Color Confinement and Bose-Einstein Condensation

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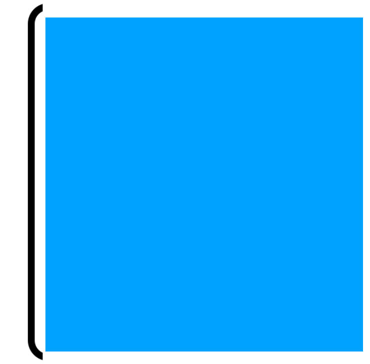
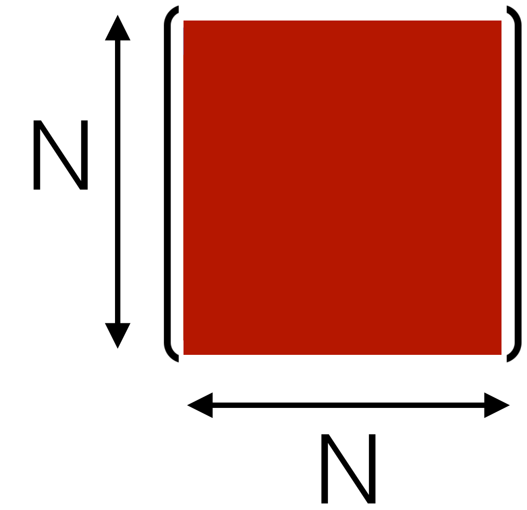
- ‘kinematical’ confinement/deconfinement at large N
- Partial confinement/deconfinement
- Explicit demonstration at weak coupling
- Confinement = BEC $\rightarrow$ generalization to strong coupling
- (Numerical evidence at strong coupling)



- Confinement phase: E, S ~ N^0



- Deconfinement phase: E, S ~ N^2

⟷ Black Hole

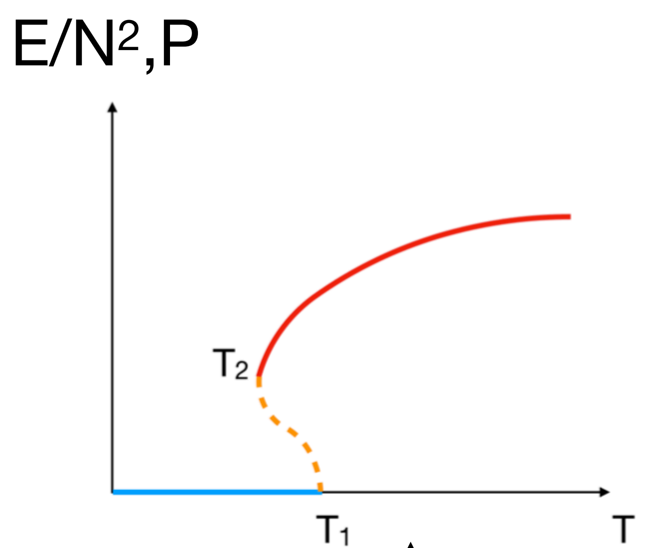
This ‘kinematical’ characterization **(Witten 1998)**

works even at weak coupling and/or small volume.

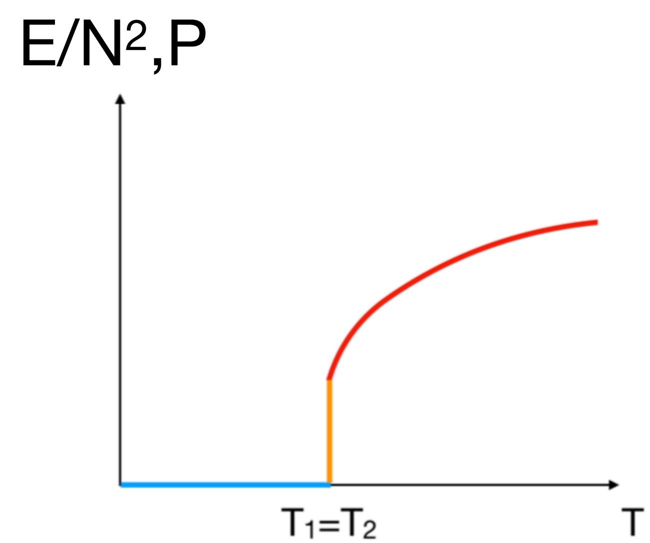
(Sundborg 1998; Aharony et al 2003)

We consider the <u>large-N</u> limit.

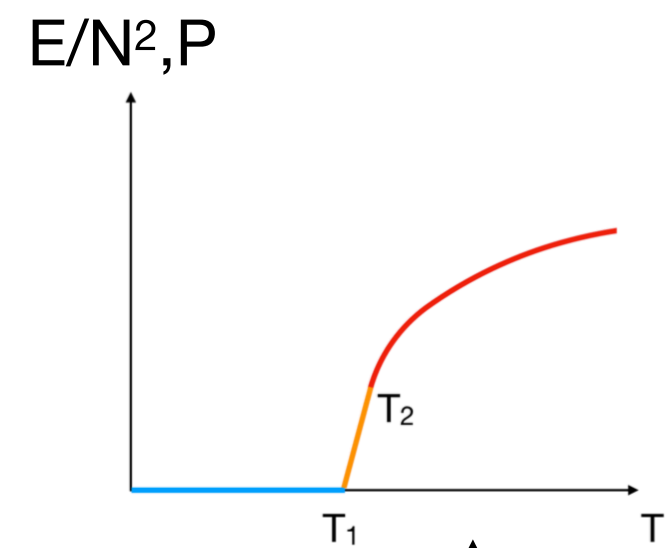
Three patterns of the phase diagram in gauge theory (Aharony et al, 2003)



Strongly-coupled Yang-Mills, black hole



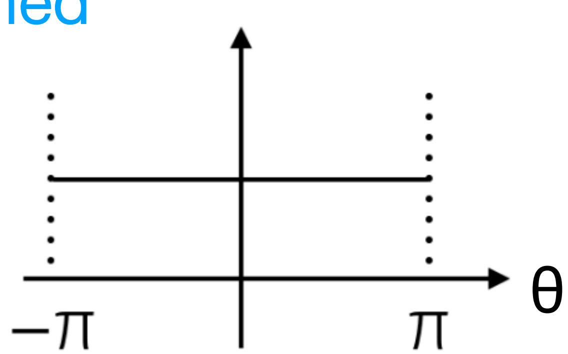
Free Yang-Mills



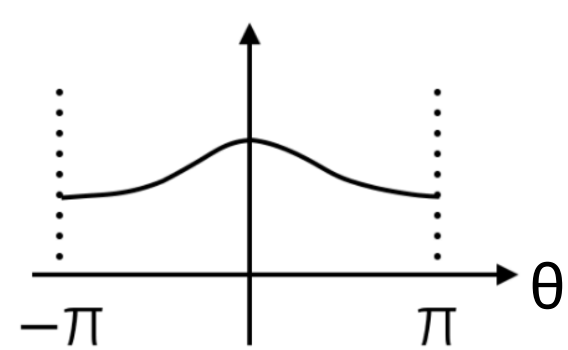
QCD looks like this

Polyakov loop $P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$

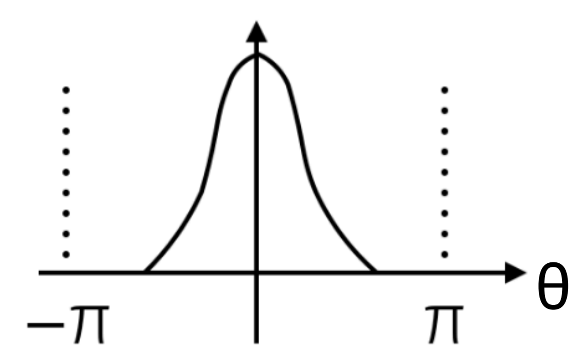
Confined



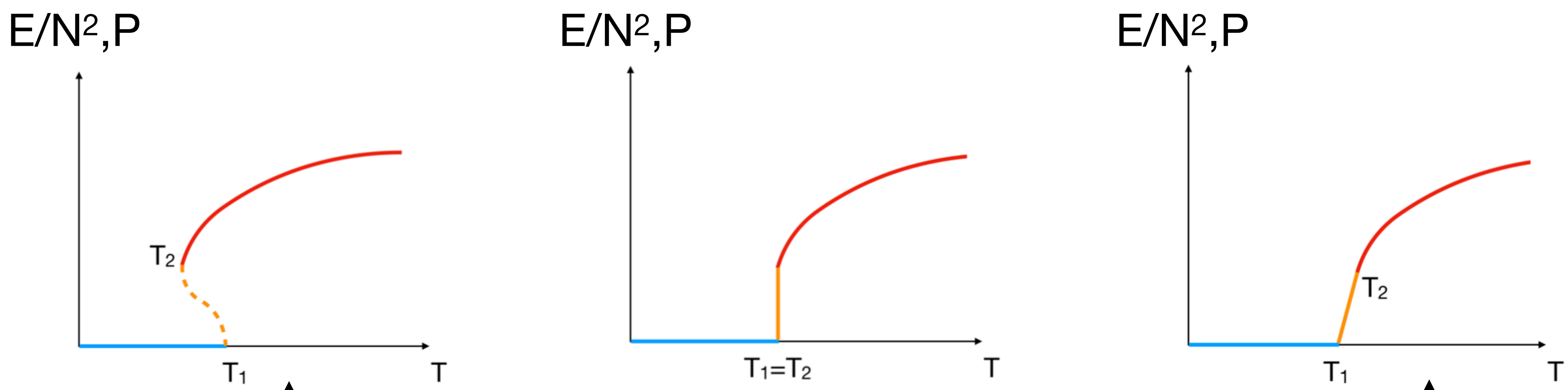
Deconfined



Deconfined



Three patterns of the phase diagram in gauge theory (Aharony et al, 2003)



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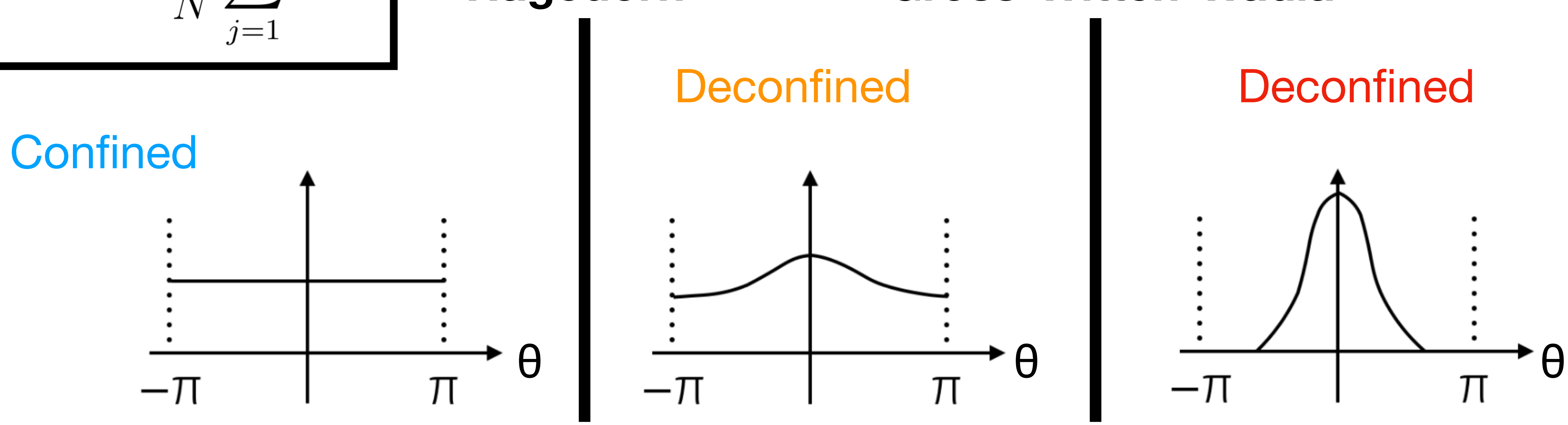
Hagedorn

Gross-Witten-Wadia



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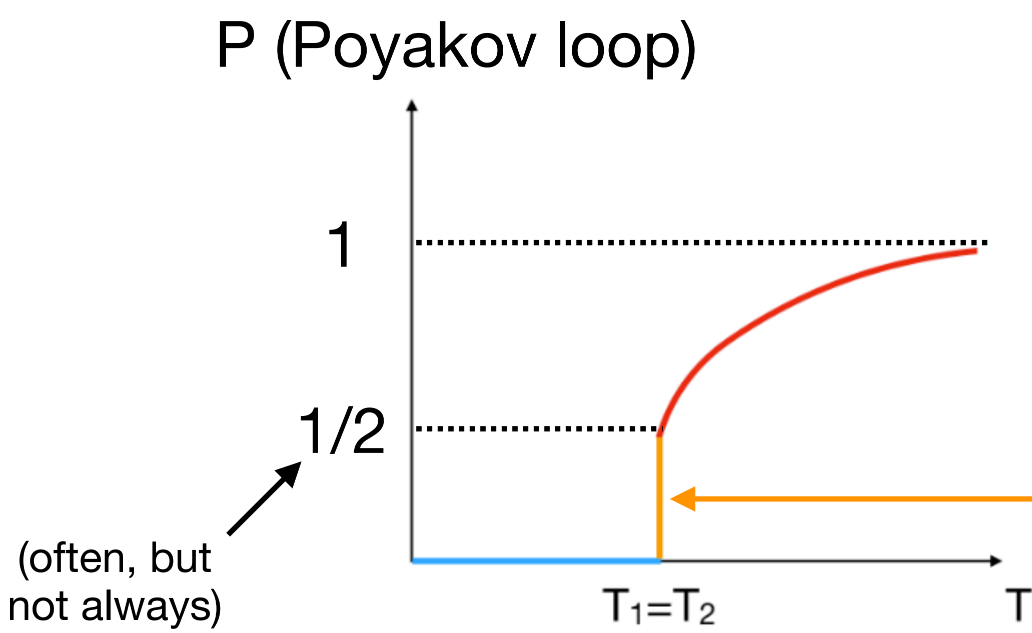


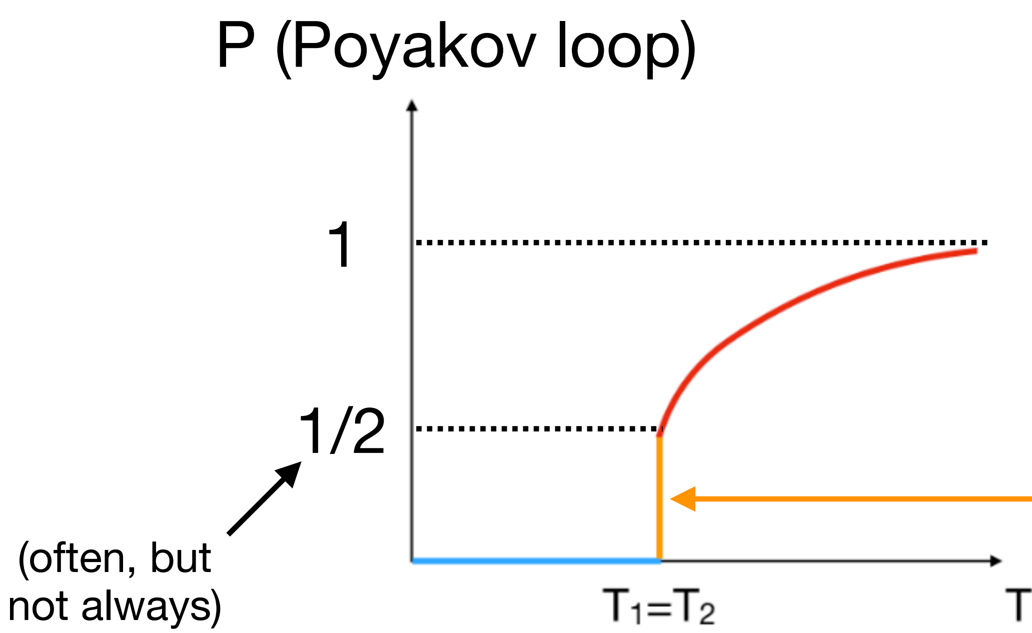
Free energy F = E−TS = 0

Degenerate vacua in *canonical ensemble*

(minimize F at each T)

Z=Z(T), P=P(T), E=E(T)

Three patterns of the phase diagram in gauge theory (Aharony et al, 2003)



Free energy F = E−TS = 0

Degenerate vacua in *canonical ensemble*

(minimize F at each T)

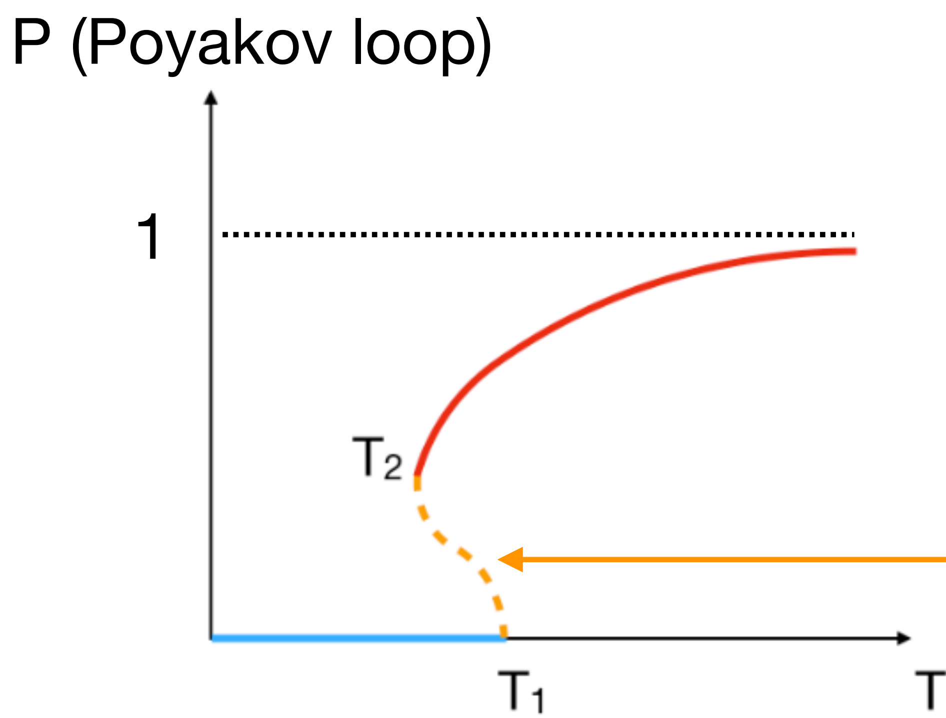
Z=Z(T), P=P(T), E=E(T)

Unique vacuum in *micro-canonical ensemble*

(maximize S at each E)

S=S(E), P=P(E), T=T(E)

Three patterns of the phase diagram in gauge theory
(Aharony et al, 2003)



Maximum of Free energy

Unstable in *canonical ensemble*

(minimize F at each T)

Stable in *micro-canonical ensemble*

(maximize S at each E)

Thermodynamics 101

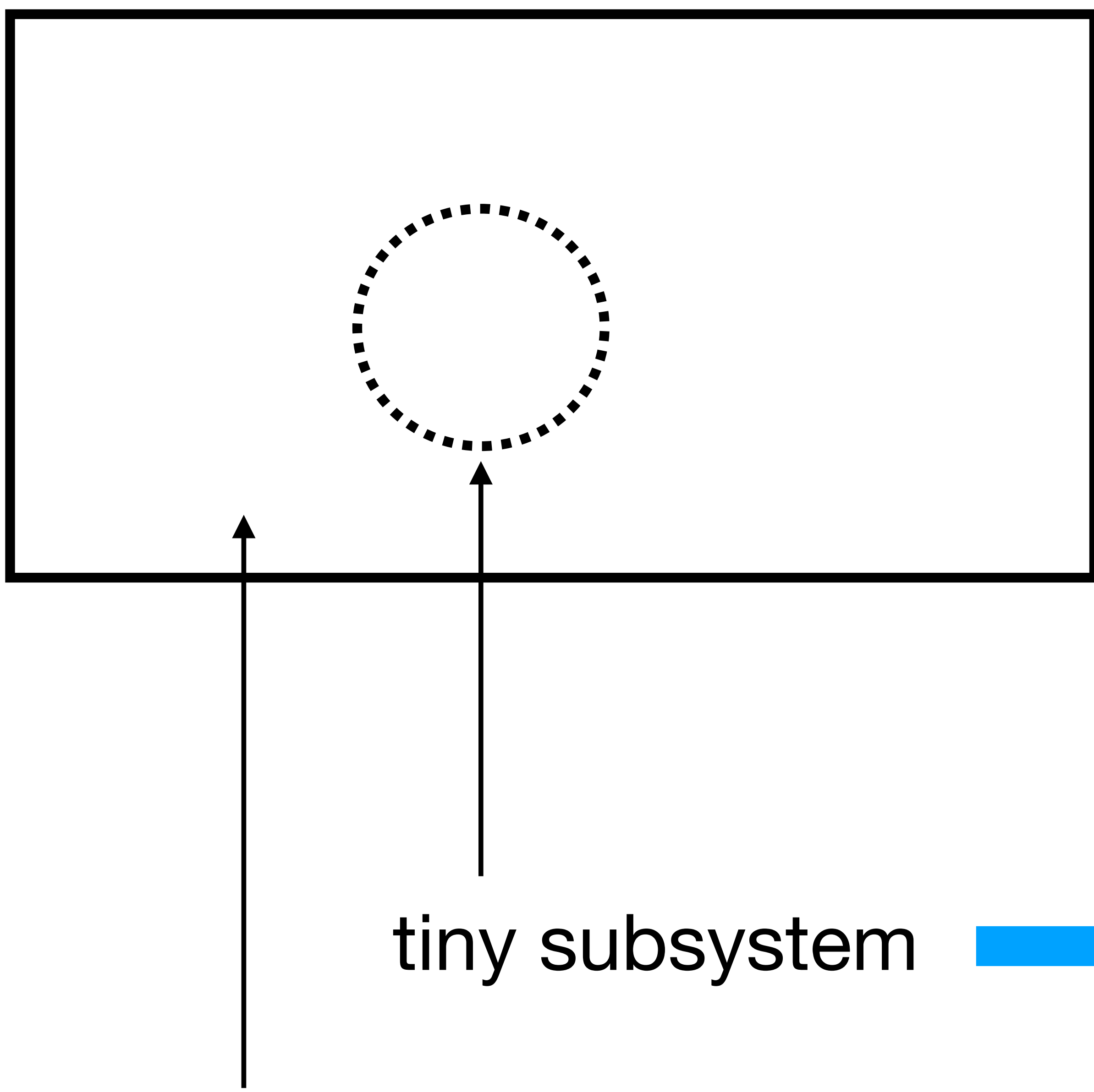
isolated system (E conserved)

→ microcanonical ensemble

maximize entropy S(E)

$T^{-1} = dS/dE$

Thermodynamics 101



isolated system (E conserved)

→ microcanonical ensemble

maximize entropy S(E)

$T^{-1} = dS/dE$

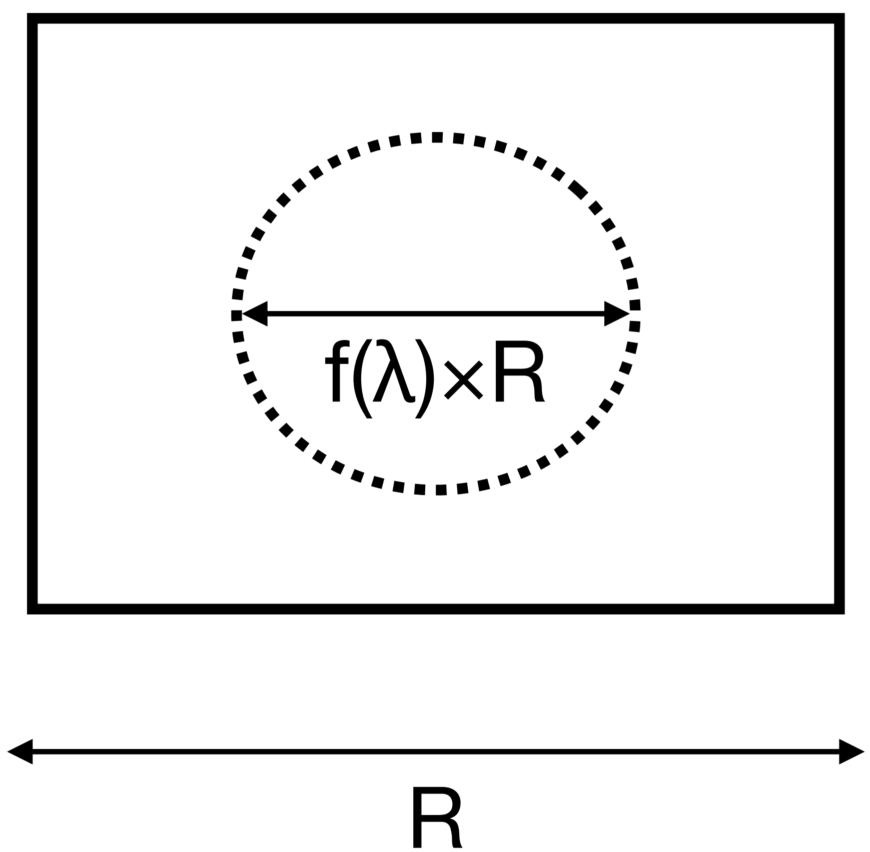
tiny subsystem → canonical ensemble

minimize free energy F(T)

gigantic heat bath
(temperature T)

Thermodynamics 102

Canonical ensemble may or may not make sense at finite volume.



4d SYM on S^3

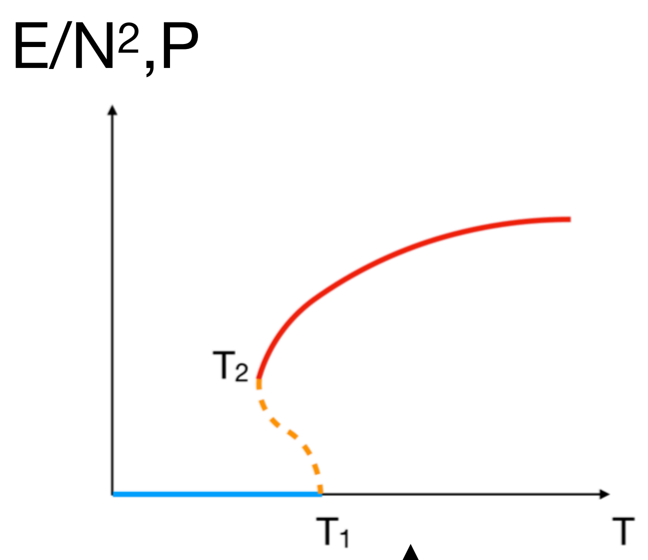
radius R

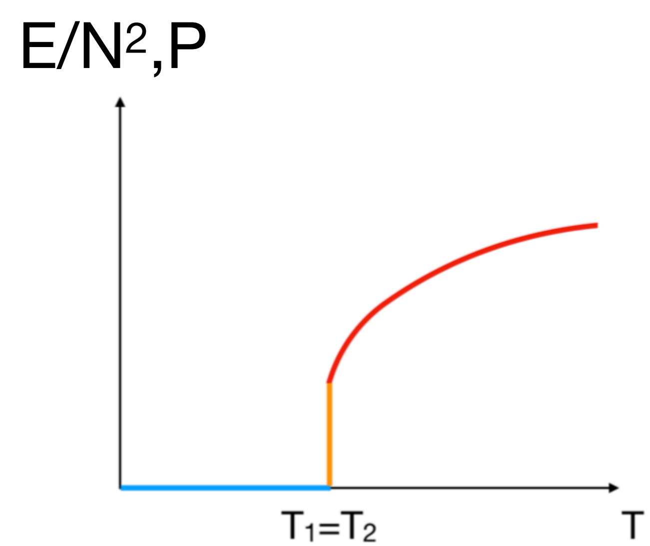
typical length scale $f(\lambda)\times R$

➡ 'tiny subsystem' may not make sense.

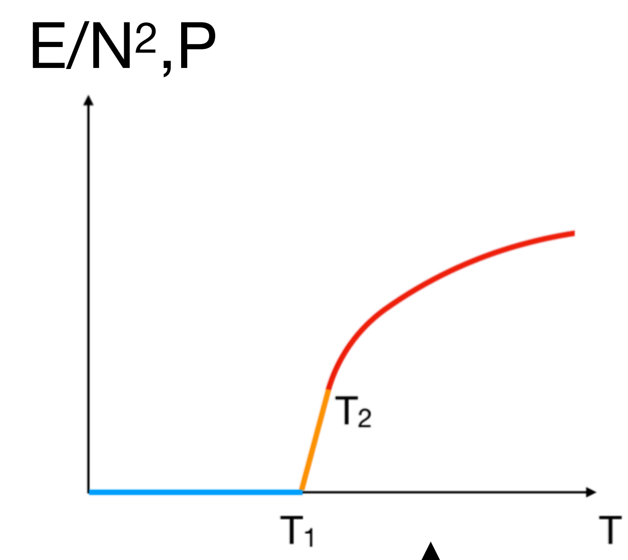
Micorocanonical ensemble always makes sense.

Three patterns of the phase diagram in gauge theory (Aharony et al, 2003)





Free Yang-Mills



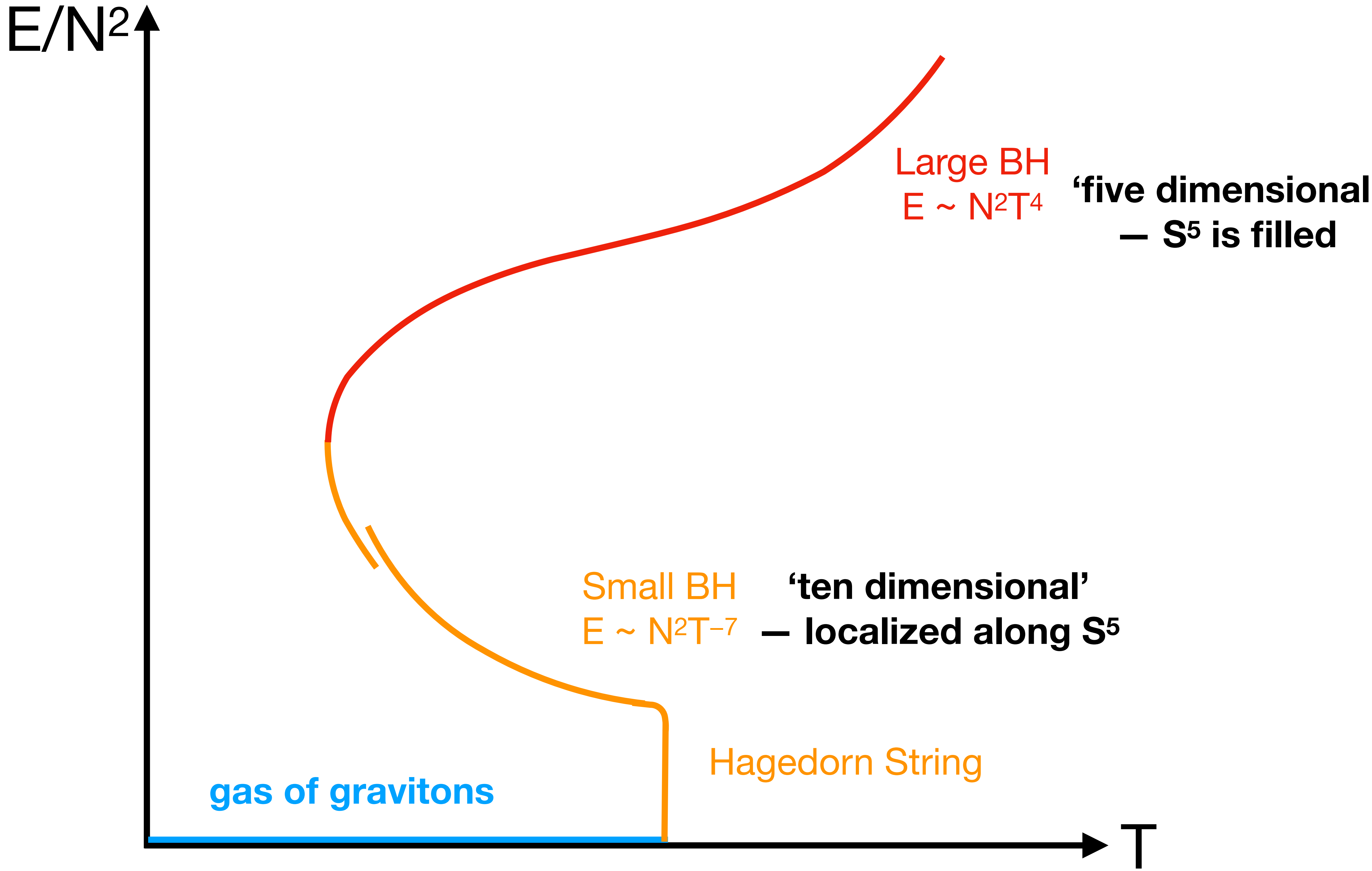
Strongly-coupled Yang-Mills, black hole

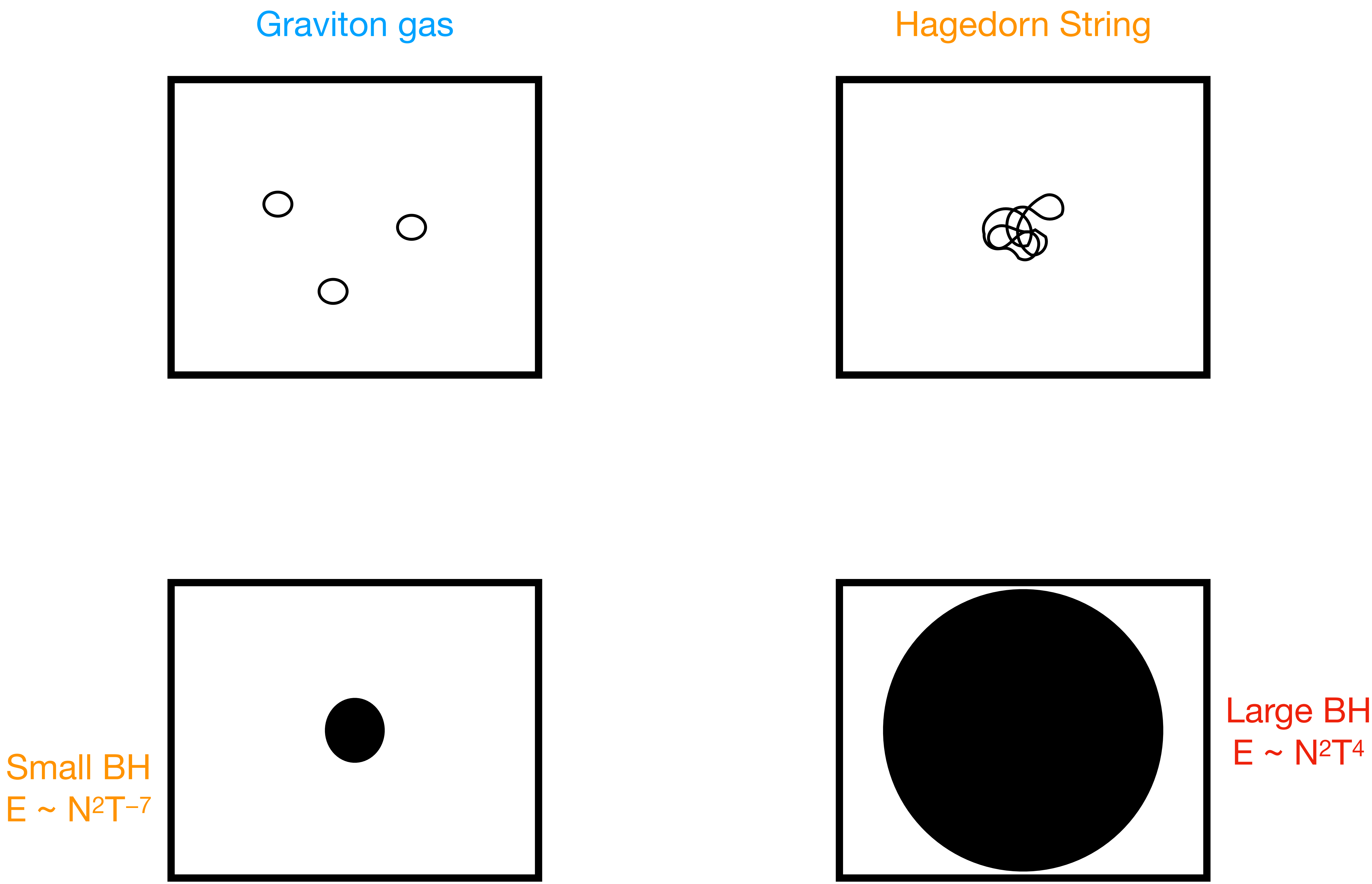
QCD looks like this

canonical ≠ microcanonical

canonical = microcanonical

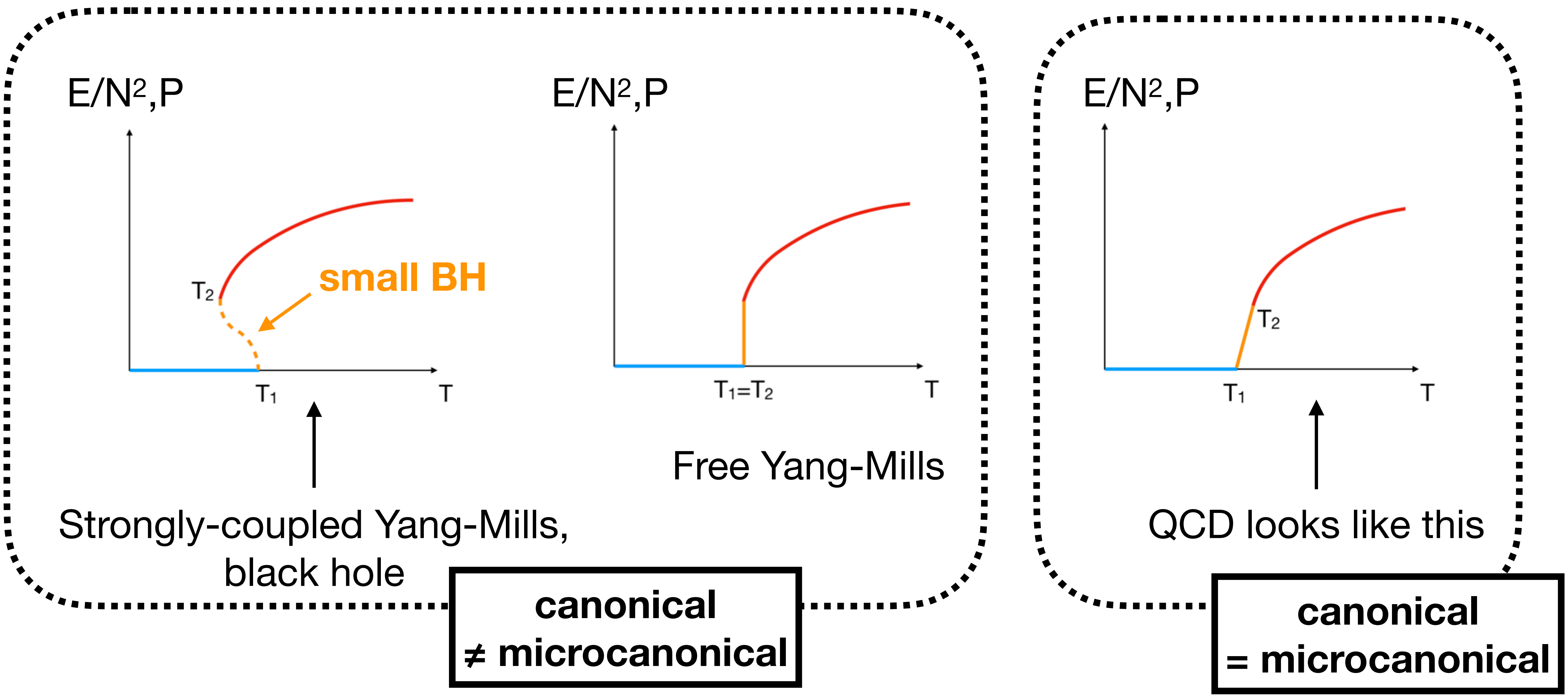
Black Hole in $AdS_5 \times S^5$ = 4d N=4 SYM on S^3





Schwarzschild BH in D-dim spacetime → $T^{-(D-3)}$

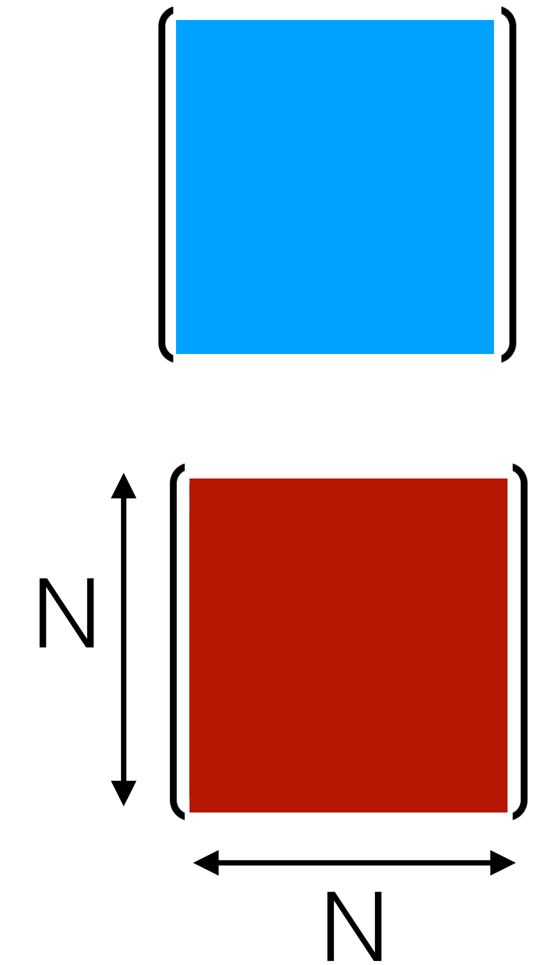
Three patterns of the phase diagram in gauge theory (Aharony et al, 2003)



- Small BH-like phase between QGP and hadron phase?

- What is happening there?

- ‘kinematical’ confinement/deconfinement at large N
- Partial confinement/deconfinement
- Explicit demonstration at weak coupling
- Confinement = BEC → generalization to strong coupling?
- (Numerical evidence at strong coupling)

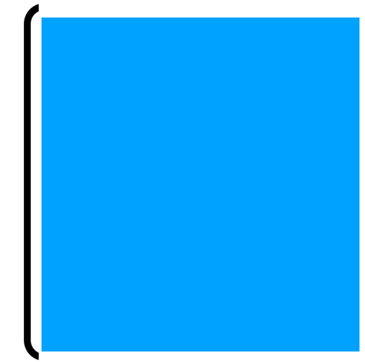


- Confinement phase: E, S ~ N^0
- Deconfinement phase: E, S ~ N^2

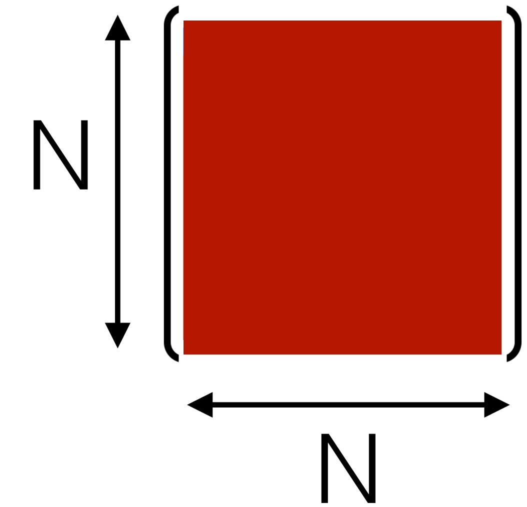
$\Longleftrightarrow$ Black Hole

What if E ~ $N^2/100$?

- Confinement phase: E, S ~ N^0

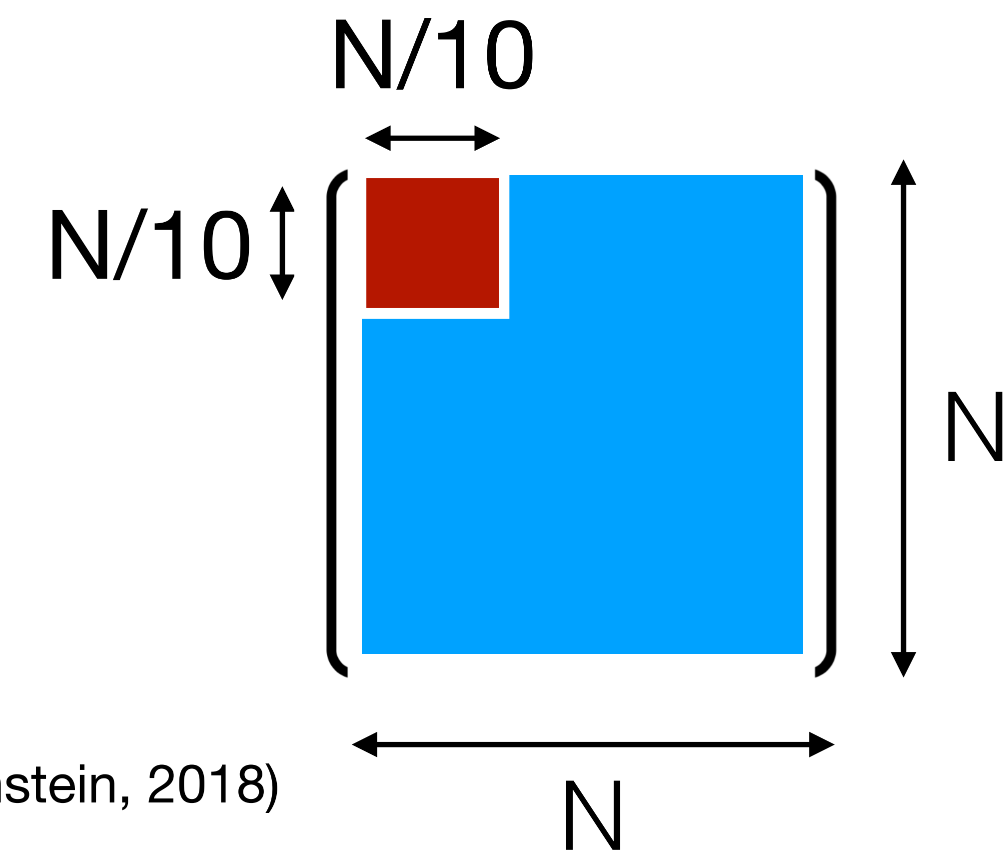


- Deconfinement phase: E, S ~ N^2



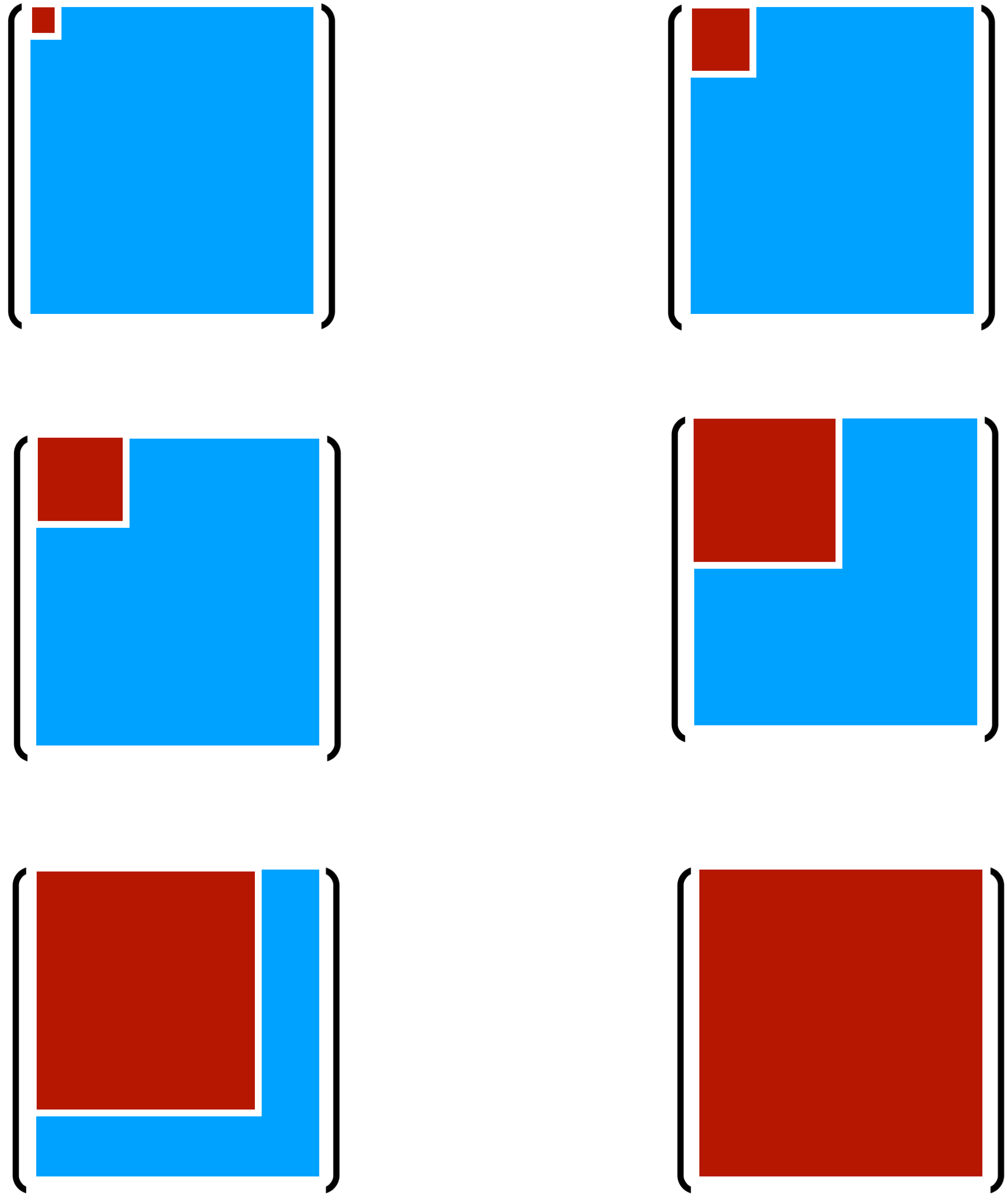
⟷ Black Hole

What if E ~ $N^2/100$?

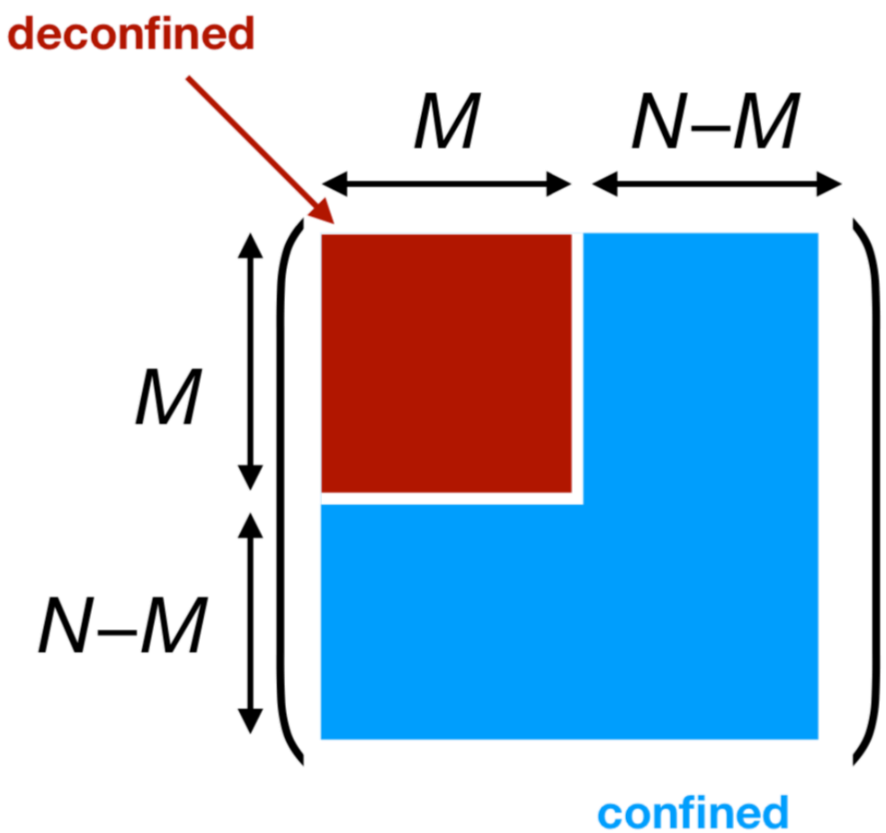


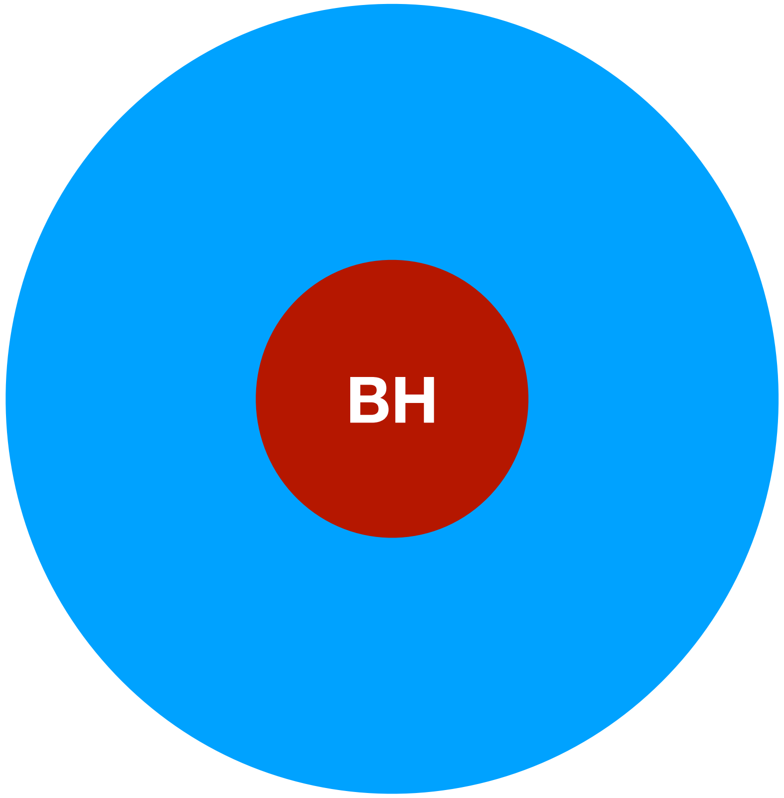
<u>‘partially’ deconfine</u>

(MH-Maltz, 2016; Berenstein, 2018)



Intuitive picture in gravity (no proof yet)

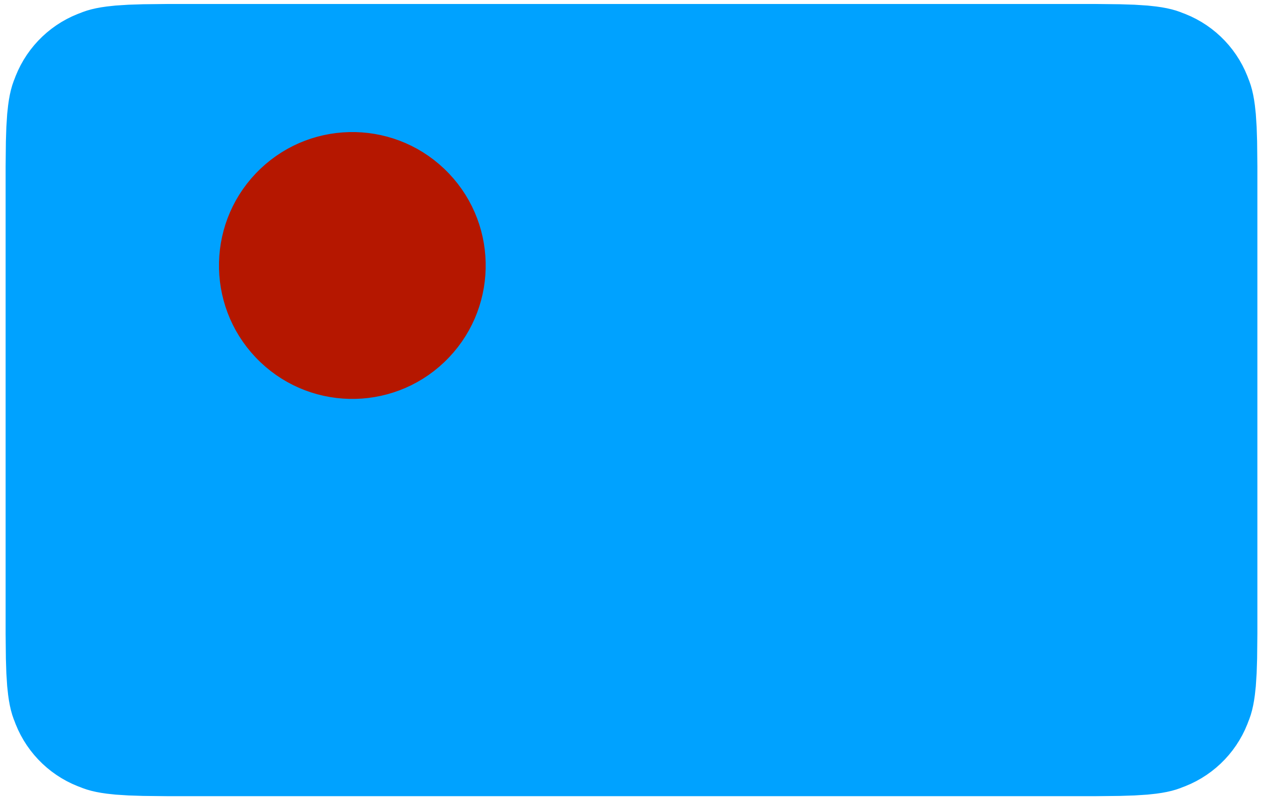




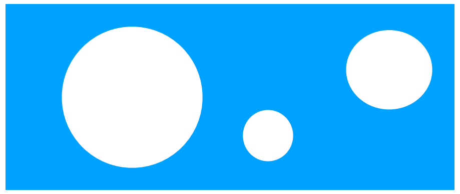
Heuristic justification

(more precise argument is given later)

Why doesn't a part of the volume deconfine?

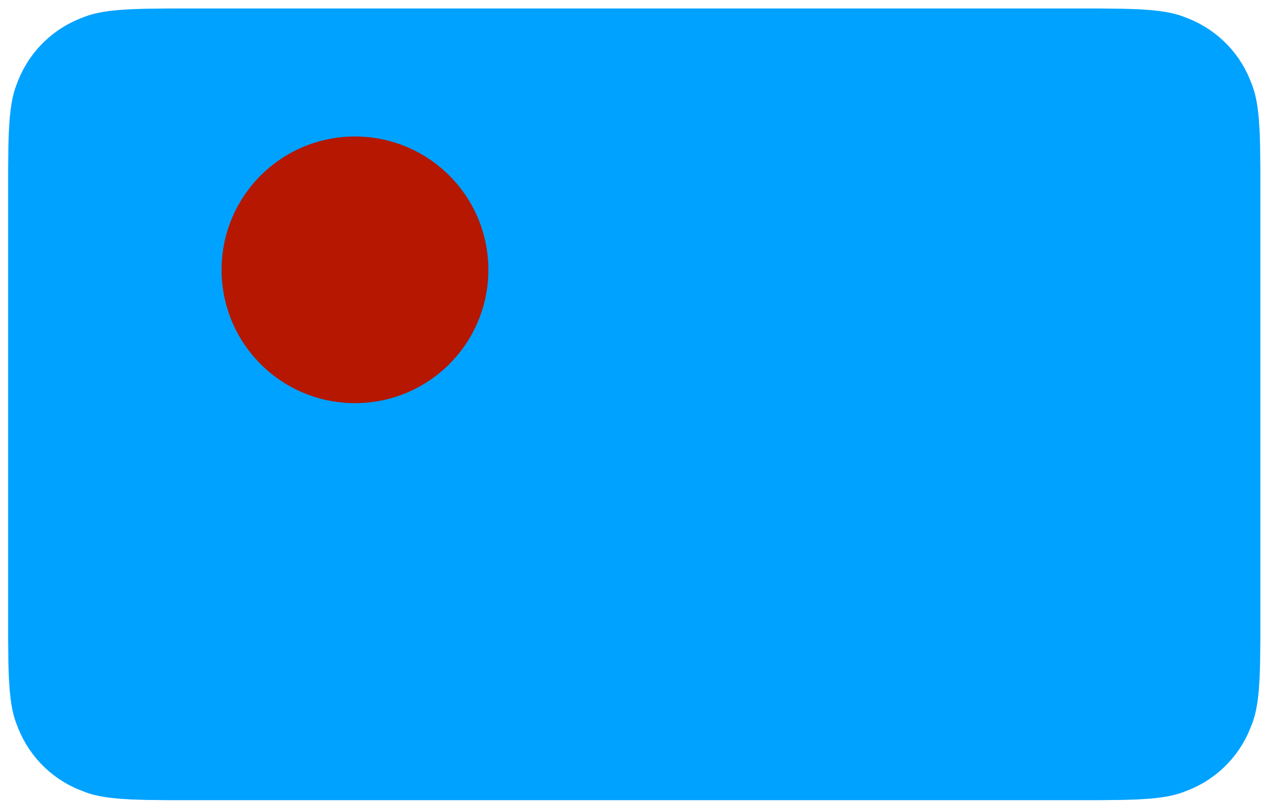


cf) water/ice



(Exception: first order transition, large volume)

Why doesn’t a part of the volume deconfine?



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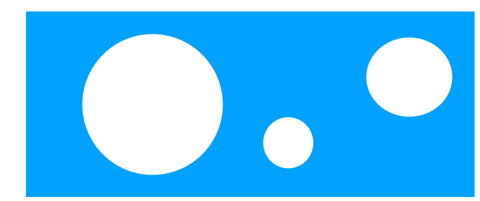
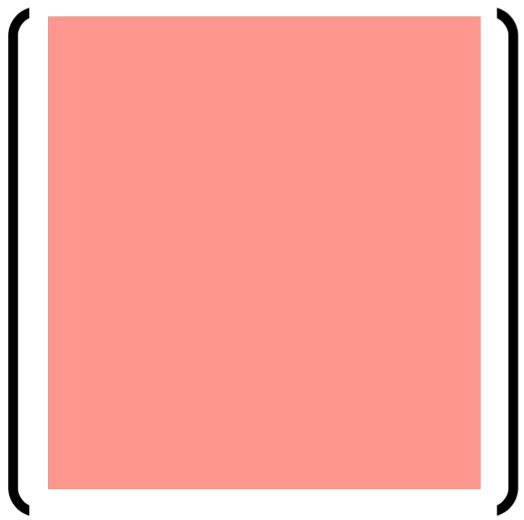
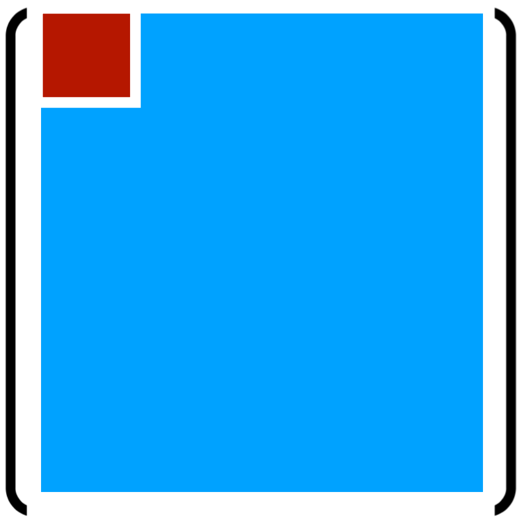


Deconfinement takes place even in matrix model,
which has no spatial dimension.

(Exception: first order transition, large volume)

Why don't all N^2 d.o.f. gently deconfine?

 instead of  ?

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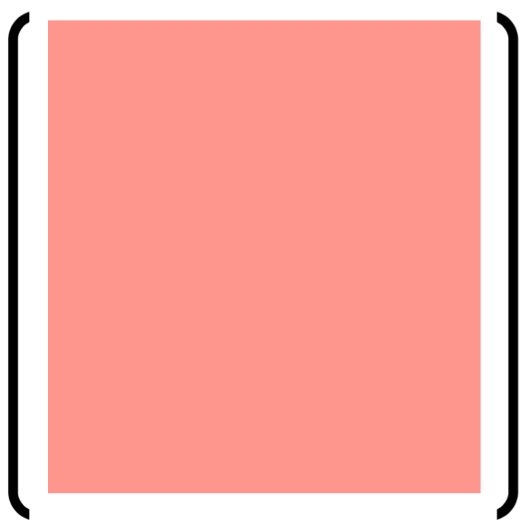
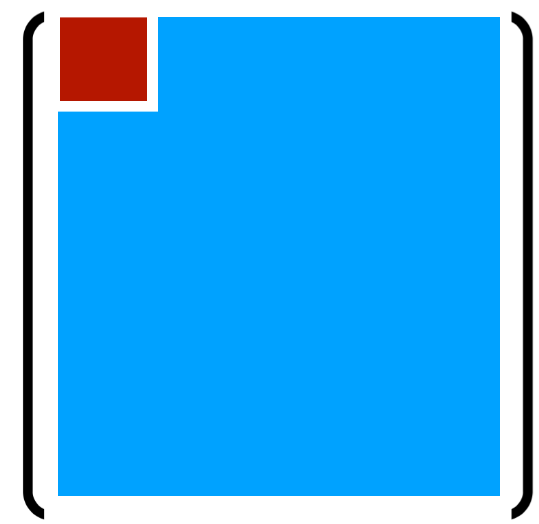


instead of

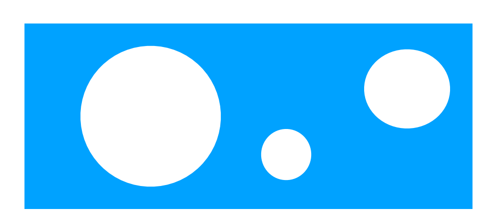


?

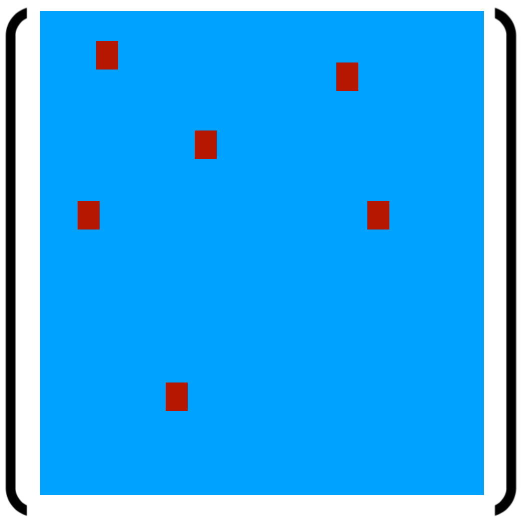
In quantum mechanics, parametrically small excitation is impossible.

minimal energy quantum $\sim$ latent heat

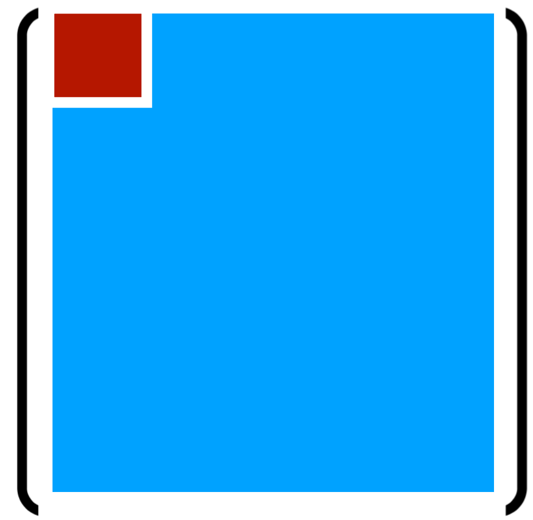
cf) water/ice



Why should large symmetry be preserved?



instead of

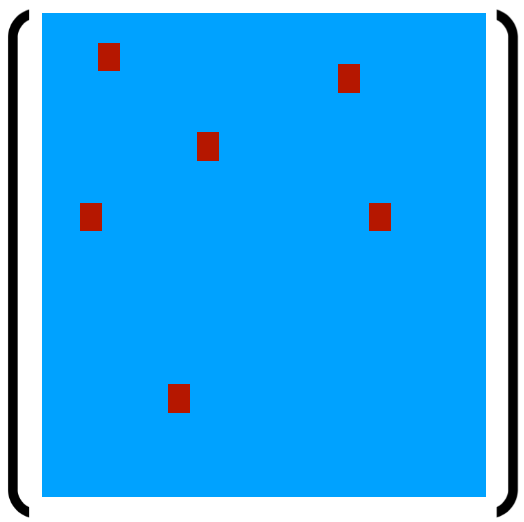
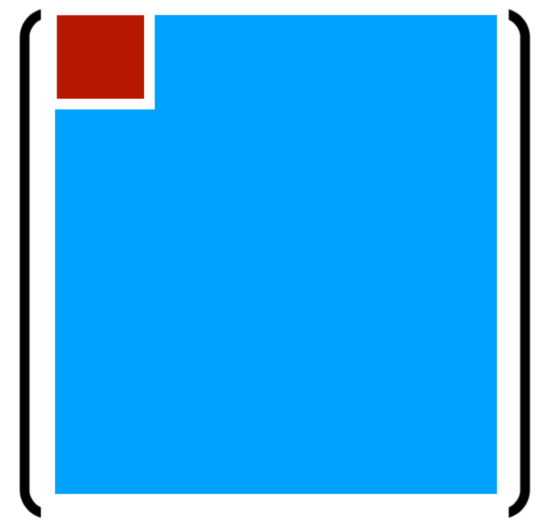


?

no symmetry

SU(M)×SU(N-M)

Why should large symmetry be preserved?

 instead of  ?

no symmetry

$SU(M)\times SU(N-M)$

It is natural to expect a large symmetry at saddle point.

“Confinement = BEC” will justify this expectation.

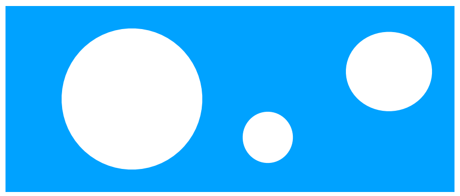
Phase Diagrams

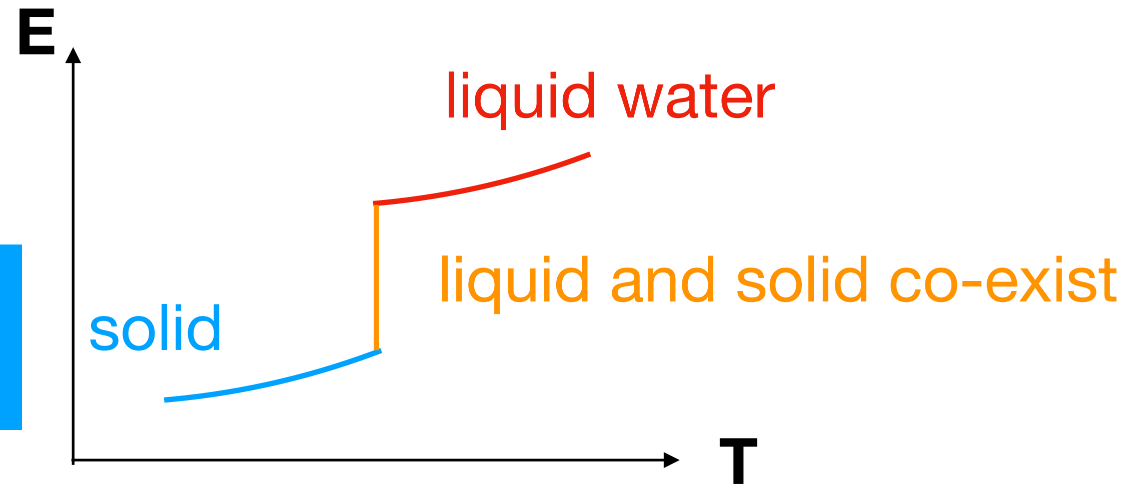
MH-Ishiki-Watanabe,
arXiv:1812.05494 [hep-th]

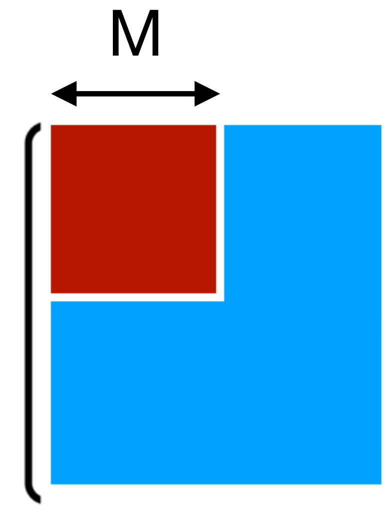
— Hagedorn transition
— Gross-Witten-Wadia transition
— “Gauge symmetry breaking”

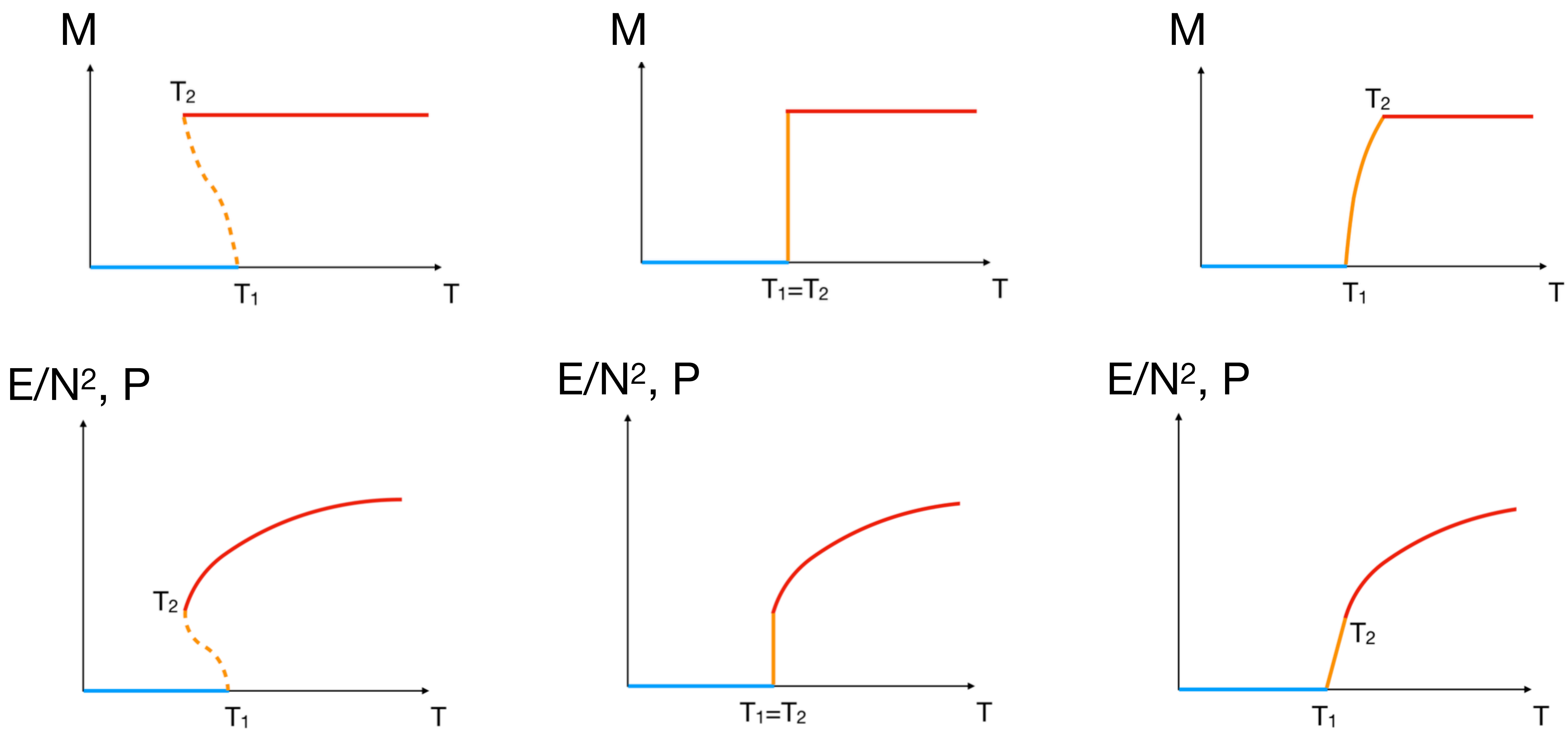
(more precise argument is given later)

cf) water/ice



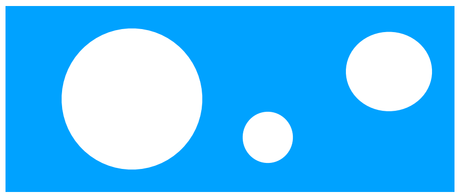


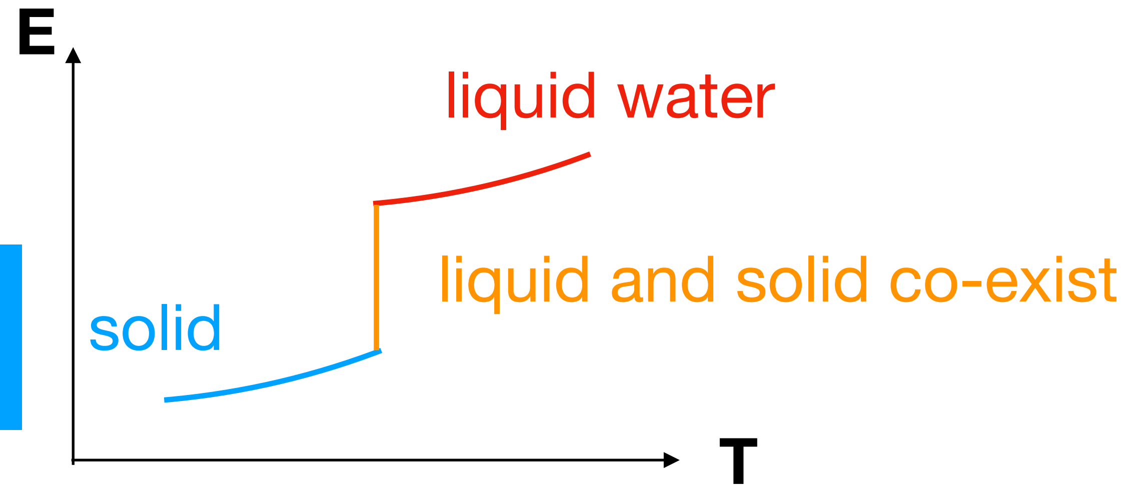


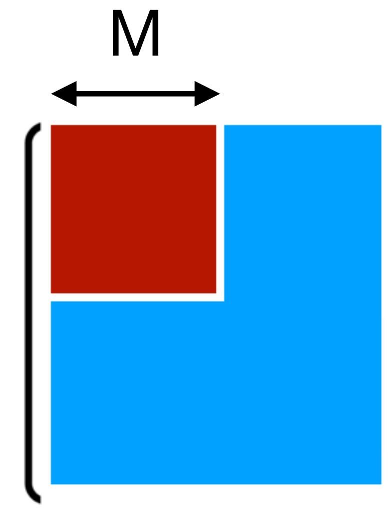


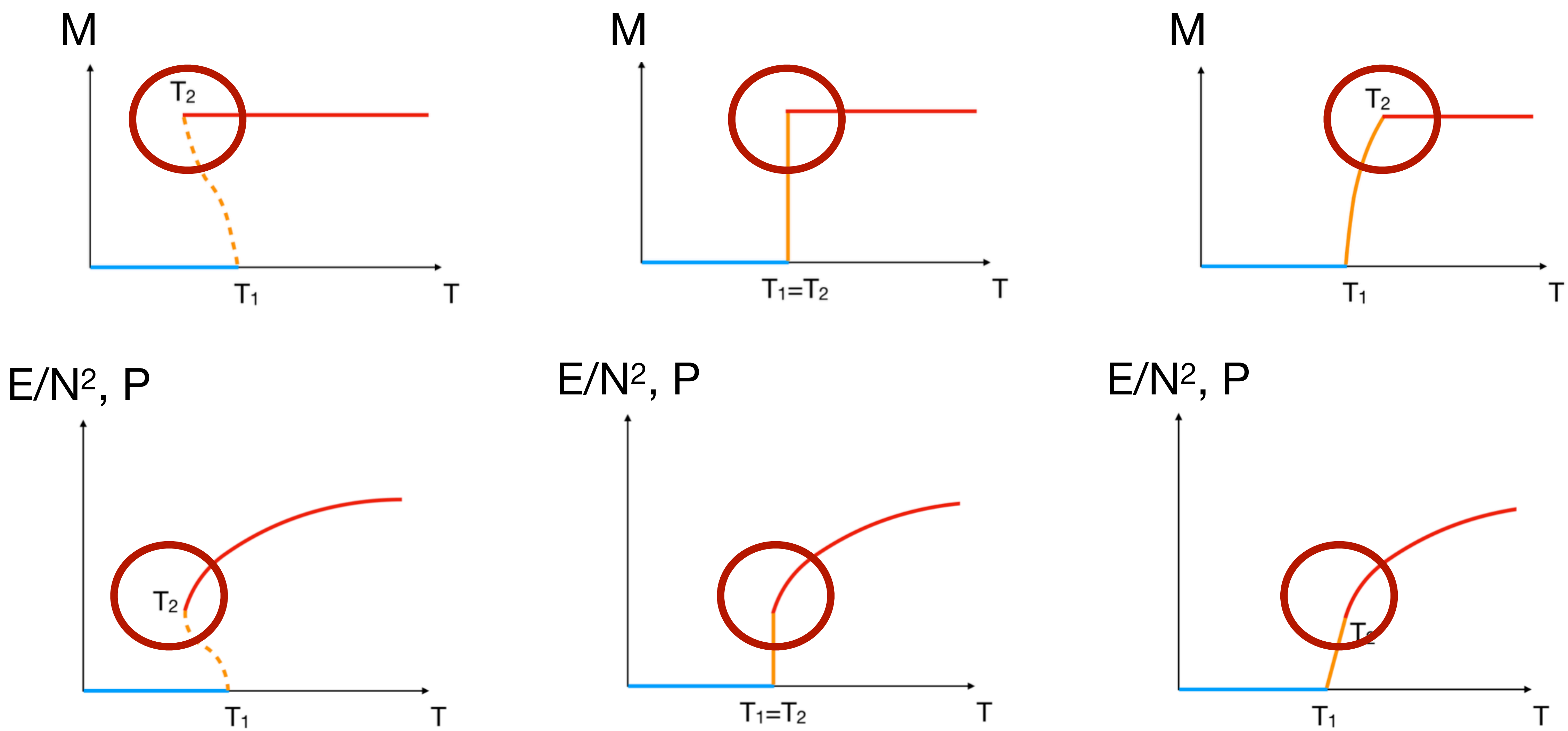
All-to-all interaction → Nontrivial T-dependence

cf) water/ice









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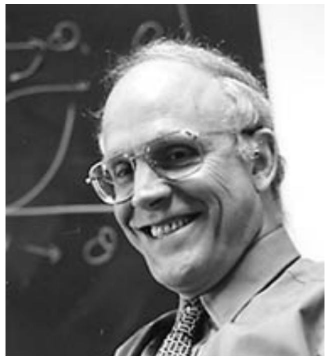
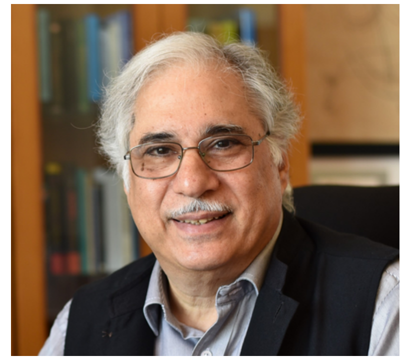
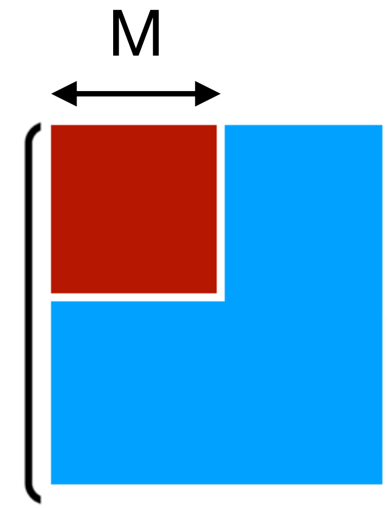
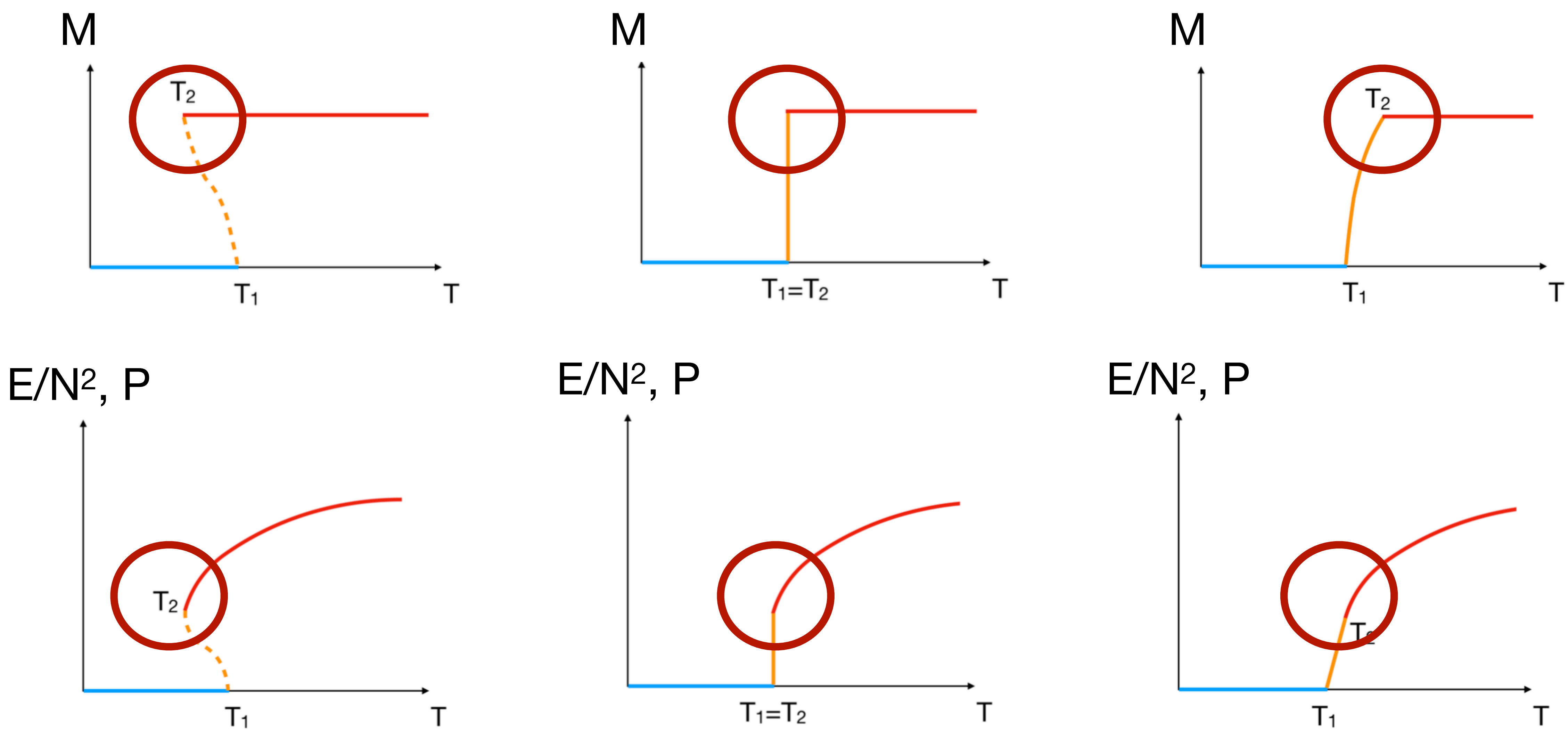




Gross-Witten-Wadia transition





All-to-all interaction → Nontrivial T-dependence

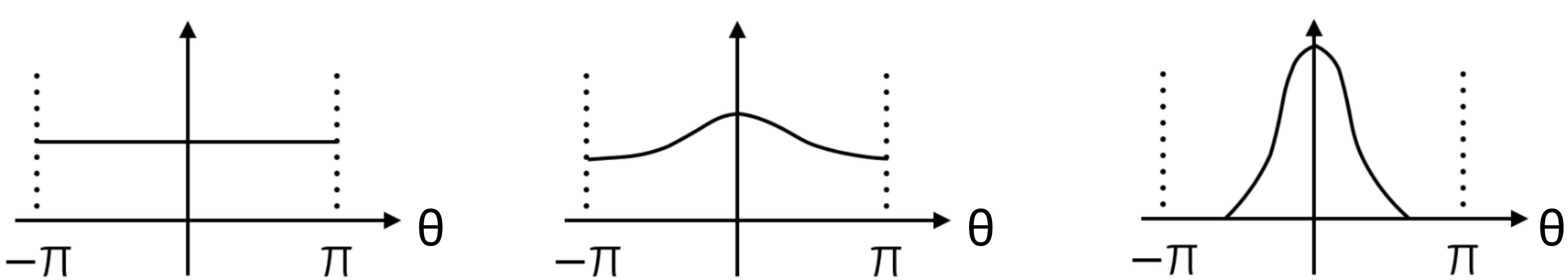
- Polyakov loop $$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

  (Wilson loop wrapped on the temporal circle)

- Phase distribution:

confined phase
P=0

deconfined phase
P ≠ 0



‘partially’ deconfined **‘completely’ deconfined**

- Polyakov loop $$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

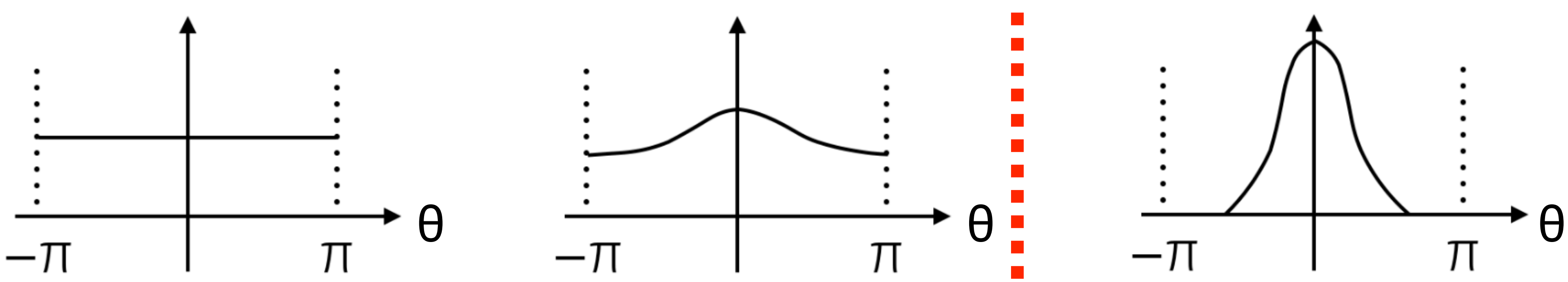
(Wilson loop wrapped on the temporal circle)

- Phase distribution:

Gross-Witten-Wadia transition (GWW)

confined phase
P=0

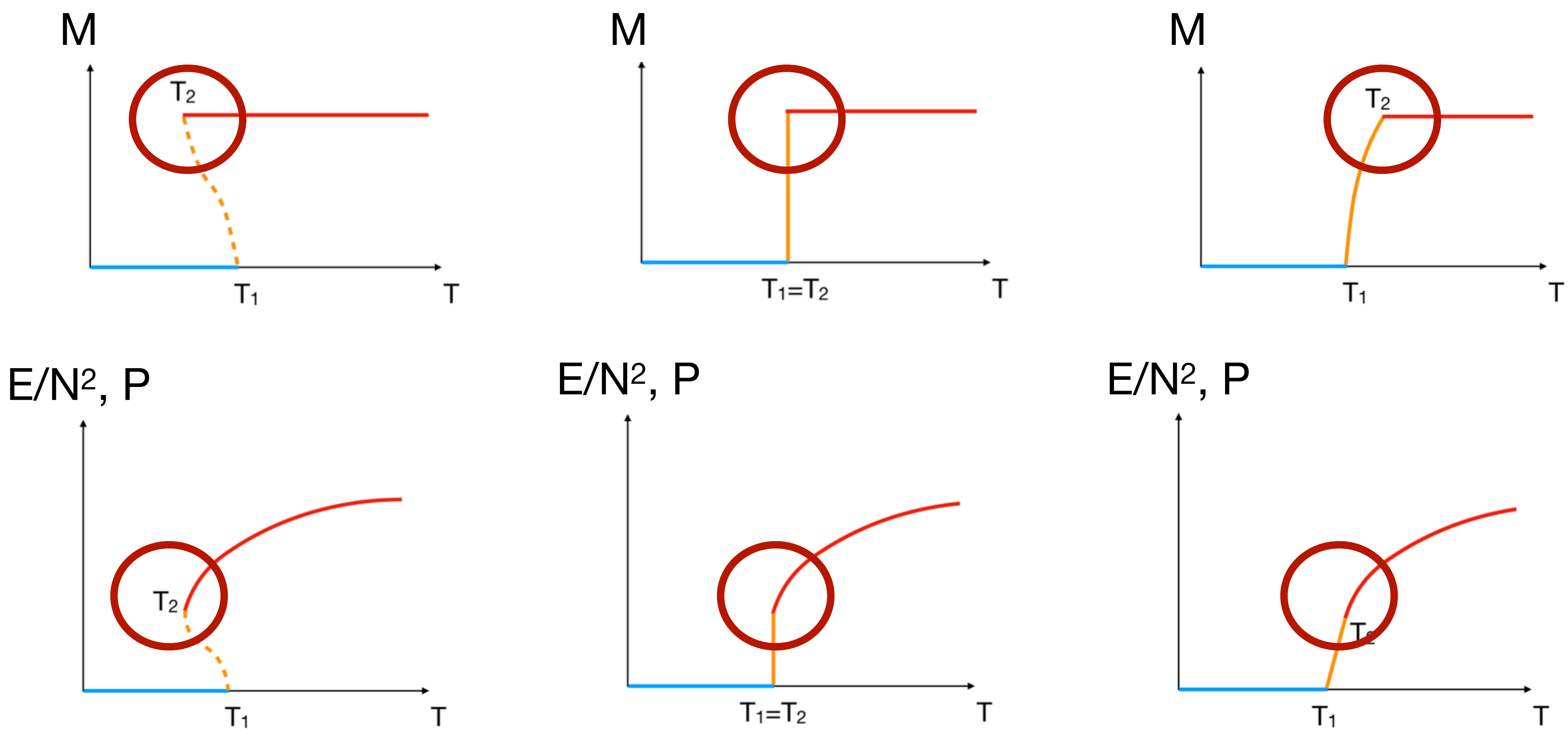
deconfined phase
P ≠ 0



‘partially’ deconfined

‘completely’ deconfined

Gross-Witten-Wadia transition = "partial deconfinement → complete deconfinement" transition



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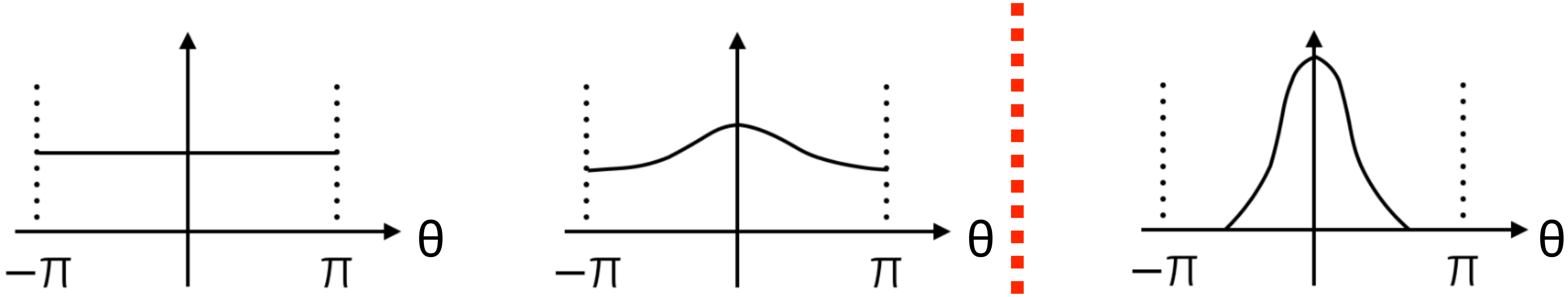

'partially' deconfined **'completely' deconfined**

- Polyakov loop

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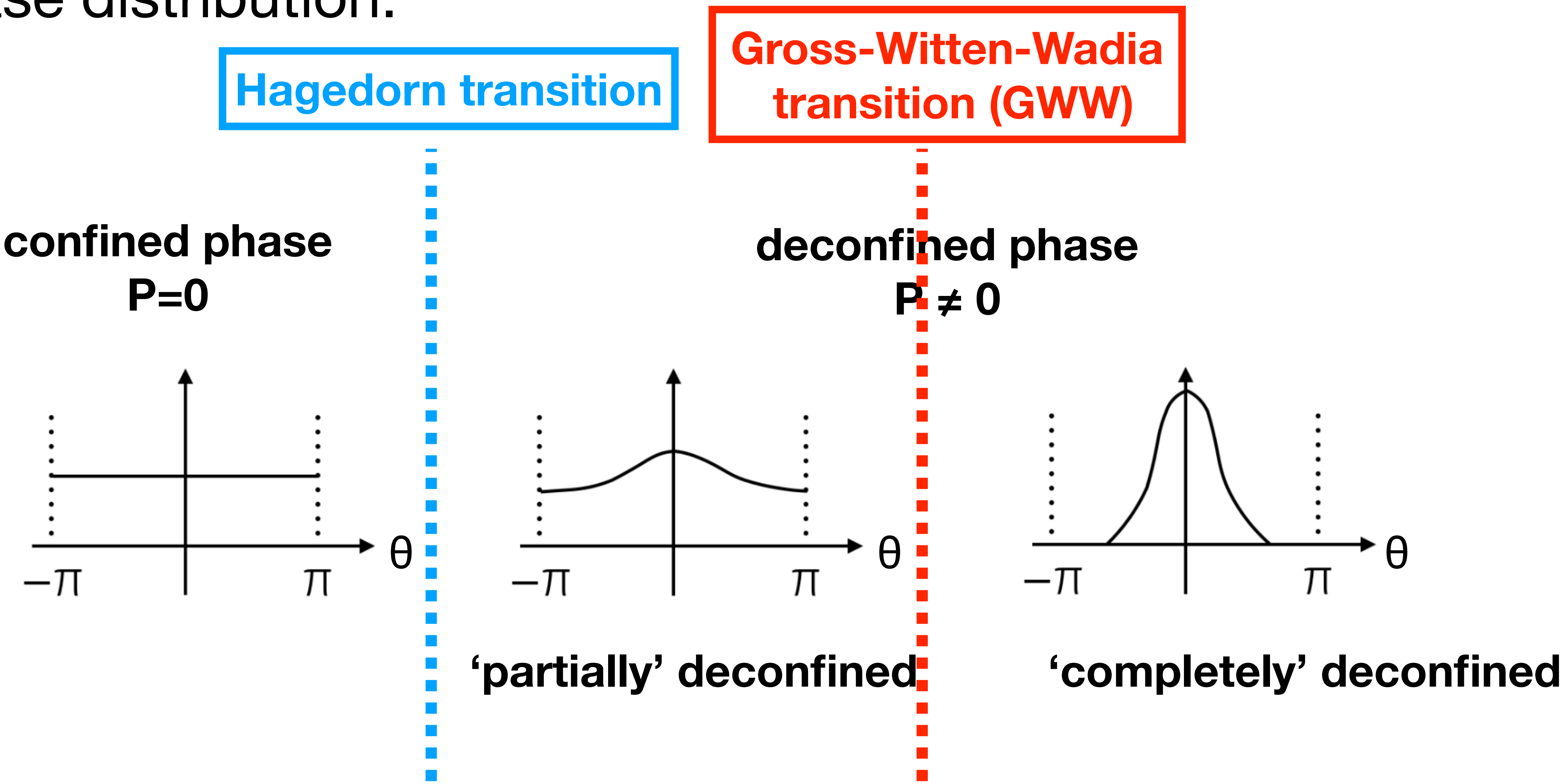
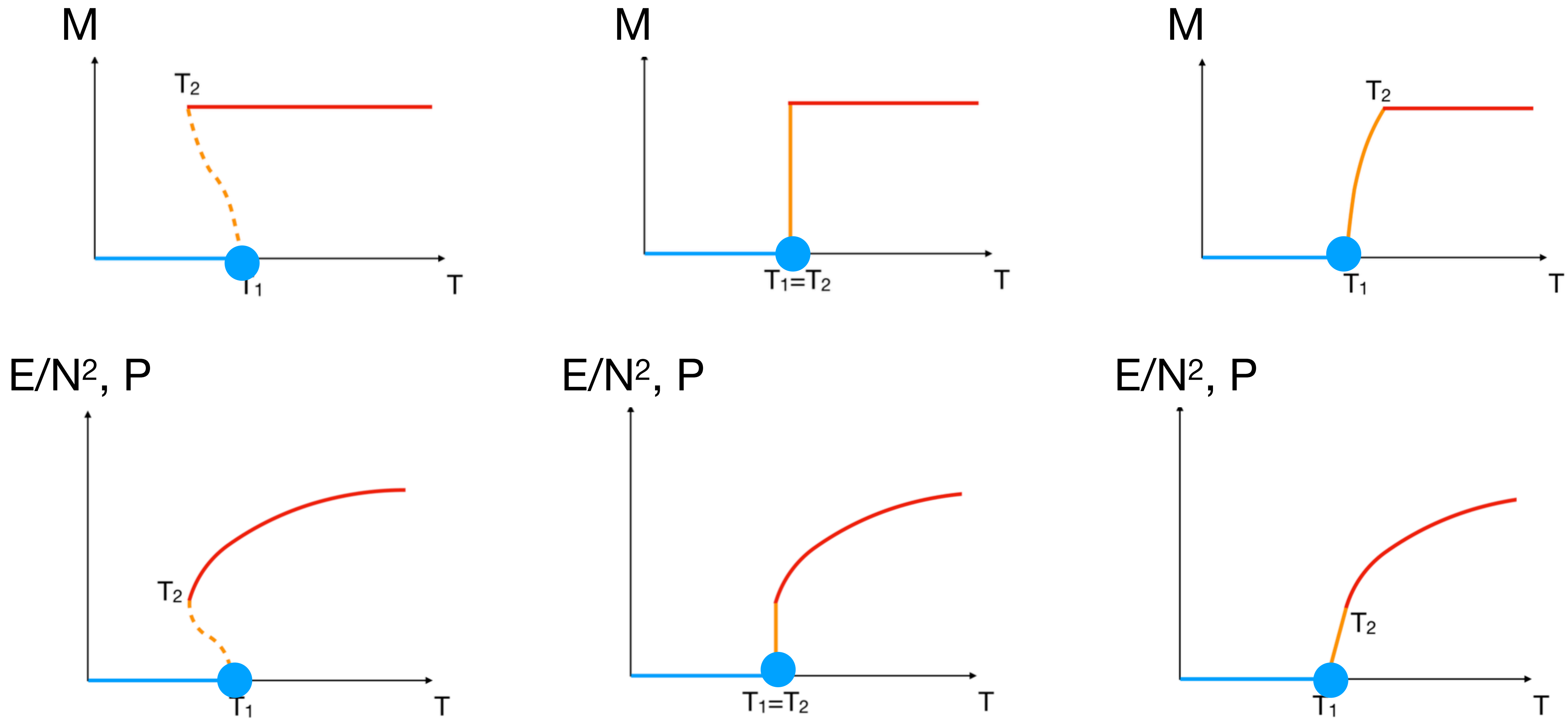
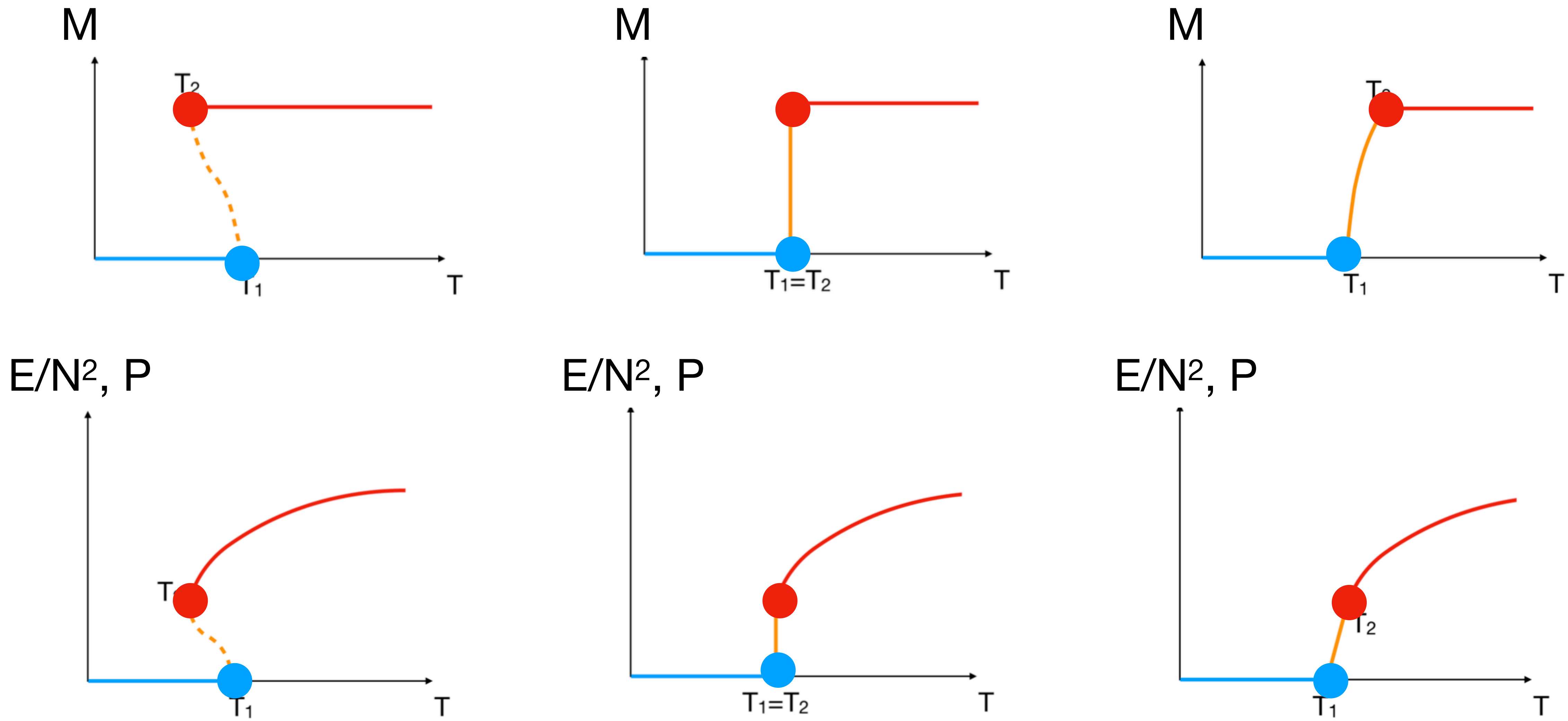
- Phase distribution:



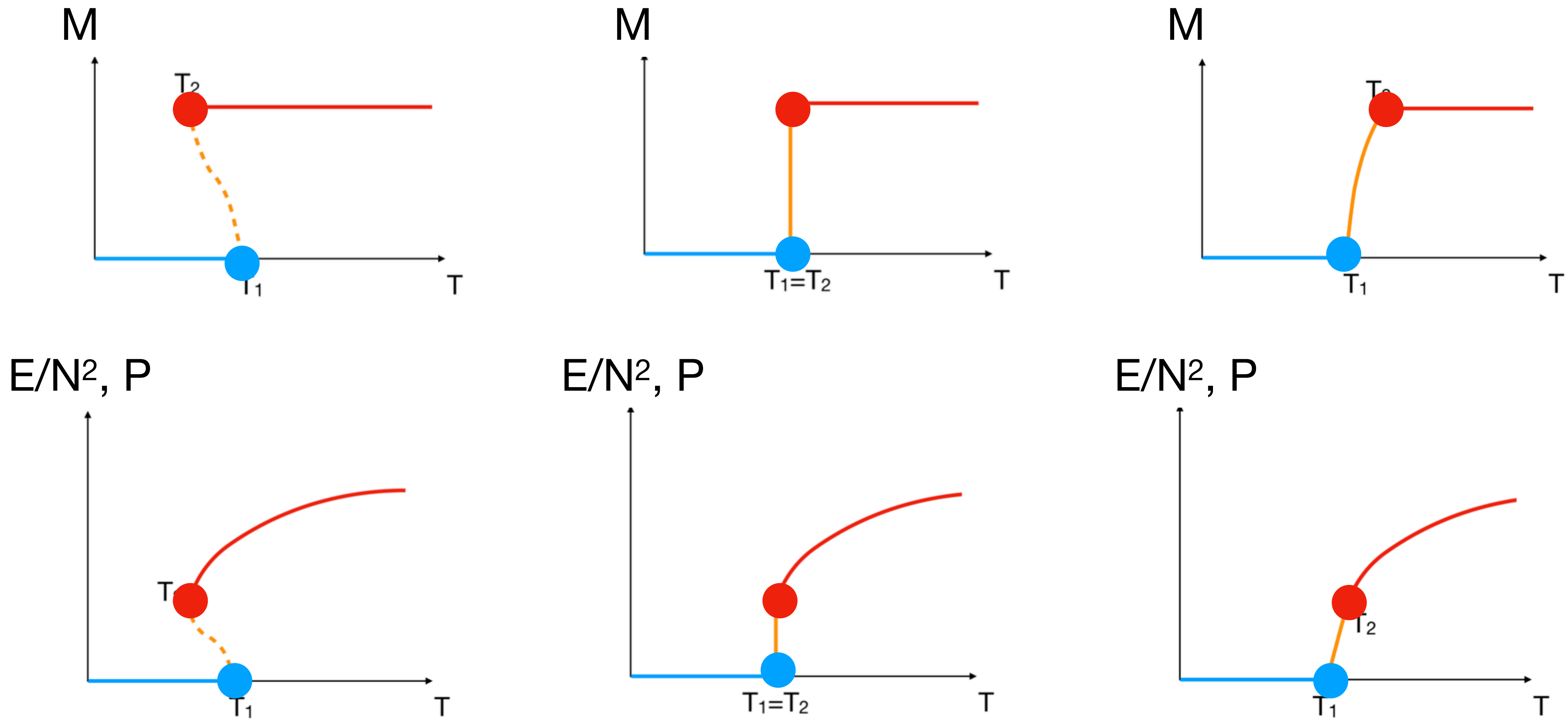


transition 1: confinement to partial deconfinement
(black hole formation begins)



transition 1: confinement to partial deconfinement
(black hole formation begins)

transition 2: partial deconfinement to complete deconfinement
(black hole formation ends)



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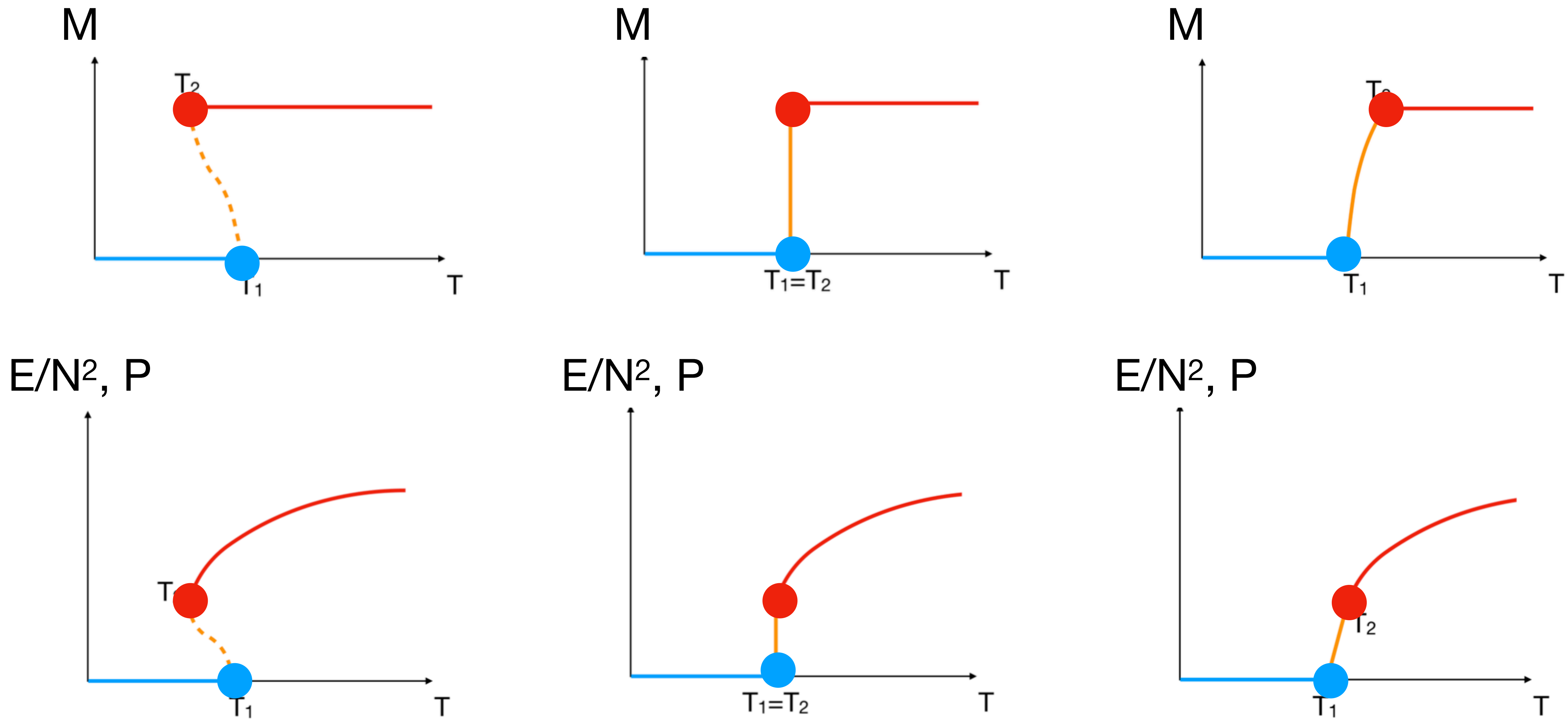
$$\boxed{SU(N) \to SU(M)\times SU(N-M) \to SU(N)}$$



transition 1: confinement to partial deconfinement
(black hole formation begins)

transition 2: partial deconfinement to complete deconfinement
(black hole formation ends)

$$\boxed{SU(N) \rightarrow SU(M) \times SU(N-M) \rightarrow SU(N)}$$

No need for center symmetry → Applicable to QCD.

Simplest Example:

Gauged Gaussian Two Matrix Model

$$\hat{H} = \frac{1}{2}\mathrm{Tr}\left(\hat{P}_X^2 + \hat{X}^2 + \hat{P}_Y^2 + \hat{Y}^2\right)$$

(Other cases are very similar)

M.H., Jevicki, Peng, Wintergerst, 1909.09118 [hep-th]

$$\hat{H} = \frac{1}{2}\mathrm{Tr}\left(\underbrace{\hat{P}_X^2 + \hat{X}^2}_{\hat{A},\,\hat{A}^\dagger} + \underbrace{\hat{P}_Y^2 + \hat{Y}^2}_{\hat{B},\,\hat{B}^\dagger}\right)$$

$$\underbrace{\mathrm{Tr}\left(\hat{A}^\dagger \hat{A}^\dagger \hat{B}^\dagger \hat{A}^\dagger \cdots\right)}_{L} |0\rangle$$

$E = L$ (up to zero-pt energy)

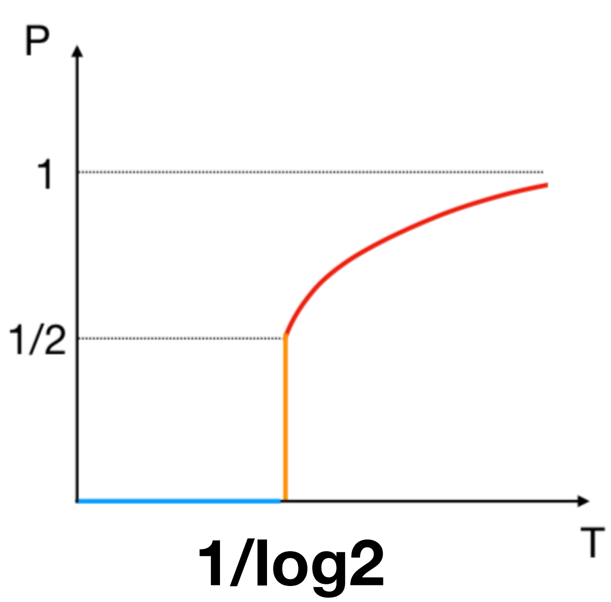
$S = L \log 2$ (# of states ~ 2^L)

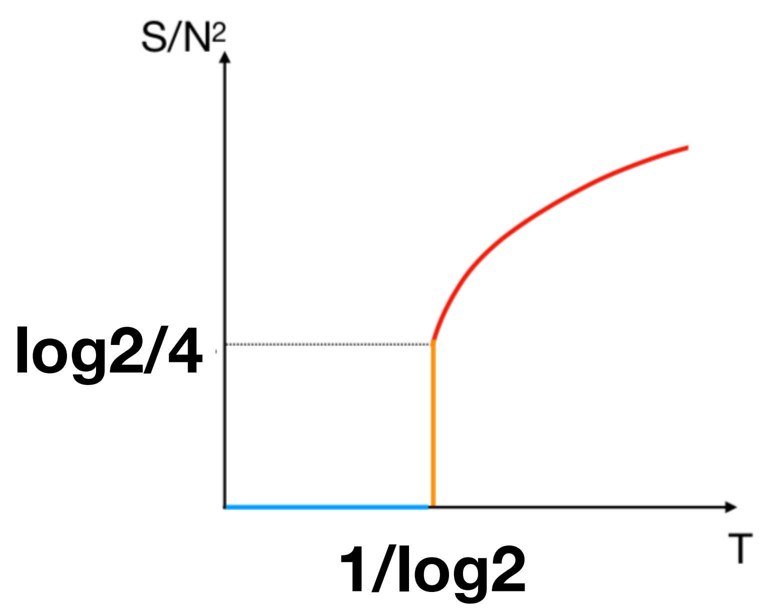
(valid at $L \ll N^2$)

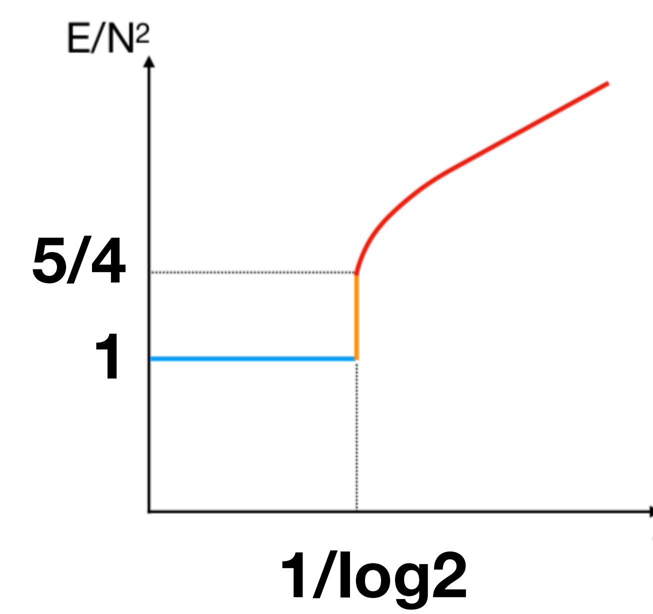
$$F = E - TS = L(1 - T\log 2)$$

(up to zero-pt energy; valid at L ≪ N^2)

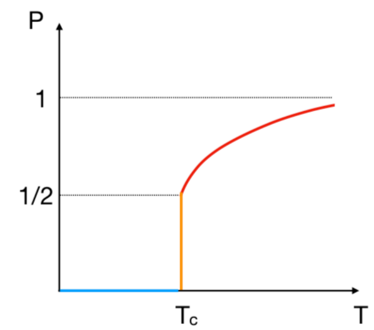
$$F = 0 \ \ @ \ \ T = \frac{1}{\log 2}$$

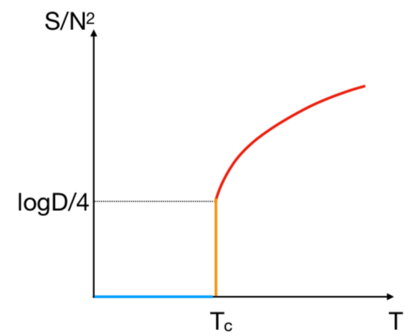


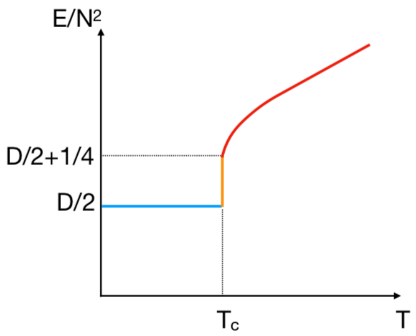




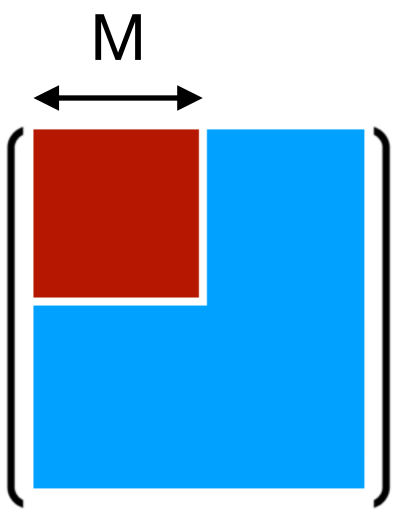
$$\boxed{T = T_c}$$







$$E = N^2 + N^2 P^2$$

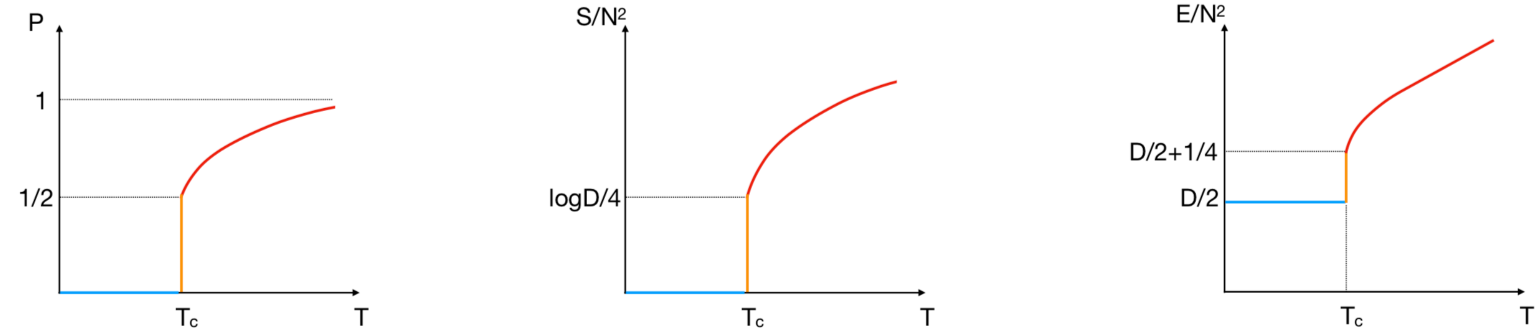


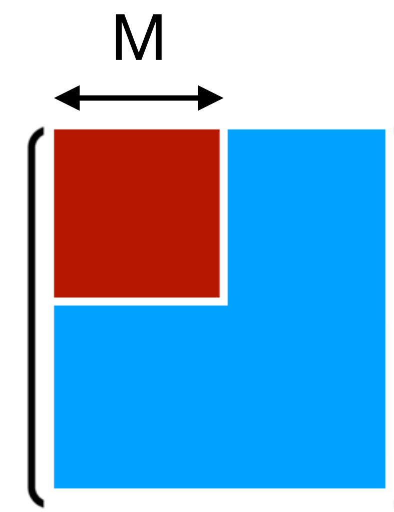
$$S = N^2 P^2 \log 2$$

$$\rho(\theta) = \frac{1 + 2P\cos\theta}{2\pi}$$

$$\boxed{T = T_c}$$

$$\boxed{P = \frac{M}{2N}}$$





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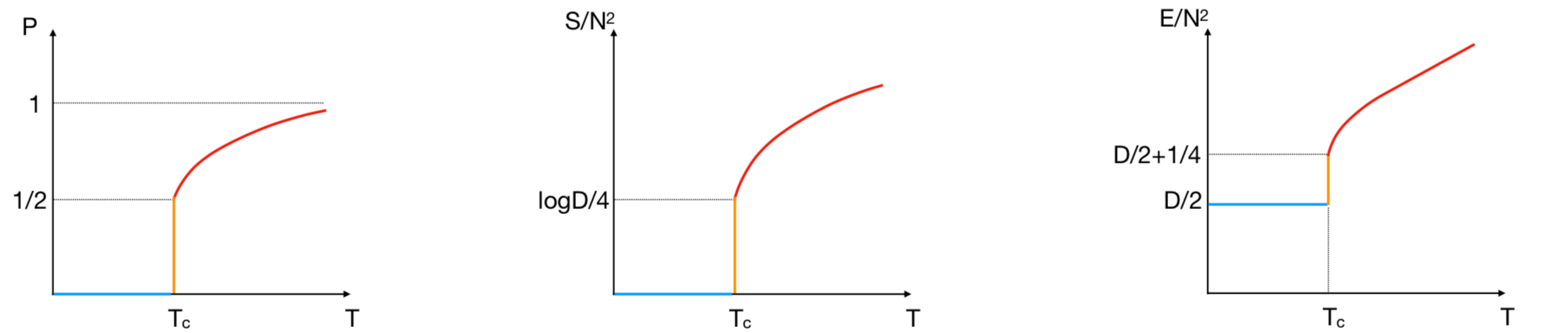

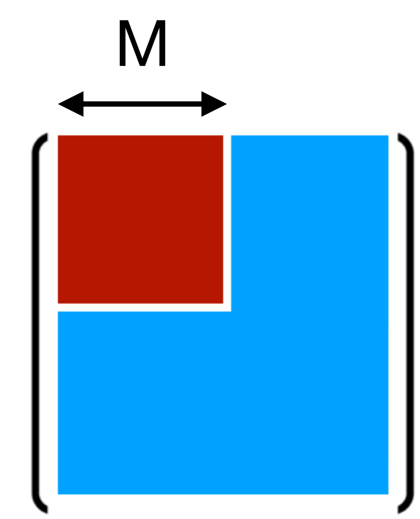

$$E = N^2 + N^2 P^2 = \frac{5}{4} M^2 + (N^2 - M^2)$$

$$S = N^2 P^2 \log 2 = \frac{M^2}{4} \log 2 + (N^2 - M^2) \times 0$$

$$\rho(\theta) = \frac{1 + 2P\cos\theta}{2\pi} = \frac{M}{N} \cdot \frac{1+\cos\theta}{2\pi} + \left(1 - \frac{M}{N}\right) \cdot \frac{1}{2\pi}$$

$$\boxed{T = T_c}$$

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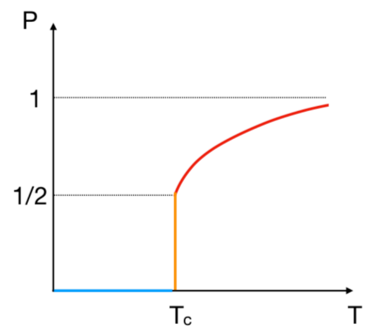

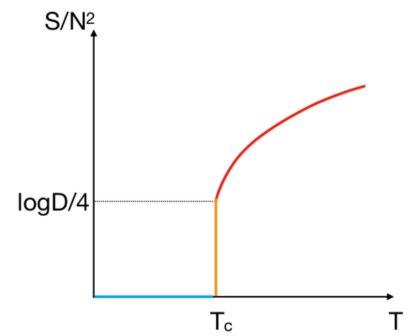

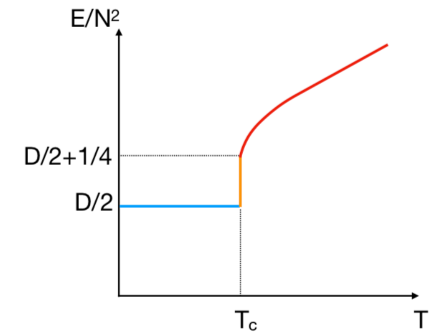

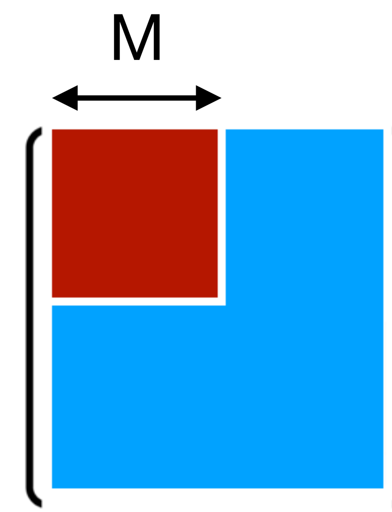

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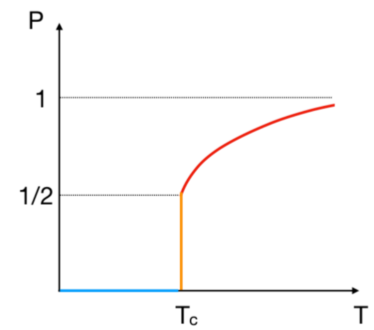
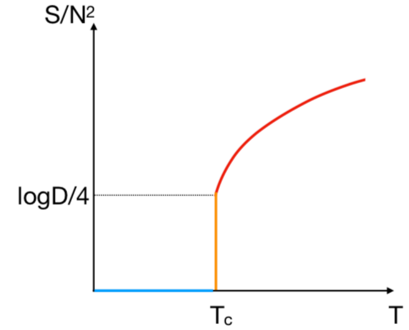
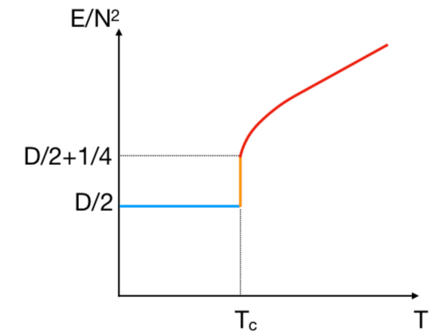
GWW-point of SU(M) theory

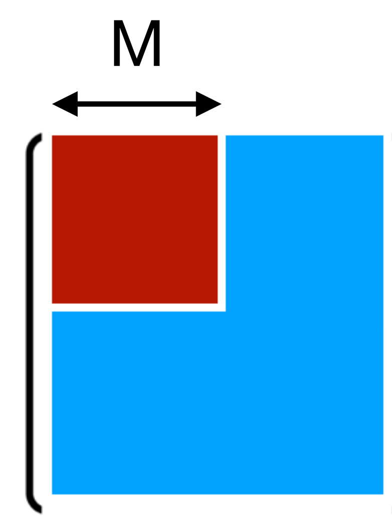
$$\boxed{T = T_c}$$

$$\boxed{P = \frac{M}{2N}}$$









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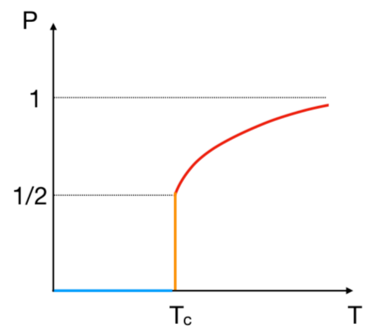
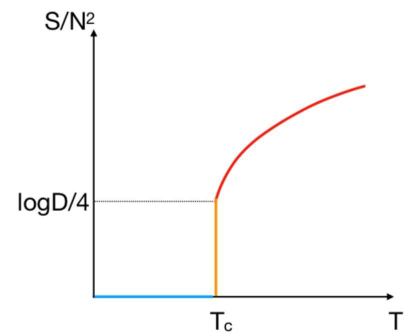
GWW-point of SU(M) theory

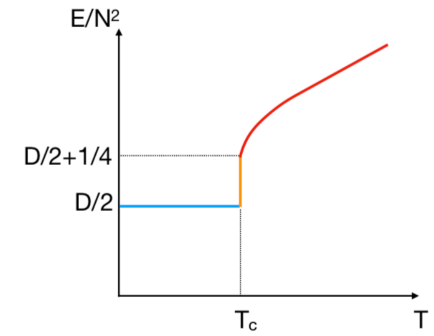
ground state (confining)

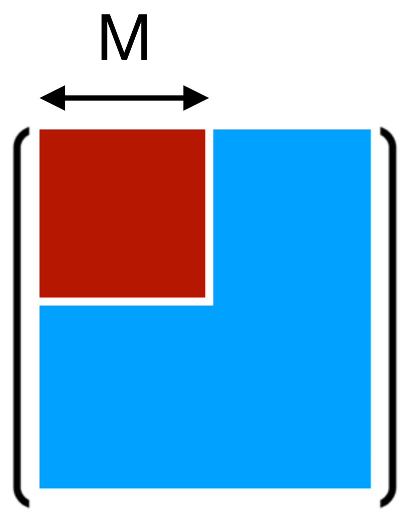
$$\boxed{T = T_c}$$

$$\boxed{P = \frac{M}{2N}}$$









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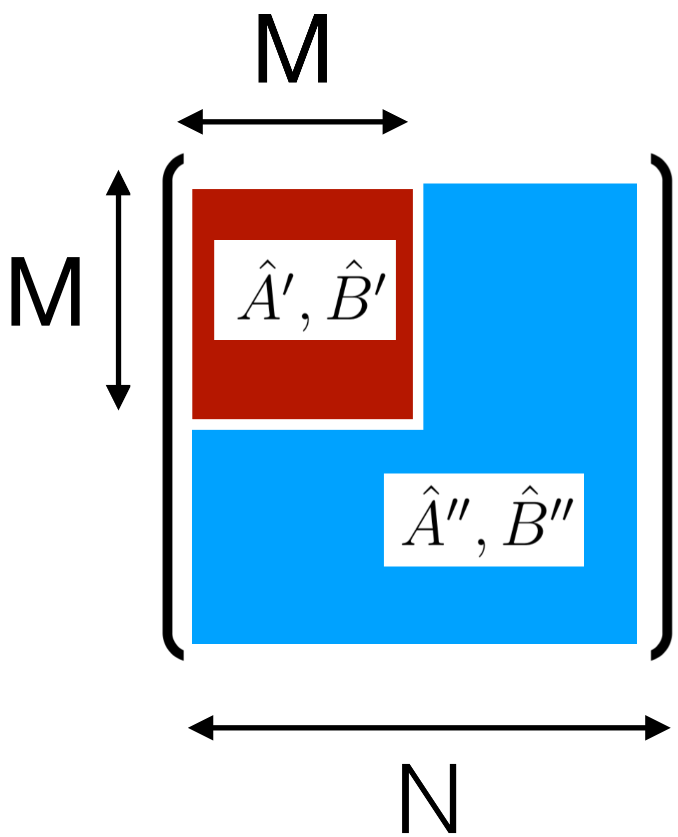
GWW-point of SU(M) theory

ground state (confining)

Free theory → no interaction term

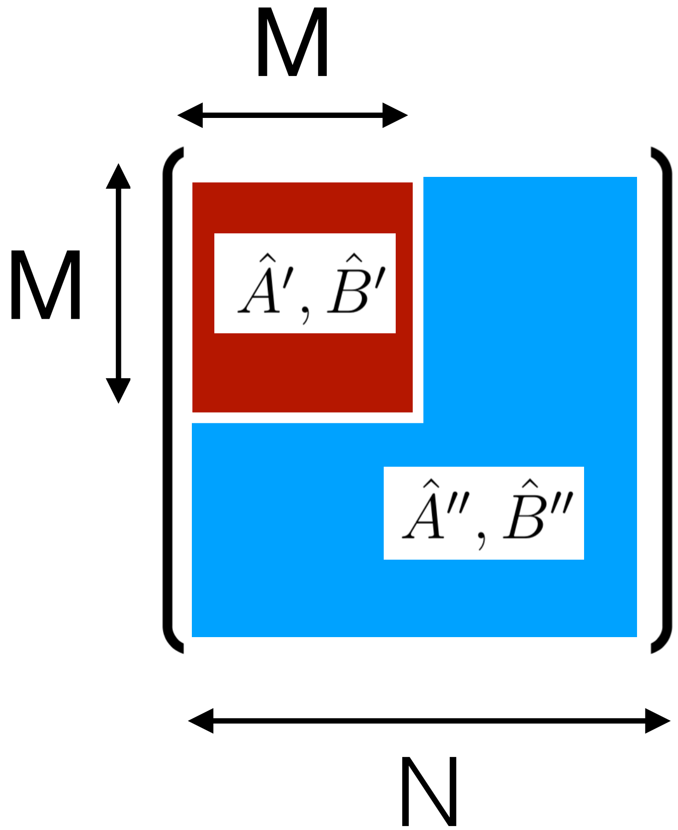


not SU(N)-invariant

$$|E; \mathrm{SU}(M)\rangle = \mathrm{Tr}\left(\hat{A}'^{\dagger}\hat{A}'^{\dagger}\hat{B}'^{\dagger}\hat{A}'^{\dagger}\cdots\right)|0\rangle$$

$$E = E_{\mathrm{GWW}}(M) \longrightarrow S = S_{\mathrm{GWW}}(M)$$

M.H.-Jevicki-Peng-Wintergerst, 2019



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$$E = E_{\mathrm{GWW}}(M) \longrightarrow S = S_{\mathrm{GWW}}(M)$$

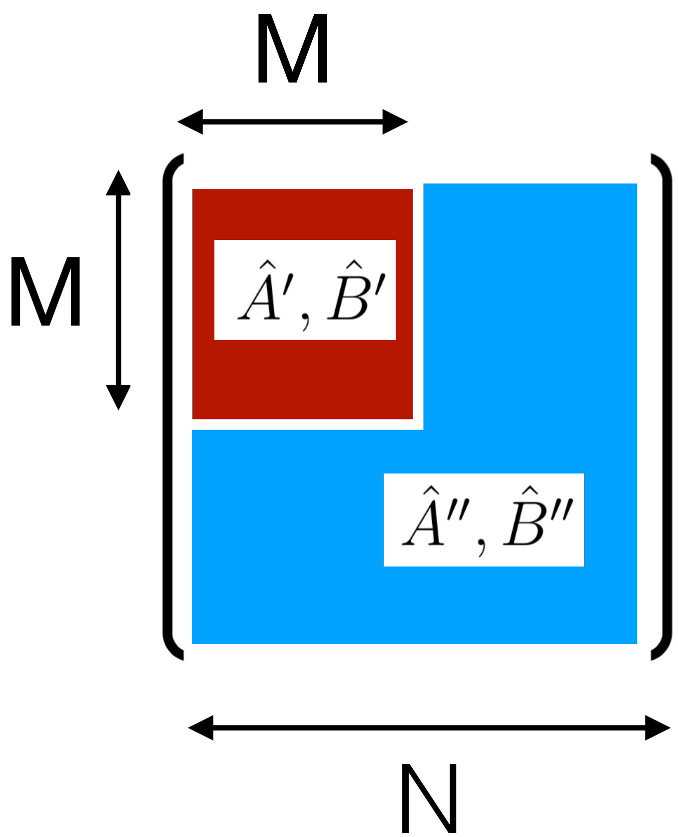
$$\updownarrow$$

one-to-one correspondence

SU(N)-invariant

$$|E\rangle_{\mathrm{inv}} \equiv \mathcal{N}^{-1/2}\int dU\,\mathcal{U}\left(|E; \mathrm{SU}(M)\rangle\right)$$

M.H.-Jevicki-Peng-Wintergerst, 2019



$\boxed{\text{not SU(N)-invariant}}$

$$|E; \mathrm{SU}(M)\rangle = \mathrm{Tr}\left(\hat{A}'^{\dagger}\hat{A}'^{\dagger}\hat{B}'^{\dagger}\hat{A}'^{\dagger}\cdots\right)|0\rangle$$

$$E = E_{\mathrm{GWW}}(M) \longrightarrow S = S_{\mathrm{GWW}}(M)$$

$\Updownarrow$ **one-to-one correspondence**

$\boxed{\text{SU(N)-invariant}}$

$$|E\rangle_{\mathrm{inv}} \equiv \mathcal{N}^{-1/2}\int dU\,\mathcal{U}\left(|E; \mathrm{SU}(M)\rangle\right)$$

<u>These states explain the entropy precisely.</u>

M.H.-Jevicki-Peng-Wintergerst, 2019

‘Spontaneous gauge symmetry breaking’

(It is a controversial phrase. The meaning will be clarified.)

(Consistent with Elitzur’s theorem)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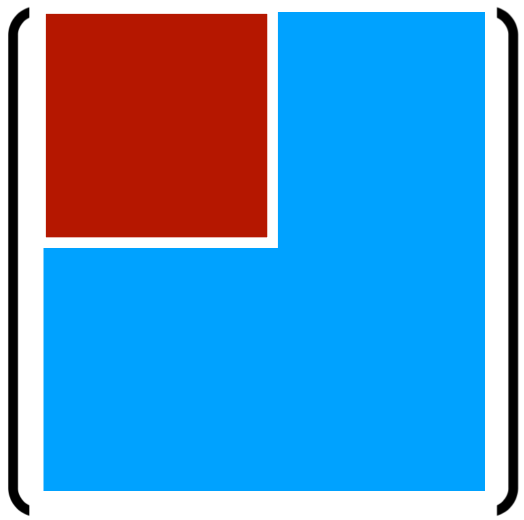
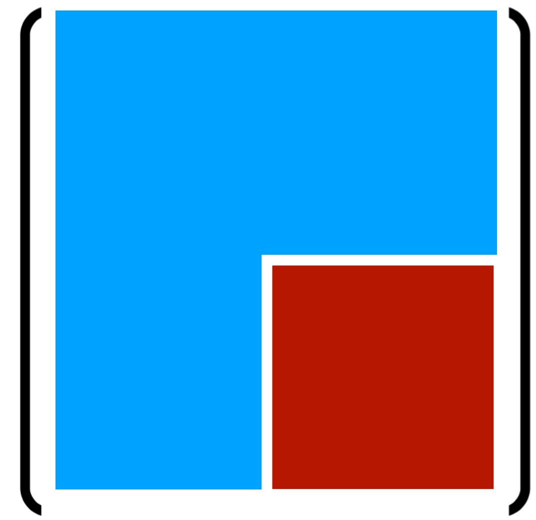
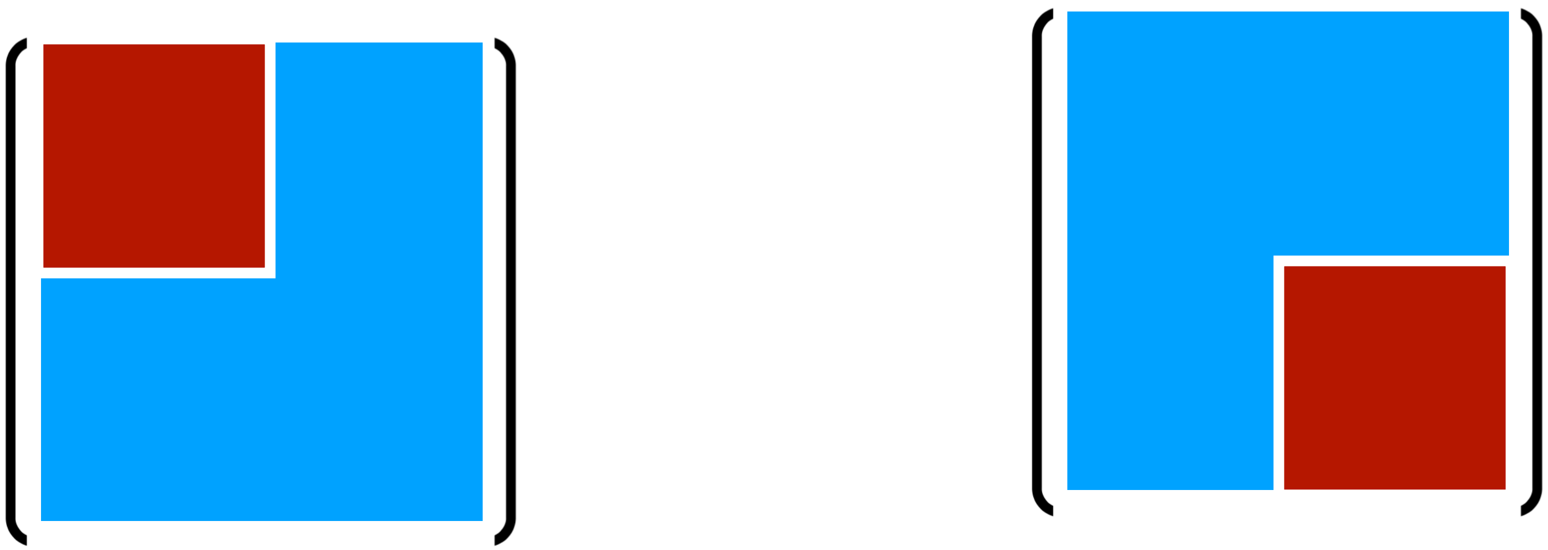
Impossibility of spontaneously breaking local symmetries

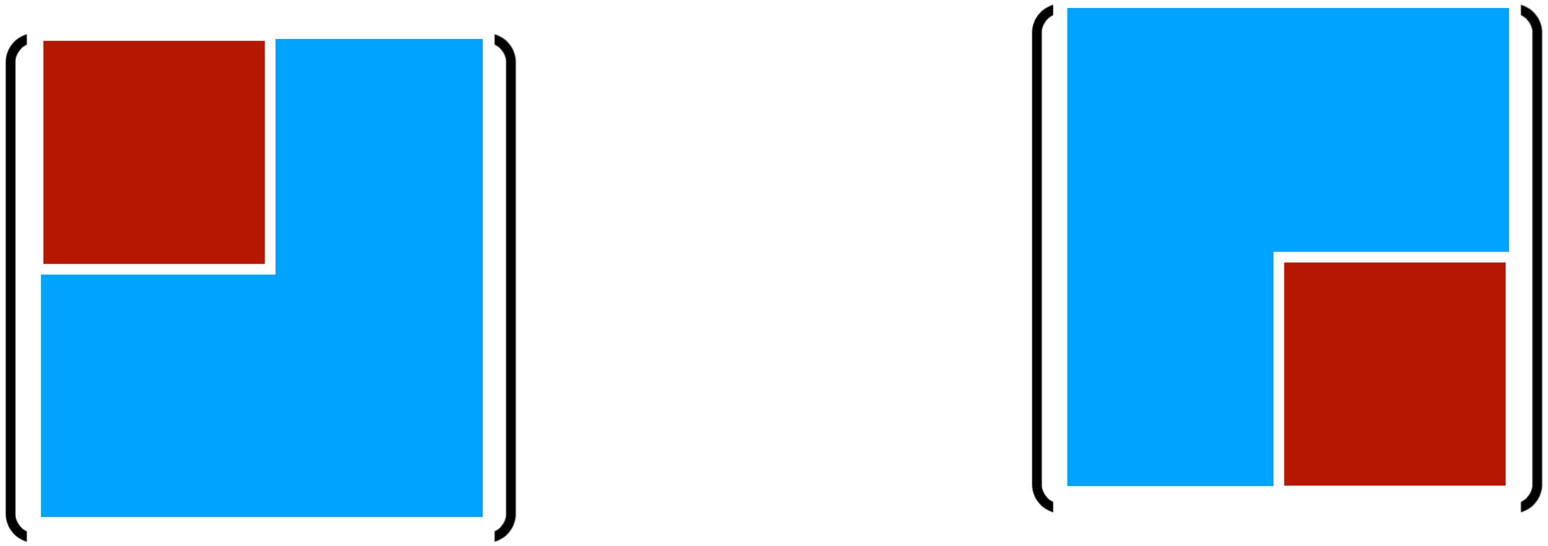
S. Elitzur
Phys. Rev. D **12**, 3978 – Published 15 December 1975







$\hat{O} = \mathrm{Tr}\left(\hat{X}\hat{Y}\cdots\right)$: length ~ N^0



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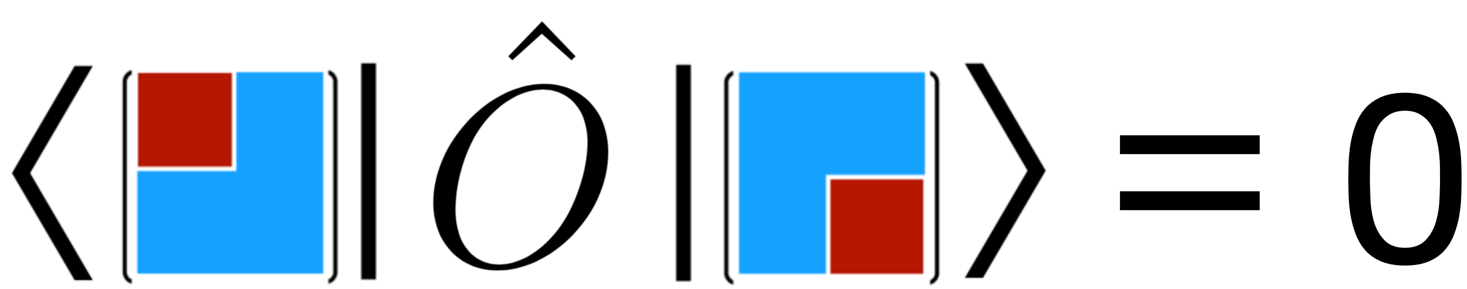

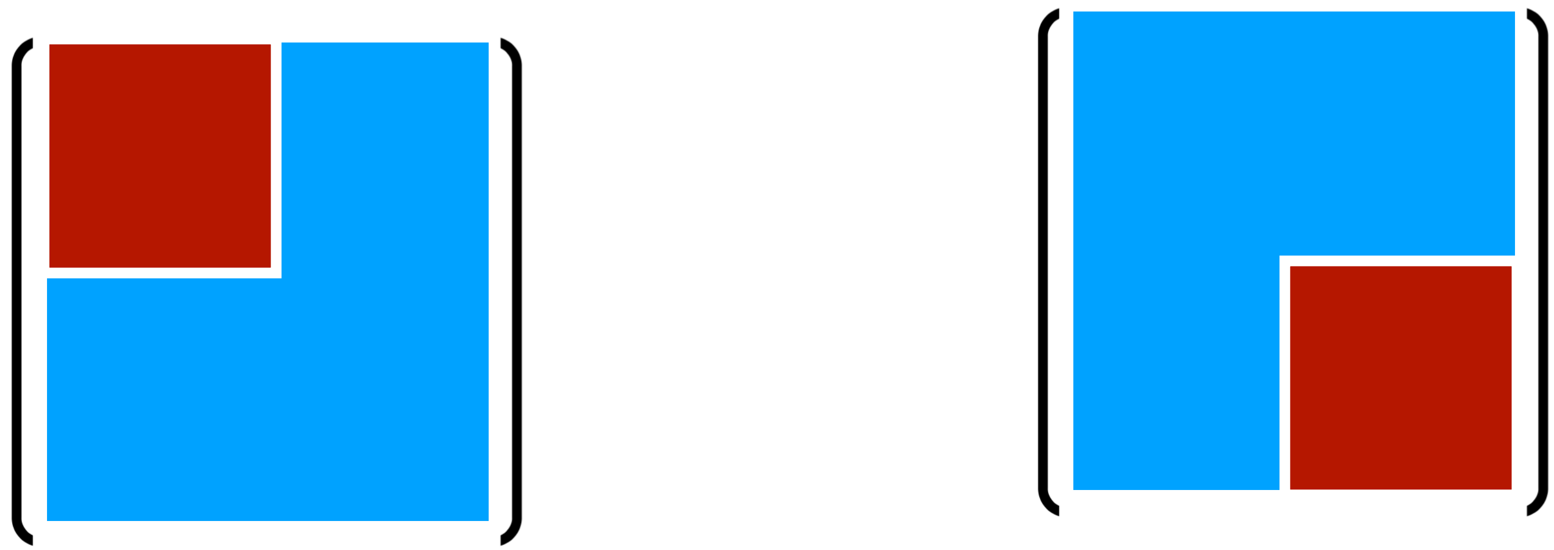

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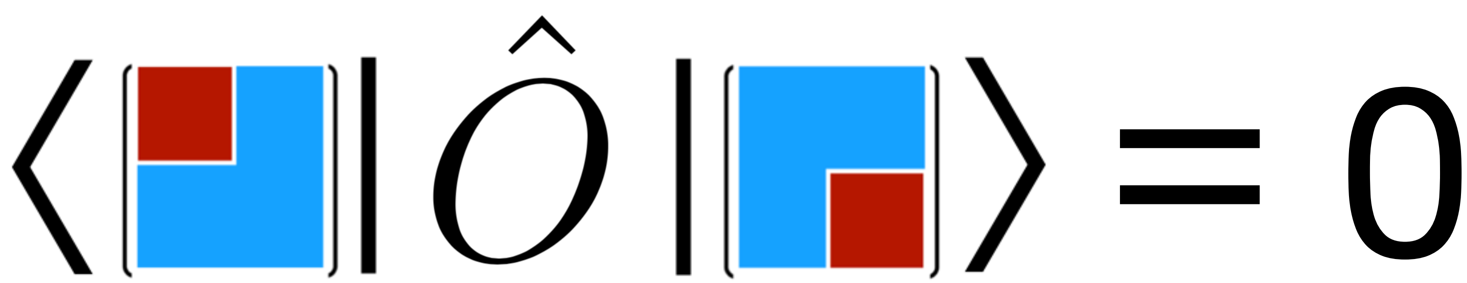
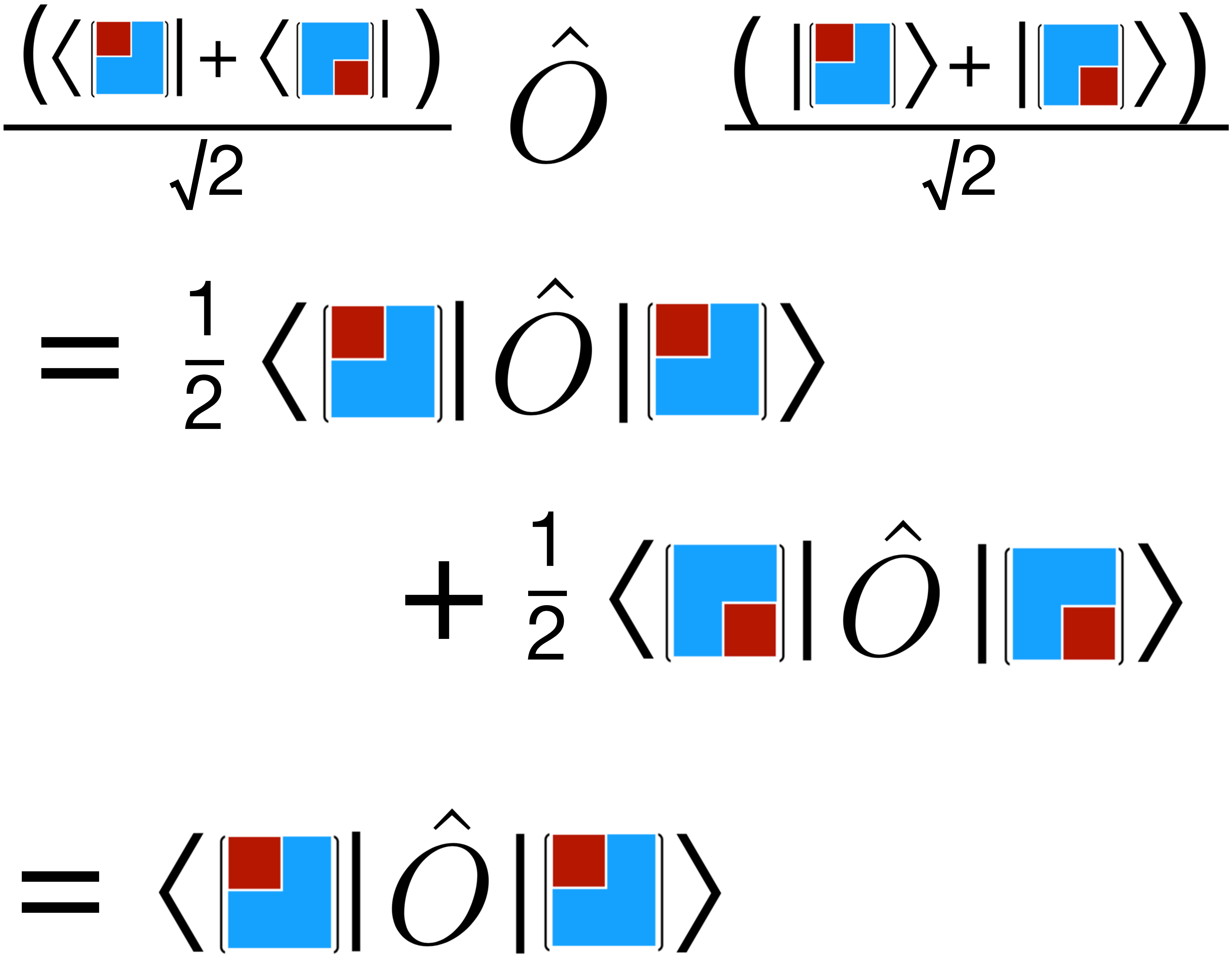
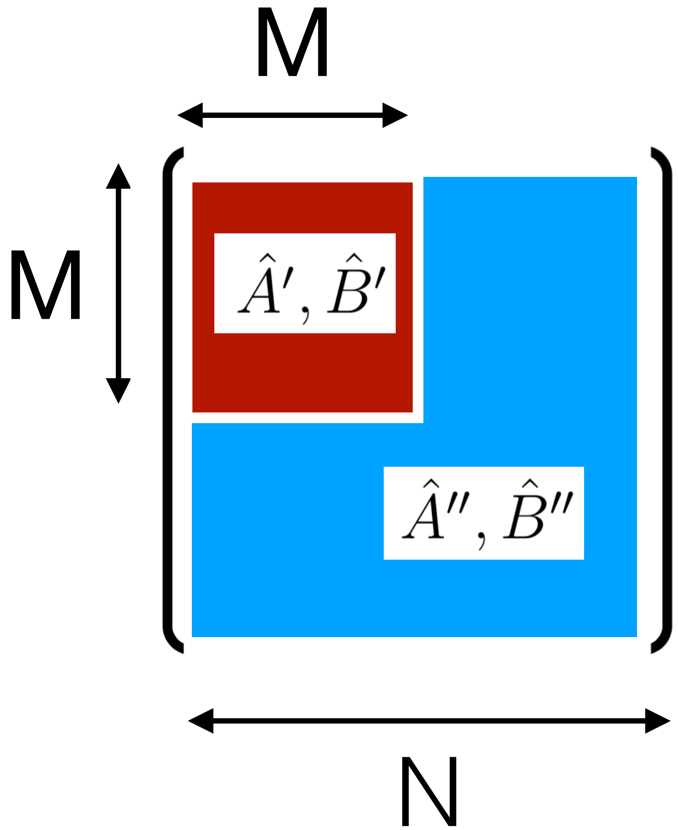
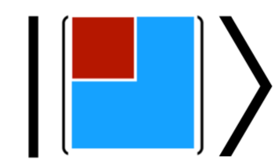


Analogous to super-selection



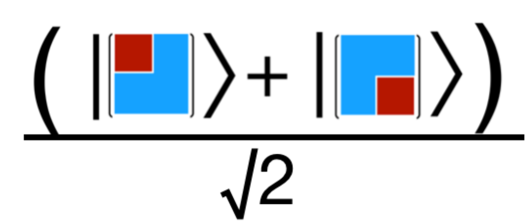


$$|E;\mathrm{SU}(M)\rangle = \mathrm{Tr}\left(\hat{A}'^{\dagger}\hat{A}'^{\dagger}\hat{B}'^{\dagger}\hat{A}'^{\dagger}\cdots\right)|0\rangle$$

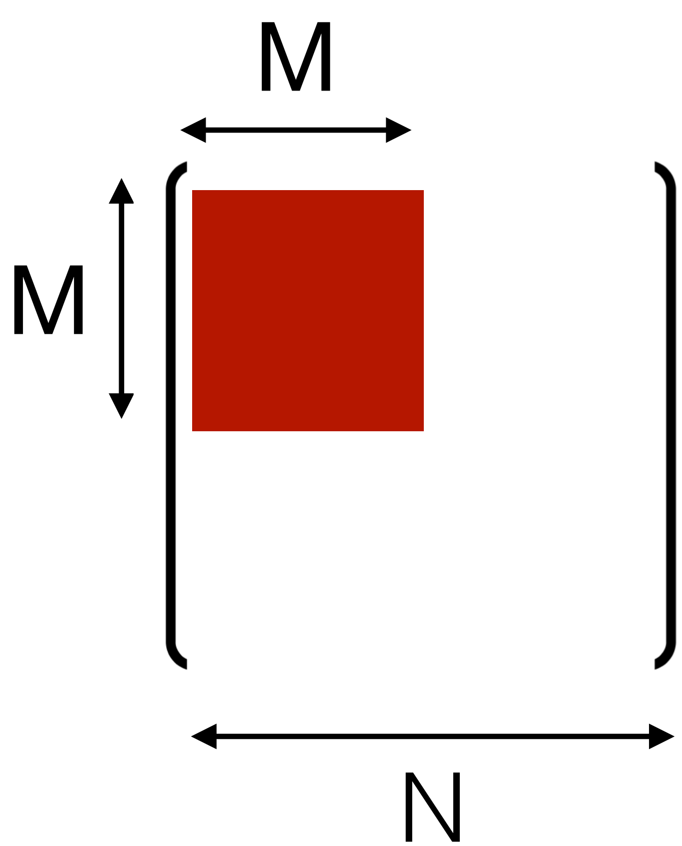


Indistinguishable (unless very long operators are used)

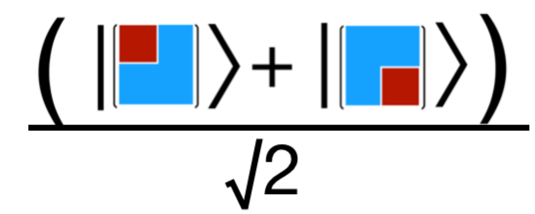
$$|E\rangle_{\mathrm{inv}} \equiv \mathcal{N}^{-1/2}\int dU\,\mathcal{U}\left(|E;\mathrm{SU}(M)\rangle\right)$$



QFT (with spatial dimension)

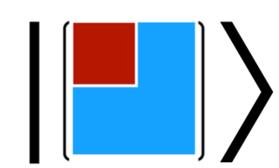


$$|E\rangle_{\rm inv} \equiv \mathcal{N}^{-1/2} \int dU\, \mathcal{U}\left(|E; \mathrm{SU}(M)\rangle\right)$$

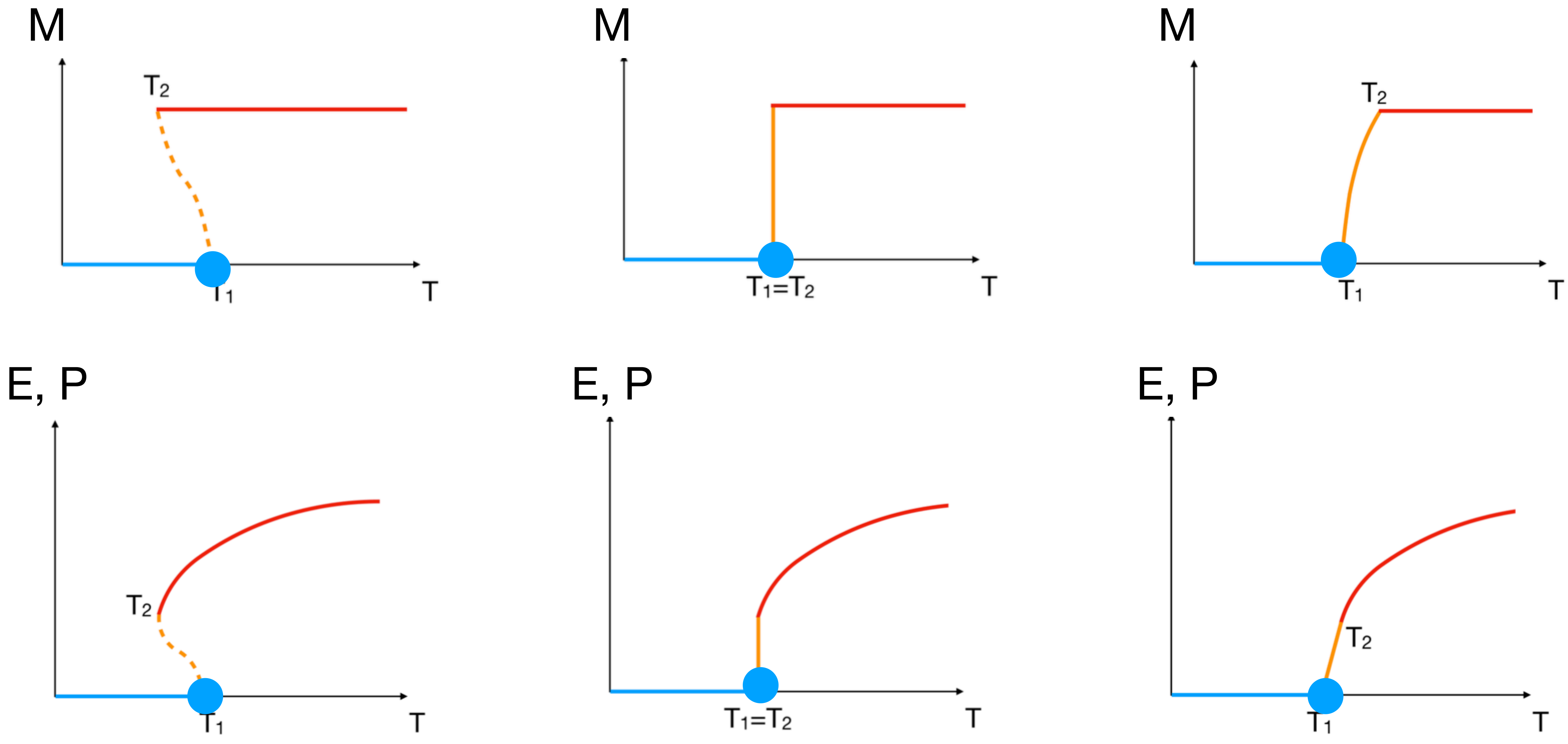


$$\Big\downarrow \text{gauge fixing}$$

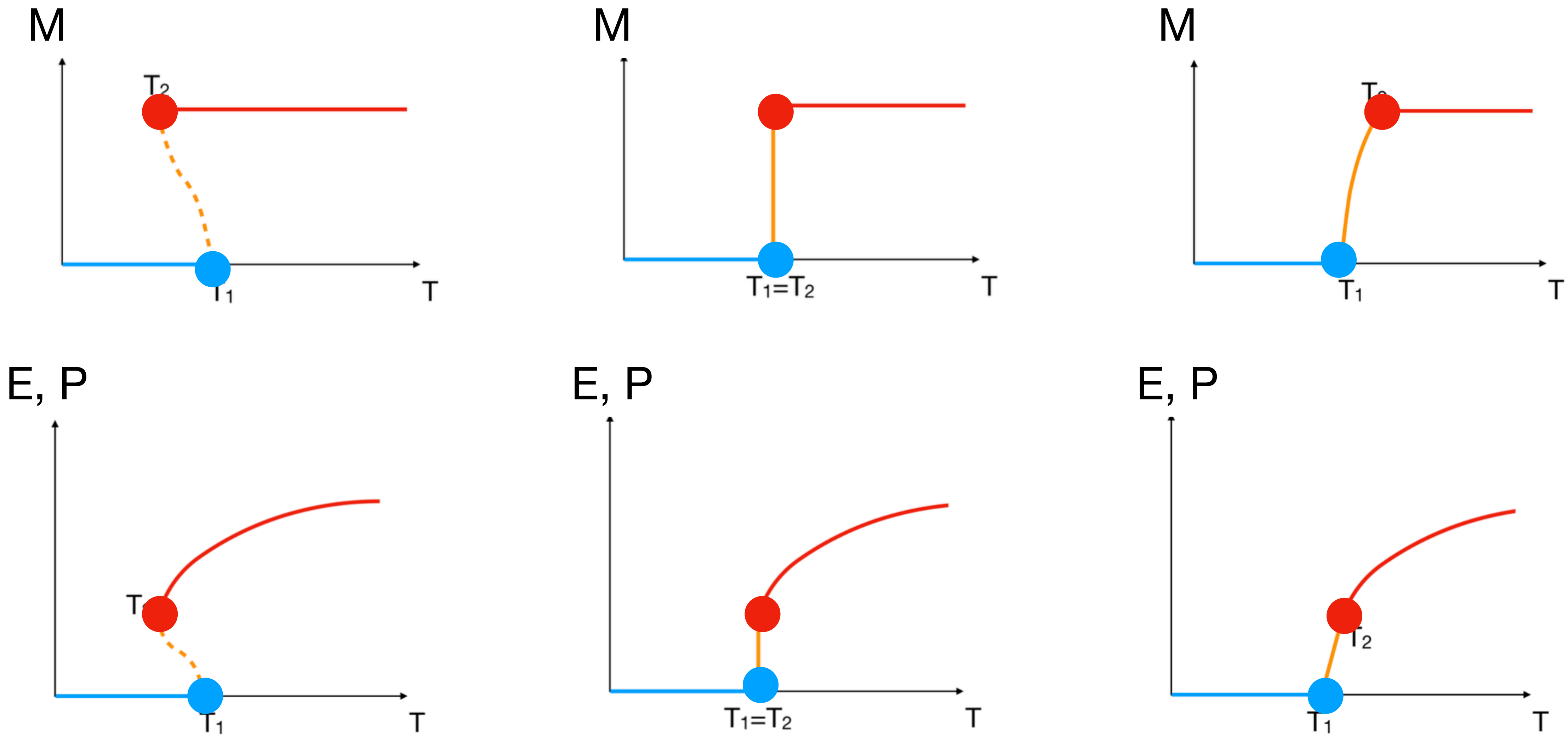
$$|E; \mathrm{SU}(M)\rangle$$



- Global part of gauge symmetry breaks spontaneously.
- It is convenient to fix the local part, like usual Higgsing.
- Gauge fixing of the local part makes physics more easily understandable.

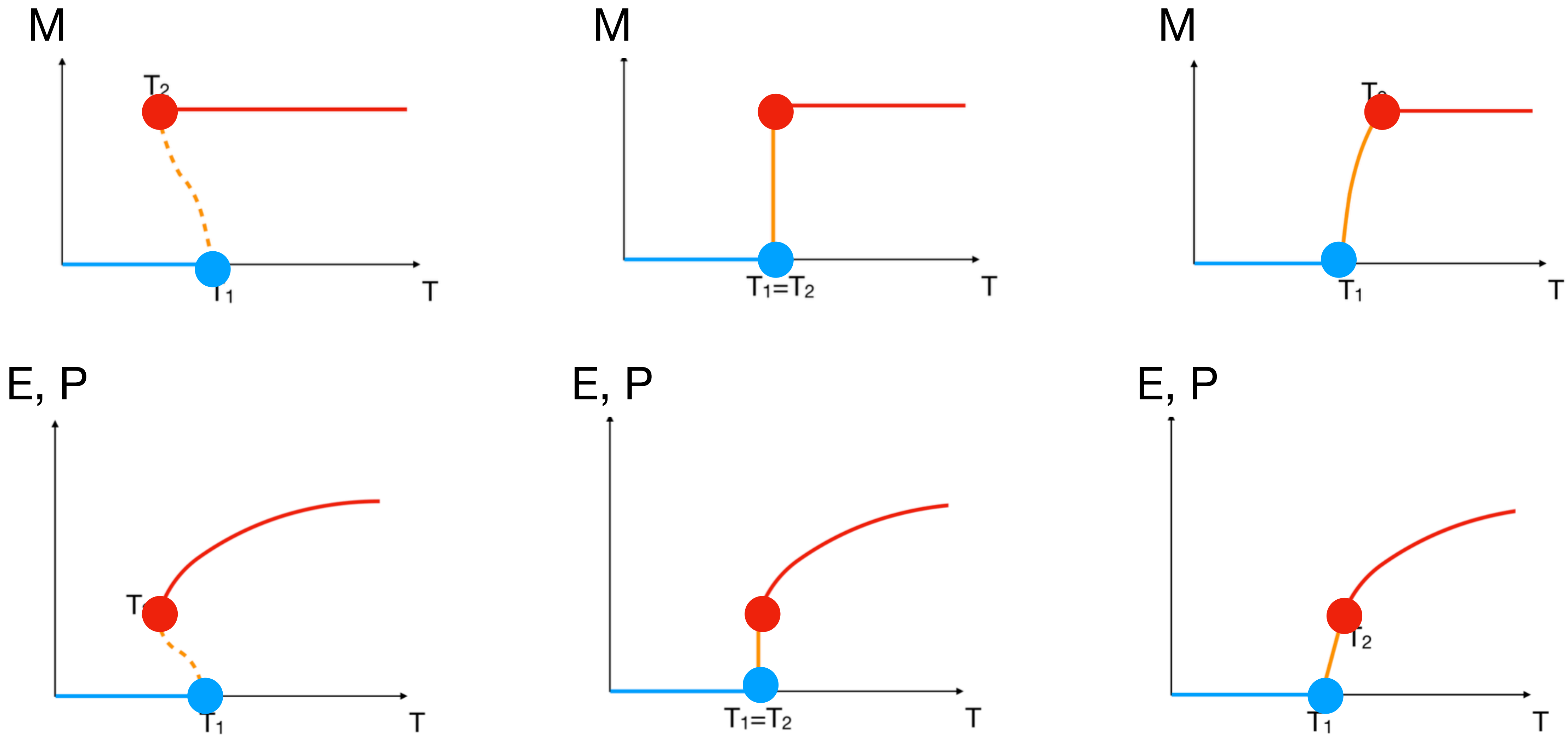


transition 1: confinement to partial deconfinement
(black hole formation begins)



transition 1: confinement to partial deconfinement
(black hole formation begins)

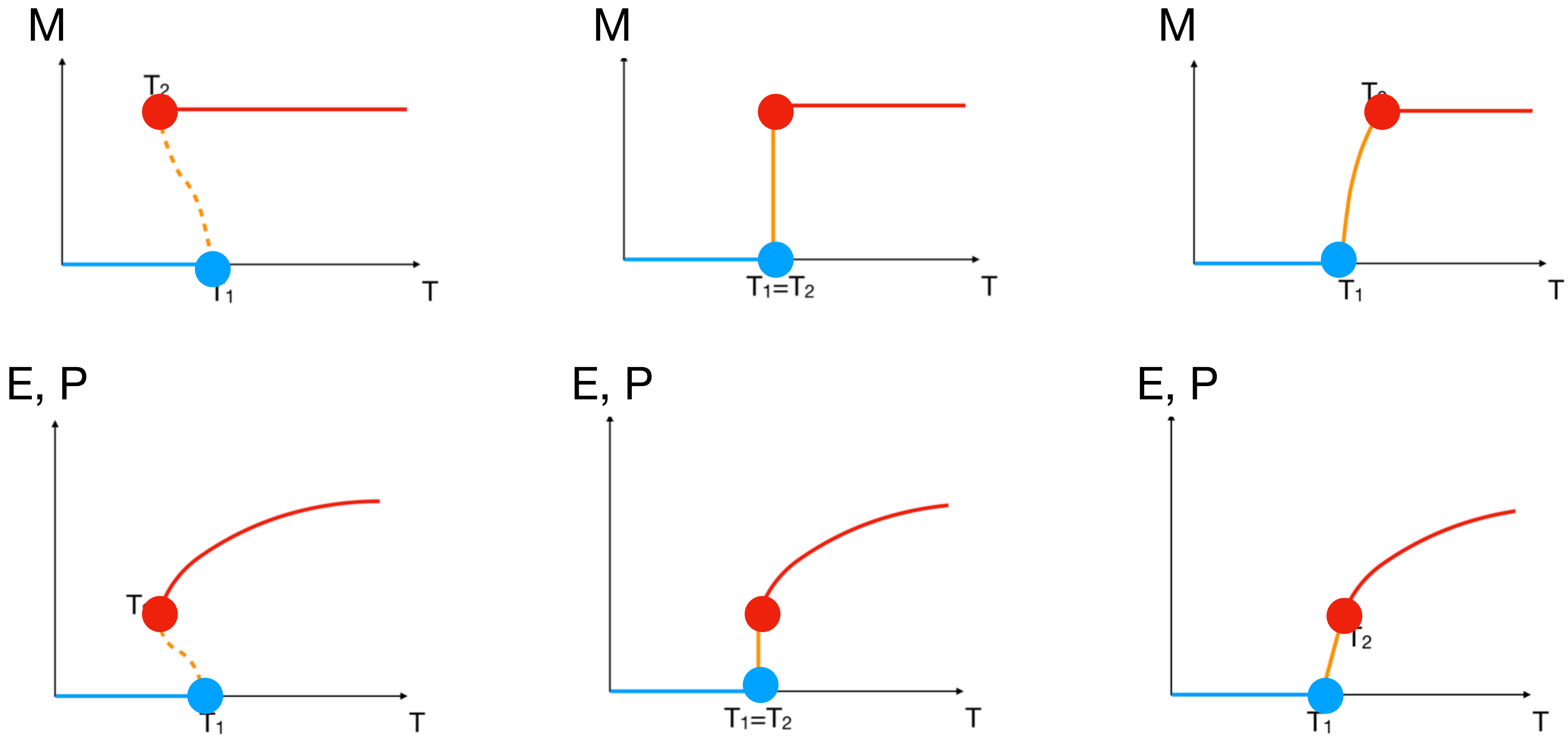
transition 2: partial deconfinement to complete deconfinement
(black hole formation ends)



transition 1: confinement to partial deconfinement
(black hole formation begins)

transition 2: partial deconfinement to complete deconfinement
(black hole formation ends)

$$\boxed{SU(N) \to SU(M)\times SU(N-M) \to SU(N)}$$



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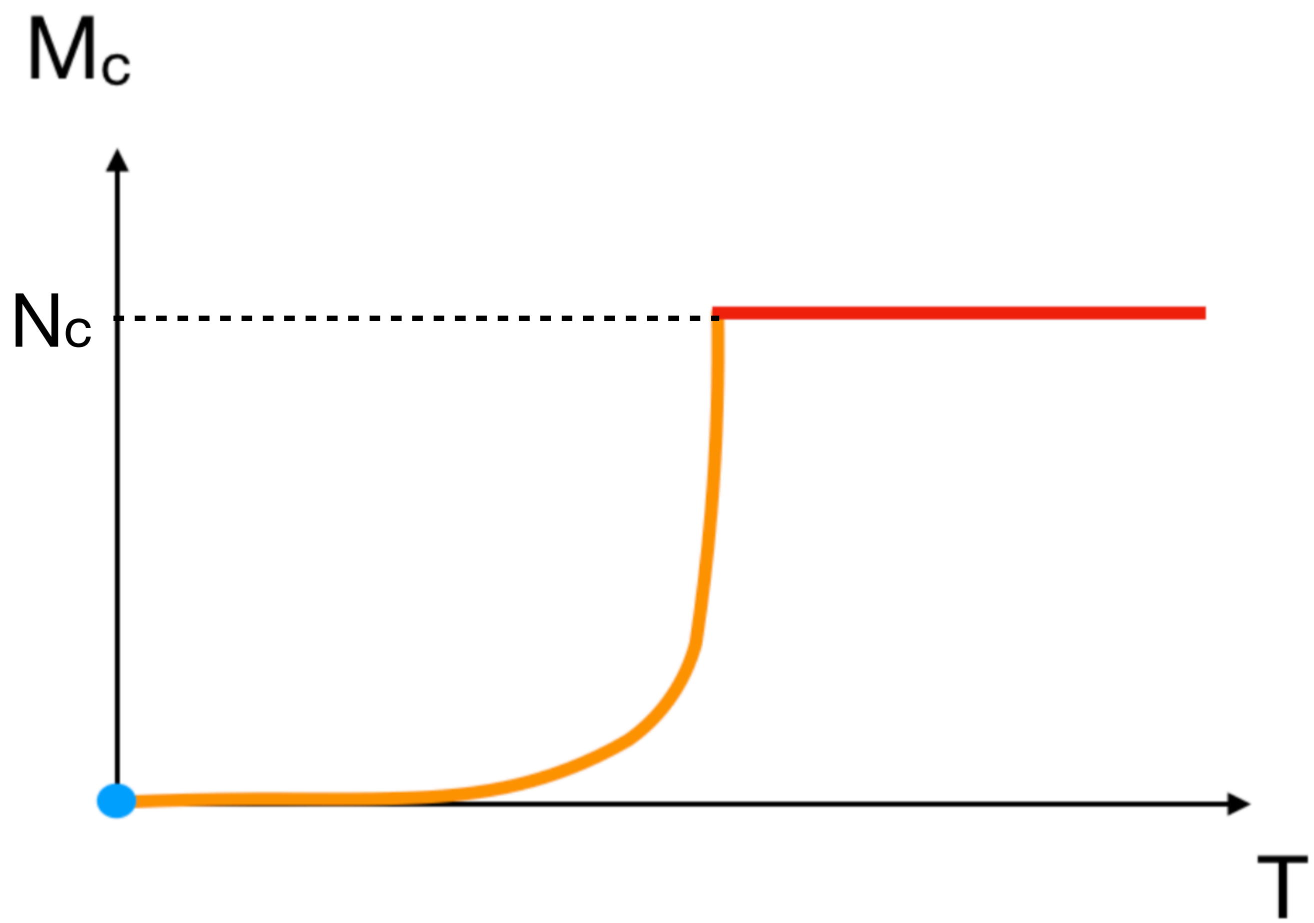
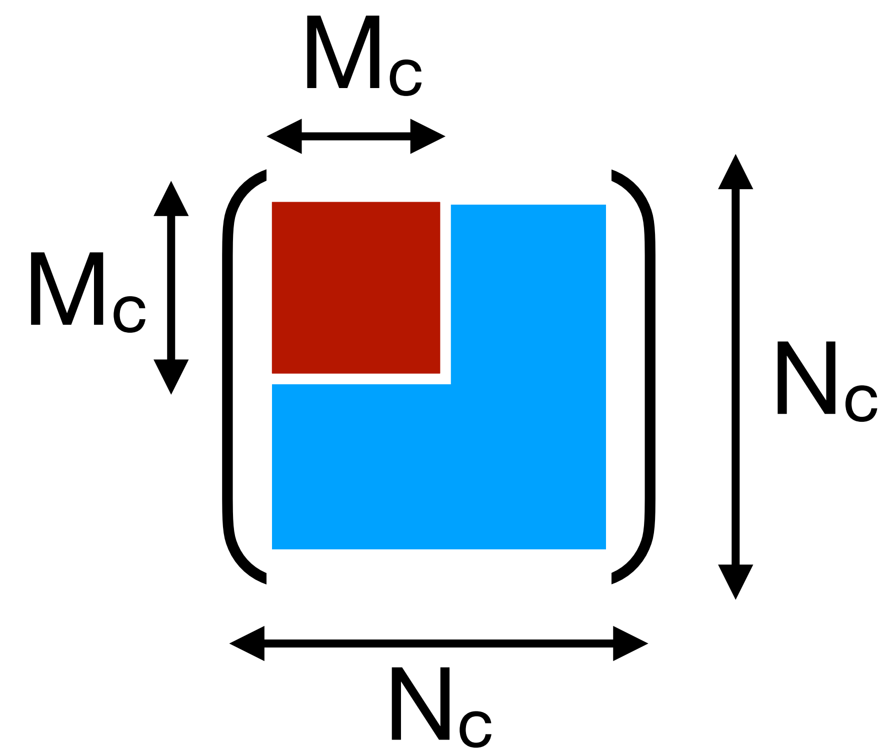
$$\boxed{SU(N) \to SU(M)\times SU(N-M) \to SU(N)}$$

No need for center symmetry → Applicable to QCD.

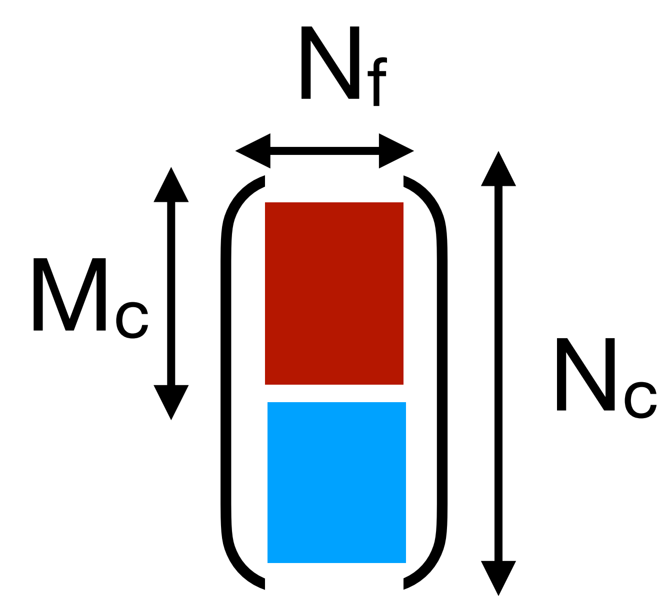
Weakly-coupled QCD on S^3





MH-Robinson, 1911.06223
(re-interpretation of Schnitzer's calculation in 2004)

Free, but N_f/M_c changes
→ nontrivial T-dependence

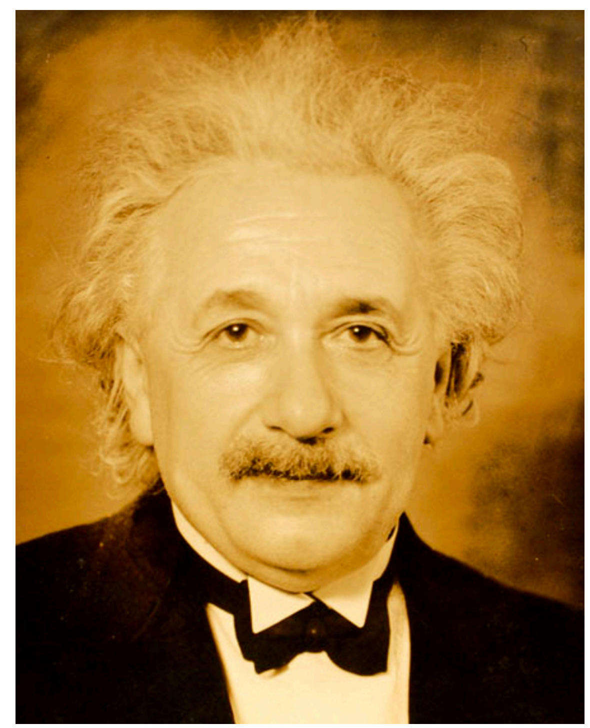


($\rho(\theta)$ is very complicated)

Confinement and BEC

MH-Shimada-Wintergerst, 2020

Bose



Einstein

Partial Deconfinement = Partial Confinement

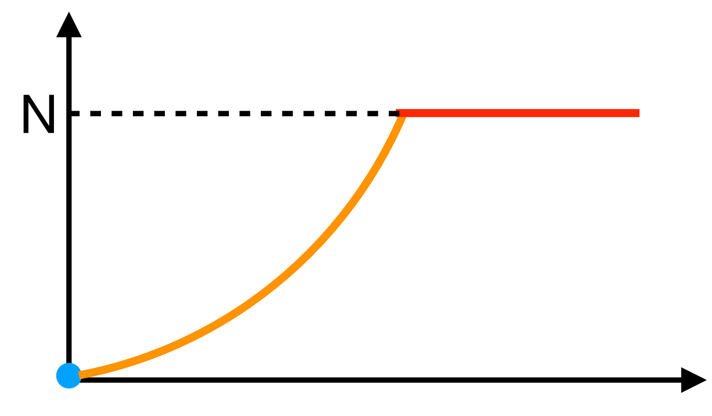
- Many colors fall into ground state (confined sector).
- Ground state and excited state can coexist.
- Happens even at zero-coupling limit.
- Gauge redundancy is crucial.

Bose-Einstein Condensation

- Many particles fall into ground state (BE condensate).
- Ground state and excited state can coexist.
- Happens even at zero-coupling limit.
- Permutation redundancy is crucial.

Free O(N) vector model on S^d

re-interpretation of Shenker-Yin, 2011



$\phi_1(x), \ldots, \phi_N(x)$

Non-interacting atoms in harmonic trap in R^d

'S_N vector quantum mechanics'

$x_1, \ldots, x_N; y_1, \ldots, y_N; z_1, \ldots, z_N$

Free O(N) vector model on S^d

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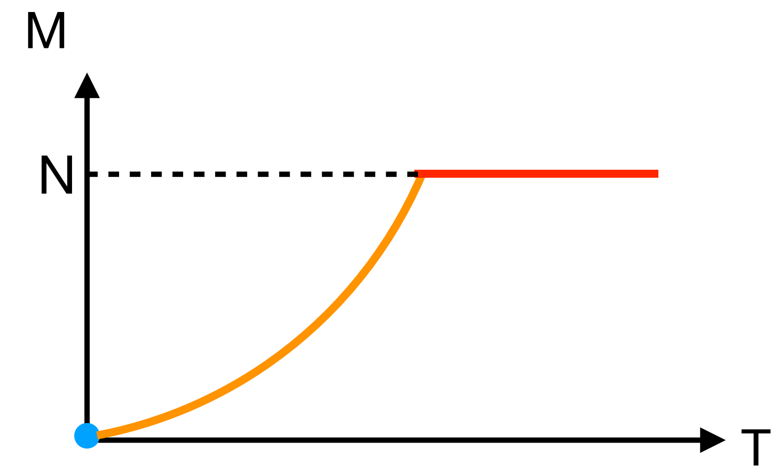

$\phi_1(x), \ldots, \phi_N(x)$

$$\frac{M}{N} = \left(\frac{T}{T_c}\right)^d$$

$$T_c = \left(\frac{N}{4(1-2^{1-d})\zeta(d)}\right)^{1/d} \cdot \frac{1}{R}$$

Non-interacting atoms in harmonic trap in R^d

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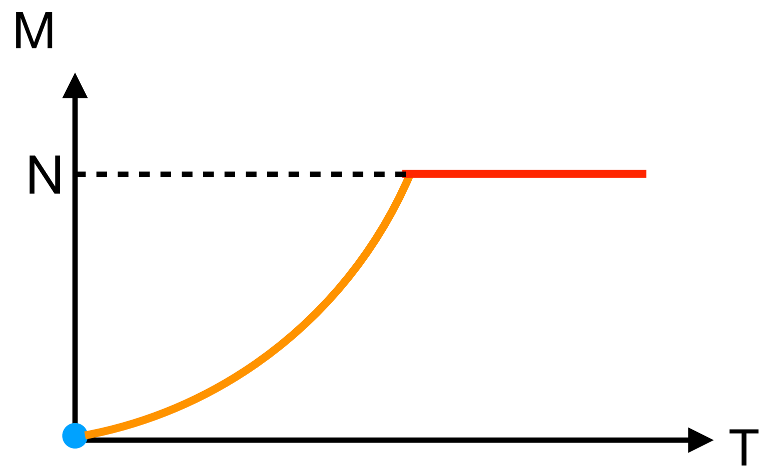
$x_1, \ldots, x_N;\ y_1, \ldots, y_N;\ z_1, \ldots, z_N$

$$\frac{M}{N} = \left(\frac{T}{T_c}\right)^d$$

$$T_c = \left(\frac{N}{\zeta(d)}\right)^{1/d} \omega$$

Free O(N) vector model on S^d

re-interpretation of Shenker-Yin, 2011



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$$E(T = T_c(M)) = E_c(M)$$

$$S(T = T_c(M)) = S_c(M)$$

Non-interacting atoms in harmonic trap in R^d

'S_N vector quantum mechanics'

$x_1, \ldots, x_N;\ y_1, \ldots, y_N;\ z_1, \ldots, z_N$

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$$E(T = T_c(M)) = E_c(M)$$

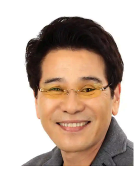
$$S(T = T_c(M)) = S_c(M)$$

Positive interference of wave function

(e.g. Feynman 1953)

$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$

permutation group



or

gauge group

$$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$

$$
\begin{aligned}
Z &= \sum_{g\in \mathrm{S}_N} \sum_{\vec{n}_1,\cdots,\vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{g} e^{-\beta \hat{H}} | \vec{n}_1, \cdots, \vec{n}_N \rangle \\
&= \sum_{\vec{n}_1,\cdots,\vec{n}_N} e^{-\beta\left(E_{\vec{n}_1} + \cdots E_{\vec{n}_N}\right)} \sum_{g\in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{g} | \vec{n}_1, \cdots, \vec{n}_N \rangle \\
&= \sum_{\vec{n}_1,\cdots,\vec{n}_N} e^{-\beta\left(E_{\vec{n}_1} + \cdots E_{\vec{n}_N}\right)} \sum_{g\in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{g(1)}, \cdots, \vec{n}_{g(N)} \rangle
\end{aligned}
$$

$$
|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle
$$

Positive interference of wave function

$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$

- Ground state → all N particle are in the same state
  → all g's returns the same value
  → factor N! enhancement
  (compared to classical Boltzmann statistics)
- All N particles are in different states
  → only g=1 gives nonzero value

$$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$

Positive interference of wave function

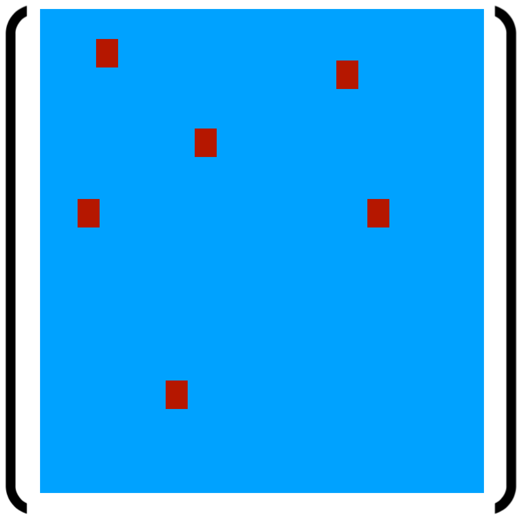
$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$

- Ground state → all N particle colors are in the same state
  → all g's returns the same value
  → factor N! enhancement volume of O(N), SU(N)
  (compared to classical Boltzmann statistics)

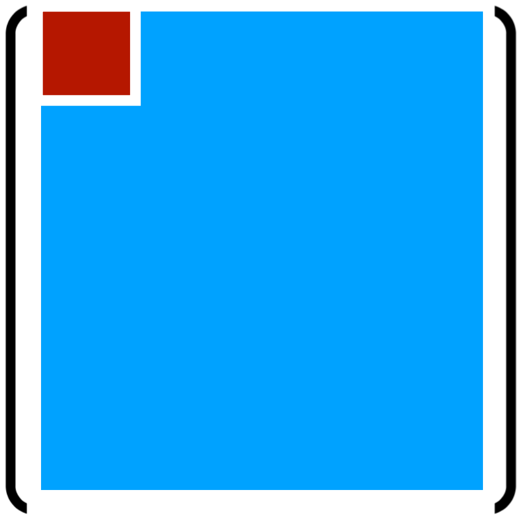
- All N particles colors are in different states
  → only g=1 gives nonzero value

$$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$

Why should large symmetry be preserved?



instead of

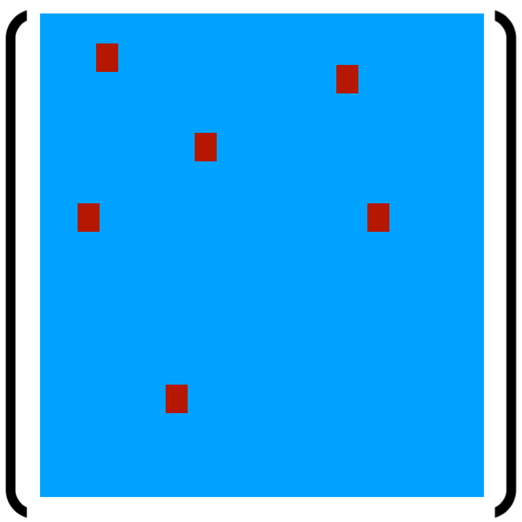


?

no symmetry

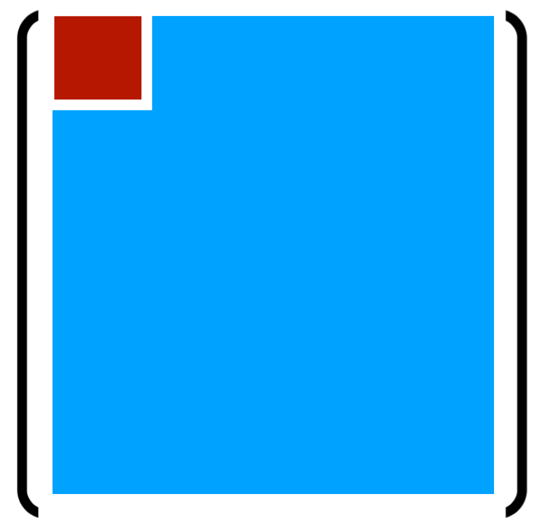
SU(M)×SU(N-M)

Why should large symmetry be preserved?



no symmetry

instead of



SU(M)×SU(N-M)

?

Larger enhancement factor (volume of SU(N-M))

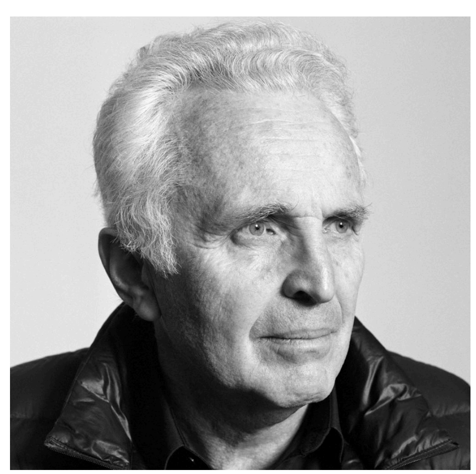
(compared to classical Boltzmann statistics)

L. Onsager

O. Penrose

C. N. Yang

Off-Diagonal Long Range Order (ODLRO)
vs
Polyakov Loop



A. Polyakov

L. Susskind

$$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$

Reduced density matrix $\hat{\rho}_1 = N \cdot \mathrm{Tr}_{2,3,\cdots,N}\hat{\rho}$

$$\hat{\rho}_1 = \underbrace{n_{\max}|\Psi\rangle\langle\Psi|}_{\text{O(N) term} \to \text{BEC}} + \sum_i n_i |\Psi_i\rangle\langle\Psi_i|$$

$\langle x|\hat{\rho}_1|y\rangle$ non-vanishing at long distance

Off-Diagonal Long Range Order

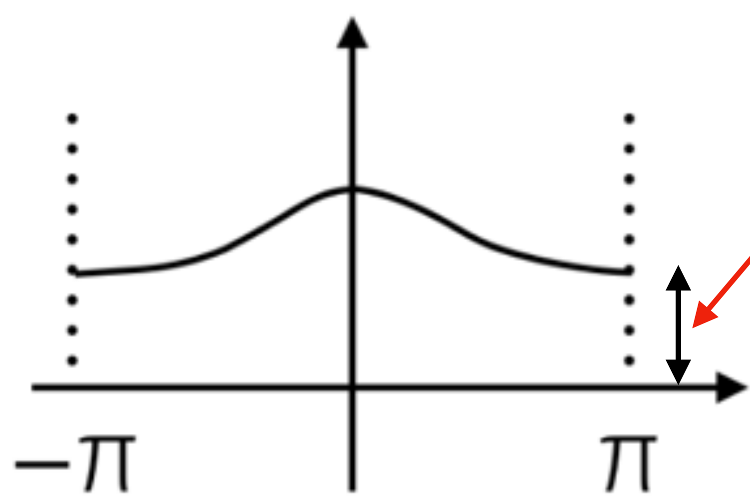
Works even with interaction! (e.g. superfluid helium)

Off-Diagonal Long Range Order

$$\hat{\rho}_1 = \underline{n_{\max}}|\Psi\rangle\langle\Psi| + \sum_i n_i|\Psi_i\rangle\langle\Psi_i|$$

# of d.o.f. in BEC

Polyakov loop phases



$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$

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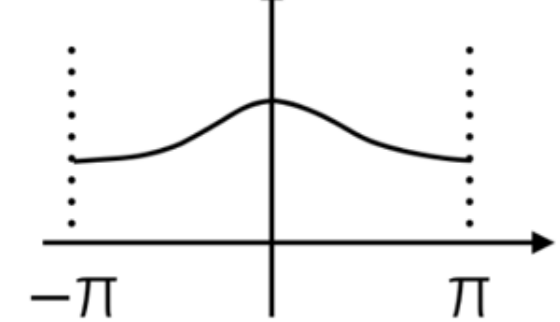
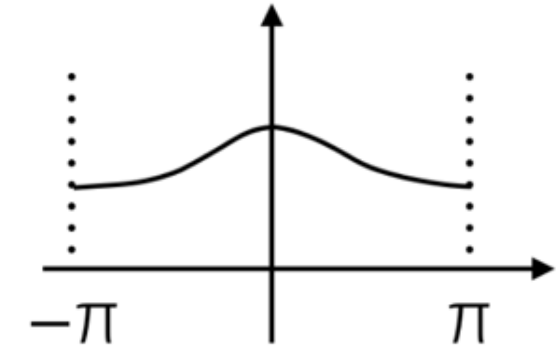
Polyakov loop



$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$



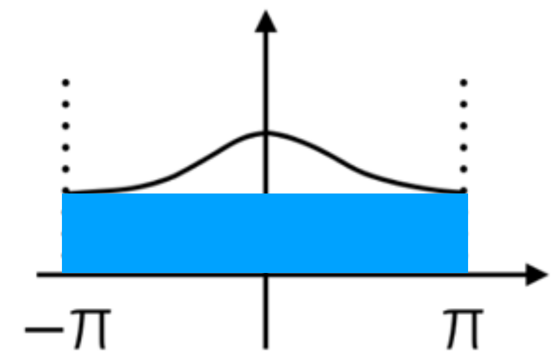
Polyakov loop

- Choose a ‘typical’ state.

$$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$

- Permutations leaving this state invariant is dominant.

$$Z = \sum_{g \in G} \mathrm{Tr}\left(\hat{g} e^{-\beta \hat{H}}\right)$$



$\uparrow$

Polyakov loop

- Choose a ‘typical’ state. $$|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{d} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$$
- Permutations leaving this state invariant is dominant.

$$|\underline{\vec{0}, \vec{0}, \cdots, \vec{0}}, \vec{n}, \vec{n}', \cdots\rangle$$

invariant under S_{N-M} in S_N

$$\begin{array}{cc} \left(\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & & & \\ \hline & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{array}\right) & \begin{array}{c} \updownarrow \; \text{N−M} \\ \\ \updownarrow \; \text{M} \end{array} \end{array}$$

Long cyclic permutation becomes dominant (Feynman 1953)

length k (∼N−M) → eigenvalues $e^{2\pi i l/k}$, $l = 0, 1, \cdots, k-1$

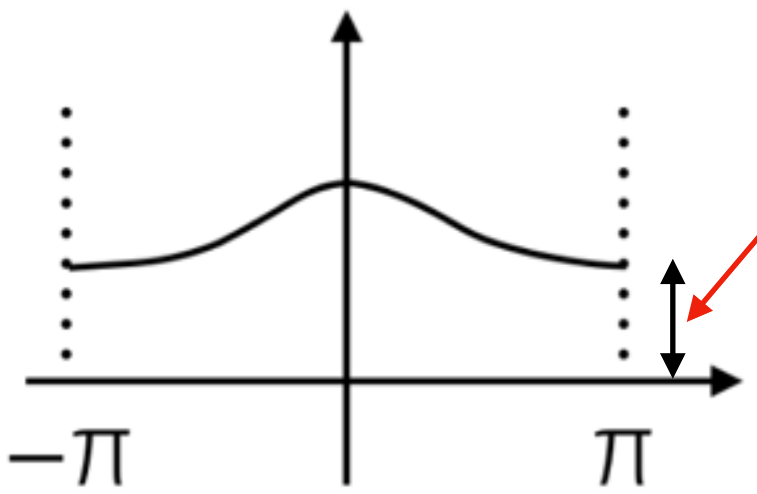
→ **constant offset**

Off-Diagonal Long Range Order

$$\hat{\rho}_1 = \underline{n_{\max}} |\Psi\rangle\langle\Psi| + \sum_i n_i |\Psi_i\rangle\langle\Psi_i|$$

# of d.o.f. in BEC

Polyakov loop phases



It works in gauge theory too.

Partial deconfinement at ***strong coupling***

Watanabe-Bergner-Bodendorfer-Funai-M.H.-Rinaldi-Schaefer-Vranas, 2005.04103 [hep-th]

PhD student @Tsukuba

渡辺展正

- Gaussian matrix model (free)

  — Analytically solvable
  — ‘Confined’ and ‘deconfined’ sectors are not interacting

$$S = N\int_0^\beta dt\ \mathrm{Tr}\left\{\frac{1}{2}(D_t X_I)^2 + \frac{1}{2}X_I^2\right\}$$

- Yang-Mills matrix model (interacting)

$$S = N\int_0^\beta dt\ \mathrm{Tr}\left\{\frac{1}{2}(D_t X_I)^2 - \frac{1}{4}[X_I, X_J]^2\right\}$$

Lattice simulation, & find *typical configuration* (～master field)

- Gaussian matrix model (free)

$$S = N \int_0^\beta dt \ \mathrm{Tr} \left\{ \frac{1}{2} (D_t X_I)^2 + \frac{1}{2} X_I^2 \right\}$$

- Yang-Mills matrix model (interacting)

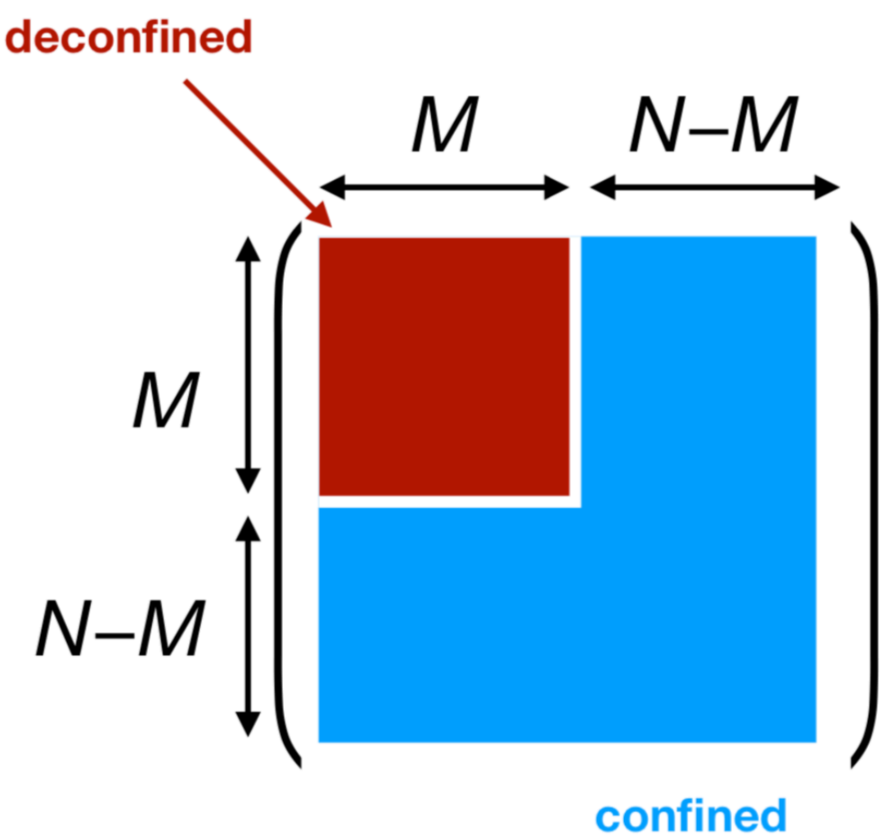
$$S = N \int_0^\beta dt \ \mathrm{Tr} \left\{ \frac{1}{2} (D_t X_I)^2 - \frac{1}{4} [X_I, X_J]^2 \right\}$$

$$\boxed{\rho(\theta) = \frac{1 + 2P\cos\theta}{2\pi} = \frac{M}{N} \cdot \frac{1 + \cos\theta}{2\pi} + \left(1 - \frac{M}{N}\right) \cdot \frac{1}{2\pi}}$$

This holds in both cases. (Not important, but makes analysis simpler.)

Static diagonal gauge $$A_t = \mathrm{diag}\left(\frac{\theta_1}{\beta}, \cdots, \frac{\theta_N}{\beta}\right)$$

$$\rho(\theta) = \frac{1 + 2P\cos\theta}{2\pi} = \underbrace{\frac{M}{N} \cdot \frac{1+\cos\theta}{2\pi}}_{\theta_1,\ldots,\theta_M} + \underbrace{\left(1 - \frac{M}{N}\right) \cdot \frac{1}{2\pi}}_{\theta_{M+1},\ldots,\theta_N}$$



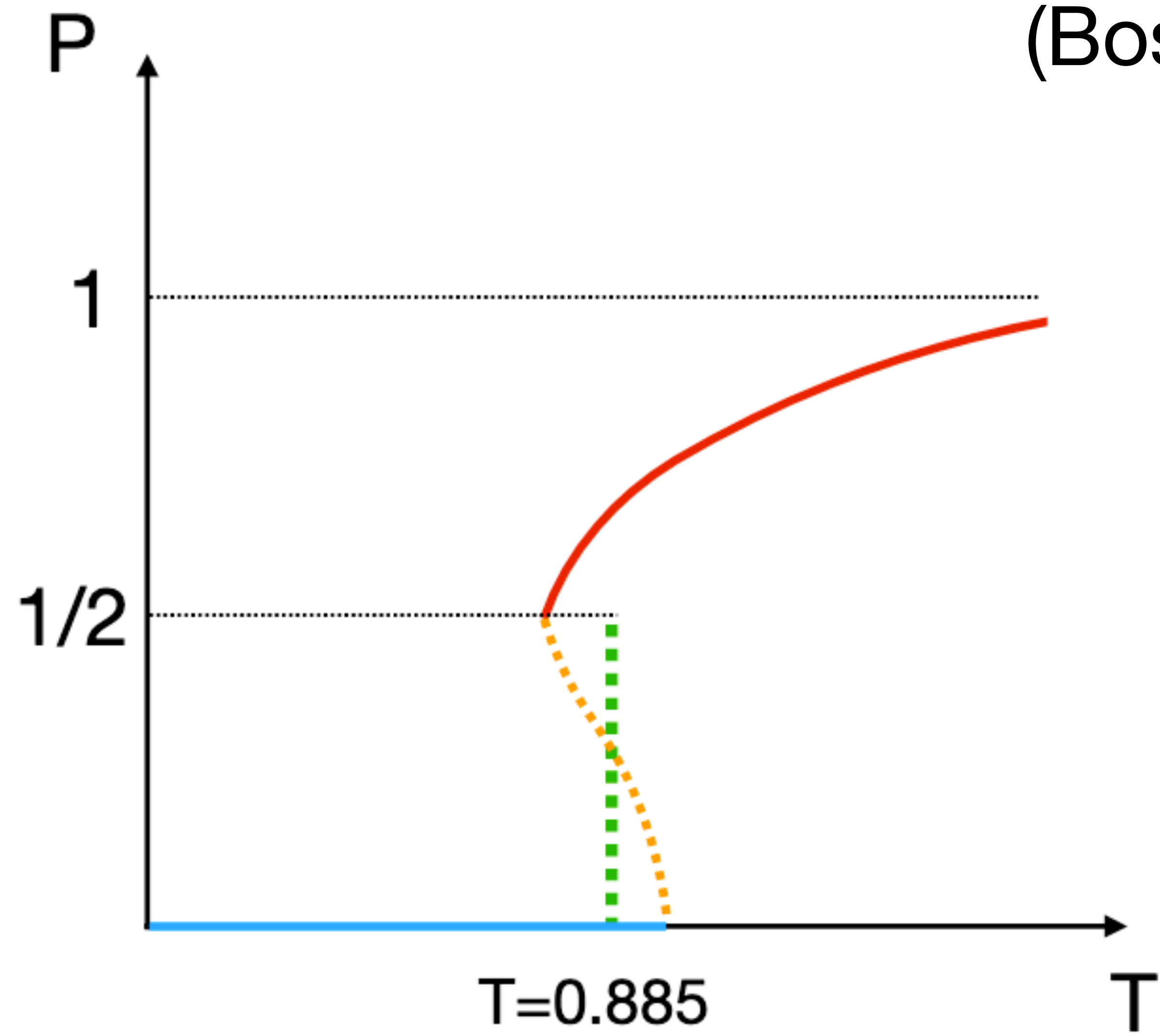
deconfined → X_{ij} large

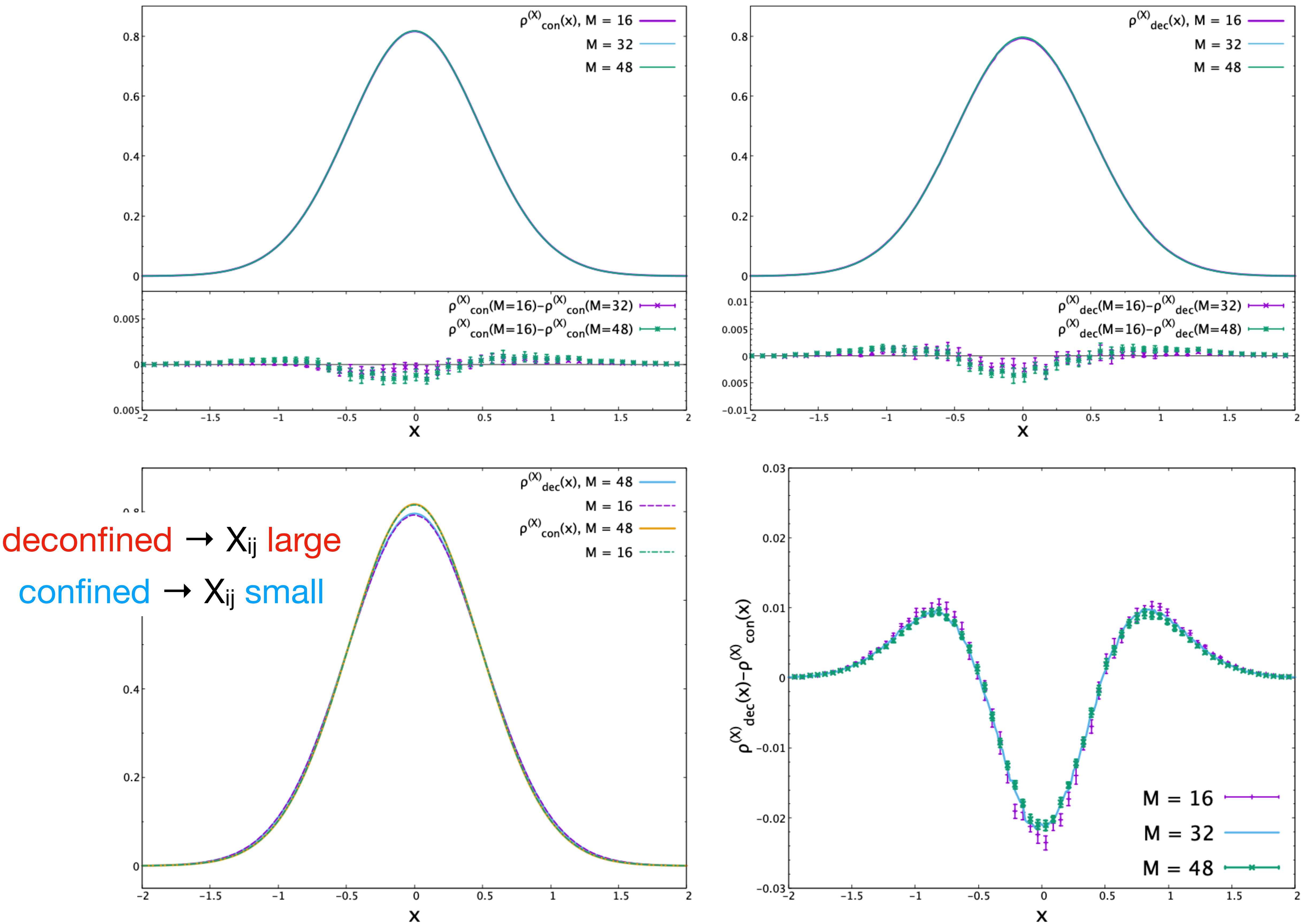
confined → X_{ij} small

$$S = N \int_0^\beta dt\ \mathrm{Tr}\left\{ \frac{1}{2}(D_t X_I)^2 - \frac{1}{4}[X_I, X_J]^2 \right\}$$

$\boxed{\text{I,J=1,…,9}}$

(Bosonic part of BFSS)



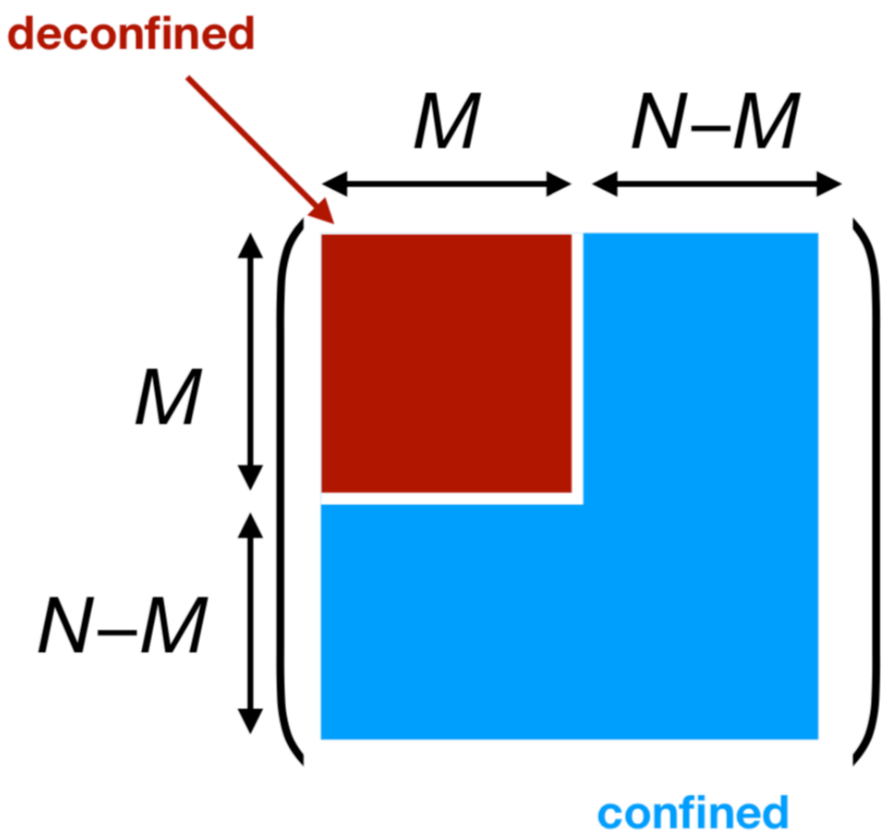


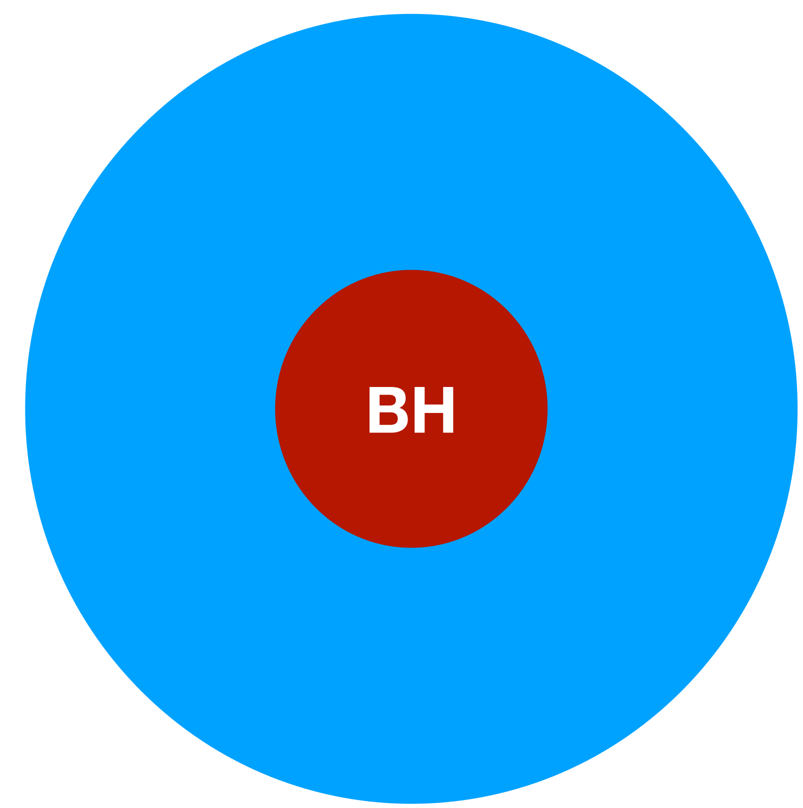
$\rho^{(X)}_{con}(x)$, M = 16
M = 32
M = 48
$\rho^{(X)}_{dec}(x)$, M = 16
M = 32
M = 48
$\rho^{(X)}_{con}(M=16)-\rho^{(X)}_{con}(M=32)$
$\rho^{(X)}_{con}(M=16)-\rho^{(X)}_{con}(M=48)$
$\rho^{(X)}_{dec}(M=16)-\rho^{(X)}_{dec}(M=32)$
$\rho^{(X)}_{dec}(M=16)-\rho^{(X)}_{dec}(M=48)$
x
$\rho^{(X)}_{dec}(x)$, M = 48
M = 16
$\rho^{(X)}_{con}(x)$, M = 48
M = 16
deconfined → X_{ij} large
confined → X_{ij} small
$\rho^{(X)}_{dec}(x)-\rho^{(X)}_{con}(x)$
M = 16
M = 32
M = 48

Summary (& Speculation)

- Confinement is (essentially) BEC (at least at large N)
- ‘Partially’ confined/deconfined phase exists
    - — Analytic demonstration at weak coupling
    - — Numerical evidence at strong coupling

Intuitive picture in gravity (no proof yet)





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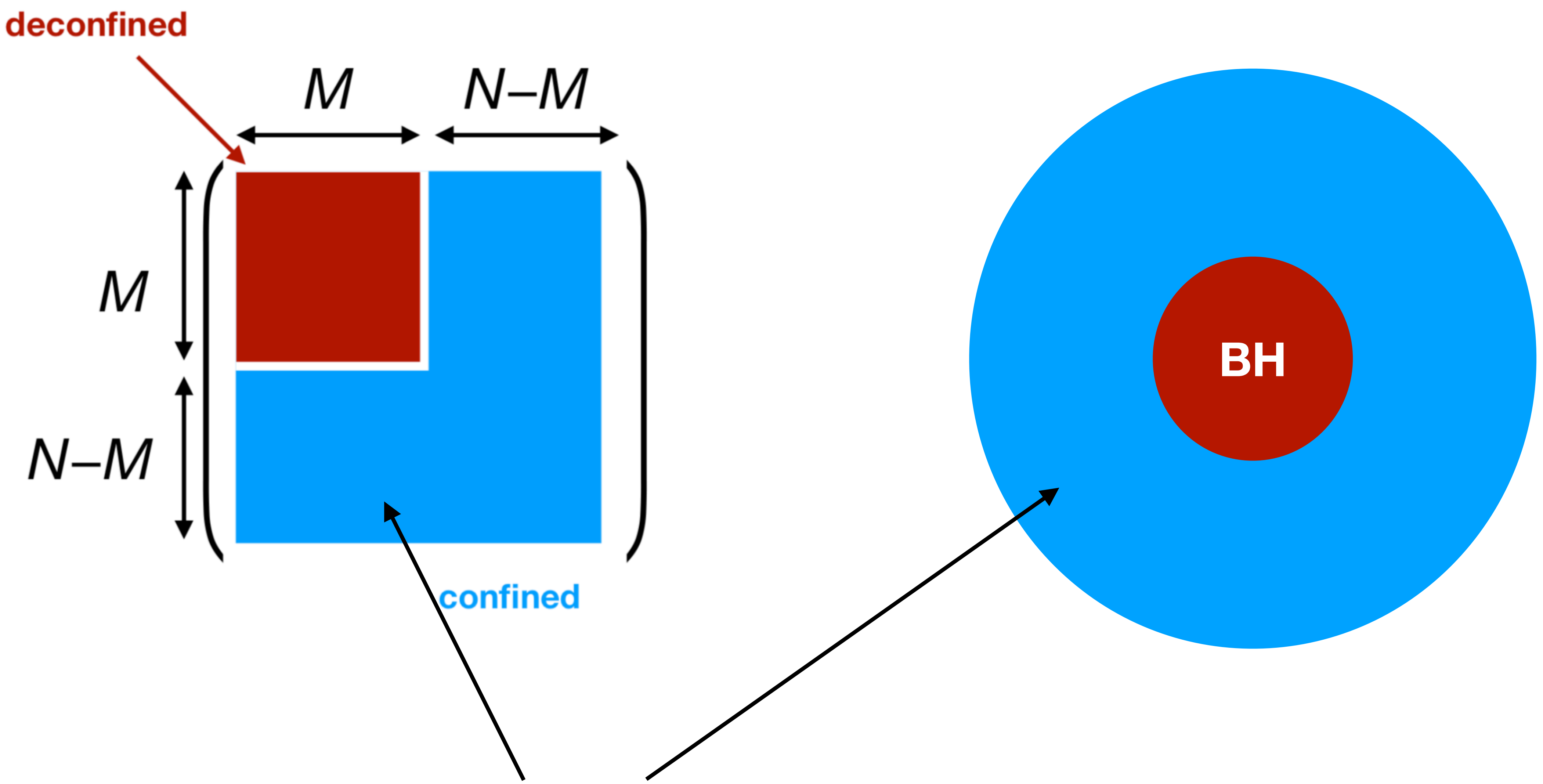
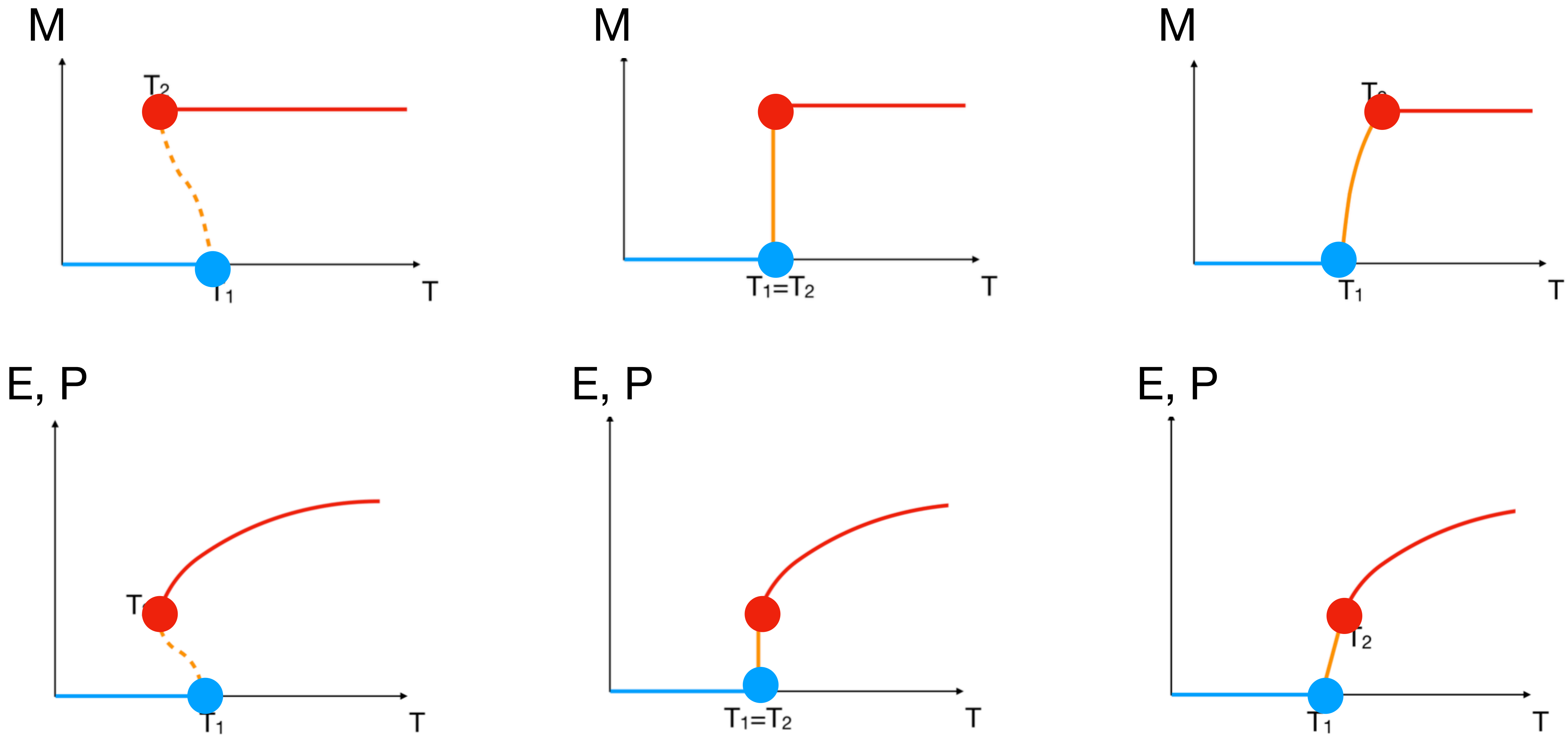


Confined sector is naturally entangled;
Emergent space from entanglement of color d.o.f?



transition 1: confinement to partial deconfinement
(black hole formation begins)

transition 2: partial deconfinement to complete deconfinement
(black hole formation ends)

$$\boxed{SU(N) \to SU(M)\times SU(N-M) \to SU(N)}$$

No need for center symmetry → Applicable to QCD.