expand a configuration around the sphaleron as

$$\Phi_A(x,s) = \Phi_A^{sph.}(x) + a_n(s)\,\chi_A^{(n)}(x)$$

- Substituting into the **modified** flow eq:

$$\dot{c}_n(s)\,|\chi^{(n)}\rangle \simeq \cancel{|\mathscr{F}\rangle_{\Phi_{sph.}}} - a_n(s)\mathscr{M}\,|\chi^{(n)}\rangle$$

$$-\beta a_n(s)\,\langle \mathscr{G}(s)\,|\,\mathscr{M}\,|\chi^{(n)}\rangle\,|\,\mathscr{G}(s)\rangle$$

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$$= -\lambda_n a_n(s)\big(1-\beta\,\underbrace{|\,\mathscr{G}(s)\rangle\langle\mathscr{G}(s)\,|}_{\text{Projection to }\chi^{(0)}}\big)\,|\chi^{(n)}\rangle$$

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$$\longrightarrow \begin{cases} \dot{a}_n(s) \simeq -\lambda_n a_n(s) & (n>0) \\ \dot{a}_0(s) \simeq \underbrace{-\lambda_0(1-\beta)}_{>0}\,a_0(s) & (\beta > 1) \end{cases}$$

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$\longrightarrow$ **Converge to the sphaleron!**

Our claim

- Modified flow eq. :

$$\partial_s |\Phi(s)\rangle = |\mathscr{F}(s)\rangle \quad + C(s)\,|\mathscr{G}(s)\rangle$$

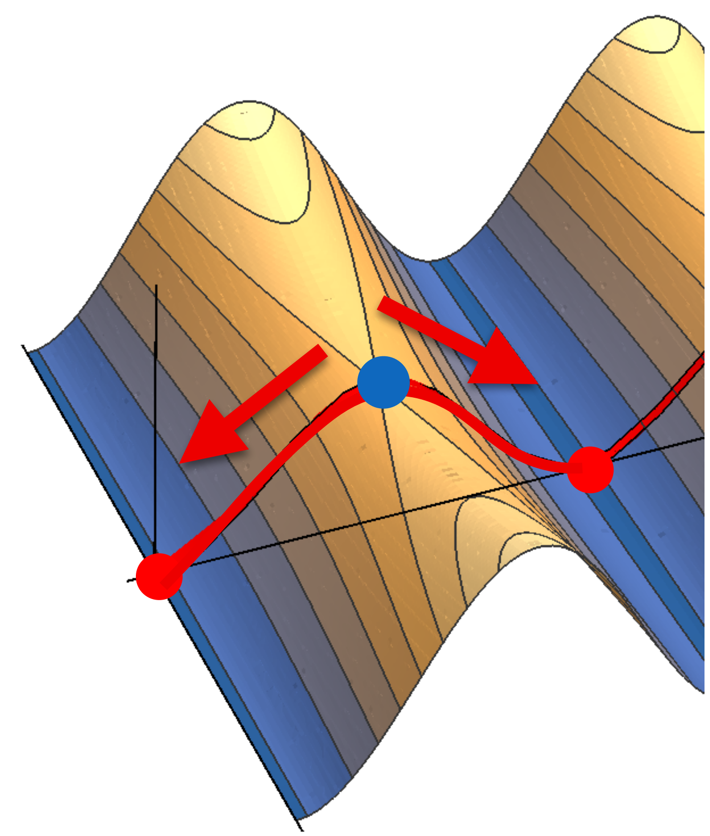
where $\mathscr{G}_A(x,s)$ should be proportional to the unstable direction $\chi_A^{(0)}(x)$.

Problem: What is a concrete expression of $\chi^{(0)}$?

Naive guess:

The unstable direction is the steepest direction changing N_{CS}

$$\chi_A^{(0)}(x) \propto \left.\frac{\delta N_{CS}}{\delta \Phi_A}\right|_{\Phi_{sph.}}$$



Our flow eq.

- Therefore, our modified flow eq. is

$$\boxed{\partial_s |\Phi(s)\rangle = |\mathscr{F}(s)\rangle \quad + C(s)\,|\mathscr{G}(s)\rangle}$$

$$C(s) \equiv -\beta\langle \mathscr{G}(s)|\mathscr{F}(s)\rangle \qquad \beta > 1 \text{ (const.)}$$

$$|\mathscr{F}(s)\rangle \equiv -\frac{\delta E[\Phi]}{\delta\Phi_A(x,s)}$$

$$\boxed{|\mathscr{G}(s)\rangle \equiv \frac{\delta N_{CS}}{\delta\Phi_A(x,s)} = \begin{cases} \frac{1}{8\pi^2}\,\epsilon^{ijk}F^a_{jk} & (\text{for } \Phi_A = A^a_i) \\ 0 & (\text{for } \Phi_A = \text{ others}) \end{cases}}$$

- In the following, we show **this flow eq. works well.**

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Result for $SU(2)$-Higgs model

$SU(2)$-Higgs model in (3+1) dim.

- $SU(2)$ gauge field A_μ and $SU(2)$ doublet Φ :

$$S = \frac{1}{g^2}\int d^4x \left[-\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + (D_\mu\Phi)^\dagger(D^\mu\Phi) - \frac{\lambda}{g^2}\left(\Phi^\dagger\Phi - \frac{1}{2}g^2v^2\right)^2 \right]$$

where we have divided Φ by g comparing to the usual convention.

- This model is equivalent to the Electroweak sector of the SM with the limit $g_Y \to 0$.
- It is known that **a spherically symmetric sphaleron solution exists for $\lambda/g^2 \lesssim 18.1$.** [Dashen-Hasslacher-Neveu '74] [Yaffe '89] [Manton '83] [Klinkhamer-Manton '84]
- For $\lambda/g^2 > 18.1$, another type of the sphaleron appears (deformed sphaleron), but we do not consider that. [Kunz-Brihaye '89] [Yaffe '89]

Spherically symmetric anzats

[Ratra-Yaffe '87]

[Yaffe '89]

$$A_0(x) = \frac{1}{2i}\left\{a_0(r,t)\hat{x}_j\sigma^j\right\}$$

$$A_i(x) = \frac{1}{2i}\left[\left\{f(r,t)-1\right\}\frac{e_i^1}{r} + h(r,t)\frac{e_i^2}{r} + a_1(r,t)e_i^3\right]$$

$$\Phi(x) = \left\{\mu(r,t) + i\nu(r,t)\hat{x}_j\sigma^j\right\}\xi$$

a_0, a_1, f, h, μ, ν are real functions

(e_i^1, e_i^2, e_i^3) are defined as
$$\begin{cases} e_i^1 = \epsilon_{ijk}\hat{x}^k\sigma^j \\ e_i^2 = (\delta_{ij} - \hat{x}_i\hat{x}_j)\sigma^j \\ e_i^3 = \hat{x}_i\hat{x}_j\sigma^j \end{cases}$$

Reduce (3+1)-dim into (1+1)-dim

[Ratra-Yaffe '87]

[Yaffe '89]

- Substituting the ansatz into the action, we can reduce the model into (1+1) dim. Abelian-Higgs model. with two scalars.

$$S = \frac{4\pi}{g^2}\int dtdr\left\{\frac{1}{4}r^2 f_{\mu\nu}f^{\mu\nu} + |D_\mu\chi|^2 + \frac{1}{2r^2}\left(|\chi|^2 - 1\right)^2 + r^2|D_\mu\phi|^2 \right.$$
$$\left. -\mathrm{Re}\left(\chi^*\phi^2\right) + \frac{1}{2}\left(|\chi|^2 + 1\right)|\phi|^2 - \frac{\lambda}{g^2}r^2\left(|\phi|^2 - \frac{1}{2}g^2v^2\right)^2\right\}$$

$$\chi \equiv f + ih \qquad \phi \equiv \mu + i\nu \qquad f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \qquad (\mu, \nu = 0 \text{ or } 1)$$

$$D_\mu\chi = (\partial_\mu - ia_\mu)\chi \qquad D_\mu\phi = (\partial_\mu - ia_\mu/2)\phi$$

two complex scalars	$U(1)$ charge	VEV
χ	$+1$	1
ϕ	$+1/2$	$gv/\sqrt{2}$

Gauge fixing

- We concentrate on static configurations:

$$\chi(r,\cancel{t}) \qquad \phi(r,\cancel{t}) \qquad a_1(r,\cancel{t}) \qquad a_0(r,t)=0$$

- Furthermore, without loss of generality, we can ``gauge out'' the gauge field $a_1(r)$ using a gauge function $\omega(r)$ as

$$a_1(r) \to a_1(r) - \partial_r \omega(r) = 0$$

- Thus we have **only two complex functions in 1 dim.**

$$\boxed{\chi(r) \qquad \phi(r)}$$

Flow eq.

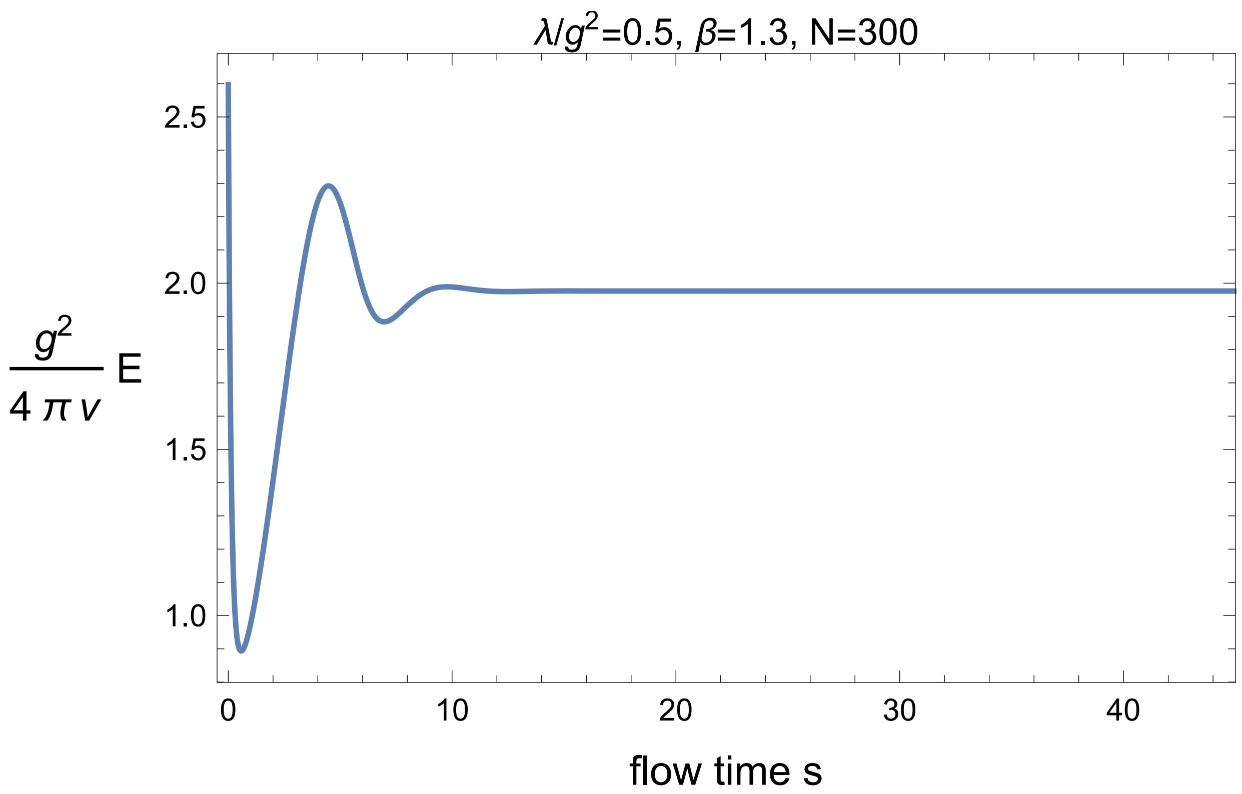
- We give an initial configuration at $s = 0$, and then evolve it by the flow equation numerically.

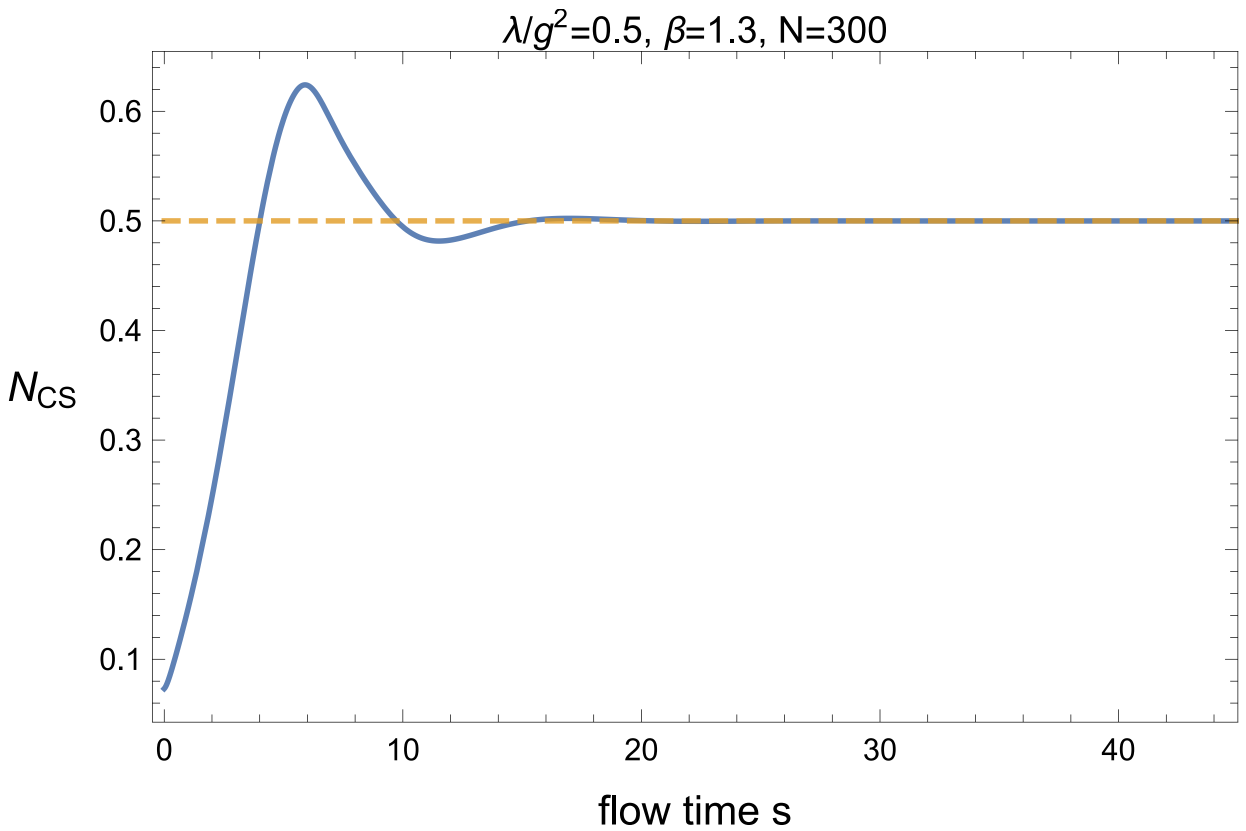
$$\boxed{\begin{aligned} \frac{\partial \chi}{\partial s} &= -\frac{\delta E}{\delta \chi^*} + C(s)\frac{\delta N_{\mathrm{CS}}}{\delta \chi^*} \\ \frac{\partial \phi}{\partial s} &= -\frac{\delta E}{\delta \phi^*} + C(s)\frac{\delta N_{\mathrm{CS}}}{\delta \phi^*} \end{aligned}}$$

$$N_{\mathrm{CS}} = \frac{1}{2\pi}\int dr \left[\mathrm{Im}\partial_1 \chi + \left\{ \frac{i}{2}\chi^*(\partial_r \chi) + h.c. \right\} \right] \qquad \frac{\delta N_{\mathrm{CS}}}{\delta \chi^*} = 2i\frac{\partial \chi}{\partial r} \qquad \frac{\delta N_{\mathrm{CS}}}{\delta \phi^*} = 0$$

- If the configuration converges to a fixed point, it should be **the sphaleron solution!**

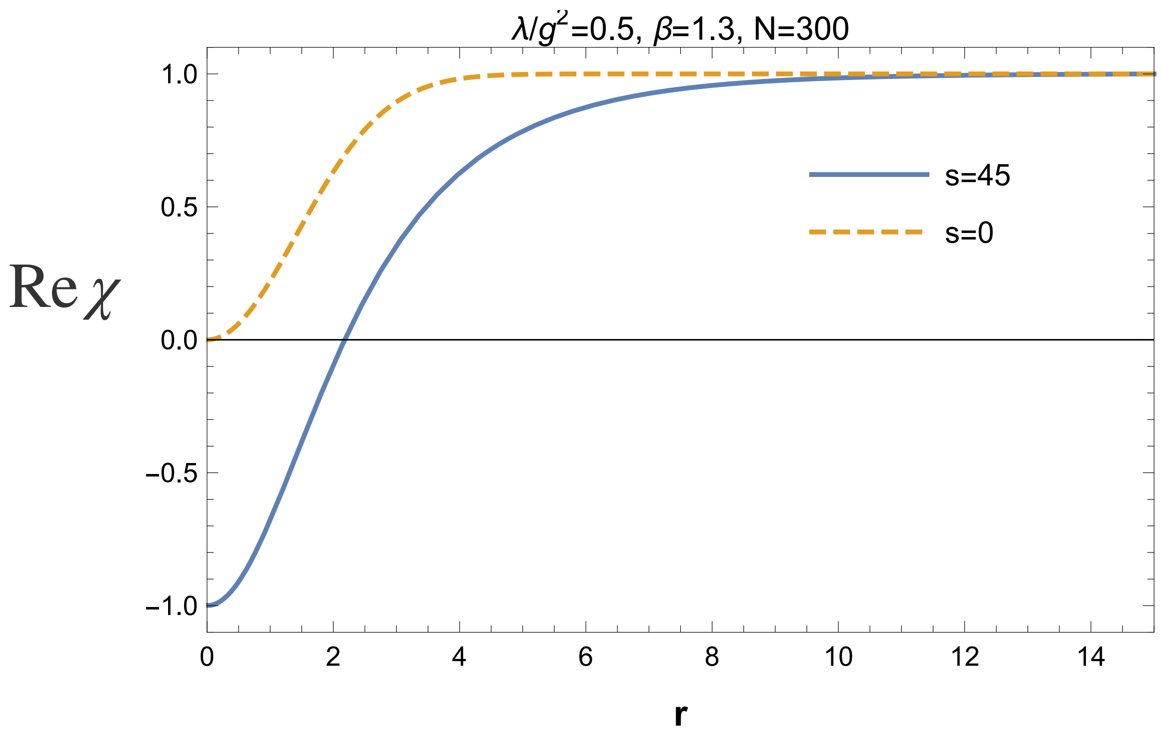
Result

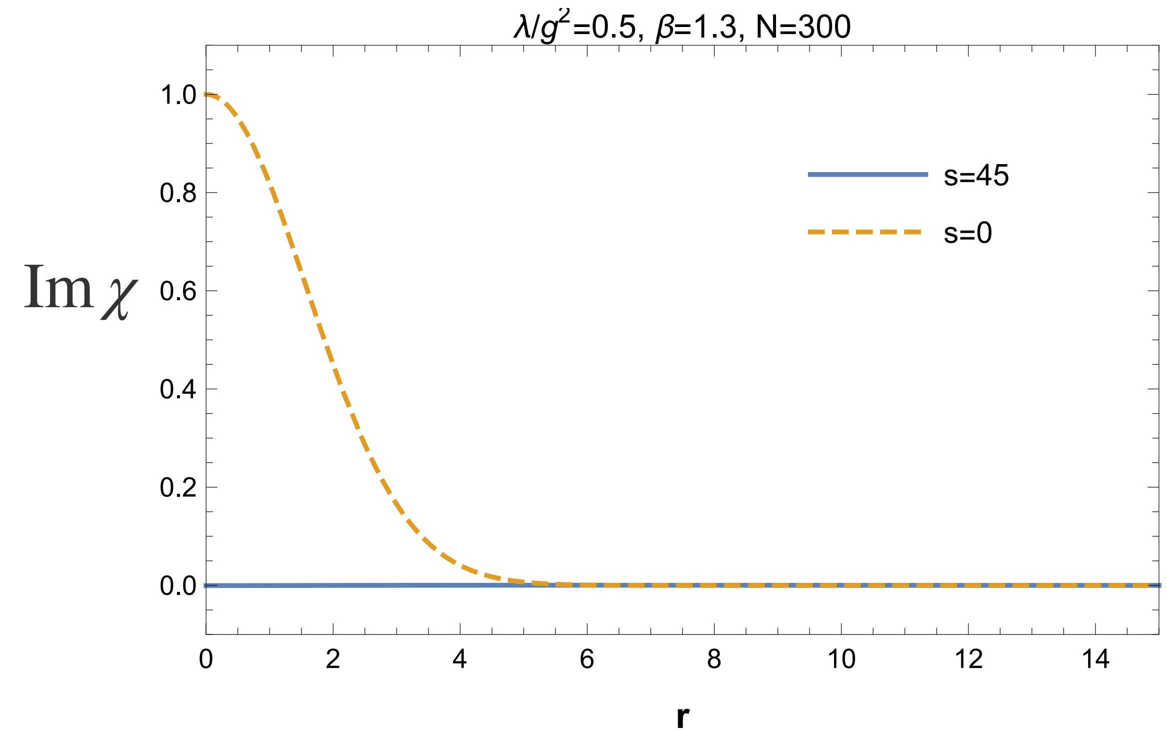


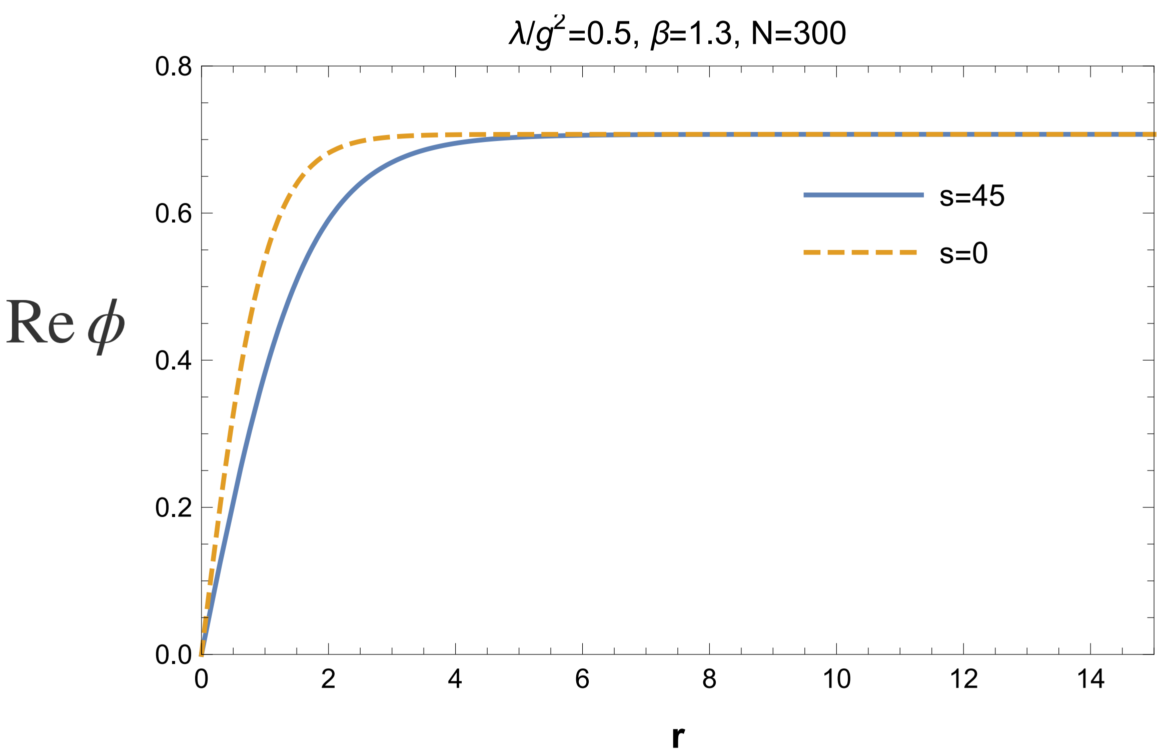


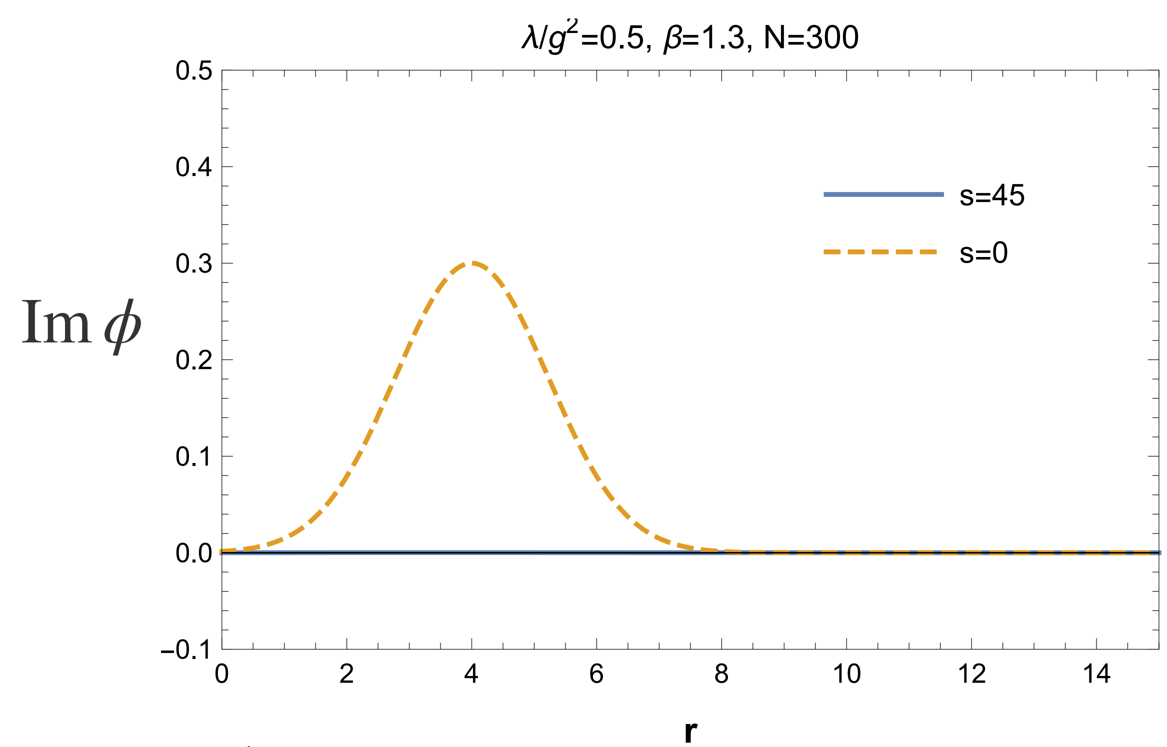
- The converged energy value is $E_{sph.} \simeq 1.976 \times 4\pi v/g^2$, **which agrees with the known value of the sphaleron energy!**
- N_{CS} **converges to 1/2.**

Result (con’t)







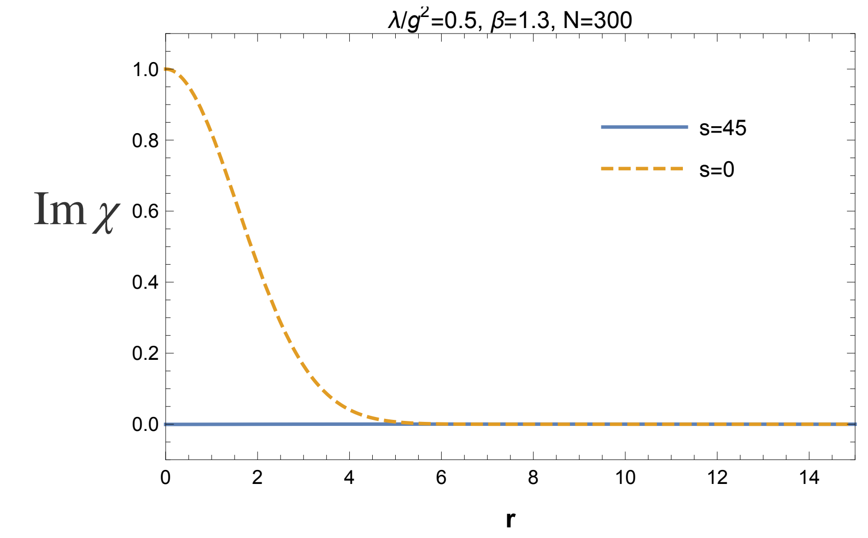


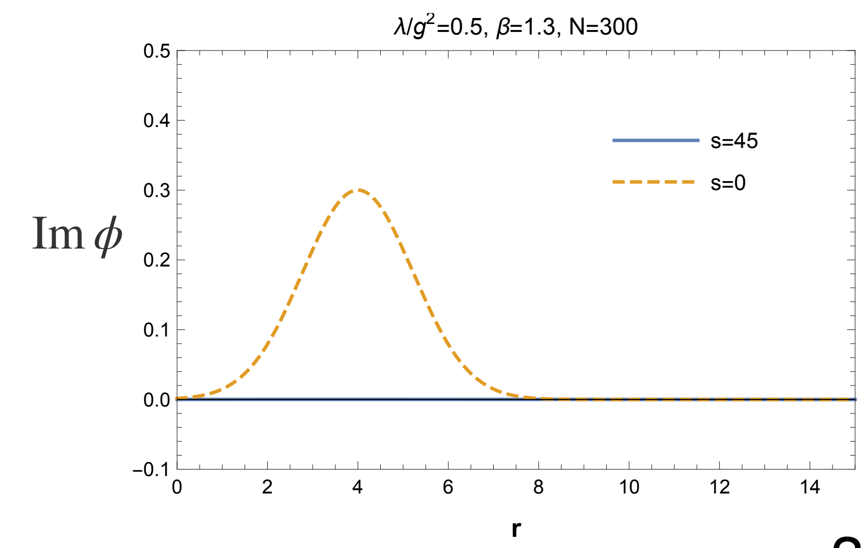
(length unit : v^{-1})

Remarks

- We **did not impose any ansatz** other than spherical symmetry and **did not fine tune** the initial configuration. The configuration automatically **converged to the sphaleron** along the flow.

- Especially, in the previous works, $\mathrm{Im}\,\chi = \mathrm{Im}\,\phi = 0$ is imposed by hand since they are unstable direction. But we did not do so.





Applications

- The most straightforward application is **the SM with $g_Y \neq 0$.** Especially, it is theoretically interesting to consider the case of $\sin\theta_W \equiv g_Y/\sqrt{g^2 + g_Y^2} \sim 1$, in which **the sphaleron has a long magnetic dipole structure.**

- **Applicable to BSMs with the same electroweak structure as the SM**, e.g., two Higgs doublet models.

- Also applicable to **BSMs in which the electroweak symmetry is extended**. There are no systematic methods to obtain the sphaleron for such models.

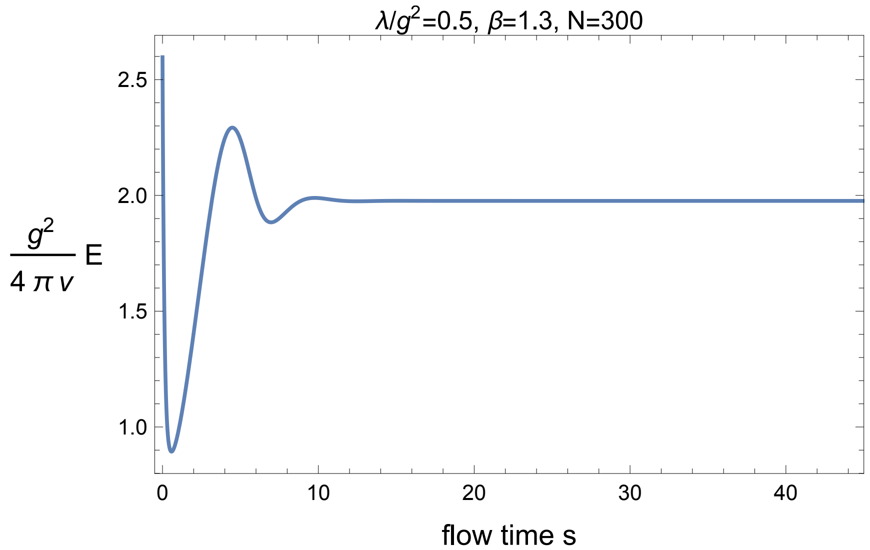
  e.g.) WIMP DM model with $SU(2)_0 \times SU(2)_1 \times SU(2)_2 \times U(1)_Y$

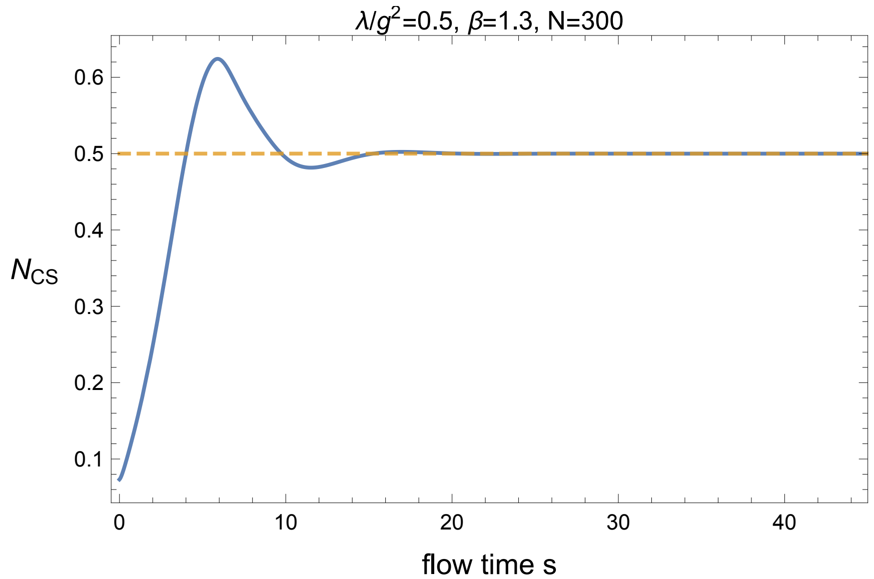
  [Abe-Fujiwara-Hisano-Matsushita, 2004.00884]

Summary

- We proposed a simple method to obtain the sphaleron solution

- We showed the modified gradient flow converges to the sphaleron solution in the $SU(2)$-Higgs model.





- A main advantage of our method is that it does not need any special ansatz and fine-tuning.

  - It has potentials for applications to various models.
    We can write many papers!!

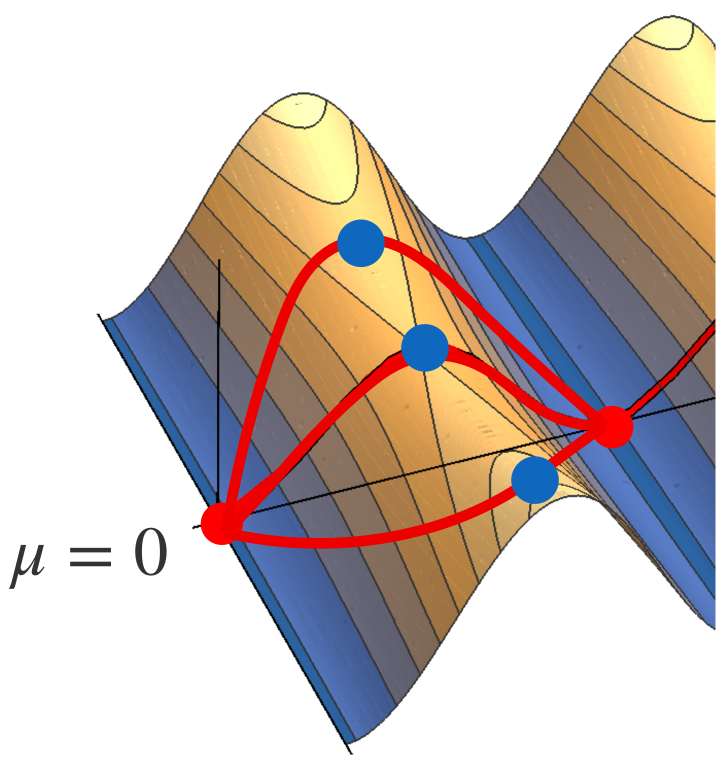
Backup

Min & Max procedure in $SU(2)$-Higgs model

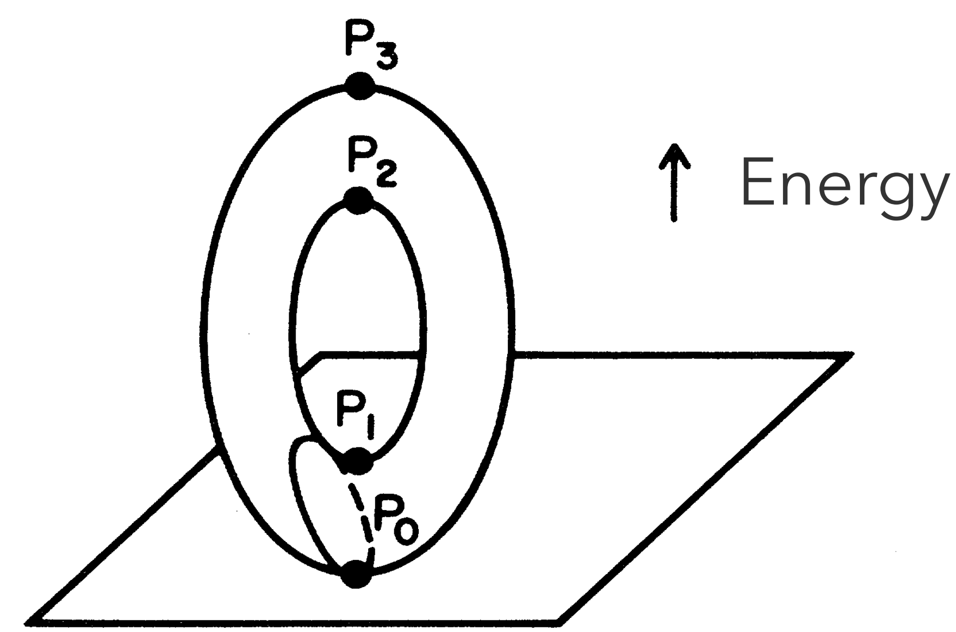
$$E_{sph.} = \min_{\text{path}} \max_{0 \le \mu \le \pi} E(\mu)$$

[Manton, 1983]

- The existence of the minimum is ensured by a topological argument.



identify $\mu = 0, \pi$



P_0 **: vacuum,** P_1 **: sphaleron**

- If there is a non-contractible loop starting and ending at P_0, the minimum point P_1 exists.
- In $SU(2)$-Higgs model, indeed $\pi_1($ config. space $) \neq 0$, and hence such a loop exists.

Min & Max procedure in $SU(2)$-Higgs model

[Manton, 1983]

$$S = \frac{1}{g^2}\int d^4x \left[-\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + (D_\mu\Phi)^\dagger(D^\mu\Phi) - \frac{\lambda}{g^2}\left(\Phi^\dagger\Phi - \frac{1}{2}g^2v^2\right)^2\right]$$

- Consider a path connecting two vacua $N_{CS} = 0,1$

$$\Phi = (1-h(r))\begin{pmatrix} 0 \\ e^{-i\mu}\cos\mu \end{pmatrix} + h(r)\Phi^\infty$$

$$A_i = -f(r)\partial_i U^\infty (U^\infty)^{-1} \quad (i = \theta, \varphi) \qquad \mu \in [0,\pi]$$

$$A_r = 0$$

$$\Phi^\infty = \begin{pmatrix} \sin\mu\sin\theta e^{i\varphi} \\ e^{-i\mu}(\cos\mu + i\sin\mu\cos\theta) \end{pmatrix} \qquad U^\infty = \begin{pmatrix} \Phi_2^{\infty *} & \Phi_1^\infty \\ -\Phi_1^{\infty *} & \Phi_2^\infty \end{pmatrix}$$

Min & Max procedure in $SU(2)$-Higgs model

[Manton, 1983]

$$\Phi = (1 - h(r))\begin{pmatrix} 0 \\ e^{-i\mu}\cos\mu \end{pmatrix} + h(r)\Phi^\infty$$

$$A_i = -f(r)\partial_i U^\infty (U^\infty)^{-1} \quad (i = \theta, \varphi)$$

$$A_r = 0$$

- Independently of $h(r), f(r)$, the energy is maximized when $\mu = \pi/2$ due to the spherical symmetry.
- In other words, the maximization and the minimization decouple.
- Firstly set $\mu = \pi/2$, and then minimize with respect to $h(r), f(r)$.

Spheleron in SM with $\theta_W \neq 0$

- In [Brihaye-Kleihaus-Kunz '92], the sphaleron solution with $\theta_W \neq 0$ is obtained, but an ansatz is imposed (parity sym.). It is unknown whether the solution is a saddle point with one unstable direction.
- Further, the sphaleron with large Weinberg angle $\theta_W \sim \pi/2$ is still unknown.

- In [Klinkhamer-Laterveer '91], a non-contractible loop is constructed, but the solution is no exact. It provides an upper bound of the sphaleron energy.

- In [Hindmarsh-James '93], it is shown that the magnetic dipole moment of the sphaleron originated from a pair of magnetic monopole and antimagnetic monopole connected by a Z flux tube (Nambu monopole) based on perturbation.
- But the large θ_W case is still unknown.

Results for other parameter choises

SM value ~ 0.2

λ / g^2	0.001	0.01	0.5	1	5	10	13
$\dfrac{E_{sph.} g^2}{4\pi v}$	1.563	1.644	1.976	2.066	2.280	2.364	2.395

~ 7.5 TeV

~ 12 TeV

$$\begin{cases} v \sim 246 \text{ GeV} \\ M_W = \frac{1}{2} g v \sim 80 \text{ GeV} \end{cases}$$

Flow eq.

$$C(s) \equiv -\beta\langle \mathcal{G} \,|\, \mathcal{F} \rangle$$

$$\langle \mathcal{G}(s) \,|\, \mathcal{F}(s) \rangle = \int dr \left\{ \left(\frac{\delta N_{\mathrm{CS}}}{\delta \chi} \right)^* \frac{\delta E}{\delta \chi} + \left(\frac{\delta N_{\mathrm{CS}}}{\delta \chi^*} \right)^* \frac{\delta E}{\delta \chi^*} \right\}$$

$$\langle \mathcal{G}(s) \,|\, \mathcal{G}(s) \rangle = \int dr \left(\left| \frac{\delta N_{\mathrm{CS}}}{\delta \chi} \right|^2 + \left| \frac{\delta N_{\mathrm{CS}}}{\delta \chi^*} \right|^2 \right)$$

$$\frac{\delta E}{\delta \chi^*} = \left[-\partial_1^2 + \frac{1}{r^2} \left(|\chi|^2 - 1 \right) + \frac{1}{2} |\phi|^2 \right] \chi - \frac{1}{2} \phi^2$$

$$\frac{\delta E}{\delta \phi^*} = \left[\partial_1 r^2 \partial_1 + \frac{1}{2} \left(|\chi|^2 + 1 \right) - \frac{\lambda}{g^2} r^2 \left(2|\phi|^2 - g^2 v^2 \right) \right] \phi - \chi \phi^*$$

Numerical setup

- We take a length unit such that $v = 1$

- The spatial lattice:

$$r = i \times \Delta r \qquad \Delta r = 5.0 \times 10^{-2}$$

$$(i = 1, 2, \cdots, N) \qquad L \equiv N \times \Delta r$$

- The flow time lattice: $\Delta s = 1.5 \times 10^{-3}$

Boundary conditions

regularity

$$\begin{cases} \partial_r \chi(0) = 0,\ \phi(0) = 0 & \text{(at the origin)} \\ \chi(L) = 1,\ \phi(L) = gv/\sqrt{2} & \text{(at large distances)} \end{cases}$$

finite energy

$$\Updownarrow$$

$$\begin{cases} f'(0) = h'(0) = 0,\ \mu(0) = \nu(0) = 0 & \text{(at the origin)} \\ f(L) = 1,\ \mu(L) = gv/\sqrt{2},\ h(L) = \nu(L) = 0 & \text{(at large distances)} \end{cases}$$

L: size of the system