Singularities of thermal correlators at strong coupling

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Outline

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Motivation: the bulk point singularity

- We are interested in what kinds of singularities can arise in correlation functions in conformal field theory.
- In Euclidean signature there is just the OPE singularity,

$$\langle O(x_1)O(x_2)\cdots\rangle \sim \frac{1}{|x_1 - x_2|^{2\Delta}}.$$

- In Lorentzian signature there are singularities at noncoincident points. The obvious example is the light cone singularity. There are also Landau singularities, which occur when we can draw a Feynman diagram with lightlike lines in position space.
- Perturbatively this exhausts the list of singularities. But there could be more nonperturbatively.

- We can look for new singularities using holography.
- Lightning review: conformal theories in d dimensions are dual to string theories in $d+1$ dimensional anti de Sitter space,

$$ds^2 = -(r^2+1)\,dt^2 + (r^2+1)^{-1}\,dr^2 + r^2 d\Omega_{d-1}^2.$$

- Correlation functions in the boundary theory are obtained by extrapolating bulk correlation functions,

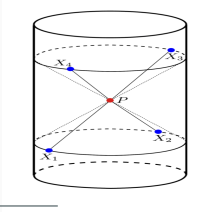
$$\langle O(x_1)O(x_2)\rangle = \lim_{r\to\infty} r^{2\Delta}\langle \Phi(r,x_1)\Phi(r,x_2)\rangle.$$

- This means that in order to study the singularities of correlation functions in a conformal field theory, it suffices to find the singularities of bulk correlators in the dual gravitational theory.

- In the holographic regime there are Landau diagrams in the bulk, which lead to singularities in boundary correlators (called bulk point singularities[1]). For example,

$$\left\langle \prod_{i=1}^{4} O(x_i) \right\rangle = \frac{1}{(z-\bar{z})^{4\Delta-3}} \text{ as } z \to \bar{z}.$$

- For these kinematics there is no Landau diagram on the boundary. So this seems like a genuinely new singularity.
- However, field theory in the bulk is only applicable if we take the strict limit $\lambda, N \to \infty$. If we instead keep λ finite and sum up the stringy corrections, we find that the singularity is smoothed out by the Gross-Mende expansion of the worldsheet[2].



[1]Gary, Giddings, Penedones '09
[2]Maldacena, Simmons-Duffin, Zhiboedov '15

- In general, the only singularities of correlators in holographic theories occur when there is a boundary Landau diagram[3]. We can try to extend this result to non-vacuum states, like thermal ensembles.
- At finite temperature conformal invariance is broken so the form of the two-point function is no longer determined. We are interested in the singularities of the function

$$\langle O(t,\phi)O(0,0)\rangle_\beta \,.$$

- At infinite volume in 1+1 dimensions, we can conformally map to the plane so the only singularity is at $t = \pm\phi$. In free field theory, one can directly compute this function and again there are no new singularities.
- In the rest of this talk we will discuss the holographic limit of this correlator. Are there any new singularities, and if so, are they resolved at finite λ?

[3] MD, Ooguri '19

The light cone of an AdS black hole

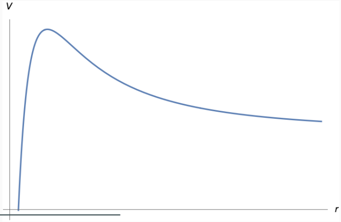
- Singularities of the boundary two point function occur when the boundary points are null separated[4]. So we should study null geodesics in the AdS$_{d+1}$ black hole,

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega_{d-1}^2, \qquad f(r) = r^2 + 1 - \frac{M}{r^{d-2}}.$$

- The conserved quantities are E, L. Null geodesics are parameterized by E/L. The radial motion is determined by

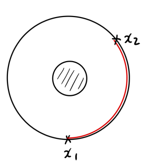
$$\frac{1}{2}\dot{r}^2 = \frac{1}{2}E^2 - V(r), \qquad V(r) = \frac{L^2}{2}\left(1 + \frac{1}{r^2} - \frac{M}{r^{d-2}}\right).$$

- The photon sphere is at the maximum of this potential. For any $d > 2$ there are geodesics with both endpoints at $r = \infty$. These give singularities in the two point function.

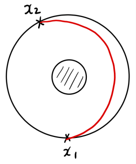


[4]Hubeny, Liu, Rangamani '06

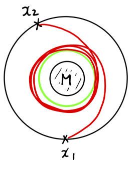
- Using the potential, we can understand the schematic behavior of the geodesics. First, there are geodesics that stay at the boundary:



- Second, there are geodesics that stay far away from the black hole. These have $\Delta\phi \sim \pi$, since they are approximately AdS geodesics.



- Finally there are geodesics that wrap the photon sphere many times,



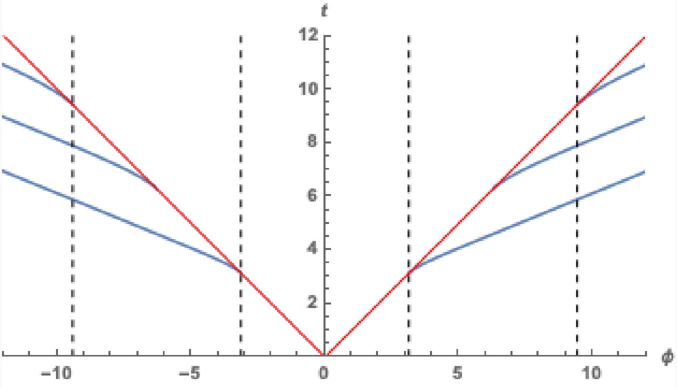
- We can compute the location of the singularities by integrating the geodesic equations,

$$\frac{d\phi}{dr} = \frac{1}{\sqrt{(E/L)^2 - 1}\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} = \frac{Ef(r)}{L}\frac{dt}{dr}.$$

  Here $r_\pm$ are the zeroes of $\dot{r}$.
- The integrals are elliptic. For instance

$$\Delta\phi = \frac{2r_-}{\sqrt{M}} K\left(\frac{r_-^2}{r_+^2}\right),$$

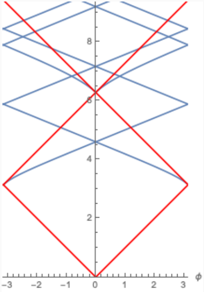
- After reaching the boundary, the geodesic can bounce back into the bulk, leading to infinitely more branches of the singularity.


t
12
10
8
6
4
2
ϕ
−10
−5
0
5
10

- To understand the singularities better, let us take some limits. In the early time limit, the turning point r_+ approaches infinity. Then we have $\Delta t \sim \Delta\phi \sim \pi$.
- In the late time limit, the geodesic wraps around the photon sphere many times. Therefore the singularity approaches a straight line, with velocity

$$v_{\text{photon}} = \sqrt{\frac{g_{tt}}{g_{\phi\phi}}}\Big|_{r_{\text{photon}}} = \sqrt{1 + \frac{1}{4M}}.$$

- Identifying $\phi \sim \phi + 2\pi$, we see that the singularities can intersect at caustic points.



- Now we will study the form of the correlator as we approach the singularity. When the two points are connected by a slightly spacelike geodesic, the geodesic approximation gives

$$\langle O(t,\phi)O(0,0)\rangle_\beta = e^{-ml_{\text{ren}}}$$

- Using the radial geodesic equation, we can compute the renormalized length,

$$l_{\text{ren}} = 2\int_{r_+}^{r_{\max}} \frac{dr}{\dot{r}} - \log r_{\max}^2 = -\log(E^2 - L^2).$$

- Finally, we can trade E, L for boundary variables by integrating the geodesic equations for slightly spacelike geodesics,

$$\Delta\phi = \Delta\phi_{\text{null}}(E/L) - \frac{2L}{E^2 - L^2}, \qquad \Delta t = \Delta t_{\text{null}}(E/L) - \frac{2E}{E^2 - L^2}.$$

- These equations are easily solved in various limits. For example at late times we have a simple power law behavior,

$$\langle O(t,\phi)O(0,0)\rangle_\beta \sim (v_{\text{photon}}\, t - \phi)^{-2m}$$

Review of string theory in the Penrose limit

- We have found new singularities in the thermal two point function which are absent in free field theory. Should these be trusted, or are they an artifact of infinite λ like the bulk point singularity?
- It is not immediately obvious how to address this question, since string theory in a black hole background is not yet solved. Luckily, we only need to study the worldsheet theory in the vicinity of a given null geodesic.
- We need some way to zoom in on a null geodesic. This is known as taking the Penrose limit. Given a metric $g_{\mu\nu}$ and a null geodesic γ, the Penrose metric is given by

  $$ds^2 = 2du\, dv + A_{ab}(u) x^a x^b\, du^2 + d\vec{x}^2, \qquad A_{ab}(u) = -\left(R_{iuju} E^i_a E^j_b\right)|_\gamma$$

  Here E is a pseudo-orthonormal frame for γ, with ∂_u tangent to γ.
- The plane-wave matrix A_{ab} captures the effects of tidal forces on extended objects near the null geodesic. In our case, these extended objects will be strings.

- There are several well-known cases of the Penrose limit. First, the Penrose limit of a maximally symmetric spacetime has $A_{ab} = 0$, implying that there is no tidal force near a null geodesic.
- Second, in the plane-wave limit of AdS/CFT, one considers a null geodesic in S^5 for the spacetime $\mathrm{AdS}_5 \times S^5$. In this case A_{ab} is constant.
- For point particles, the worldline Lagrangian becomes

$$L = \dot{u}\dot{v} + \frac{1}{2} A_{ab}(u) x^a x^b \dot{u}^2 + \frac{1}{2} \dot{x}^a \dot{x}^a$$

- This still looks tricky. To make the simplicity of the Penrose limit manifest, we fix light cone gauge, $u = p_v \tau$. Then the equations of motion become

$$\ddot{x}^a = p_v^2 A^a{}_b(p_v \tau) x^b.$$

- This is a collection of harmonic oscillators with time-dependent frequency. For vacuum solutions we have $\mathrm{Tr}\, A = 0$, so some of the eigenvalues of A are positive, leading to unstable directions.

- Now let us move on to strings[5]. Expanding into Fourier modes, there are now an infinite number of harmonic oscillators,

$$\ddot{X}_n^a = \left(p_v^2 A^a{}_b(p_v\tau) - n^2\delta^a{}_b\right) X_n^b.$$

- In general, several approximations might be applicable. First, if $p_v^2 A^a{}_b(p_v\tau)$ is localized in a small range of τ, then it can be approximated by a delta function. This will be the case at early times. The analysis is then similar to strings in a shockwave[6].
- Second, if the frequency is approximately constant, then we can apply the adiabatic approximation. This will be the case at late times. Note that if an eigenvalue of $p_v^2 A^a{}_b(p_v\tau)$ is large and positive, then there are many unstable modes.

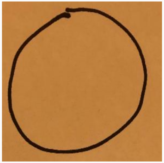
[5] Horowitz and Steif '90
[6] Giddings, Gross, Maharana '07

- For an AdS_5 black hole,

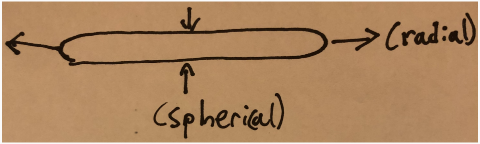
$$A_{11} = \frac{4L^2 M}{r^6} = -2A_{22} = -2A_{33}.$$

  Here A_{22} and A_{33} are along the S^3 directions, and A_{11} is along a combination of the (r, t, ϕ) directions.

- For instance, suppose we start with a circular string near the null geodesic,



- It will be tidally disrupted by the curvature of the black hole,



Singularity resolution at early times

- We are interested in the behavior of the boundary two point function near $(t, \phi) = (\pi, \pi)$. In this talk I will compute a slightly simpler quantity, the bulk-to-bulk propagator near the light cone.
- The propagator is Weyl invariant since it is the path integral with two D-instanton boundary states.[7] The worldsheet path integral is

$$\int_{X^a(\tau_i,\sigma)=x_i^a}^{X^a(\tau_f,\sigma)=x_f^a} dX^a \, e^{iS[X^a]} = \frac{G_0(p_v, x_f^a, x_i^a, \tau_f, \tau_i)}{\prod_{a=1}^{3} \prod_{n=1}^{\infty} \det\left(-\partial_\tau^2 - n^2 + p_v^2 A_{aa}(p_v \tau)\right)}$$

- The determinants can be calculated using the Gelfand-Yaglom theorem. Instead I will compute the magnitude of the determinants, which will give a bound on the propagator on the light cone.
- The magnitude of the determinants has a simple interpretation. We have time dependent harmonic oscillators, so there is particle production. Then we can represent

$$|\langle \text{out}, n, a | \text{in}, n, a \rangle| = \frac{1}{|\det\left(-\partial_\tau^2 - n^2 + p_v^2 A_{aa}(p_v \tau)\right)|}$$

[7] Cohen, Moore, Nelson, Polchinski 1986

- Let us now compute the particle production. At early times, we have

$$p_v^2 A_{aa}(p_v\tau) \sim \frac{M}{(\epsilon r_+)^2}\left(\frac{\epsilon^2}{\tau^2+\epsilon^2}\right)^3, \qquad \epsilon = \frac{r_+^2}{p_v L}.$$

  So for large p_v, we can use the shockwave approximation. The equations of motion for X_1 are

$$\lim_{\delta\to 0}\left(X'_{1n}(\delta) - X'_{1n}(-\delta)\right) = \frac{3\pi p_v L M}{2r_+^4} X_{1n}(0).$$

- Making the ansatz

$$X_{1n} = a_n^\dagger e^{in\tau} + a_n e^{-in\tau} \text{ for } \tau < 0,$$
$$X_{1n} = b_n^\dagger e^{in\tau} + b_n e^{-in\tau} \text{ for } \tau > 0,$$

  we find

$$b_n = \left(1 + \frac{3\pi i M p_v L}{4nr_+^4}\right) a_n + \frac{3\pi i M p_v L}{4nr_+^4} a_n^\dagger.$$

- It follows that

$$|\langle \text{out}, n, a=1|\text{in}, n, a=1\rangle| = \left(1 + \left(\frac{3\pi M p_v L}{4nr_+^4}\right)^2\right)^{-1/2} \ll 1 \text{ for small } n.$$

- Multiplying all the determinants gives

$$\prod_{n,a} |\langle \text{out}, n, a=1|\text{in}, n, a=1\rangle| \sim \frac{(p_v L)^{3/2}}{\sqrt{\sinh\left(\frac{3\pi M p_v L}{4r_+^4}\right)\sinh\left(\frac{3\pi M p_v L}{8r_+^4}\right)}}.$$

- Finally, we need the zero mode dependence on p_v. Since $G_0(x^2) \sim (x^2)^{-3/2}$ near the light cone, we have $G_0(p_v) \sim \sqrt{p_v}$. Said differently, the propagator is singular on the light cone because

$$\int_0^\infty dp_v \sqrt{p_v} = \infty.$$

- Once we include the stringy modes, we get

$$|G(x^2=0)| \leq \int_0^\infty dp_v \sqrt{p_v} \frac{(p_v L)^{3/2}}{\sqrt{\sinh\left(\frac{3\pi M p_v L}{4r_+^4}\right)\sinh\left(\frac{3\pi M p_v L}{8r_+^4}\right)}}$$

  The integrand is exponentially suppressed at large p_v, so the answer is finite and the light cone singularity is resolved.

- The interpretation is that at very large p_v, many stringy modes are produced, so the probability for a particle to remain localized on the light cone is small.

Singularity resolution at late times

- We now turn to the opposite limit, where the geodesic winds around the photon sphere many times. This geodesic is approximately circular for a long time, so we can apply the WKB approximation.
- From the unstable direction, we have a large number of growing modes at large p_v, with imaginary frequencies

$$\omega_n^2 = n^2 - \frac{4p_v^2 L^2 M}{r_{\text{photon}}^2} < 0.$$

This is negative for $n < n_{\max} \sim p_v$.

- The growing solution in the adiabatic approximation is

$$x_{1n}(u) \sim \exp\left(\int^u du' \, |\omega_n(u')|\right)$$

The number of produced particles near the photon sphere is then

$$\langle N_n \rangle = \exp\left(2\int_{u_{\text{in}}}^{u_{\text{out}}} du' \, |\omega_n(u')|\right).$$

- Plugging in ω_n, we find

$$\langle N_n \rangle \sim (r_+/r_{\text{photon}} - 1)^{-4\sqrt{1-n^2/n_{\text{max}}^2}}$$

  Since the geodesic passes very close to the photon sphere, this is large.

- The overlap between the in and out state is then

$$|\langle \text{out}|\text{in}\rangle| = \left(\prod_{n=1}^{n_{\text{max}}} \langle N_n \rangle\right)^{-1} = \exp\left(-\frac{\pi |p_v| L}{4\sqrt{2}M} \log(1/(r_+ - r_{\text{photon}}))\right)$$

  This is exponentially suppressed at large p_v.

- Finally we can bound the magnitude of the propagator on the light cone,

$$|G(x^2 = 0)| \leq \int_0^\infty dp_v \, G_0(p_v) \exp\left(-\frac{\pi |p_v| L}{4\sqrt{2}M} \log(1/(r_+ - r_{\text{photon}}))\right).$$

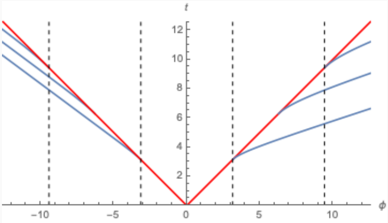
  The exponential suppression again smooths out the divergence.

More general black holes

- We can generalize the previous analysis to Kerr black holes. This corresponds to a boundary ensemble at finite temperature and nonzero rotation parameter.
- Let us consider equatorial geodesics. Then we have two photon spheres, one for prograde and one for retrograde orbits. For extremal rotation parameter, the prograde photon sphere is on the horizon.
- We take the four-dimensional case for simplicity. For equatorial geodesics, the Penrose plane wave matrix is

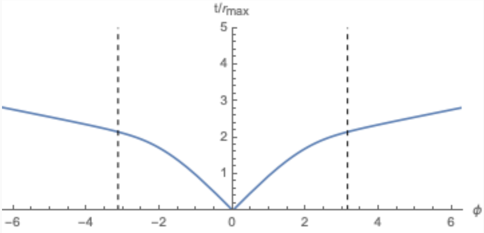
$$A_{11} = \frac{3(L - aE)^2 M}{r^5} = -A_{22}.$$

For nonequatorial geodesics the matrix is no longer diagonal, so there is mixing between the oscillators.



- So far we described the situation for AdS black holes, because we were primarily motivated by singularities in the boundary field theory. What happens in the asymptotically flat case, if we compute the propagator at some large $r_{\max}$?
- In fact everything is essentially the same, except that geodesics cannot bounce off the boundary, so the singularity has only one branch.
- The singularity is again resolved by string theory. This leads to a sharp criterion for stringy behavior near a black hole:

$$\alpha' \neq 0 \Rightarrow G(x^2 = 0) < \infty.$$



Summary and future directions

- Our goal was to understand the singularities of the thermal two point function. We first derived nontrivial singularities in the holographic setting, and then showed how they were resolved by strings. This suggests that the only singularity is on the boundary light cone.
- We only dealt with the one-sided case here. There is a similar singularity in the two-sided correlator, and the corresponding geodesic probes the black hole singularity[8]. We are currently trying to understand if this singularity is resolved in string theory.
- For extremal Kerr black holes, particles shot in from infinity collide at high energies near the horizon[9]. Does this have some visible signature in the bulk point singularity?
- The black hole image is obtained by convolving the Green's function with a source, and is dominated by light rays from the source to the observer[10]. If the light cone singularity is resolved by string theory, the black hole image could be modified. Is this consistent with data?

[8] Fidkowski, Hubeny, Kleban, Shenker '03
[9] Banados, Silk, West '09
[10] Hashimoto, Kinoshita, Murata '19

Questions?