Tensor renormalization group approach to four-dimensional lattice field theories

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Based on

S. A., D. Kadoh, Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP09(2020)177

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, arXiv:2009.11583[hep-lat]

2020.11.10 @ Osaka University

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2. 4d lattice field theories w/ the TRG approach

   ✓ Nambu—Jona-Lasinio model at finite density

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Tensor renormalization group approach

Tensor Network (TN)

-> Contractions of the tensors locating on a real-space lattice

Tensor Renormalization Group (TRG)

-> A variant of real-space renormalization group to coarse grain tensor networks

Procedures

1) **TN representation for $\boldsymbol{X}$** : (# of tensors in TN) $=$ (# of lattice sites)

$$X \rightarrow \Sigma_{abcd\cdots} T_{aiw\cdots} T_{bjx\cdots} T_{cky\cdots} T_{dlz\cdots} \cdots$$

2) **TRG** : Block-spin trans. for T to reduce # of tensors in TN

$$\approx \Sigma_{a'b'c'd'\cdots} T'_{a'i'w'\cdots} T'_{b'j'x'\cdots} T'_{c'k'y'\cdots} T'_{d'l'z'\cdots} \cdots$$

1) Construct the TN representation for the target function X defined on lattice

ex. Partition function, Path integral

2) Approximately perform the tensor contraction with TRG

TN rep. for 2d Ising model w/ PBC

Decompose nearest-neighbor interactions

$$Z=\Sigma_{\{\sigma=\pm1\}}\Pi_{n,\mu}\exp\left[\beta J\sigma_n\sigma_{n+\hat{\mu}}\right] \quad\Rightarrow\quad Z=\mathrm{Tr}\left[\Pi_n T_{x_n y_n x'_n y'_n}\right]$$

$T_{x_n y_n x'_n y'_n}$ specifies the details of the model

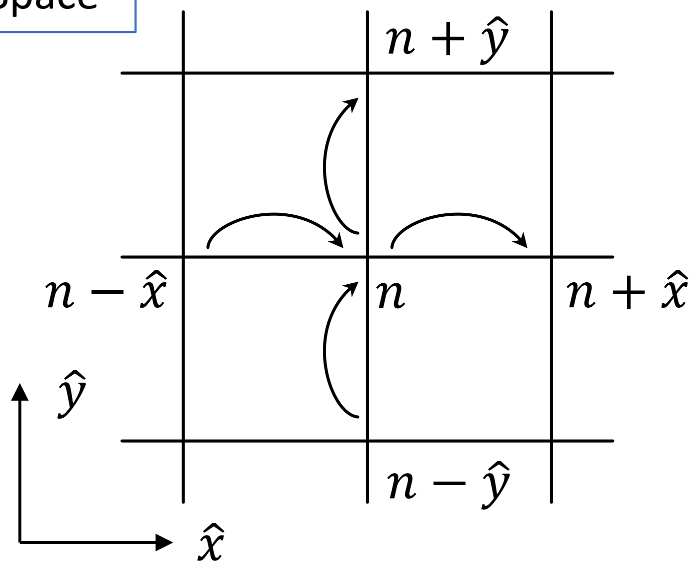
$$\exp\left[\beta J\sigma_n\sigma_{n+\hat{\mu}}\right]=\sum_{l_n}\sqrt{\lambda_{l_n}}U(\sigma_n,l_n)\sqrt{\lambda_{l_n}}U(\sigma_{n+\hat{\mu}},l_n)=\sum_{l_n}W(\sigma_n,l_n)W(\sigma_{n+\hat{\mu}},l_n)$$

$$W(a,b)\coloneqq\sqrt{\lambda_b}U(a,b)$$

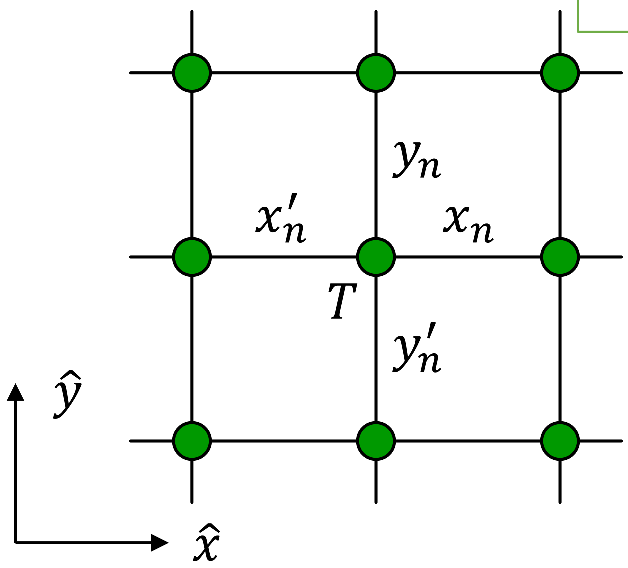
$$T_{x_n y_n x'_n y'_n}\coloneqq\sum_{\sigma_n=\pm1}W(\sigma_n,x_n)W(\sigma_n,y_n)W(\sigma_n,x'_n)W(\sigma_n,y'_n)$$

$$x'_n\coloneqq x_{n-\hat{x}},\ y'_n\coloneqq y_{n-\hat{y}}$$

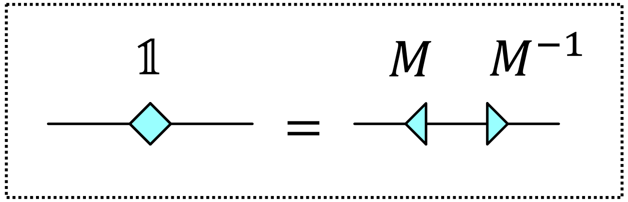
Real Space



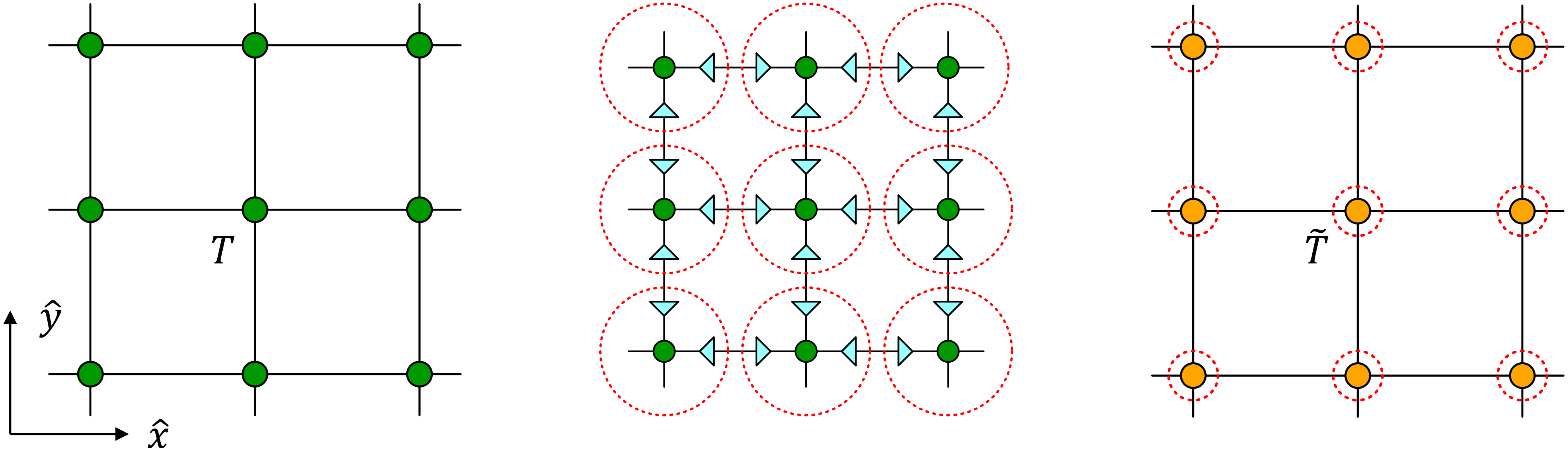
TN rep. for Z



Remark TN rep. is not unique



Inserting a matrix and its inverse,
one obtains a different TN rep. for the same Z



$Z = \mathrm{Tr}\left[\Pi_n T_{x_n y_n x'_n y'_n}\right]$

Both represent the same Z

$Z = \mathrm{Tr}\left[\Pi_n \tilde{T}_{\tilde{x}_n \tilde{y}_n \tilde{x}'_n \tilde{y}'_n}\right]$

$$\Sigma_{x_n, y_n, x'_n, y'_n} T_{x_n y_n x'_n y'_n} M_{x_n \tilde{x}_n} M_{y_n \tilde{y}_n} M^{-1}_{x'_n \tilde{x}'_n} M^{-1}_{y'_n \tilde{y}'_n} = \tilde{T}_{\tilde{x}_n \tilde{y}_n \tilde{x}'_n \tilde{y}'_n}$$

How to construct TN rep.

Partition function w/ discrete dof (Ref. Talks by Tao Xiang in TNQMP2016)

-> Exact TN rep. is easily available (ex. Ising, Potts, ……)

Path integral w/ continuous dof (Ref. Liu et al, PRD88(2013)056005)

-> Some approximation is necessary to derive TN rep.

Path integral in fermion systems w/ Grassmann numbers

-> No approximation is necessary to derive TN rep.

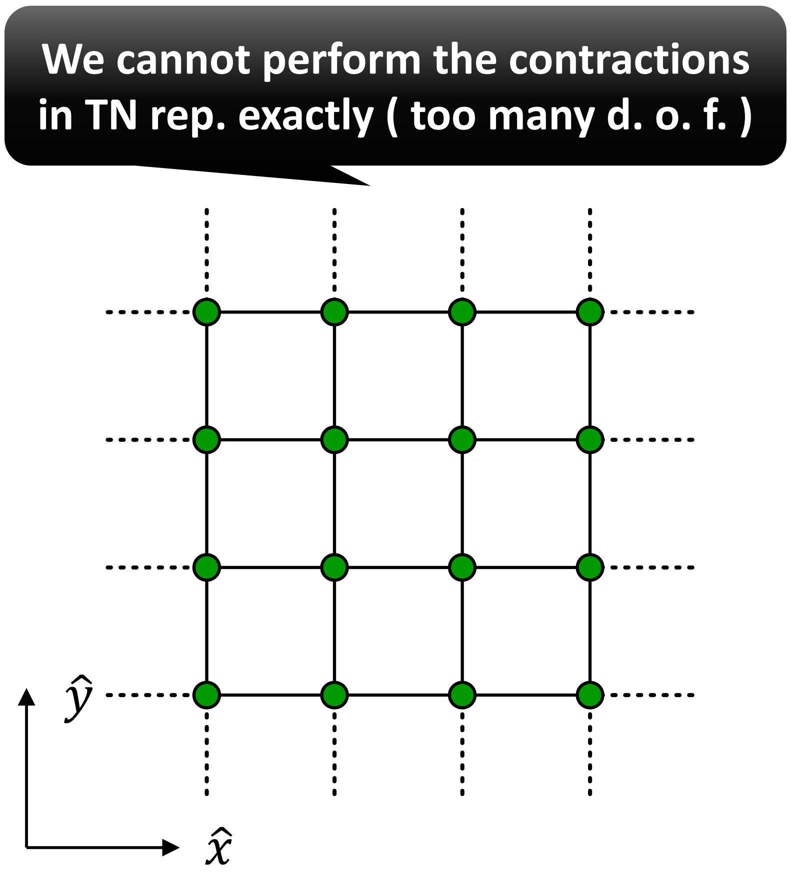
- Taylor expansion: $e^{\bar{\psi}\psi} = \Sigma_m (\bar{\psi}\psi)^m$ (discussed later)

  Shimizu-Kuramashi, PRD90(2014)014508, Takeda-Yoshimura, PTEP2015(2015)043B01

- Auxiliary fermion fields

  SA-Kadoh, arXIv:200507570 [hep-lat]

Basic concept of TRG algorithm



Idea of real-space renormalization group

Iterate a simple transformation w/ approximation
and we can investigate thermodynamic properties

$+$

Information compression
w/ the Singular Value Decomposition (SVD)

$$A_{ij} = \Sigma_k U_{ik}\sigma_k V_{jk} \approx \Sigma_{k=1}^{D} U_{ik}\sigma_k V_{jk}$$

w/ $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$

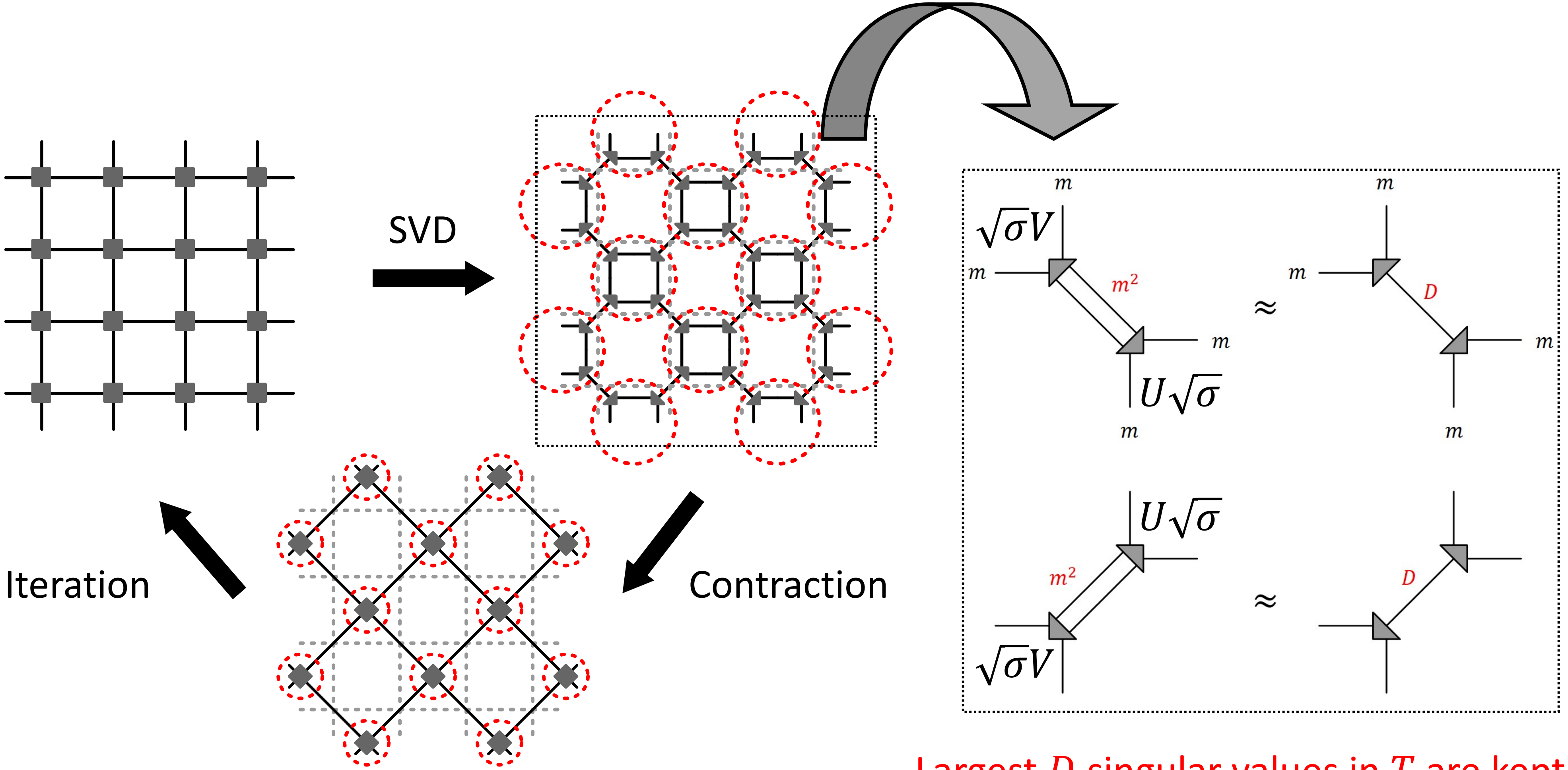
(A: $m\times n$ matrix, U: $m\times m$ unitary, V: $n\times n$ unitary)

$\downarrow$

TRG employs the SVD to reduce d. o. f.
and perform the tensor contraction approximately

TRG w/ SVD of T

Ex. Levin-Nave TRG   Levin and Nave, PRL99(2007)120601

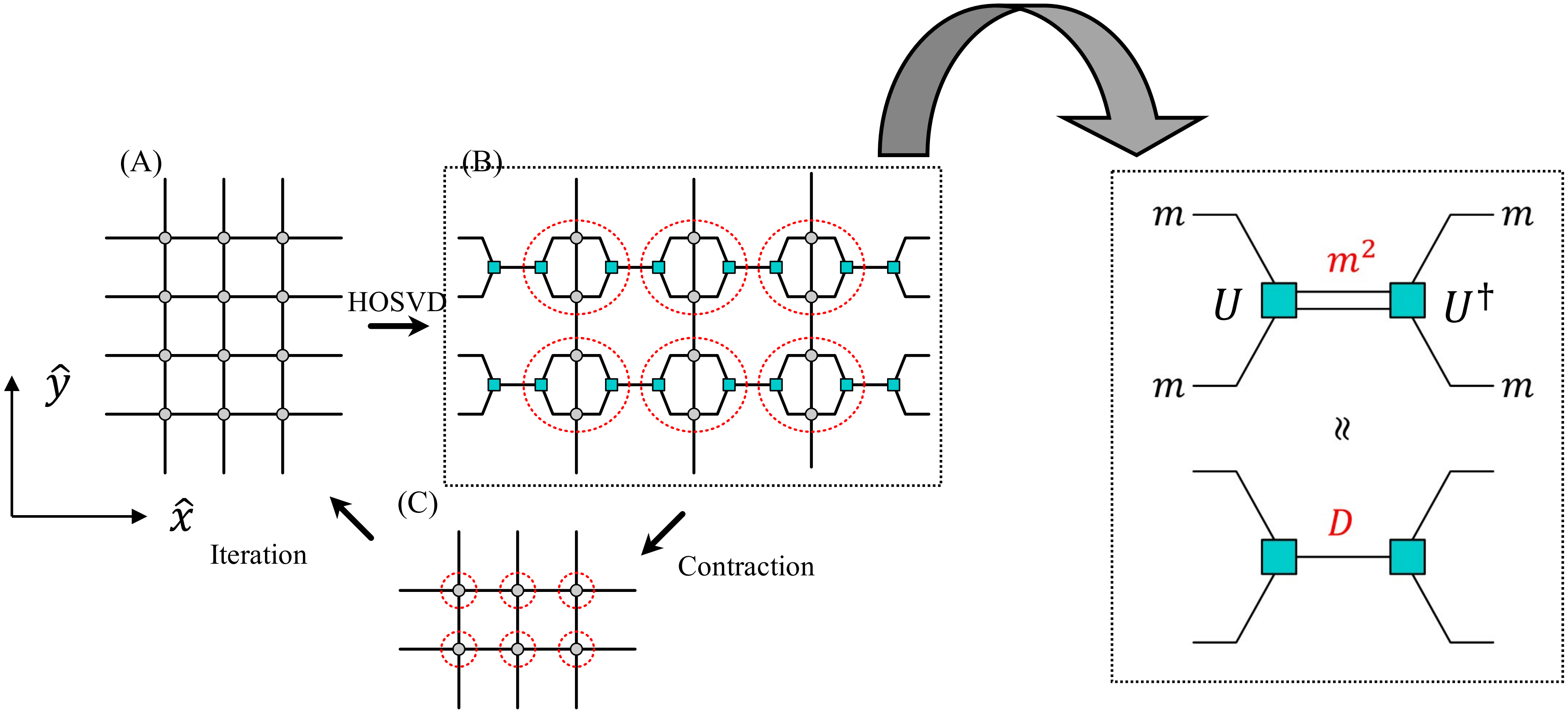


Largest D singular values in T are kept
(D: bond dimension)

# of tensors are reduced to half

TRG w/ isometry insertion

Ex. Higher-order TRG (HOTRG) Xie et al, PRB86(2012)045139

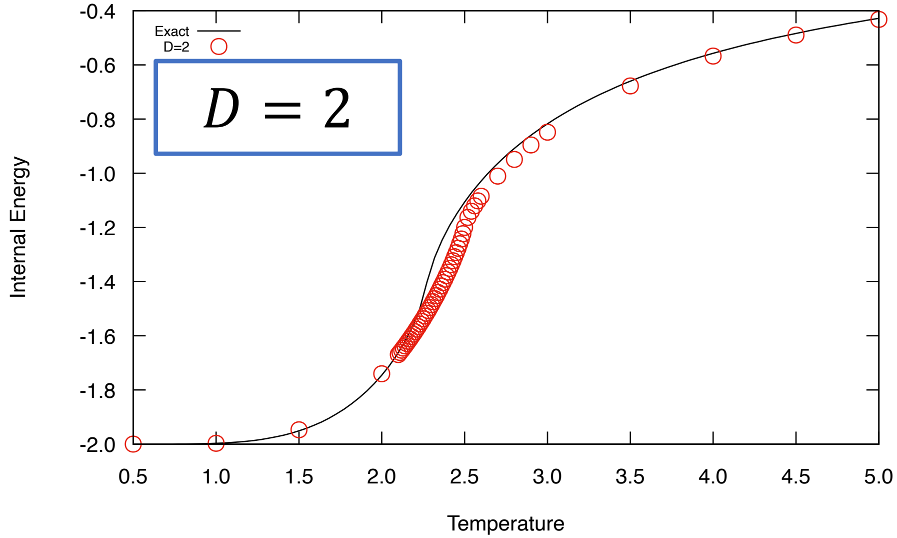


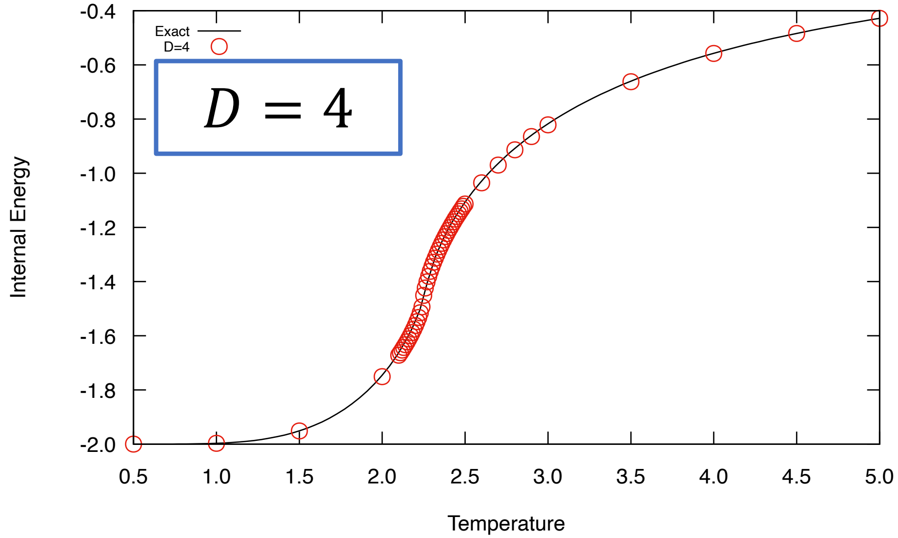
Sequential coarse-graining along with each direction

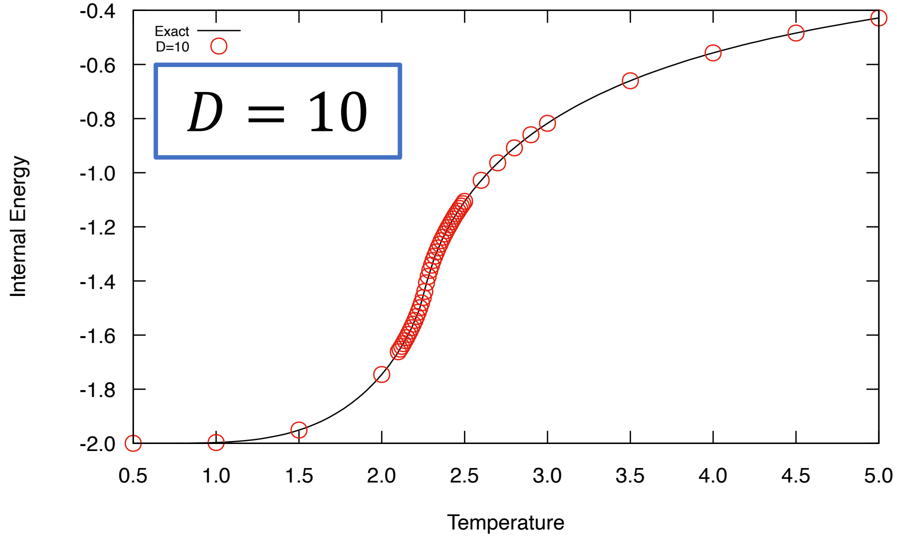
Largest D singular values in TT are kept (D: bond dimension)

# of tensors are reduced to half

Example: 2d Ising model w/ HOTRG



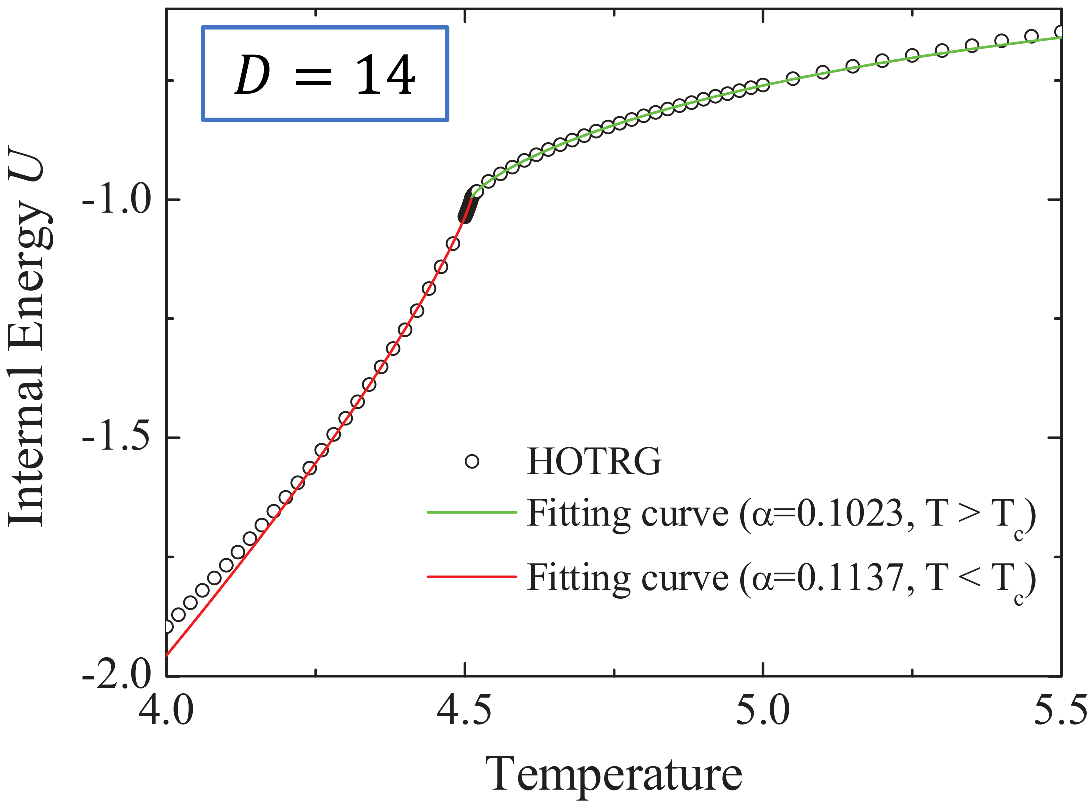




TRG (red circle) well reproduces the exact solution (solid curve)

Example: 3d Ising model w/ HOTRG

Xie et al, PRB86(2012)045139



Critical point

Method	T_c
HOTRG ($D = 16$, from U)	4.511544
HOTRG ($D = 16$, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

-> Good agreement with
the Monte Carlo results

Advantage of TRG approach

Tensor renormalization group is a deterministic numerical method

- **No sign problem**
- **The computational cost scales logarithmically w. r. t. the system size**
- **Direct evaluation of the Grassmann integrals** (w/o introducing pseudo-fermions)
- **Direct evaluation of the partition functions**

TRG has been successfully applied to various 2d or 3d models w/ or w/o the sign problem

Meurice et al, arXiv:2010.06539 (review paper)

<u>Today's message</u> **TRG is an efficient approach also in 4d !**

Computational cost of TRGs

D: bond dimension, L: linear system size

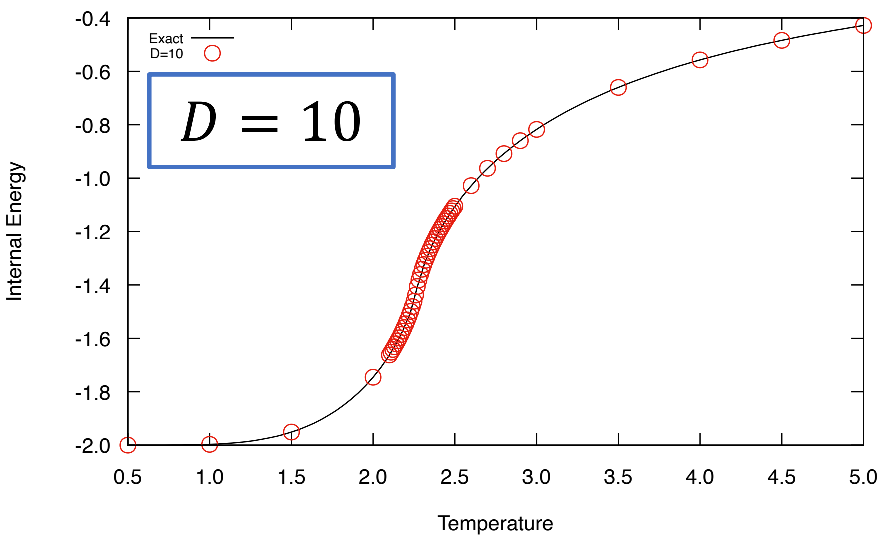
Algorithm	Computational Time	Target
Levin-Nave TRG Levin and Nave, PRL99(2007)120601	$D^6 \ln L$	2-dim
HOTRG Xie et al, PRB86(2012)045139	$D^{4d-1} \ln L$	$\boldsymbol{d}$**-dim**
Anisotropic TRG (ATRG) Adachi et al, PRB102(2020)054432	$D^{2d+1} \ln L$	$\boldsymbol{d}$**-dim**
Triad RG Kadoh-Nakayama, arXiv:1912.02414	$D^{d+3} \ln L$	$\boldsymbol{d}$**-dim**

We need economic TRG algorithms to investigate 4d systems

ATRG (= TRG w/ SVD): Benchmarking w/ 2d Ising model

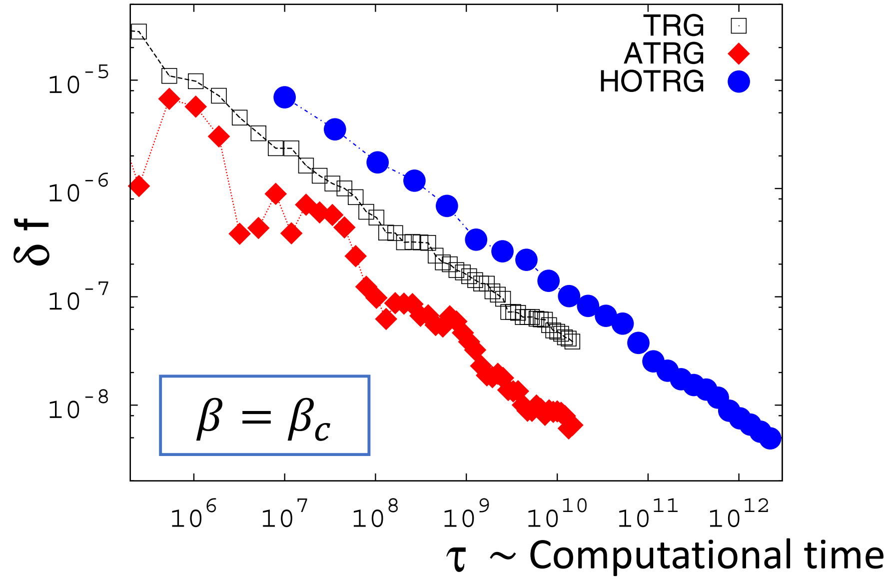
-> Sequential coarse-graining along with each direction

Internal Energy



Relative error of free energy

Adachi et al, PRB102(2020)054432



-> Accuracy with the fixed computational time is improved

	2d ATRG	**2d HOTRG**	**TRG**
Memory	$O(D^3)$	$O(D^4)$	$O(D^4)$
Time	$O(D^5)$	$O(D^7)$	$O(D^6)$

Status of TRG approach in 4d system

Model	TRG algorithm
Ising model SA et al, PRD100(2019)054510	HOTRG (Xie et al, PRB86(2012)045139) w/ parallel computation
Ising model SA et al, PoS(LATTICE2019)138	ATRG (Adachi et al, PRB102(2020)054432) w/ parallel computation
Complex ϕ^4 theory at finite density SA et al, JHEP09(2020)177	ATRG w/ parallel computation
NJL model at finite density SA et al, arXiv:2009.11583[hep-lat]	Grassmann ATRG w/ parallel computation

Parallel computation helps us carry out the large multi-linear algebra in TRG

4d ATRG with parallel computation

ATRG is a coarse-graining (direct truncation) method based on SVD

	4d ATRG	**4d HOTRG**
Memory	$O(D^5)$	$O(D^8)$
Time	$O(D^9)$	$O(D^{15})$

$O(D^9)$ calculations in 4d ATRG -> SVD and tensor contraction

Our implementation

	SVD	**contraction**
Strategy	Randomized SVD	Parallel computing
Time	$O(D^7)$	$O(D^8)$

-> **Parallel computation reduces the computational cost from $\boldsymbol{O(D^9)}$ to $\boldsymbol{O(D^8)}$**

4d Nambu–Jona-Lasinio model at finite density

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, arXiv:2009.11583[hep-lat]

Expected phase diagram of the NJL model

✓ Effective theory of QCD

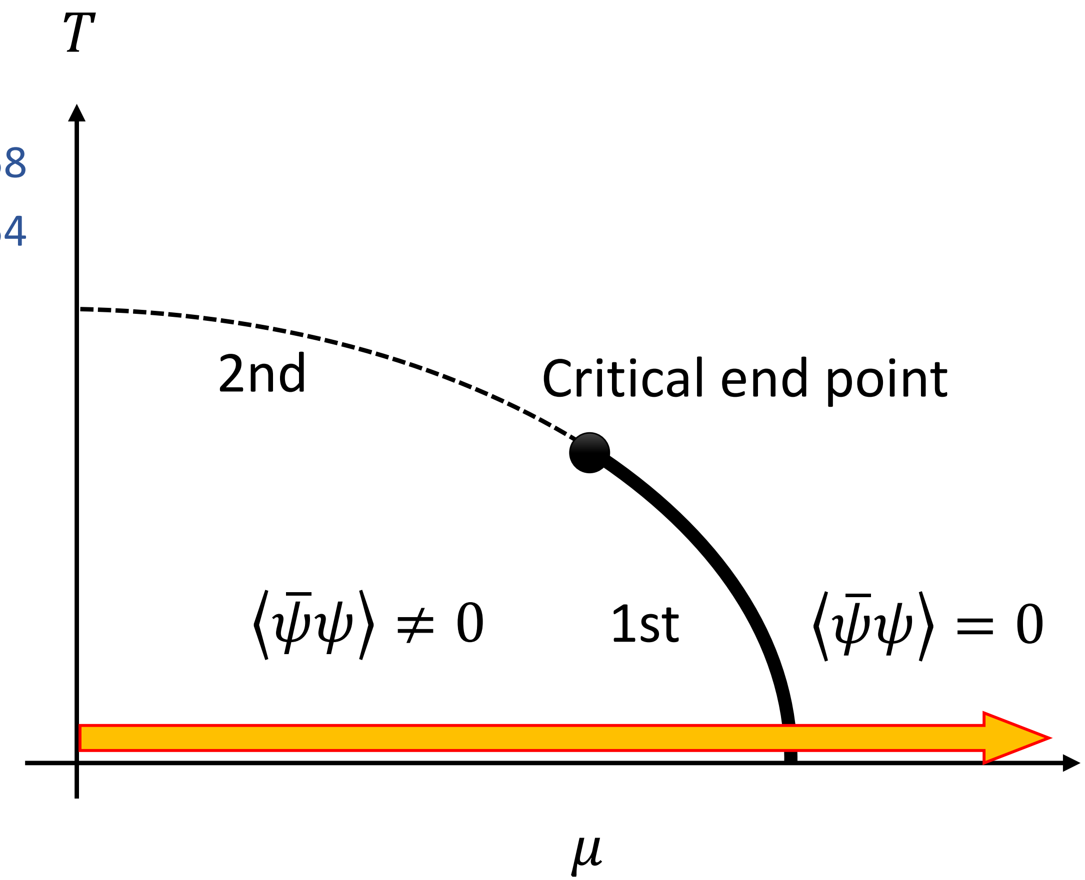
Nambu−Jona-Lasinio, PRD122(1961)345-358

Nambu−Jona-Lasinio, PRD124(1961)246-254

✓ **Chiral restoration is expected in cold & dense region**

Asakawa-Yazaki, NPA504(1989)668-684

✓ **Severe sign problem in cold & dense region**



We apply the Tensor Renormalization Group (TRG) approach to investigate the 1st order chiral phase transition in cold & dense region

NJL model at finite density

✔ w/ the Kogut-Susskind fermion

-> Single-component Grassmann variables w/o the Dirac structure

-> Staggered sign function $\eta_\nu(n) = (-1)^{n_1+\cdots+n_{\nu-1}}$ with $\eta_1(n) = 1$

✔ μ : chemical potential

$$S_{\rm lat} = \frac{1}{2}a^3\Sigma_{n\in\Lambda}\Sigma_{\nu=1}^4\eta_\nu(n)\left[\mathrm{e}^{\mu a\delta_{\nu,4}}\bar{\chi}(n)\chi(n+\hat{\nu}) - \mathrm{e}^{-\mu a\delta_{\nu,4}}\bar{\chi}(n+\hat{\nu})\chi(n)\right]$$
$$+ma^4\Sigma_{n\in\Lambda}\bar{\chi}(n)\chi(n) - g_0a^4\Sigma_{n\in\Lambda}\Sigma_{\nu=1}^4\bar{\chi}(n)\chi(n)\bar{\chi}(n+\hat{\nu})\chi(n+\hat{\nu})$$

(This formulation follows Lee-Shrock, PRL59(1987)14)

✔ Continuous chiral symmetry for vanishing m :

$$\chi(n) \to \mathrm{e}^{\mathrm{i}\alpha\epsilon(n)}\chi(n), \qquad \bar{\chi}(n) \to \bar{\chi}(n)\mathrm{e}^{\mathrm{i}\alpha\epsilon(n)}$$

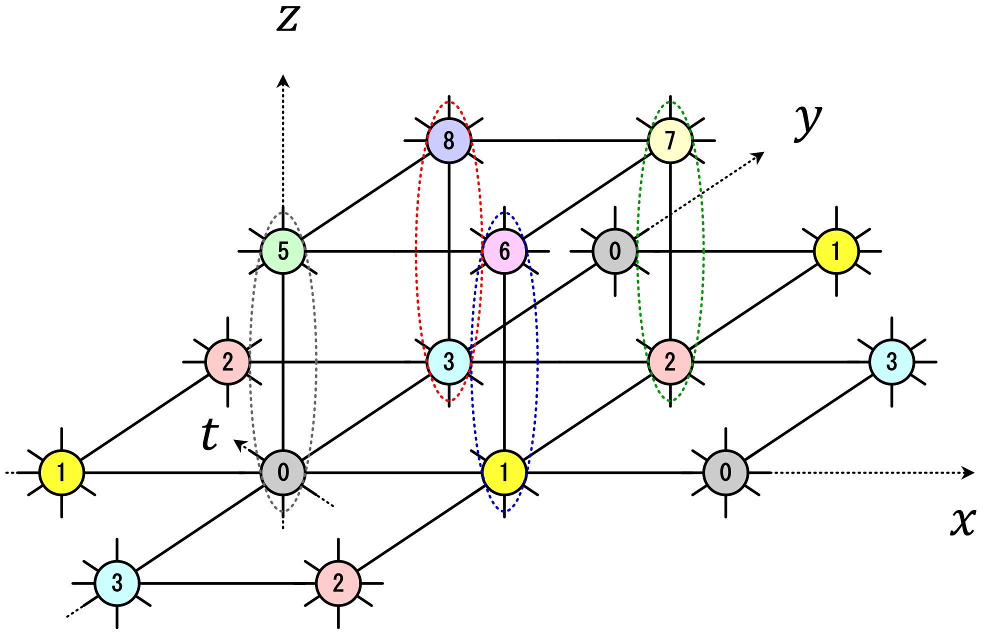
w/ $\alpha \in \mathbb{R}$ and $\epsilon(n) = (-1)^{n_1+n_2+n_3+n_4}$

Tensor network rep. & Grassmann ATRG

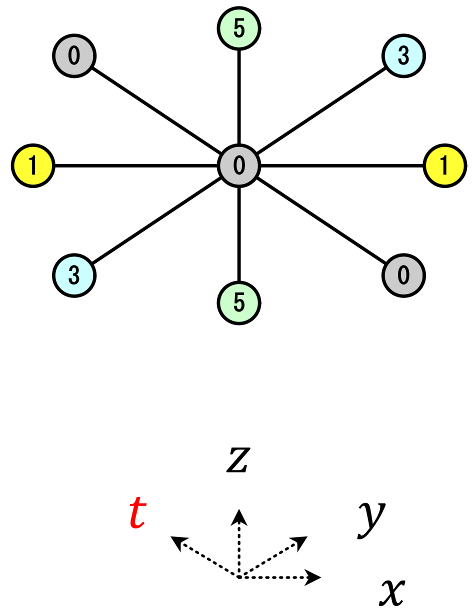
$Z = \Sigma \int \Pi_{n \in \Lambda^{(0)}} \mathcal{T}^{(0)}_{n;txyzt'x'y'z'}$ -> consists of 8 types of tensor

x-, y-, z-directions : PBC
t-directions : Anti-PBC

$\eta_\nu(n) = (-1)^{n_1 + \cdots + n_{\nu-1}}$ with $\eta_1(n) = 1$



# of lattice sites are reduced to half
$\Lambda^{(0)} \to \Lambda^{(1)}$

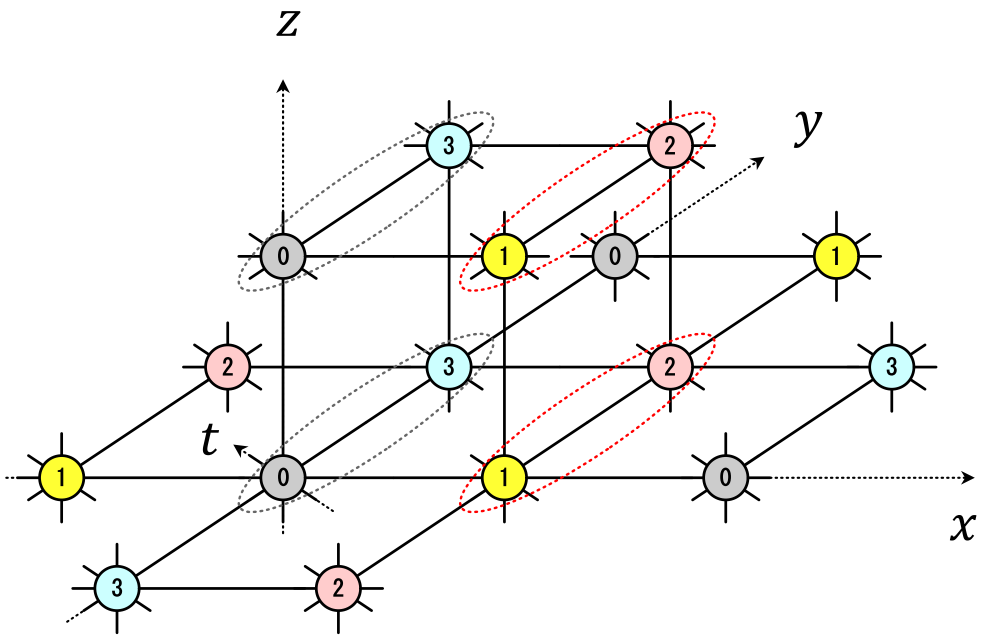


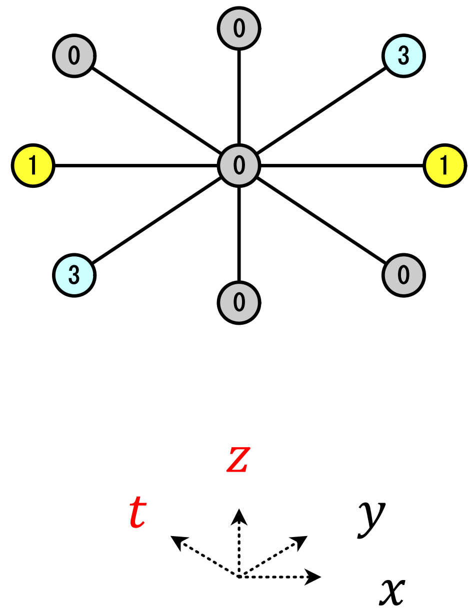
Uniform structure
in t-direction

Tensor network rep. & Grassmann ATRG

$Z = \Sigma \int \Pi_{n \in \Lambda^{(1)}} \mathcal{T}^{(1)}_{n;txyzt'x'y'z'}$ -> consists of 4 types of tensor

x-, y-, z-directions : PBC
t-directions : Anti-PBC





# of lattice sites are reduced to half
$\Lambda^{(1)} \to \Lambda^{(2)}$

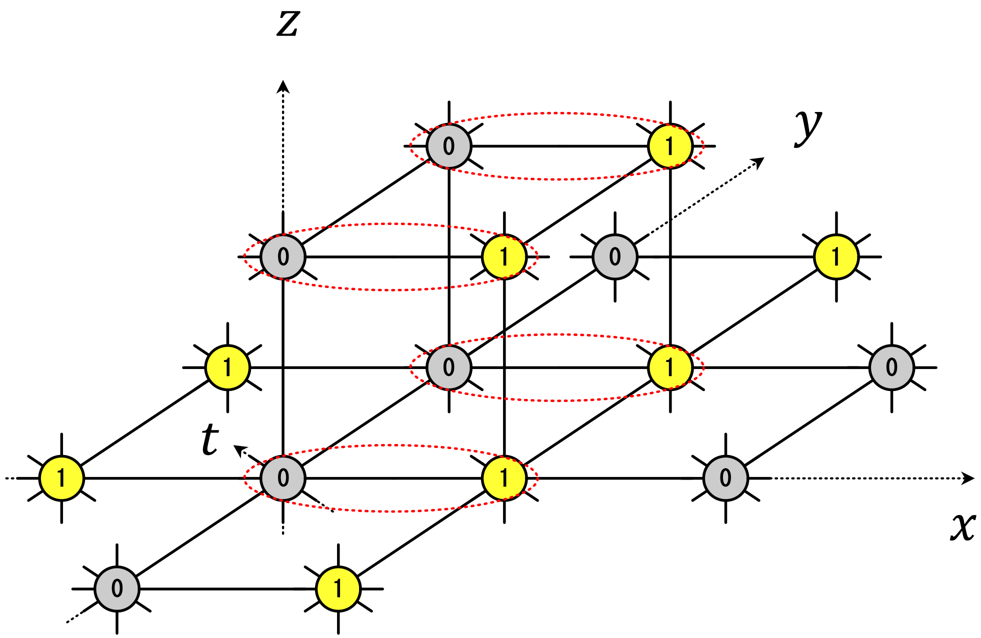
Uniform structure
in t-, z-directions

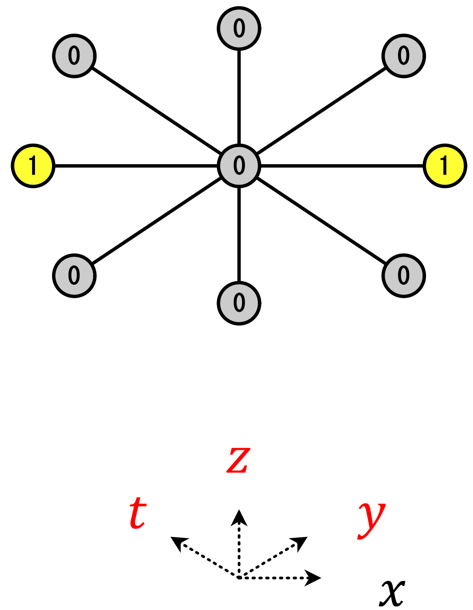
Tensor network rep. & Grassmann ATRG

$$Z = \Sigma \int \Pi_{n \in \Lambda^{(2)}} \mathcal{T}^{(2)}_{n;txyzt'x'y'z'}$$ -> consists of 2 types of tensor

x-, y-, z-directions : PBC

t-directions : Anti-PBC





# of lattice sites are reduced to half

$\Lambda^{(2)} \to \Lambda^{(3)}$

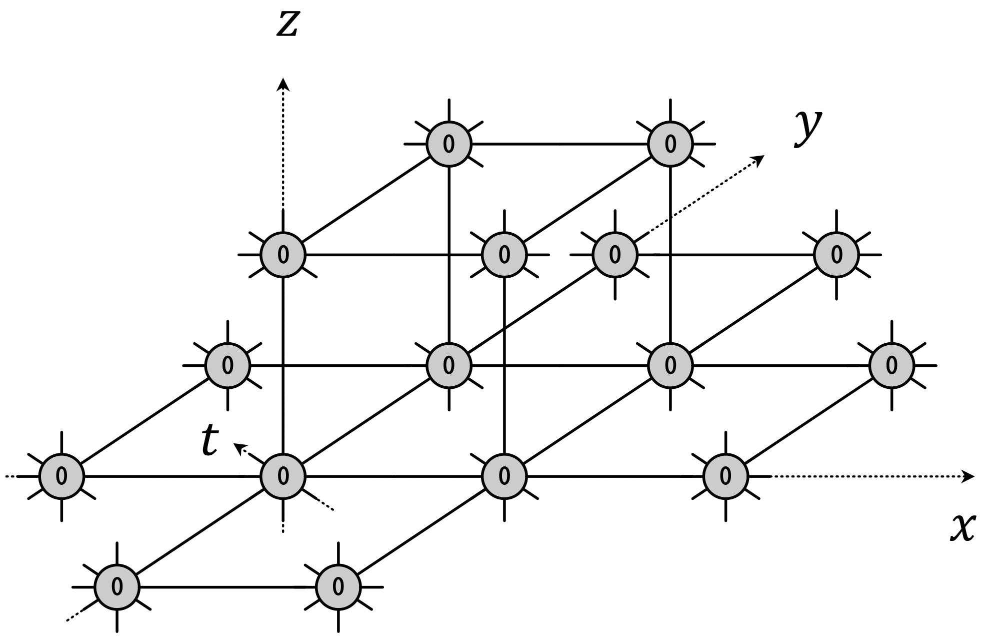
Uniform structure

in t-, z-, y-directions

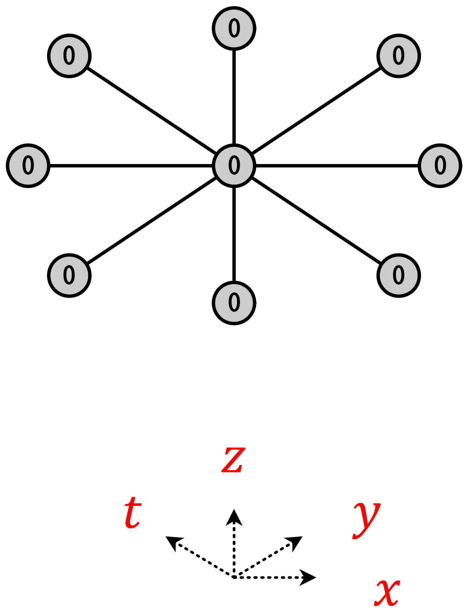
Tensor network rep. & Grassmann ATRG

$Z = \Sigma \int \Pi_{n \in \Lambda^{(3)}} \mathcal{T}^{(3)}_{n;txyzt'x'y'z'}$ -> consists of 1 type of tensor

x-, y-, z-directions : PBC
t-directions : Anti-PBC



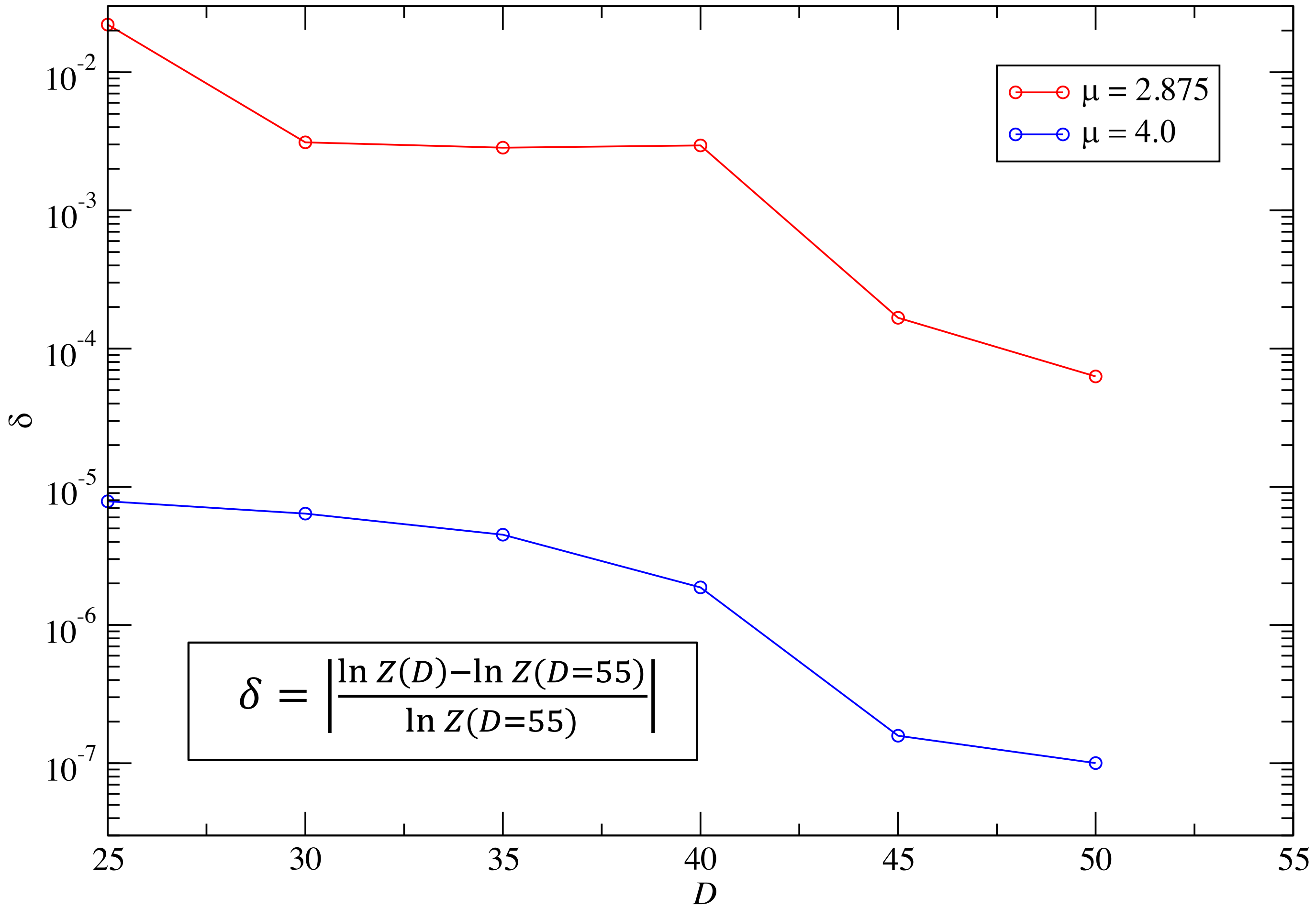
# of lattice sites are reduced to half
$\Lambda^{(3)} \to \Lambda^{(4)}$



Uniform structure
in all directions

Converging behavior in bond dimension

with $m = 0.01, g_0 = 32, L = 1024$



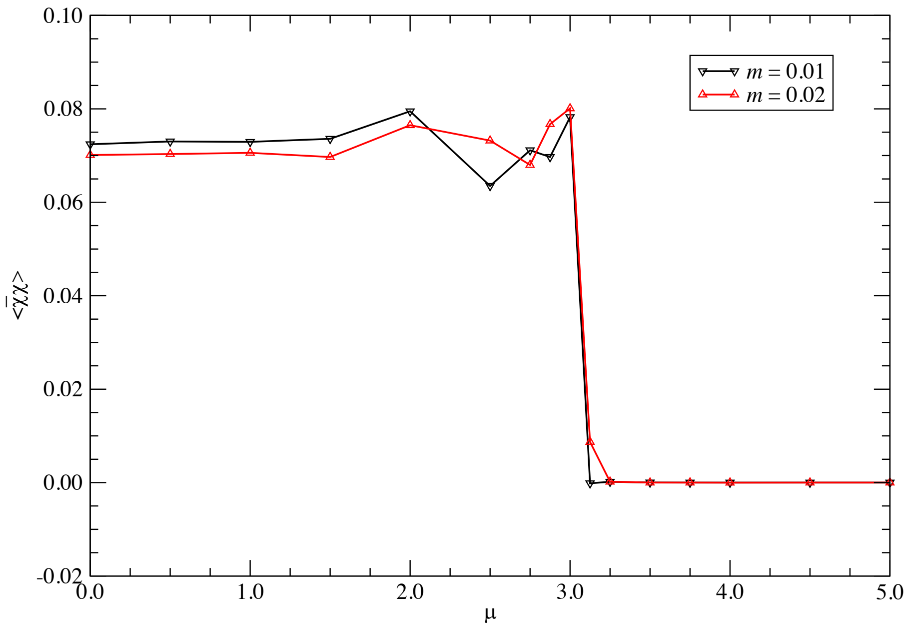
$\delta \lesssim 10^{-4}$ has been achieved up to $D = 55$ at $\mu \approx \mu_c$

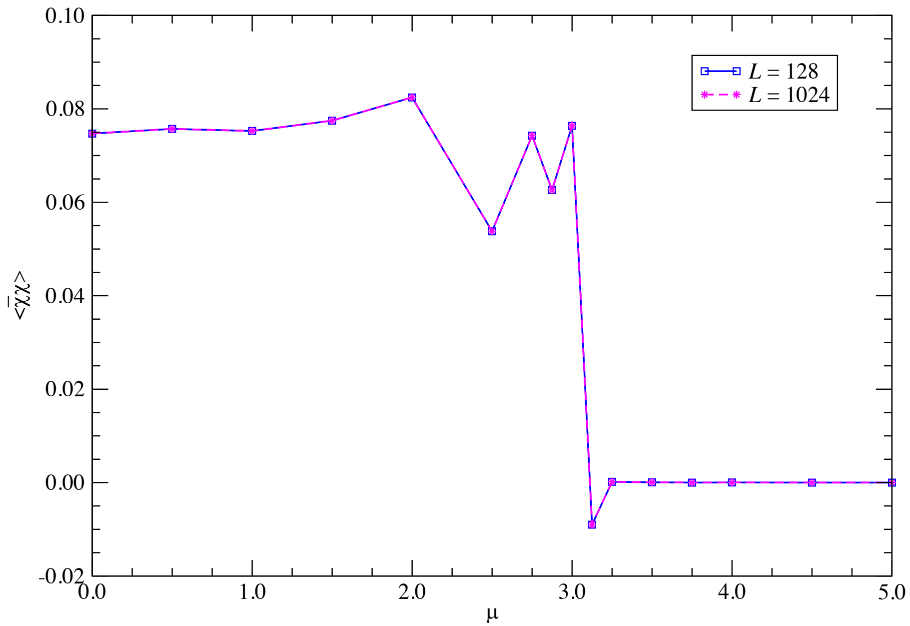
Chiral condensate

with $g_0 = 32, D = 55$

$L = 1024^4$

$m \to 0$





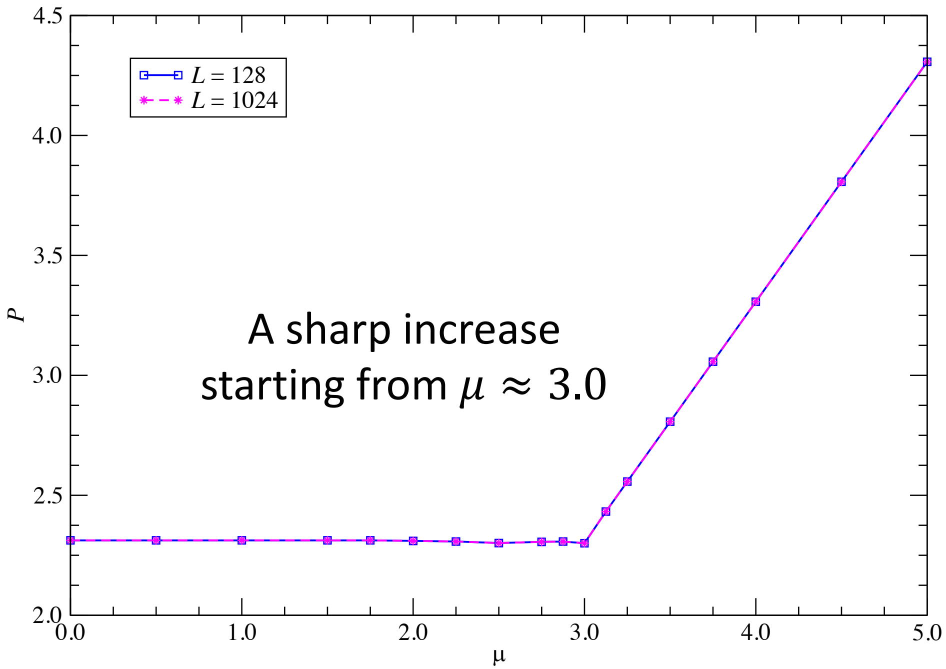
Chiral symmetry is restored in the region with $\mu \gtrsim 3.0$

A discontinuity at $\mu \approx 3.0$ indicates the 1st order transition

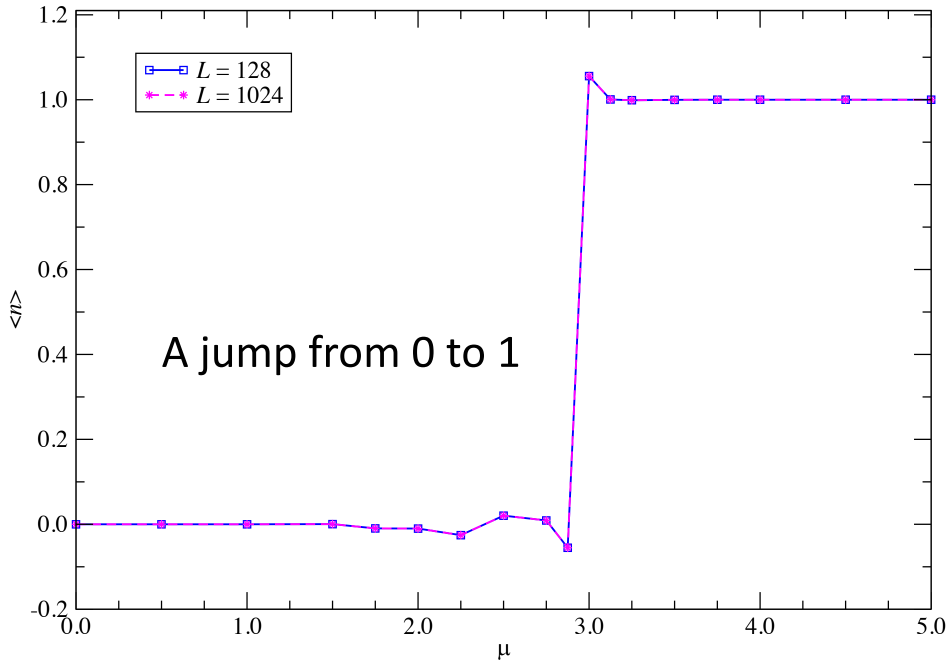
Ingredients of the equation of state

with $m = 0.01, g_0 = 32, D = 55$

Pressure
(∼ Thermodynamic potential)



Number density



Current numerical results clearly show that the chiral phase transition in cold & dense region is 1st order

Summary

- The TRG is a variant of RSRG, which is a deterministic numerical method
- 1st order chiral transition in the NJL model at finite density has been detected w/ the parallelized Grassmann ATRG
  (the first application of the TRG approach to 4d fermionic QFT)
- **This work shows that the TRG approach does not suffer from the sign problem and nicely works to evaluate the observables on almost thermodynamic lattice**
- TRG will be an efficient numerical approach to other types of 4d QFT