Pseudo Entropy

:A Measure of Quantum Information from Quantum Gravity

Kotaro Tamaoka (YITP)

References:
2005.13801, 2011.09648 + ongoing work
w/ **Ali Mollabashi**, **Yoshifumi Nakata**, **Noburo Shiba**, **Tadashi Takayanagi**, **Yusuke Taki**, and **Zixia Wei**

YITP
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It from Qubit
Simons Collaboration on Quantum Fields, Gravity and Information

Plan & Summary

After introduction, I will introduce

Pseudo Entropy = Entanglement Entropy for "Transition Matrix"

$$\rho^{\psi} = |\psi\rangle\langle\psi| \longrightarrow \mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

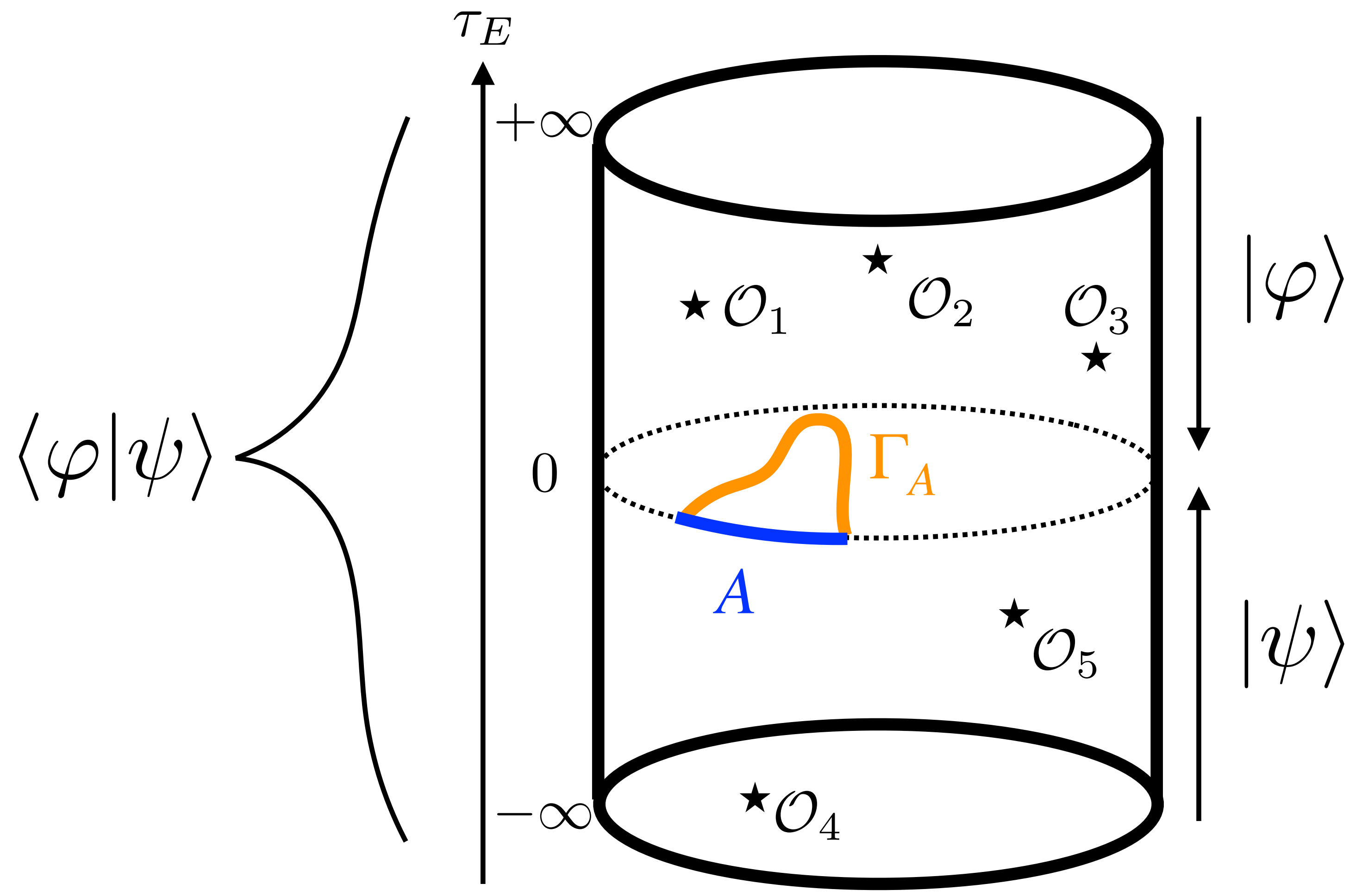
1. Interpretation

# of distillable EPR pairs
under LOCC+ **post-selection**

$|\psi\rangle \longrightarrow |\varphi\rangle$

Initial **Final**

2. Gravity dual



3. Order parameter

$|\varphi\rangle$ & $|\psi\rangle$

In the same quantum phase?

Take-home Message

String theory

(Quantum Gravity, QG)

⬇ "detect"

Brand-new measures for quantum information

→ Various application/implication to QG and other fields

This direction has just initiated quite recently.

"**Pseudo Entropy**" is just one of (interesting) examples!

Introduction

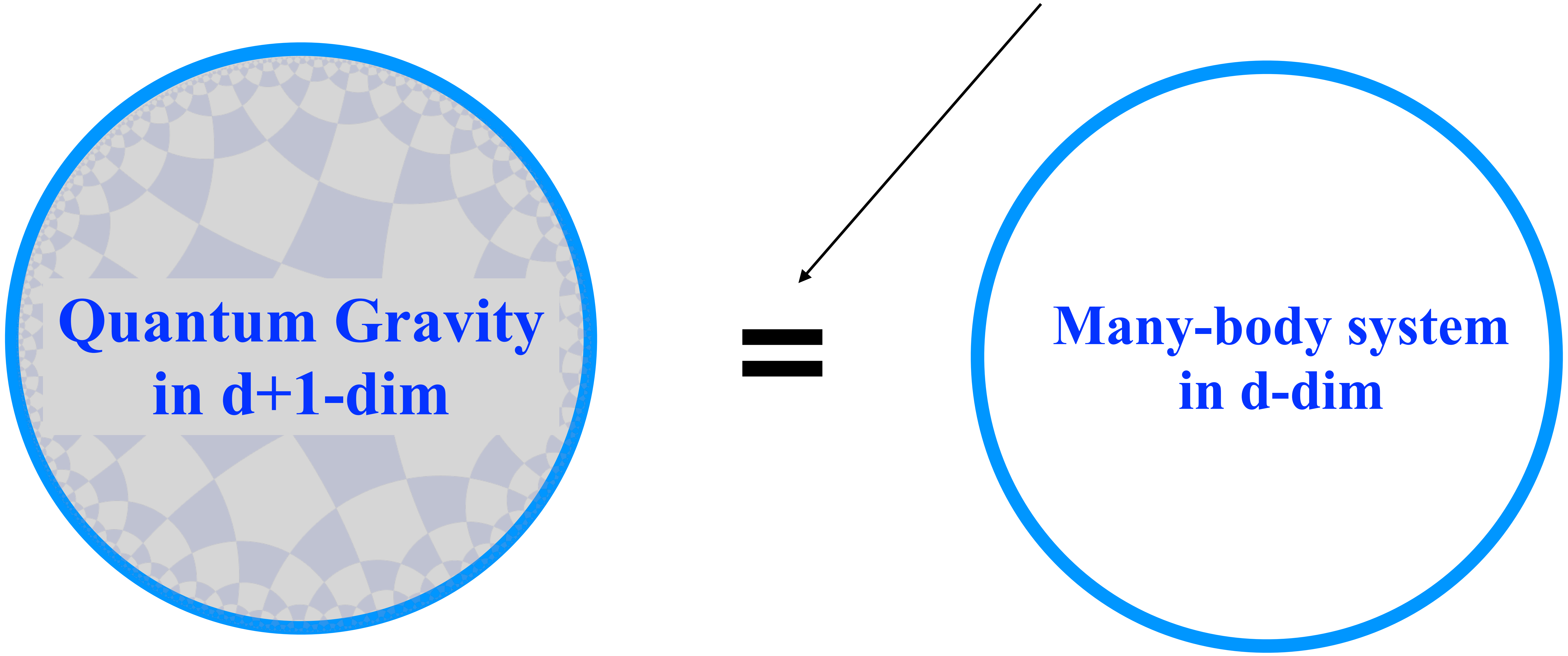
Backgrounds and Motivations

Entropy and Area in Quantum Gravity

Thermodynamical Entropy and Area [Bekenstein '72] and [Hawking '74]

$$S = \frac{A}{4G_N}$$

! Volume law in lower dimension (The idea of **holographic principle**)



A famous example: **AdS/CFT correspondence** [Maldacena '97]

Left: QG in Asymptotically AdS, Right: certain CFT

Entropy and Area in Quantum Gravity

Thermodynamical Entropy and Area [Bekenstein '72] and [Hawking '74]

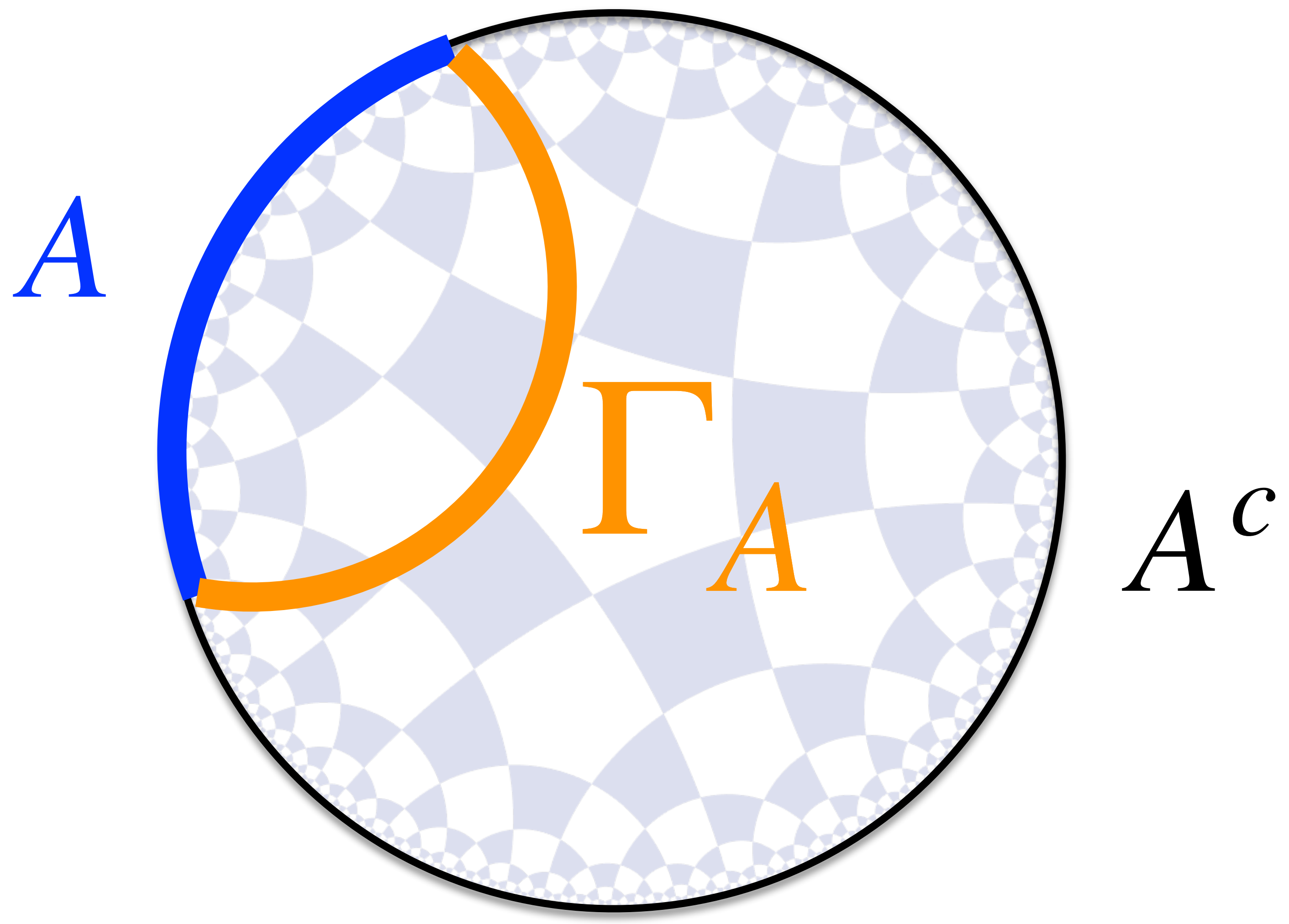
$$S = \frac{A}{4G_N}$$

! Volume law in lower dimension (The idea of **holographic principle**)

Entanglement Entropy and Area [Ryu-Takayanagi '06],…

$$S(\rho_A) \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}$$

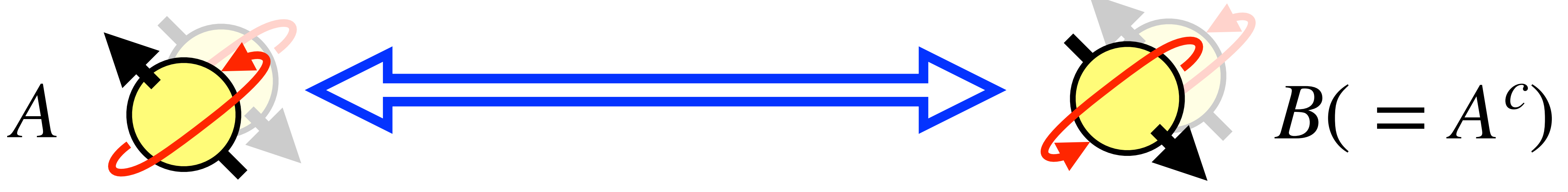
↑ Microscopic entropy



A "microscope" in quantum gravity! (next slide)

Entanglement Entropy ~ # of EPR pairs

e.g.) EPR pair $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle)$



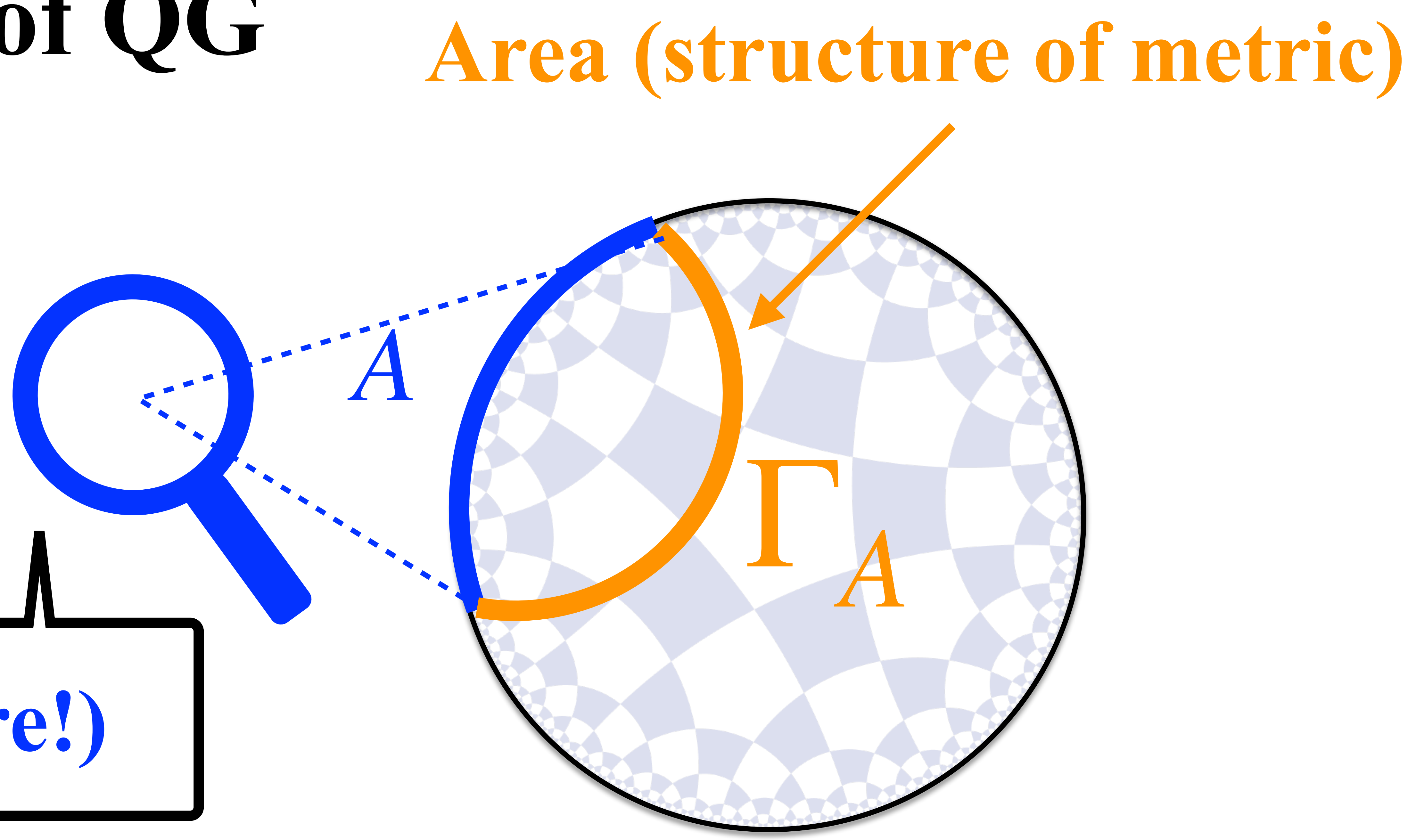
$$\boxed{S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A} = \log 2 \quad \left(\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \frac{I}{2}\right)$$

Take binary log → EE ~ # of EPR pair

Entanglement Entropy as microscope of QG

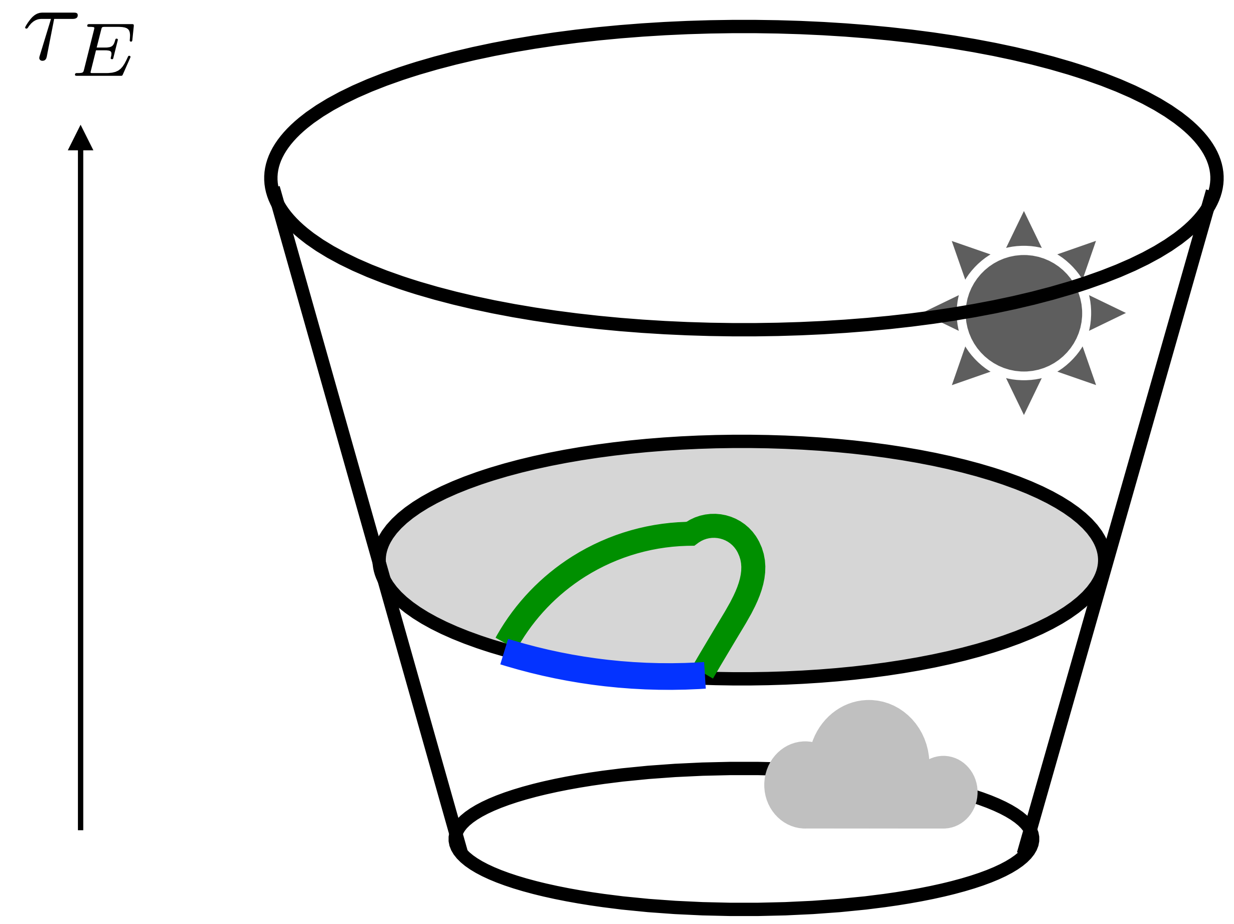
$$S(\rho_A) = \frac{\mathrm{Area}(\Gamma_A)}{4G_N}$$

# of EPR pairs! (microscopic structure!)

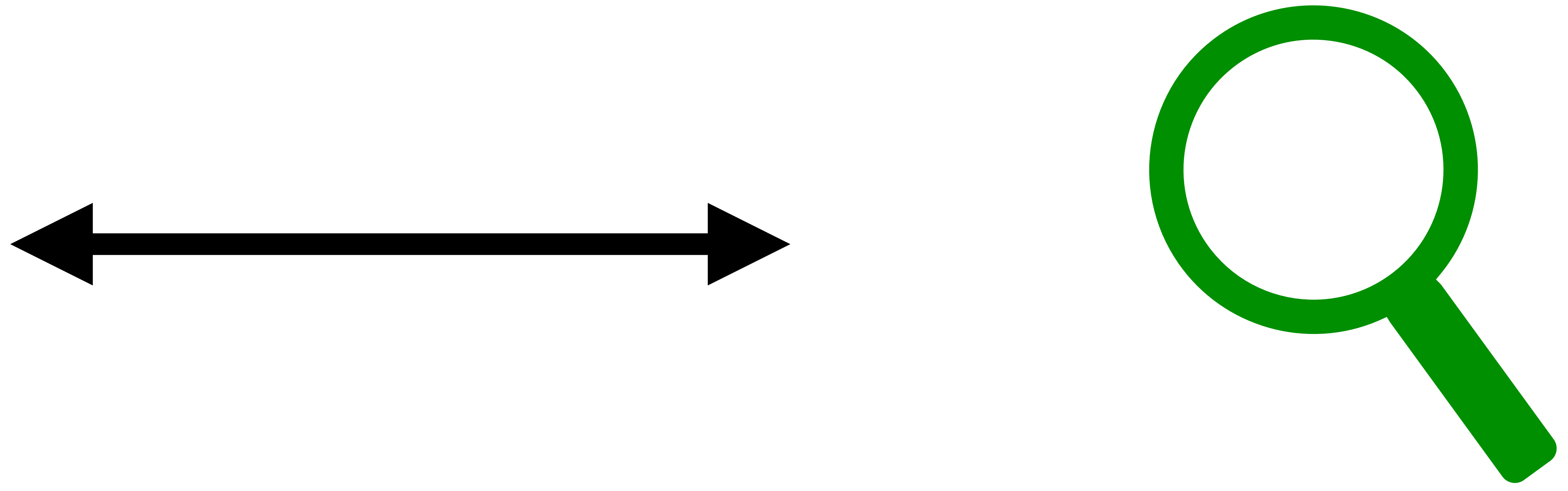


This talk: boundary dual of area in Euclidean space?

New class of surfaces



(Perhaps new) QI quantity?



Left : Minimal surfaces in Euclidean space

Right: **Pseudo entropy**

Sometimes string theory tells us brand-new QI quantities:

e.g.) Odd entropy [KT '18], Reflected entropy [Dutta-Faulkner '19],…

Many applications already! See also review in 日本物理学会誌 (2020年6月号) [KT-Umemoto'20]

Plan

Pseudo Entropy

= Entanglement Entropy for "Transition Matrix"

$$\rho^{\psi} = |\psi\rangle\langle\psi| \longrightarrow \mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

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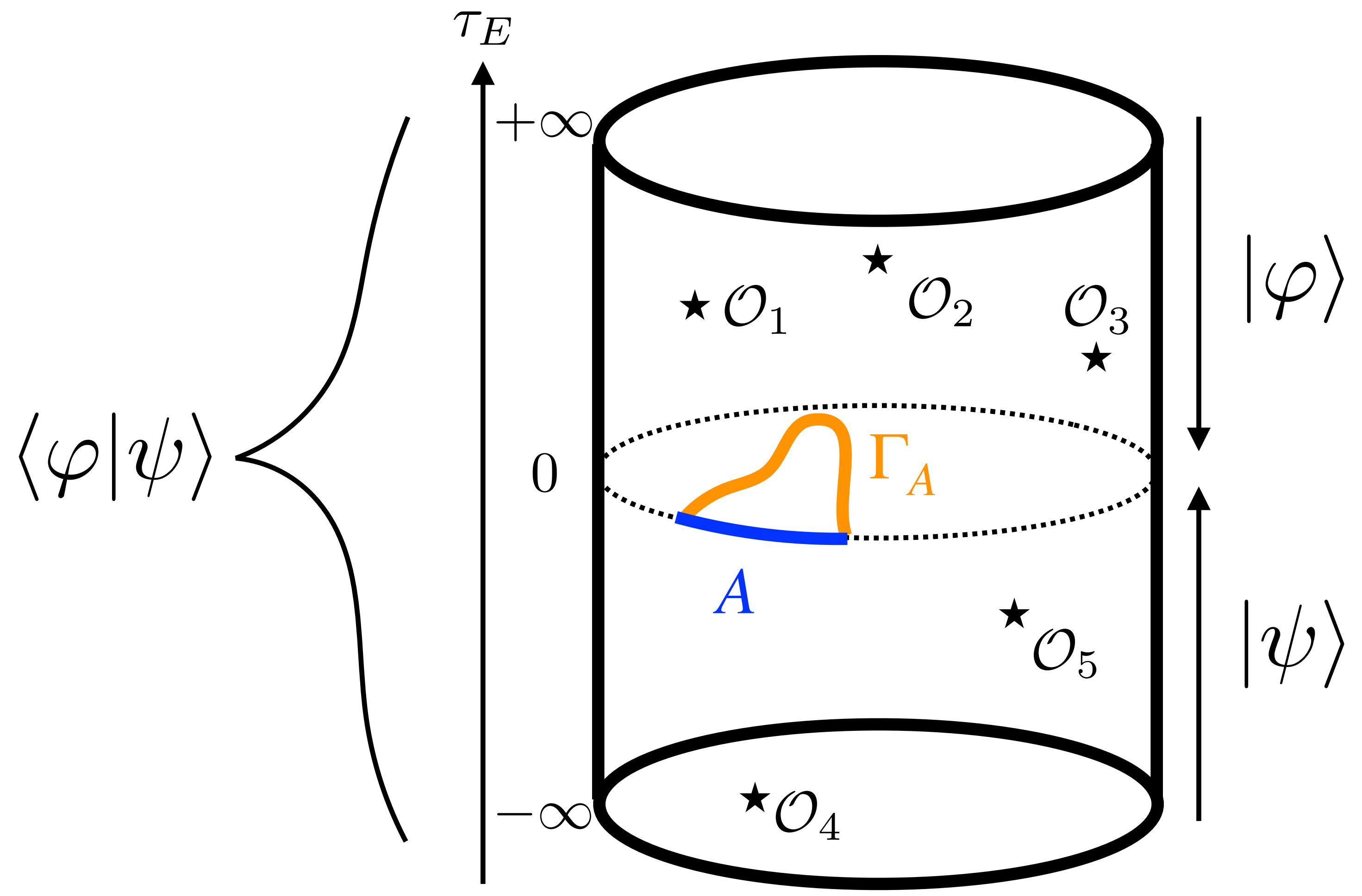


3. Order parameter

$|\varphi\rangle$ & $|\psi\rangle$

Same quantum phase?

Density Matrix (pure state)

$$\rho^{\psi} = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

Expectation value

$$\langle A\rangle_{\psi} = \mathrm{Tr}[A\cdot\rho^{\psi}] = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$$

Transition Matrix

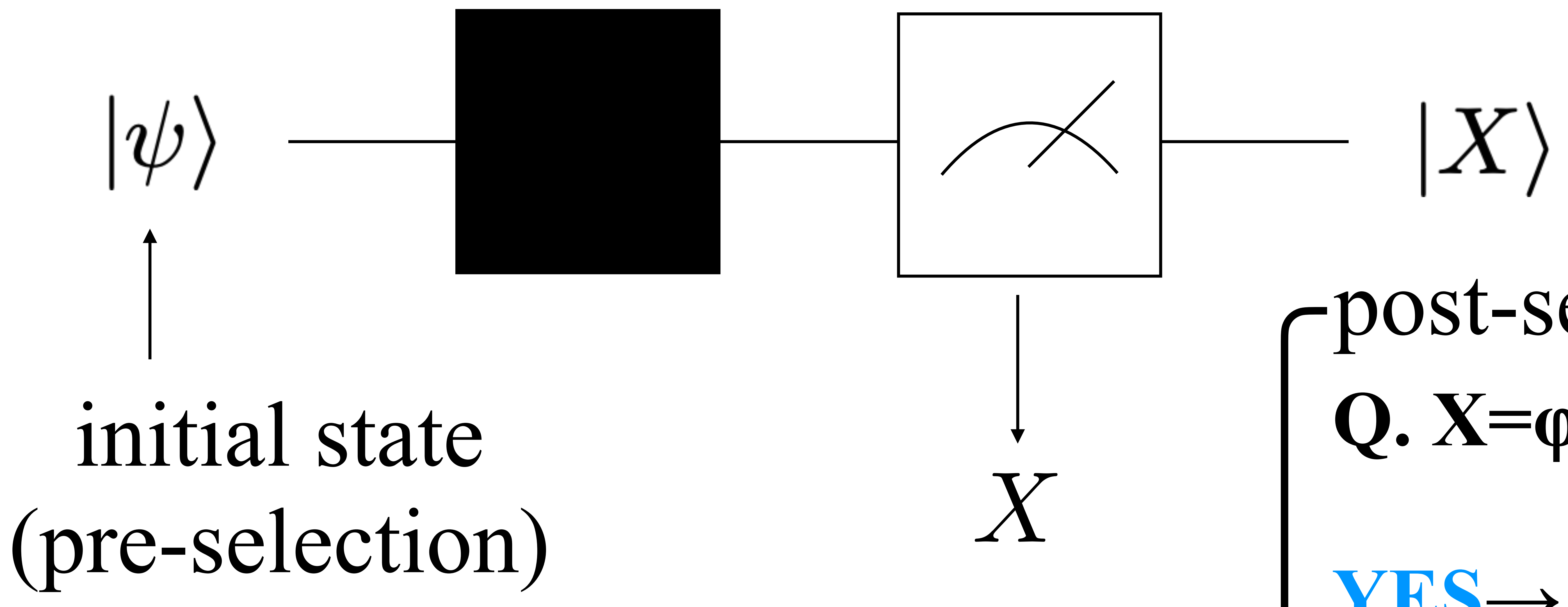
$$\mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

Weak value (next slide)

$$\frac{\langle\varphi|A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[A\cdot\mathcal{T}^{\psi|\varphi}]$$

[Aharanov-Albert-Vaidman ’88]

Post-selection & Weak value

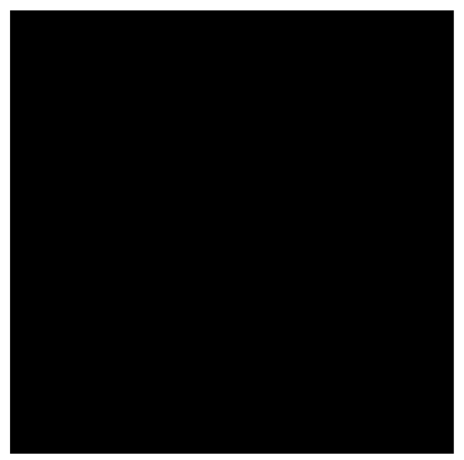


post-selection

Q. X=φ?

YES→ Storage **NO→ Abandon!**

Weak value: complex value in general [Aharanov-Albert-Vaidman '88]



$$= \boxed{A} \longrightarrow \frac{\langle\varphi|A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[A\cdot\mathcal{T}^{\psi|\varphi}]$$

Pseudo (Entanglement) Entropy

[Nakata-Takayanagi-Taki-KT-Wei '20]

$$S(\mathcal{T}_A^{\psi|\varphi}) = -\mathrm{Tr}\left[\mathcal{T}_A^{\psi|\varphi} \log \mathcal{T}_A^{\psi|\varphi}\right]$$

where $\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_{A^c}\mathcal{T}^{\psi|\varphi}$

Precise definition: defined via eigenvalues (Jordan normal form),

"Renyi entropy" : $S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) = \dfrac{1}{1-n}\log \mathrm{Tr}[(\mathcal{T}_A^{\psi|\varphi})^n]$

Pseudo Entropy can be defined as n→1 limit

PE inherits desired properties of entanglement

- $S(\mathcal{T}_A^{\psi|\varphi}) = S(\mathcal{T}_{A^c}^{\psi|\varphi})$ (A^c : complement of A)

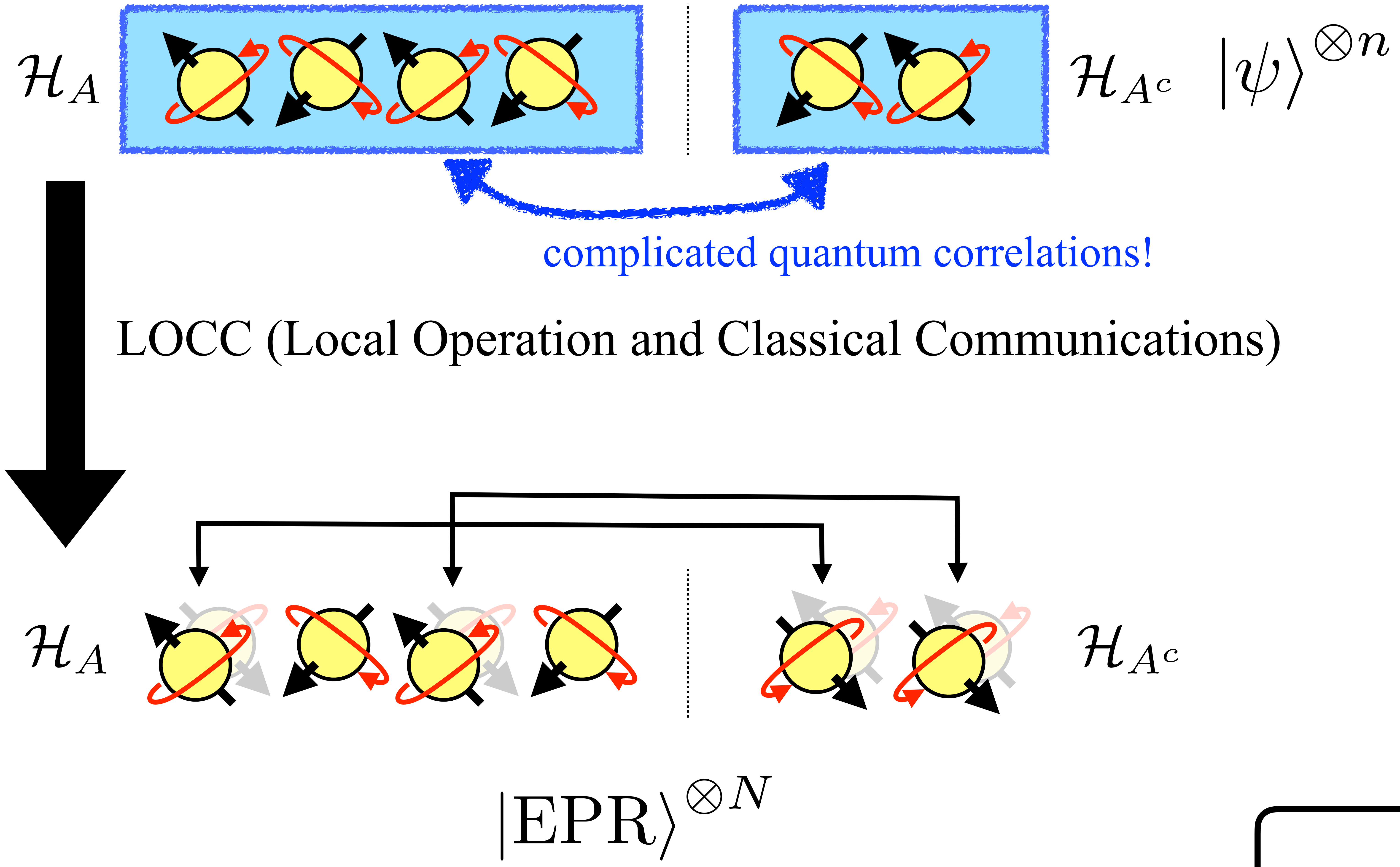
- $S(\mathcal{T}_A^{\psi|\varphi}) = 0$ ($|\varphi\rangle$ and/or $|\psi\rangle$: product state)

- $S(\mathcal{T}_A^{\psi|\varphi}) = (\#\text{ of distillable EPR pairs})$

  under LOCC + post-selection process

We will discuss the 3rd property in detail!

Entanglement Entropy = # of Distillable EPR pairs under LOCC

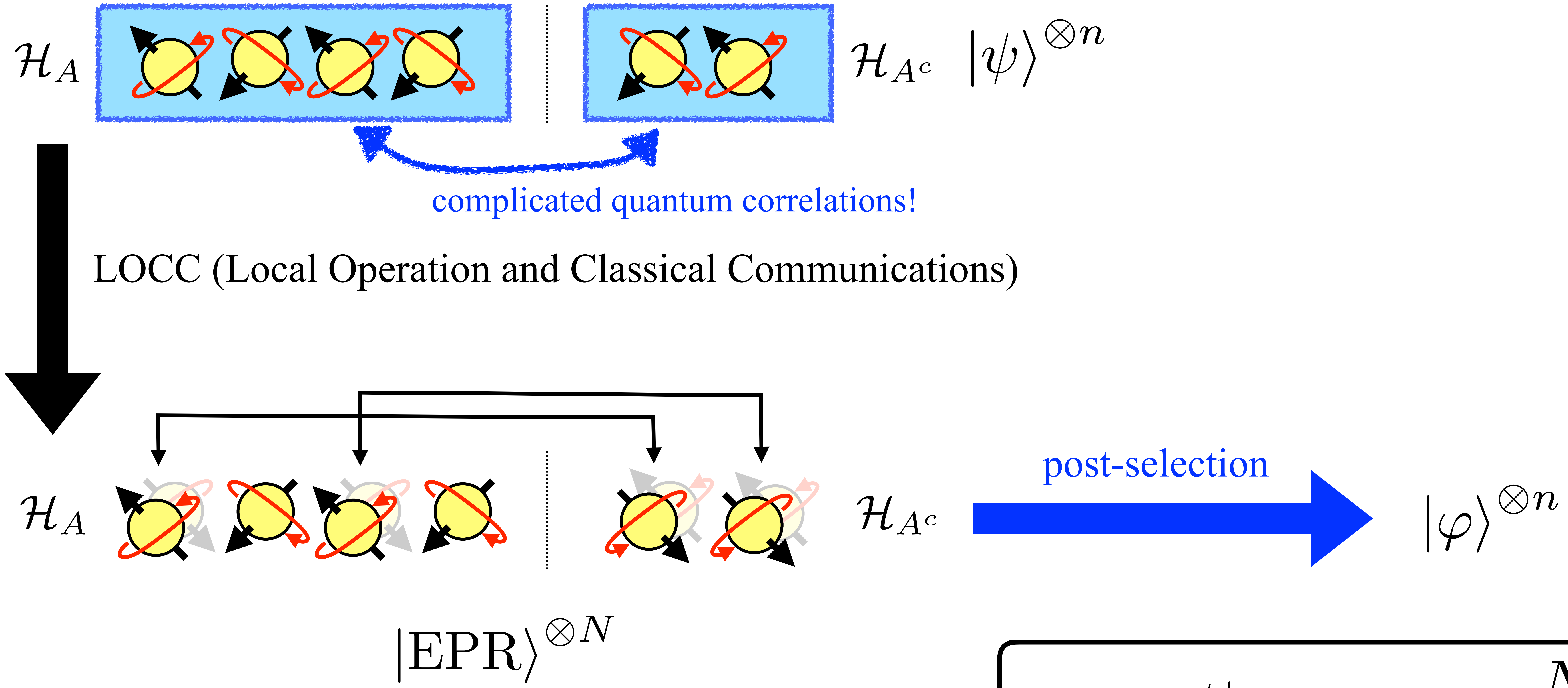


Bunch of maximally entangled states (EPR pairs)

$$S(\rho_A) = \lim_{n\to\infty} \frac{N}{n}$$

Pseudo Entropy = # of Distillable EPR pairs under LOCC + post-selection

(⚠ Proven only in specific examples. General argument is still an open-problem!)



Bunch of maximally entangled states (EPR pairs)

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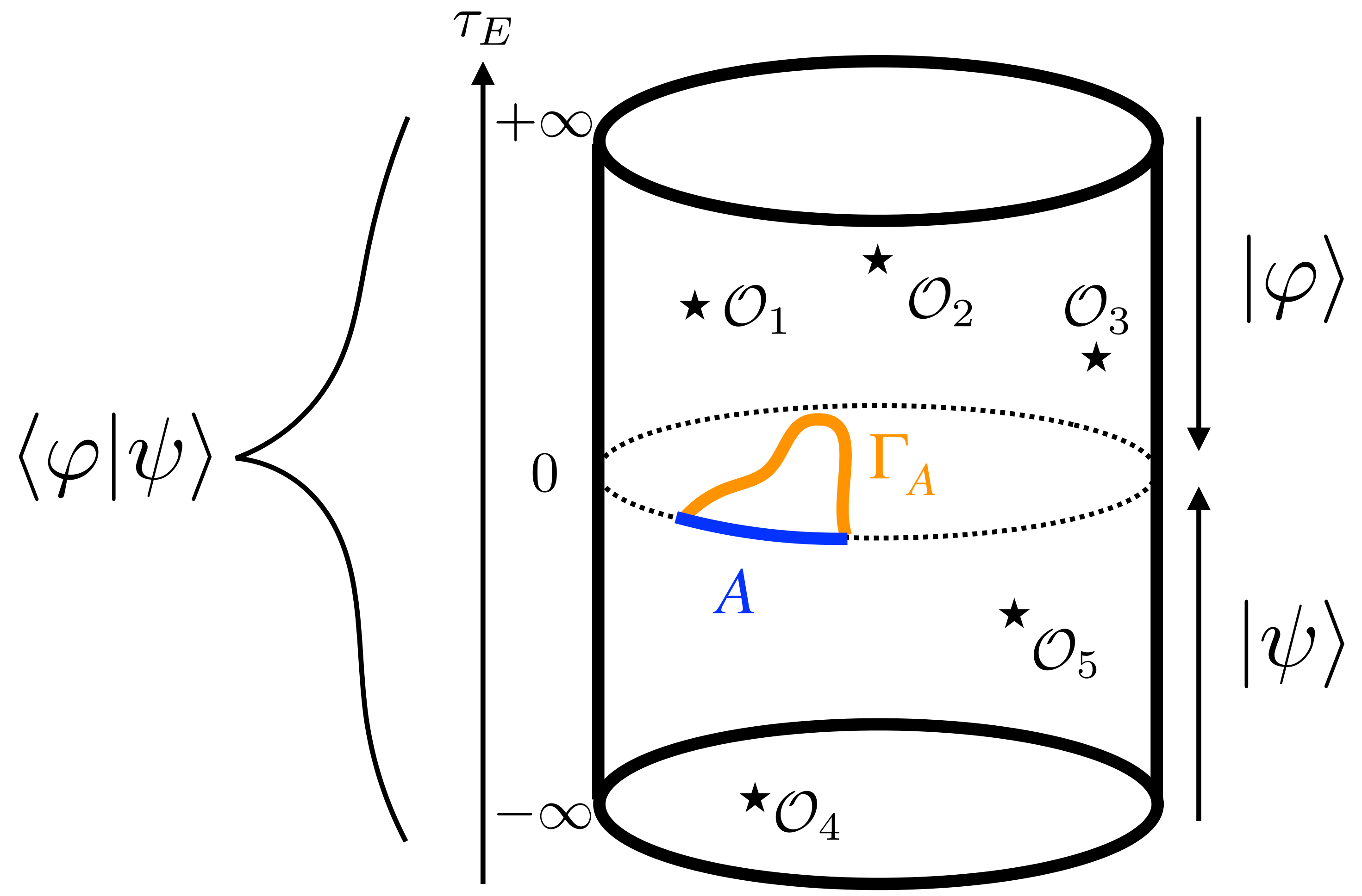


3. Order parameter

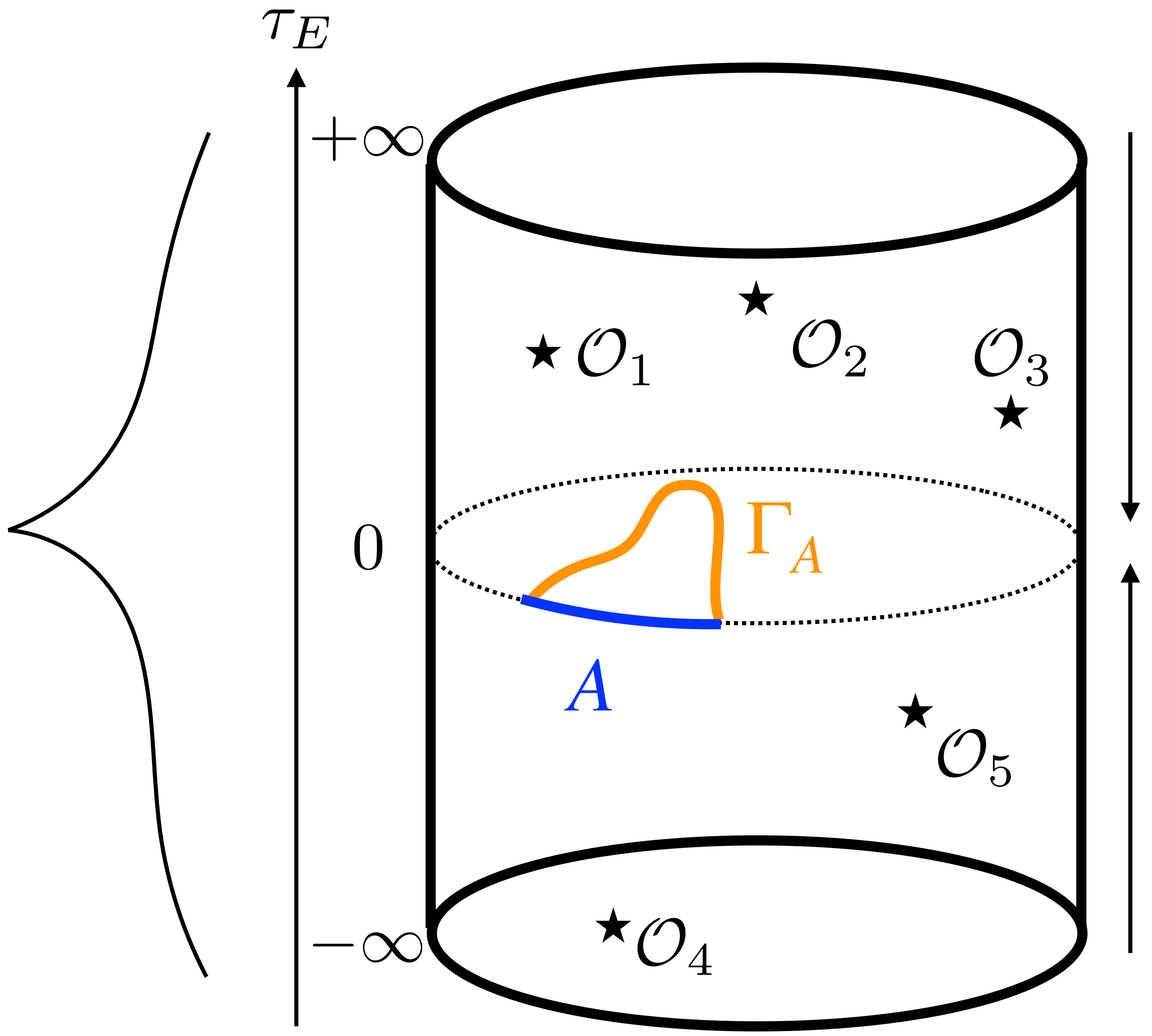
$|\varphi\rangle$ & $|\psi\rangle$

Same quantum phase?

Holographic Pseudo Entropy (HPE)

$$S(\mathcal{T}_A^{\psi|\varphi}) = \underset{\substack{\partial\Gamma_A = \partial A \\ \Gamma_A \sim A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N} \right]$$

$$Z = \mathrm{Tr}(\mathcal{T}^{\psi|\varphi}) = \langle\varphi|\psi\rangle$$



$$\langle \mathrm{final}| = \langle\varphi|$$

$$|\mathrm{initial}\rangle = |\psi\rangle$$

! Can prove by reusing [Lewkowycz-Maldacena'13] argument

(Just use GKP-Witten relation to the replica manifold)

HPE as Weak Value of Area Operator

$$S(\mathcal{T}_A^{\psi|\varphi}) = \frac{\langle\varphi|\frac{\hat{A}}{4G_N}|\psi\rangle}{\langle\varphi|\psi\rangle}$$

Can confirm **linearity of PE** in holographic CFT_2 :

$$|\psi\rangle = \sum_i c_i \, \underline{|\mathcal{O}_{H_i}\rangle}$$

$$|\varphi\rangle = \sum_j b_j \, \underline{|\mathcal{O}_{H_j}\rangle}$$

Heavy states

$$\Rightarrow \quad \frac{\sum_i b_i^* c_i \, \frac{\mathrm{Area}(\Gamma_A^{H_i})}{4G_N}}{\sum_i b_i^* c_i}$$

→ **complex-valued**

A single holographic state is rather special!

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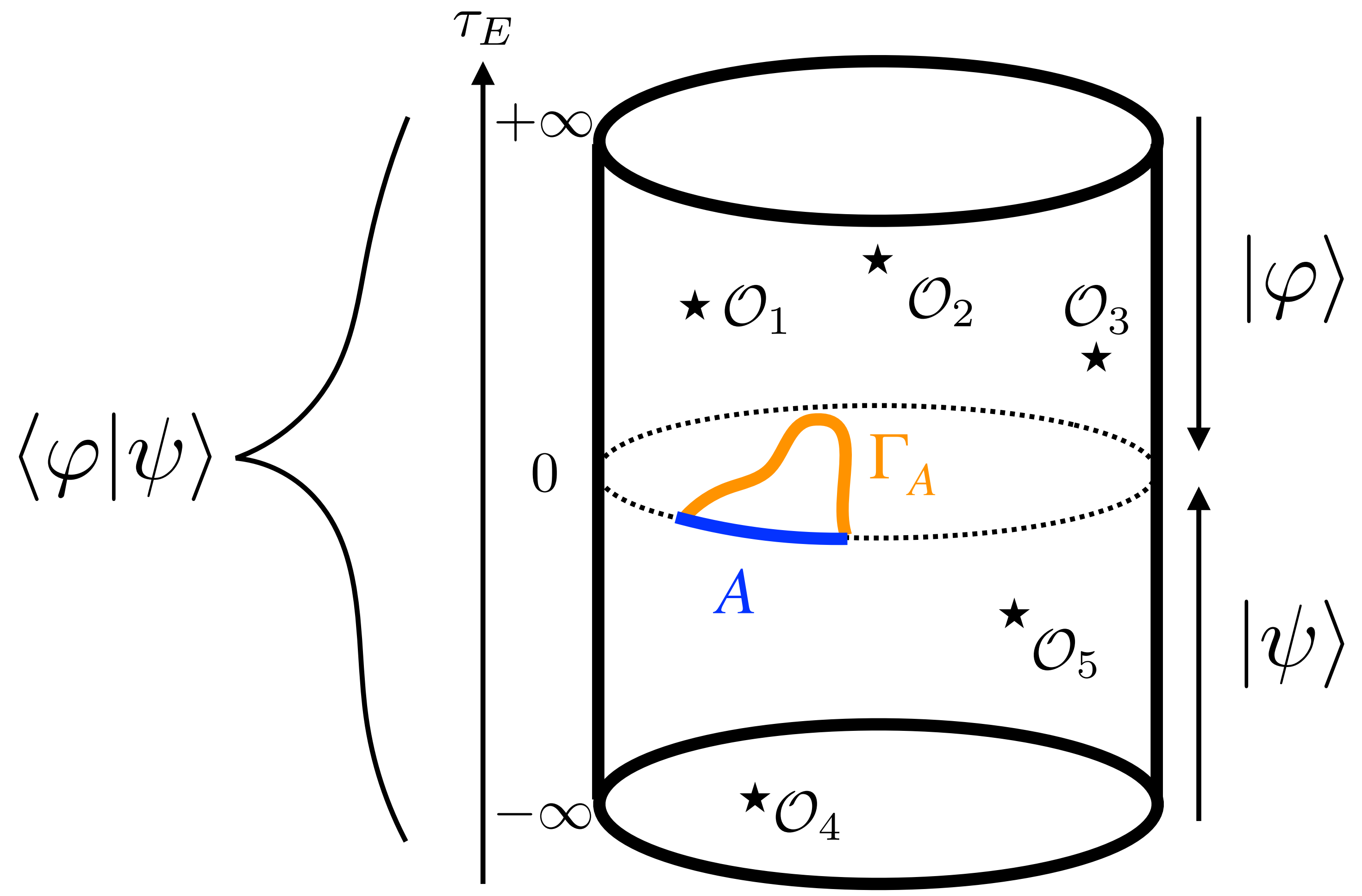

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Same quantum phase?

More general QFTs?

[Mollabashi-Shiba-Takayanagi-KT-Wei '20]

e.g.) Vacua $|0^{(1,2)}\rangle$ for 2d massive scalar fields with mass $m_{1,2}$

$$H_{1,2} = \frac{1}{2}\int dx[\pi^2 + (\partial_x\phi)^2 + m_{1,2}^2\phi^2]$$

$$\mathcal{T}_A^{1|2} = \mathrm{Tr}_{A^c}\left[\frac{|0^{(1)}\rangle\langle 0^{(2)}|}{\langle 0^{(1)}|0^{(2)}\rangle}\right]$$

$$S(\mathcal{T}_A^{1|2}) = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right) + f(m_1, m_2, \ell, L)$$

$$f(m_1, m_2, \ell, L) \simeq \frac{1}{2}\log\left[-\frac{m_1^2\log(m_1\ell) - m_2^2\log(m_2\ell)}{m_1^2 - m_2^2}\right] - \frac{1}{2}\log\left[\frac{m_1+m_2}{2}L\right]$$

★ $m_1 \to m_2$ reproduces the known result for EE (a massive free scalar)

Constant-term: [Casini-Heurta '04], subsystem size-correction: [Casini-Heurta'05]

PE as an order parameter

[Mollabashi-Shiba-Takayanagi-KT-Wei '20]

Free QFTs, holographic CFT + deformations

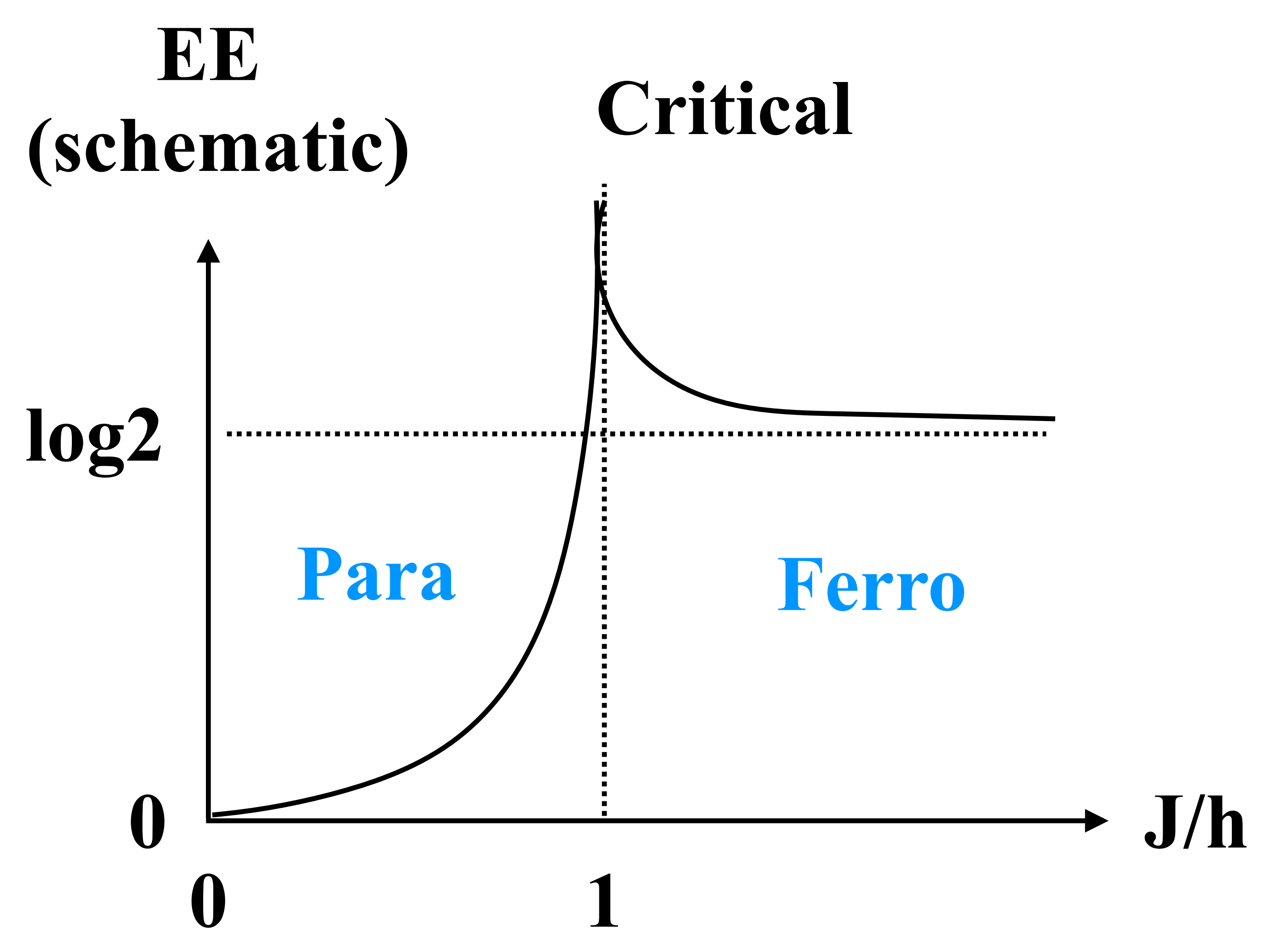
$$\Delta S_{12} \equiv S(\mathcal{T}_A^{1|2}) - S(\rho_A^{(1)})/2 - S(\rho_A^{(2)})/2 \leq 0$$

! Given QFT vacua (Area law) + small deformations
seem to satisfy this inequality

Transverse Ising model

$$H = -J_{1,2}\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z - h_{1,2}\sum_{i=0}^{N-1}\sigma_i^x$$

If two states are in the **different phases**
$\rightarrow$ ΔS_{12} becomes **positive**



(Note: ΔS_{12} is finite)

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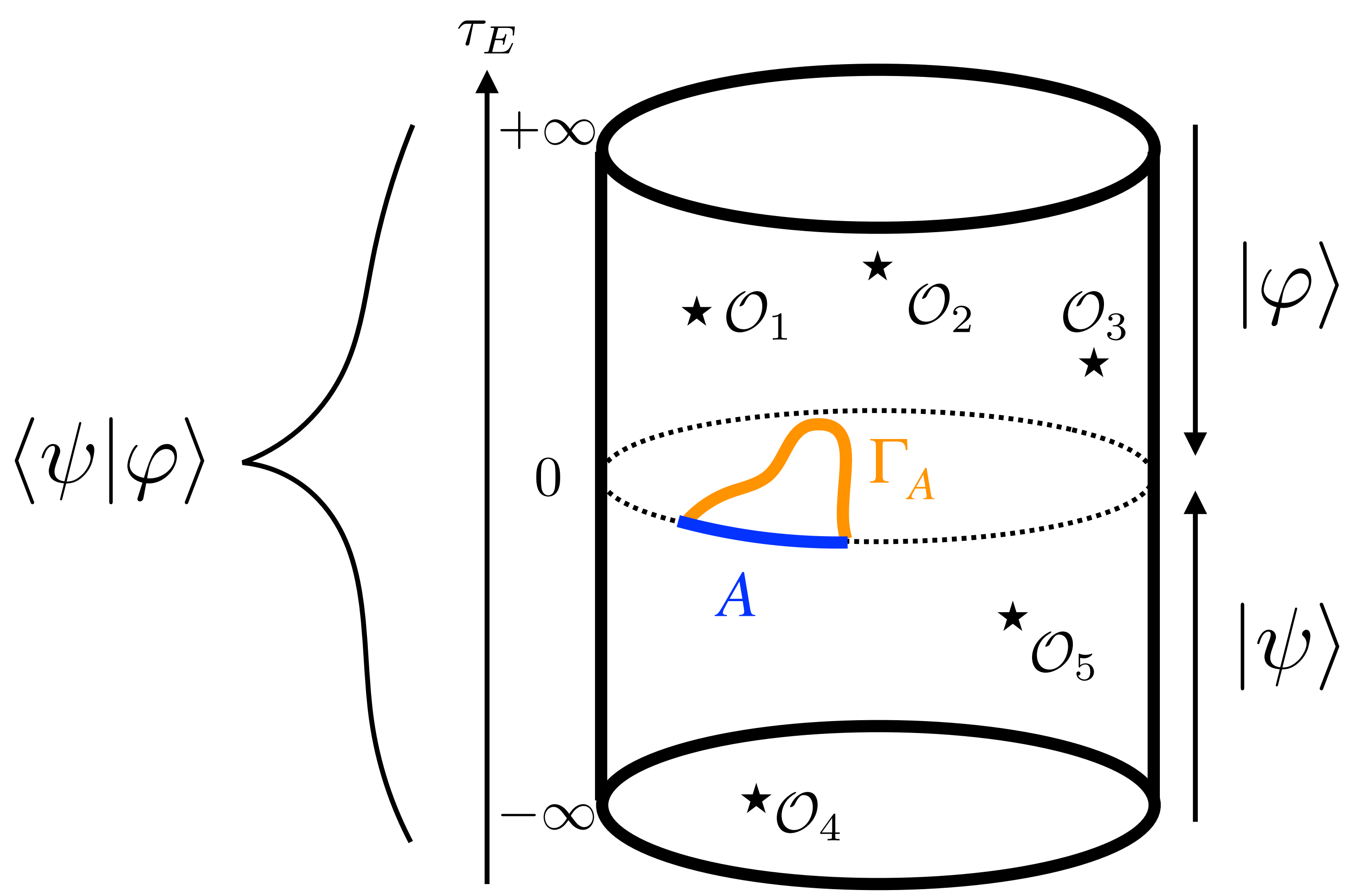

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$|\varphi\rangle$ & $|\psi\rangle$

Same quantum phase?

What I haven’t discussed so far

- Classification of transition matrix and examples
- How holographic states are special
- Mixed state-generalizations of PE
- Plateau effect of PE (Area law v.s. Volume law)

- Detail of calculations/derivations etc.

Discussion

- More rigorous argument on the negativity of ΔS_{12} in QFTs?

  (When ΔS_{12} becomes positive-valued?)

  Mollabashi-Shiba-Takayanagi-KT-Wei in progress

- Real-time evolution? Goto-Nozaki-KT in progress

- Operational interpretation in general setup?

- Further study on mixed state generalizations (Not discussed today)

Need to pursue various things !

(Case studies would be also useful)

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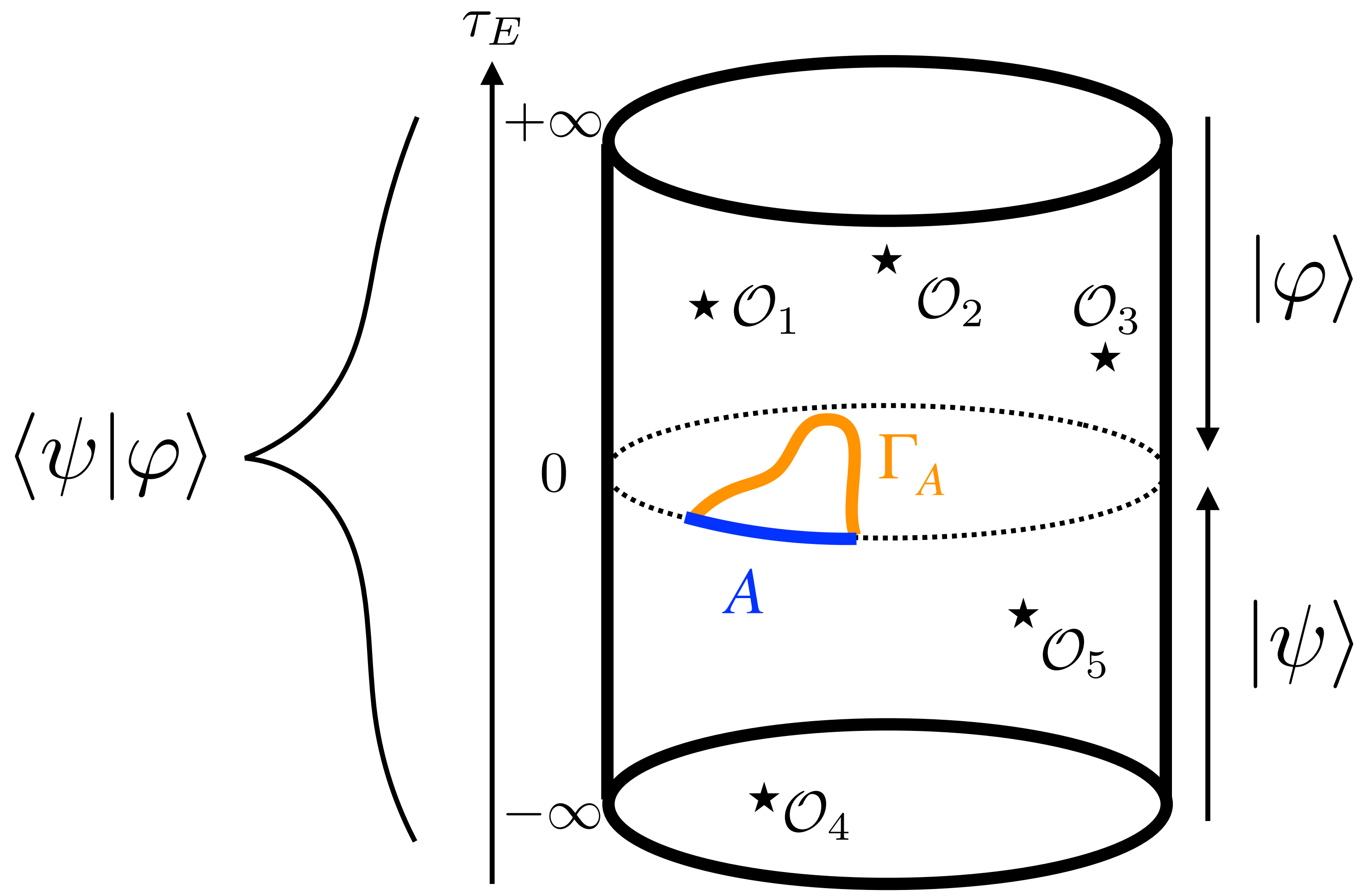

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Same quantum phase?

Backup slides

Classifications and examples

Some special classes of T_A

A	$\mathrm{Tr}[(\mathcal{T}_A^{\psi\|\varphi})^n]^* = \mathrm{Tr}[(\mathcal{T}_A^{\psi\|\varphi})^n]$
B	$S^{(n)}(\mathcal{T}_A^{\psi\|\varphi}) \geq 0, \quad (n > 0)$
C	$\mathcal{T}_A^{\psi\|\varphi} \geq 0$
D	$\mathcal{T}_A^{\psi\|\varphi} \geq 0$ and $(\mathcal{T}_A^{\psi\|\varphi})^\dagger = \mathcal{T}_A^{\psi\|\varphi}$
E	The same is also true for $\mathcal{T}_B^{\psi\|\varphi}$

Stronger condition

$S^{(n)}(\mathcal{T}_A^{\psi|\varphi})$: n-th Rényi PE

Particularly interesting classes:

A	$\mathrm{Tr}[(\mathcal{T}_A^{\psi\vert\varphi})^n]^*$
B	$S^{(n)}(\mathcal{T}_A^{\psi\vert\varphi}) \geq 0, \quad (n>0)$
C	
D	$\mathcal{T}_A^{\psi\vert\varphi} \geq 0$ and $(\mathcal{T}_A^{\psi\vert\varphi})^\dagger = \mathcal{T}_A^{\psi\vert\varphi}$
E	The same is also true for $\mathcal{T}_B^{\psi\vert\varphi}$

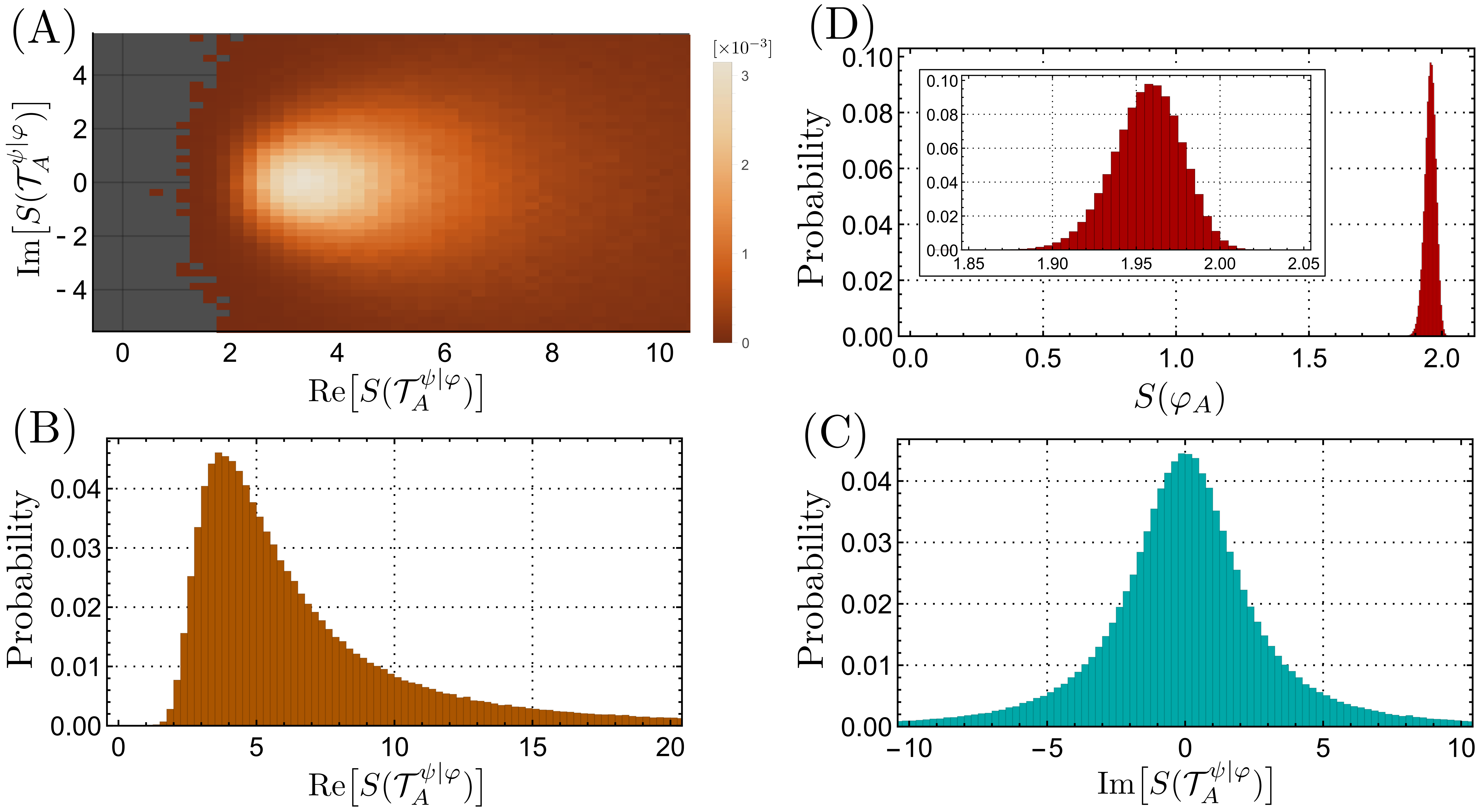
“Holographic states”

Nice Operational Interpretation

Stronger condition

$S^{(n)}(\mathcal{T}_A^{\psi|\varphi})$: n-th Rényi PE

PE for (Haar) Random States (out of class A)



$\dim \mathcal{H}_A = 8$

$\dim \mathcal{H}_{A^c} = 32$

# of sampling : 655360

Observations from random states

・∃ **Re[S(T_A)]<0**, but quite rare!

・Average of **Im[S(T_A)]=0**!

・Average of **Re[S(T_A)] > Max[S_{EE}]=log(dimH_A)**

PE behaves “very-well” for random states!

An example of class-A

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + \mathrm{e}^{i\theta}\,|11\rangle)$$

$$\Rightarrow \quad \mathrm{Spec}(\mathcal{T}_A^{\psi|\varphi}) = \left\{ \frac{\mathrm{e}^{i\theta/2}}{2\cos(\theta/2)}, \frac{\mathrm{e}^{-i\theta/2}}{2\cos(\theta/2)} \right\}$$

$$\Rightarrow \quad \mathrm{Tr}(\mathcal{T}_A^{\psi|\varphi})^n = \frac{\cos(n\theta/2)}{2^{n-1}\cos^n(\theta/2)}$$

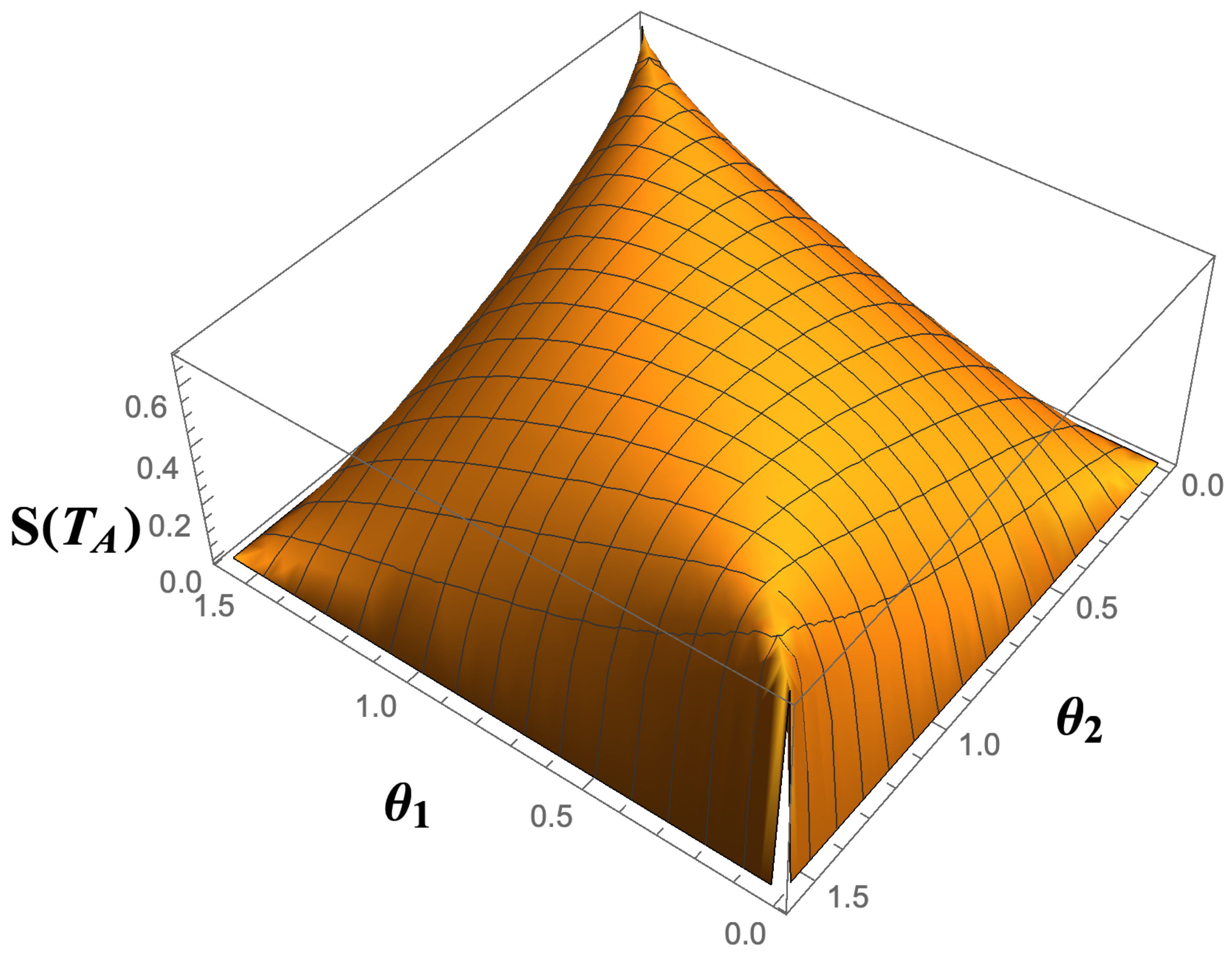
Always real-valued, but can be negative

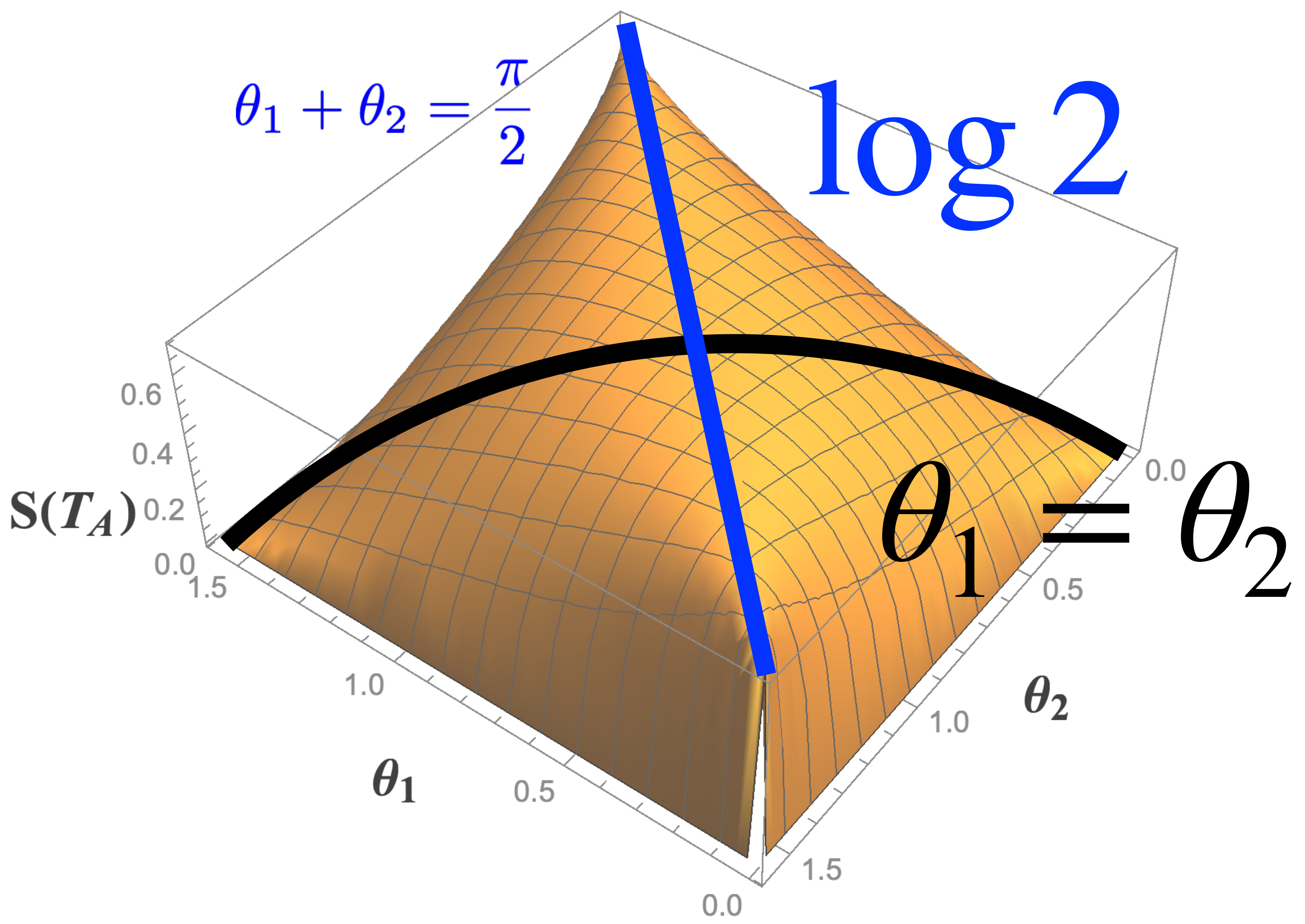
An example of class-E

$$\mathcal{T}^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle} \quad \text{with} \quad \begin{cases} |\psi\rangle = \cos\theta_1 \, |0_A 0_B\rangle + \sin\theta_1 \, |1_A 1_B\rangle \, , \\ |\varphi\rangle = \cos\theta_2 \, |0_A 0_B\rangle + \sin\theta_2 \, |1_A 1_B\rangle \, . \end{cases}$$

$$\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_B \mathcal{T}^{\psi|\varphi} \quad (B = A^c) \qquad \left(0 \leq \theta_1, \theta_2 \leq \frac{\pi}{2}\right)$$

$$\mathcal{T}_A^{\psi|\varphi} = \frac{\cos\theta_1 \cos\theta_2}{\cos(\theta_1 - \theta_2)} \, |0_A\rangle\langle 0_A| + \frac{\sin\theta_1 \sin\theta_2}{\cos(\theta_1 - \theta_2)} \, |1_A\rangle\langle 1_A|$$





Pseudo Entropy as distillable EPR pairs

A first step : prepare n-copy of initial states

$$|\psi\rangle^{\otimes n} = (\cos\theta_1 |00\rangle + \sin\theta_1 |11\rangle)^{\otimes n}$$
$$= \sum_{k=0}^{n} (\cos\theta_1)^{n-k} (\sin\theta_1)^k \sum_{a=1}^{{}_nC_k} |P(k), a_A\rangle |P(k), a_B\rangle$$

maximally entangled states with $\log {}_nC_k$ entropy

= # of distillable EPR pairs

$${}_nC_k = \frac{n!}{(n-k)!k!}$$

k : # of $|1\rangle$

a : label for position of $|1\rangle$

$|P(0), 1\rangle = |00\cdots 0\rangle$

$|P(1), 1\rangle = |10\cdots 0\rangle, |P(1), 2\rangle = |01\cdots 0\rangle, \cdots$

Pseudo Entropy as distillable EPR pairs

Perform a projection measurement on A: $\Pi_k = \sum_{a=1}^{{}_nC_k} |P(k), a\rangle\langle P(k), a|$

$$|\psi\rangle^{\otimes n} = (\cos\theta_1 |00\rangle + \sin\theta_1 |11\rangle)^{\otimes n}$$

$$= \sum_{k=0}^{n} (\cos\theta_1)^{n-k} (\sin\theta_1)^k \sum_{a=1}^{{}_nC_k} |P(k), a\rangle |P(k), a\rangle$$

maximally entangled states with $\log {}_nC_k$ entropy

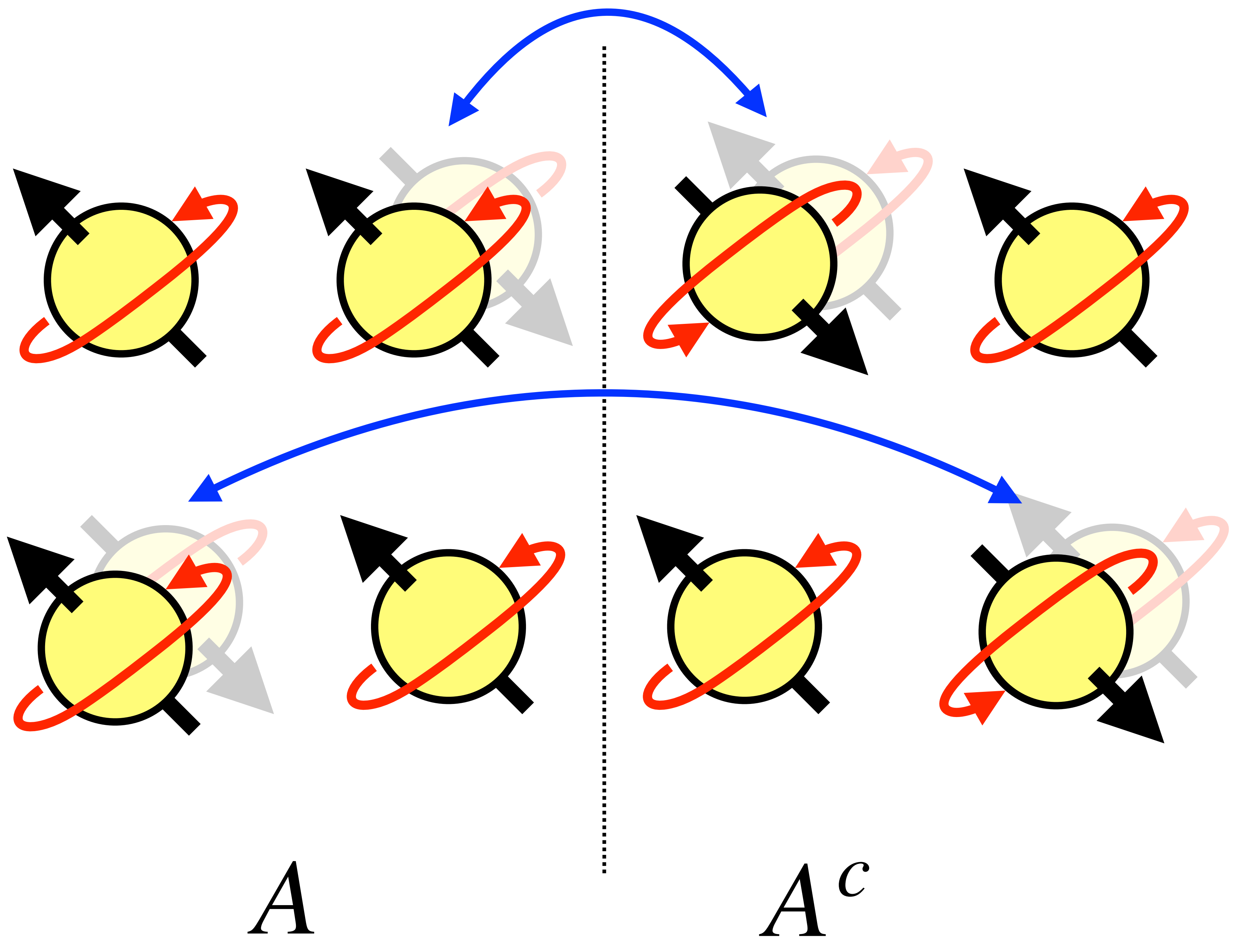
probability: $p_k = \frac{\langle\varphi|\Pi_k|\psi\rangle}{\langle\varphi|\psi\rangle} = {}_nC_k \cdot \frac{(\cos\theta_1 \cos\theta_2)^{n-k} (\sin\theta_1 \sin\theta_2)^k}{(\cos(\theta_1 - \theta_2))^n}$

Averaged number: $N = \sum_{k=1}^{{}_nC_k} p_k \log[{}_nC_k] \simeq n \cdot S(\mathcal{T}_A^{\psi|\varphi})$

PE<EE from Entanglement Swapping

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0110\rangle)$$

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1001\rangle)$$



$$S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) = 0$$

$$< \quad S^{(n)}(\rho_A^{\psi}) = S^{(n)}(\rho_A^{\varphi}) = \log 2$$

! We can also see the similar effects in free scalar CFT

(Can make EPR pair-like excitations: Nozaki-Numasawa-Takayanagi '14**)**

Replica trick

Replica trick

A way to compute EE in quantum field theories

$$S(\rho_A) = -\mathrm{Tr}_A \rho_A \log \rho_A$$

Difficult to compute it directly (in many cases)

⬍

A kind of partition function: easier! (in many cases)

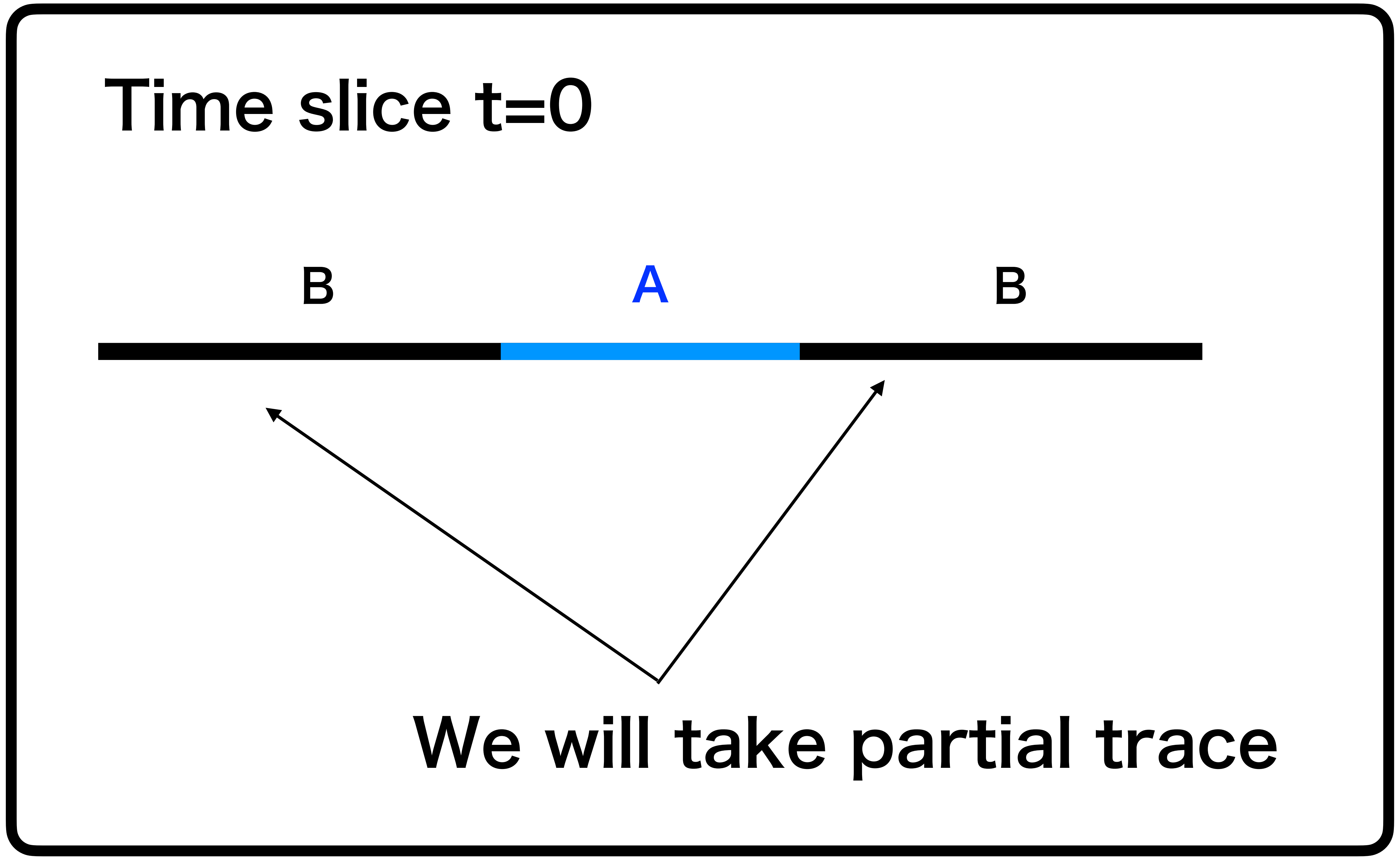
$$S(\rho_A) = -\frac{\partial}{\partial n} \log \mathrm{Tr}\rho_A^n \Big|_{n\to 1}$$

For concreteness, let us consider vacuum state in two-dimensional CFT

(May have 2d massless free fields in mind)

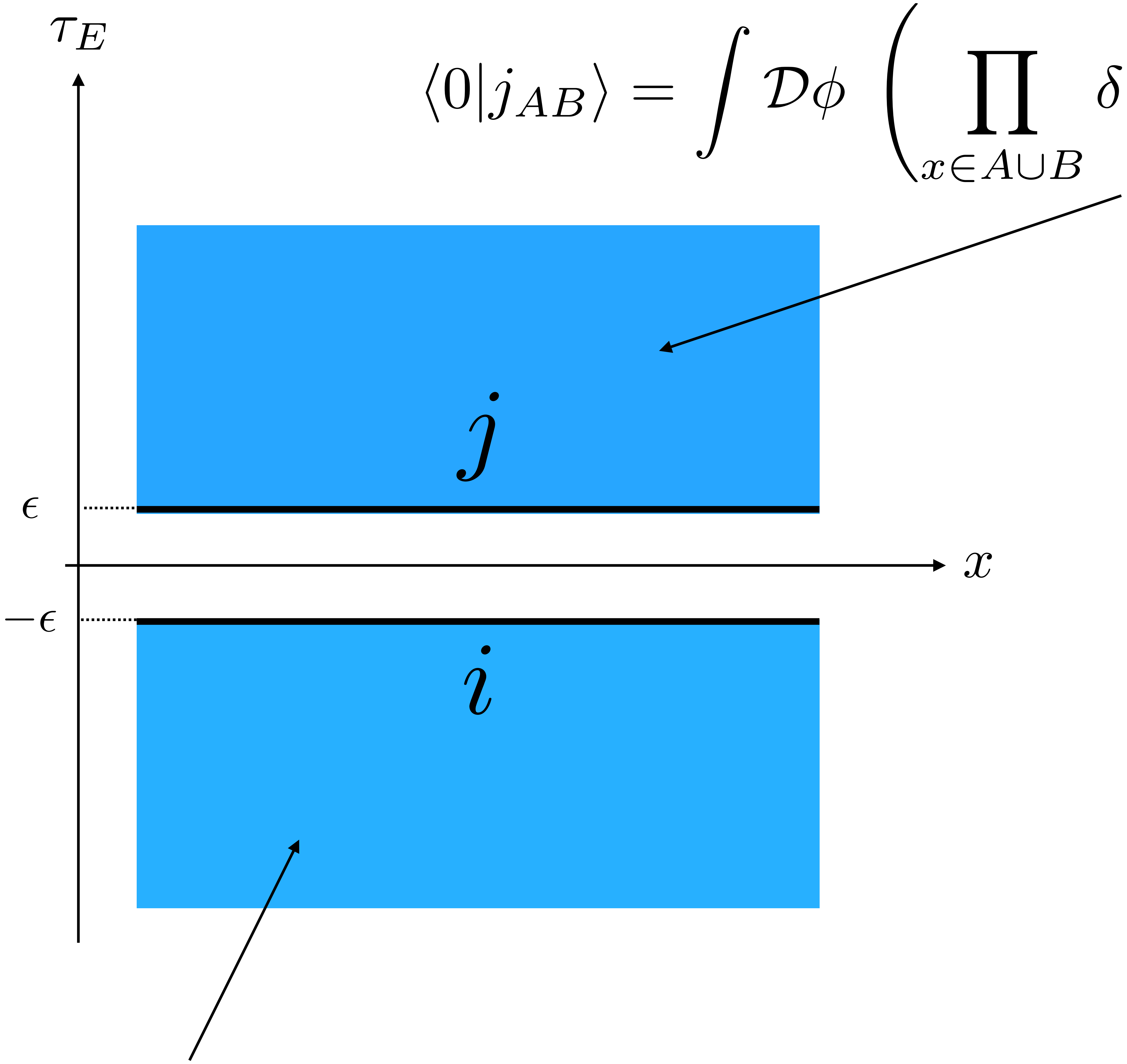
$$\rho_A = \mathrm{Tr}_B \, |0\rangle\langle 0|$$

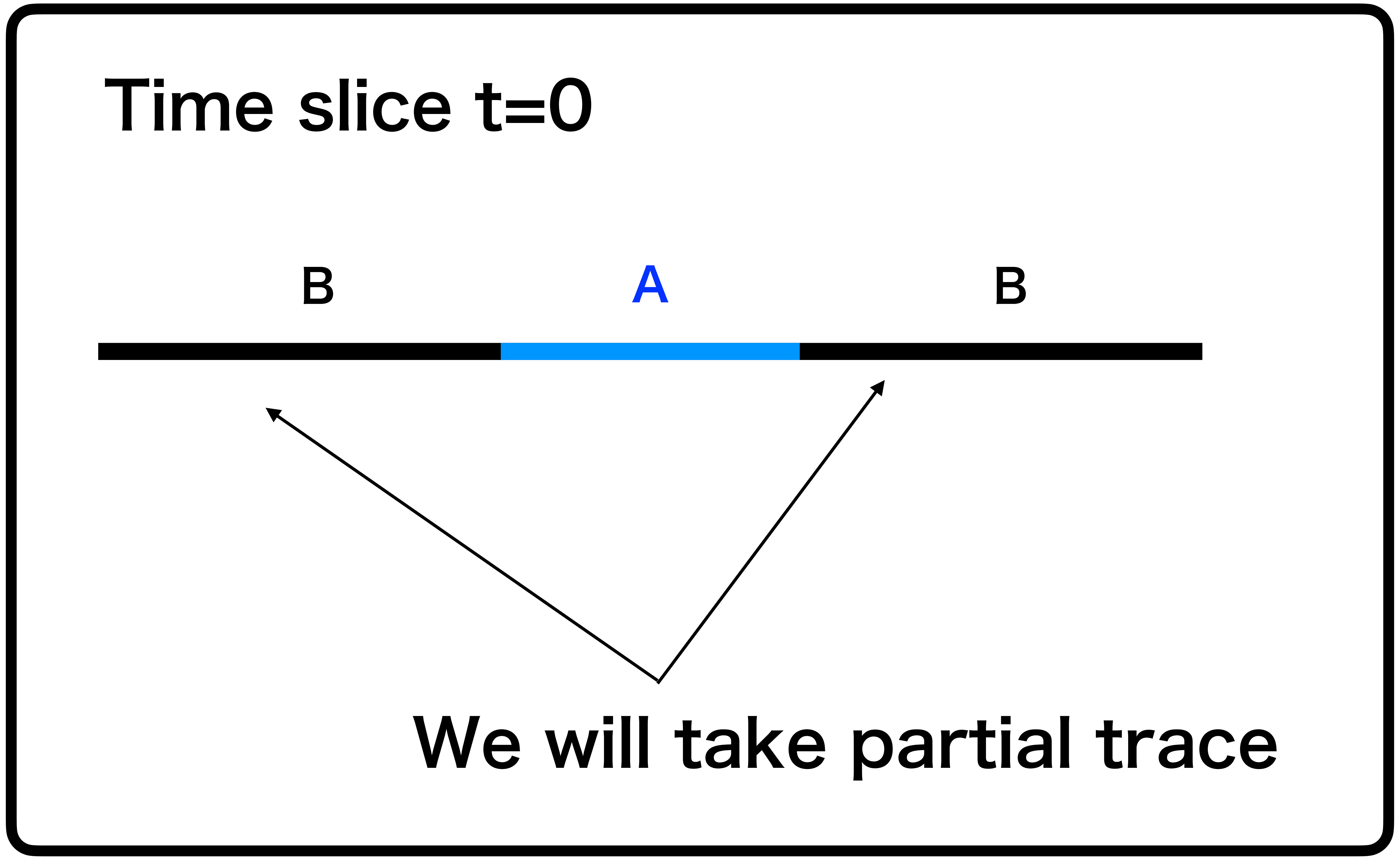
$$S(\rho_A) = -\frac{\partial}{\partial n} \log \boxed{\mathrm{Tr}\rho_A^n}\Big|_{n\to 1}$$



Reduced density matrix from Euclidean path integral

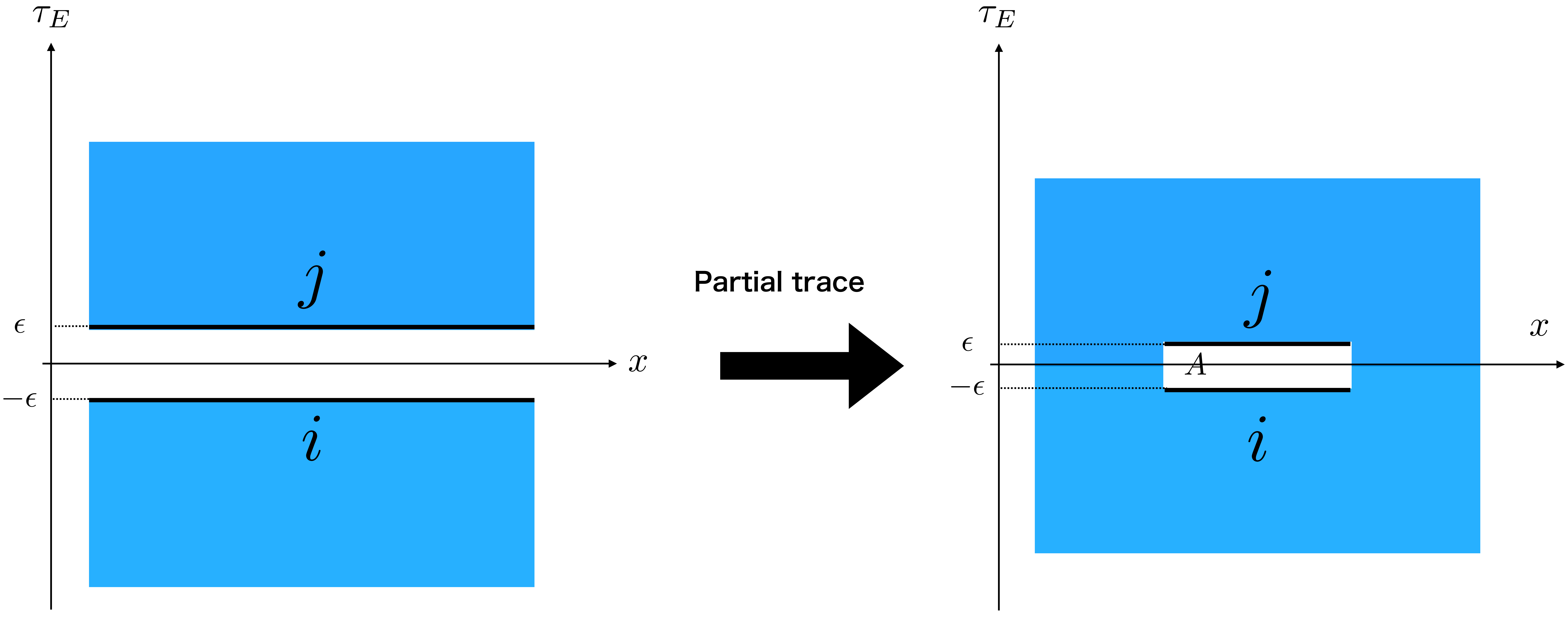
$$\langle 0 | j_{AB} \rangle = \int \mathcal{D}\phi \left(\prod_{x \in A \cup B} \delta(\phi(+\epsilon, x) - \phi_i(x)) \right) \exp\left[- \int_{+\epsilon}^{+\infty} d\tau_E \int_{A \cup B} dx \mathcal{L} \right]$$





$$\langle i_{AB} | 0 \rangle = \int \mathcal{D}\phi \left(\prod_{x \in A \cup B} \delta(\phi(-\epsilon, x) - \phi_i(x)) \right) \exp\left[- \int_{-\infty}^{-\epsilon} d\tau_E \int_{A \cup B} dx \mathcal{L} \right]$$

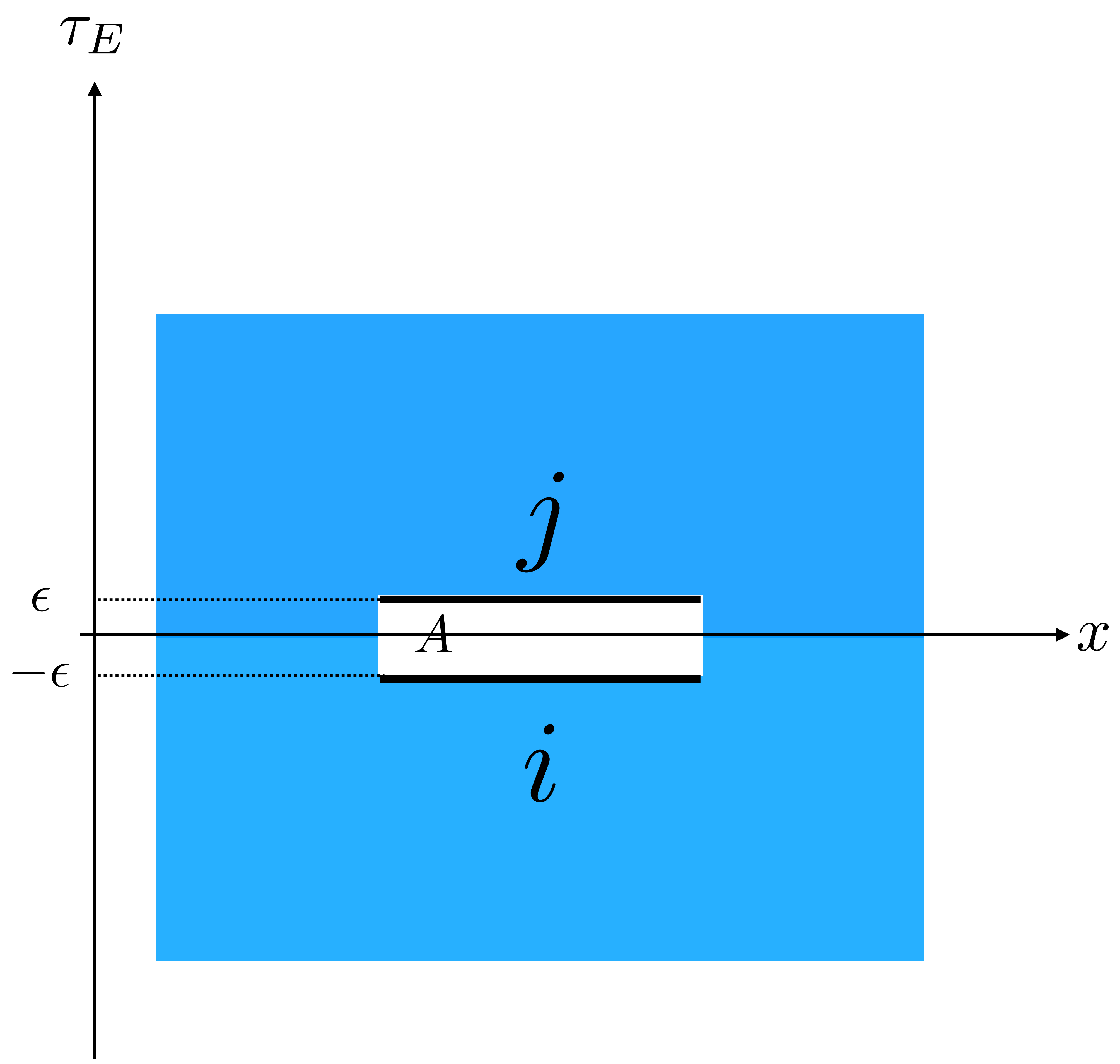
Reduced density matrix from Euclidean path integral



$$\langle i_A|\rho_A|j_A\rangle = \sum_k \langle i_A k_B|0\rangle\langle 0|j_A k_B\rangle$$

A Remark for Later Convenience

The braket can be regarded as Euclidean path integral with ⋯



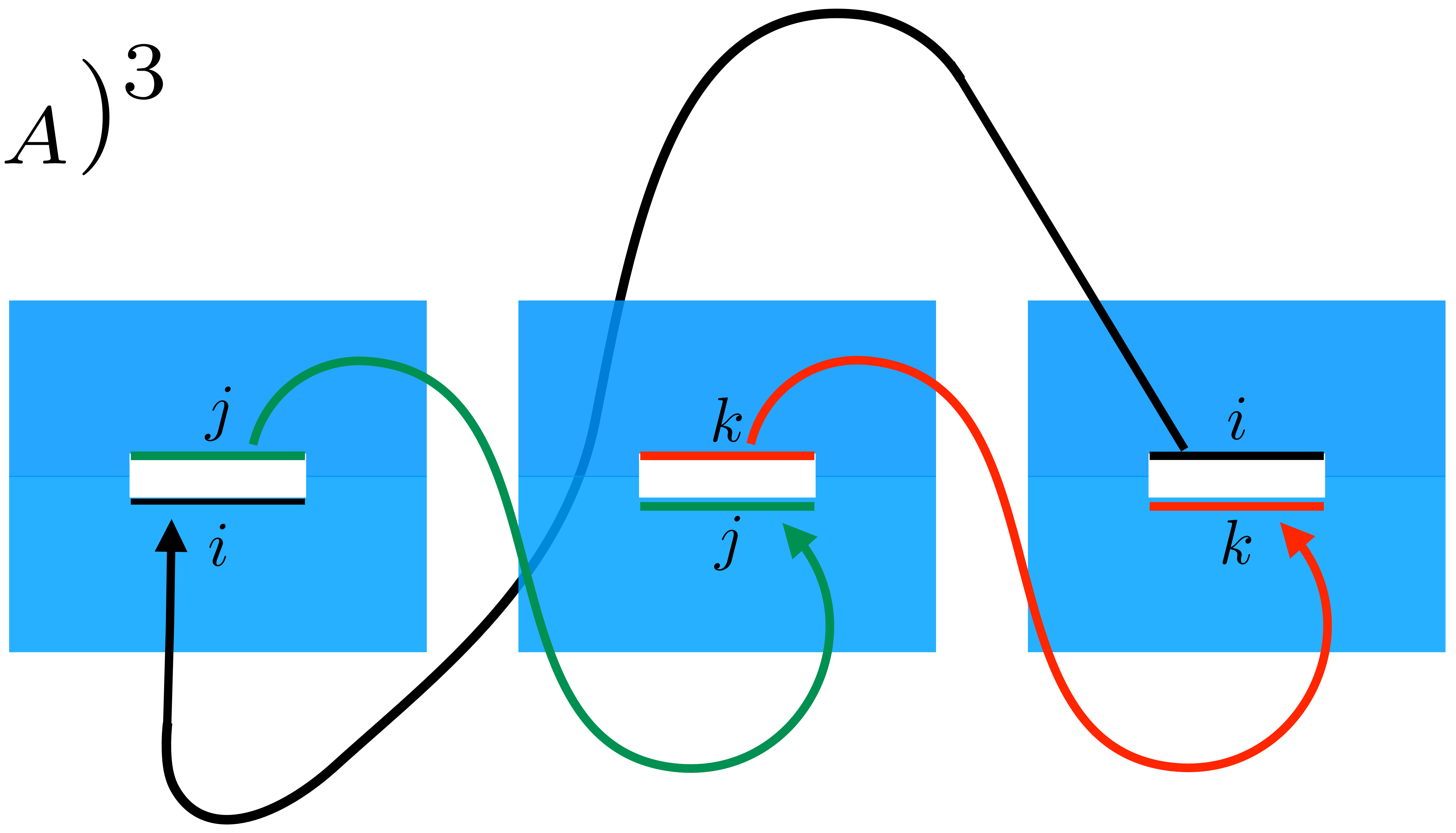
Reflection symmetry

since bra and ket are the same!

↑ Related to the hermicity of our density matrix

In AdS/CFT, this symmetry persists on the gravity side as well

$$\operatorname{Tr}_A (\rho_A)^3$$



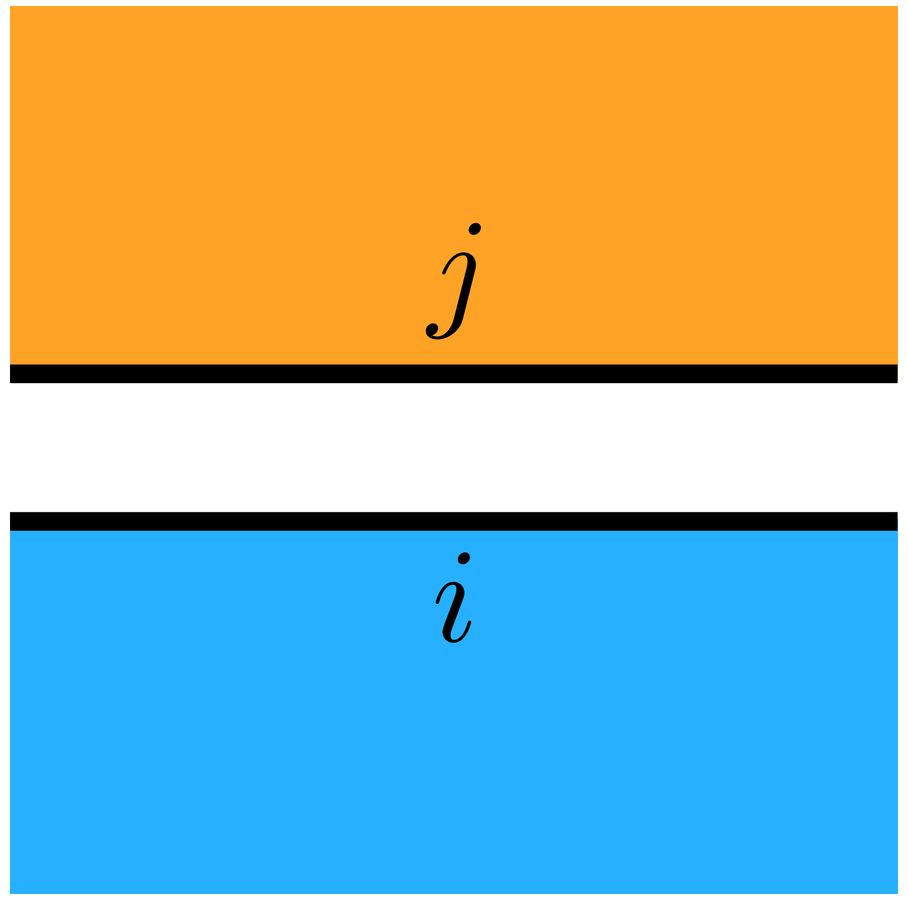
"Temperature" of partition function:

$3 \times 2\pi$ periodicity around branch cut

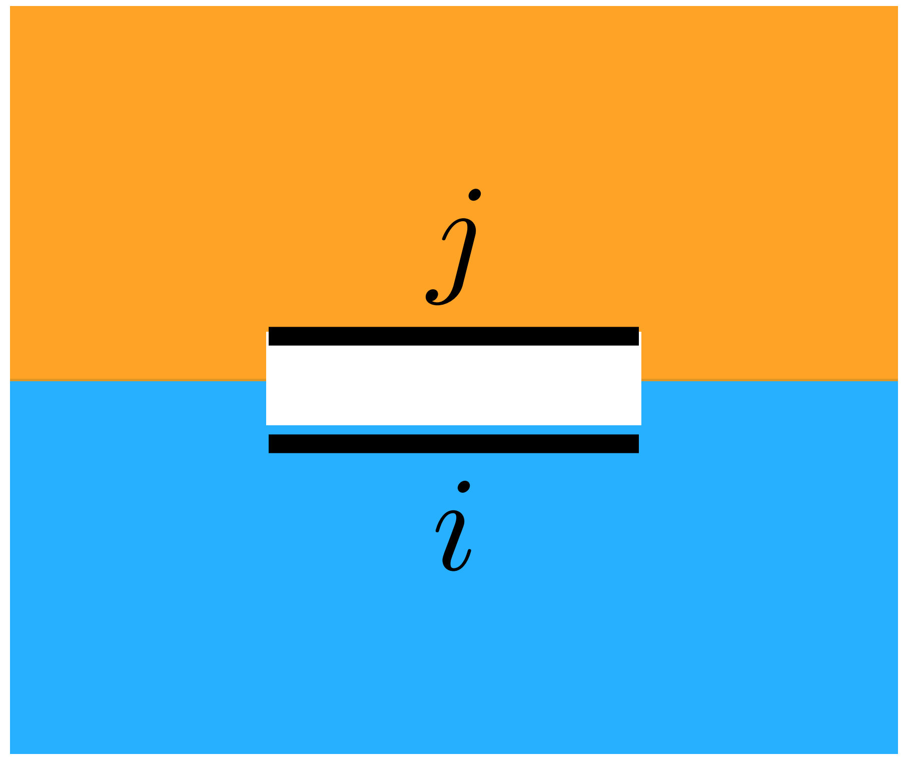
$\exists Z_3$-symmetry

States from Euclidian Path Integral

$\langle \varphi | j_{AB} \rangle$



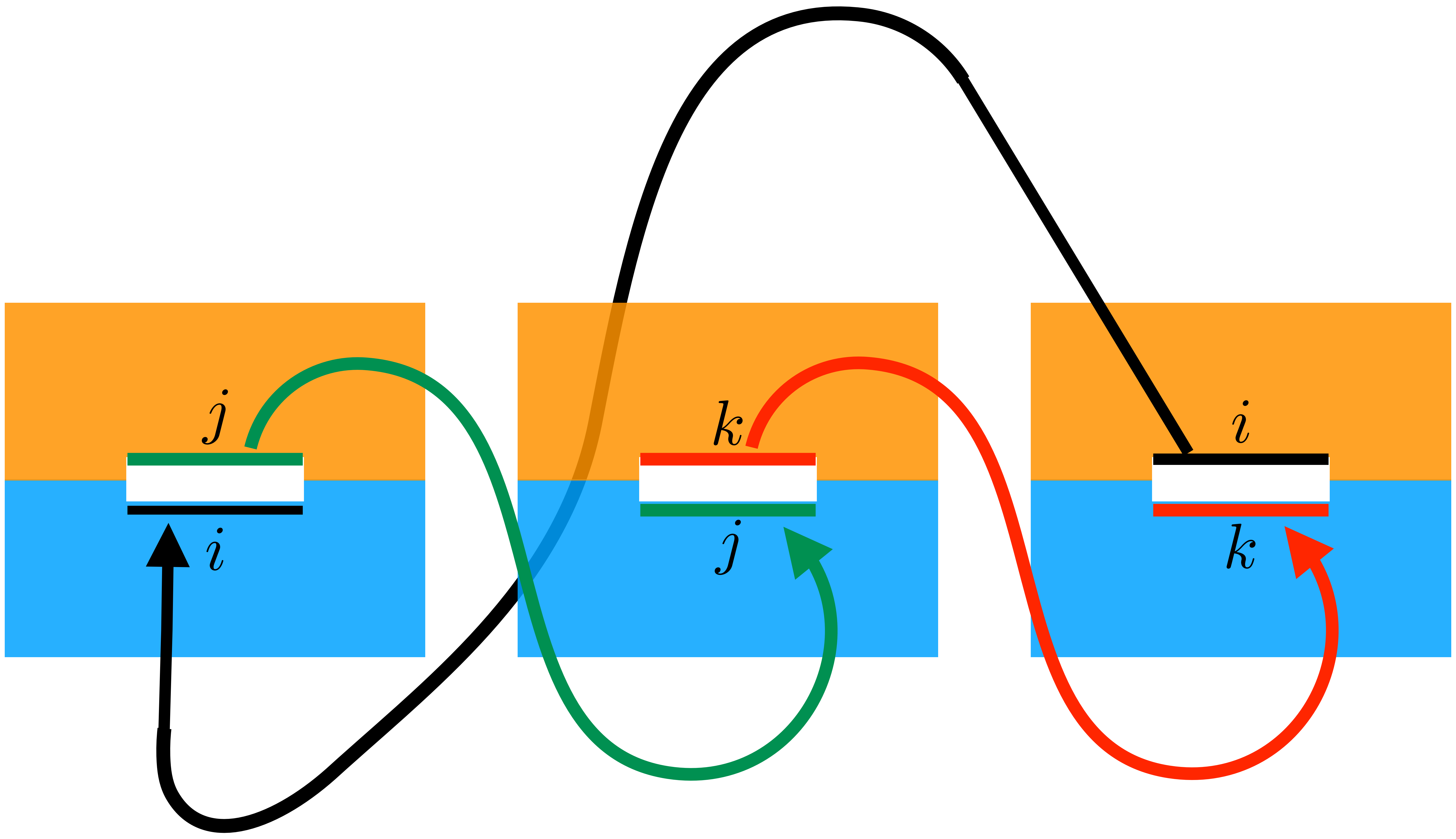
$\langle i_{AB} | \psi \rangle$



$\langle i_{AB} | \mathcal{T}^{\psi|\varphi} | j_{AB} \rangle$

$\langle i_A | \mathcal{T}_A^{\psi|\varphi} | j_A \rangle = \langle i_A | \mathrm{Tr}_B \mathcal{T}^{\psi|\varphi} | j_A \rangle$

- Vacuum and operator excitations can be easily obtained from EPI
- For simplicity, we ignored the denominator of the transition matrix



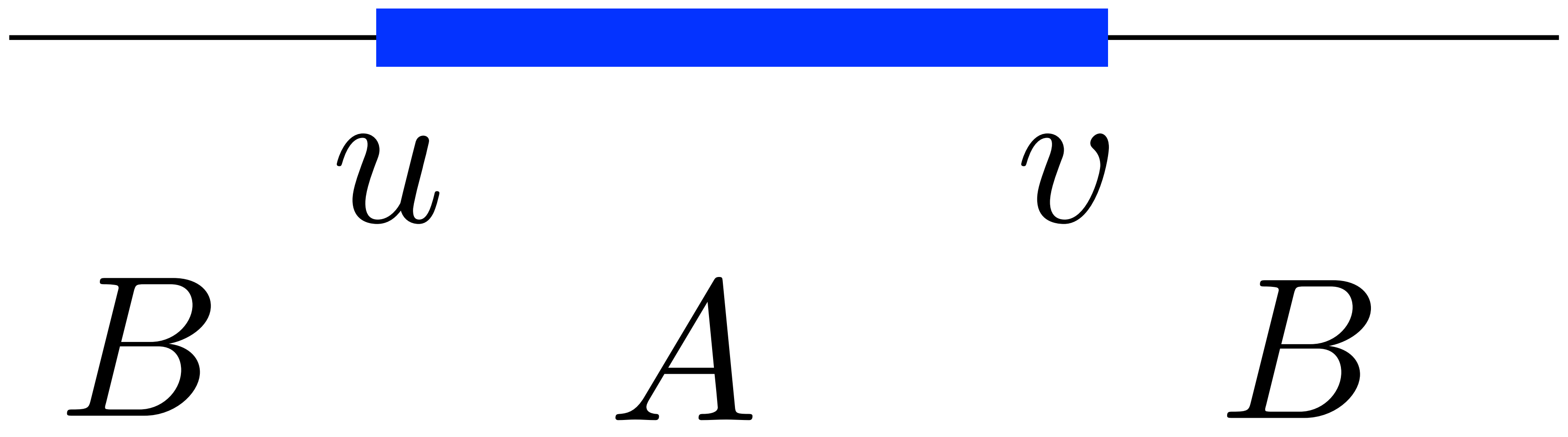
$$\mathrm{Tr}(\mathcal{T}_A^{\psi|\varphi})^3$$

EE for vacuum in 2d CFT (single interval)

Calabrese-Cardy ’04

Fixed by the conformal symmetry

$$\mathrm{Tr}(\rho_A)^n = \langle \sigma_n(u)\bar{\sigma}_n(v)\rangle \propto \frac{1}{|u-v|^{2\Delta_n}}$$



$$\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

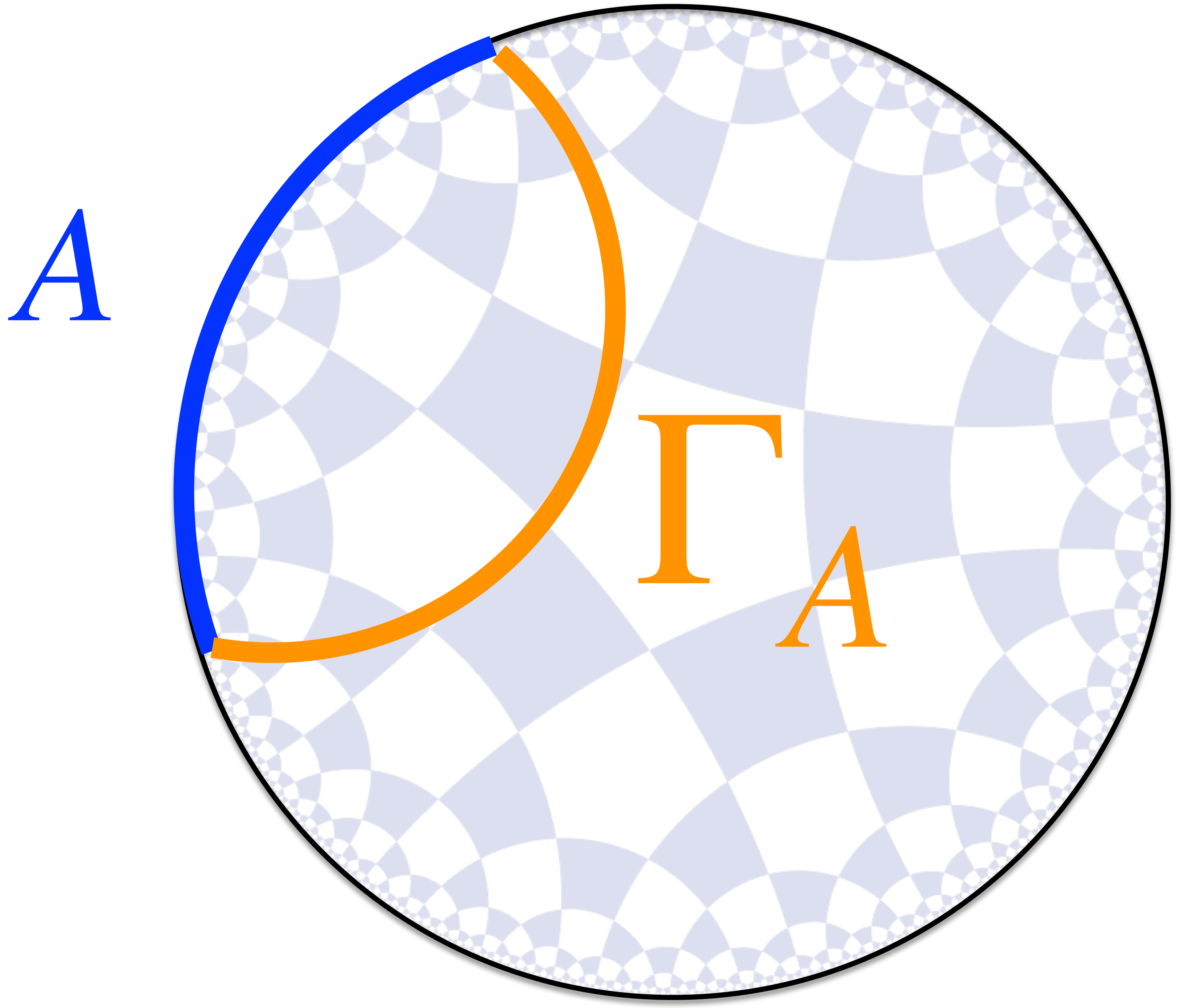
$$-\mathrm{Tr}(\rho_A \log \rho_A) = -\frac{\partial}{\partial n}\mathrm{Tr}(\rho_A^n)\Big|_{n\to 1} \quad \Longrightarrow \quad S(\rho_A) = \frac{c}{3}\log\frac{|u-v|}{\epsilon}$$

RT and LM

Ryu-Takayanagi Formula Ryu-Takayanagi '06 (static case)

Relation between Entanglement and Area in AdS/CFT

$$S(\rho_A) = \underset{\substack{\partial\Gamma_A = \partial A \\ \Gamma_A \sim A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N} \right]$$



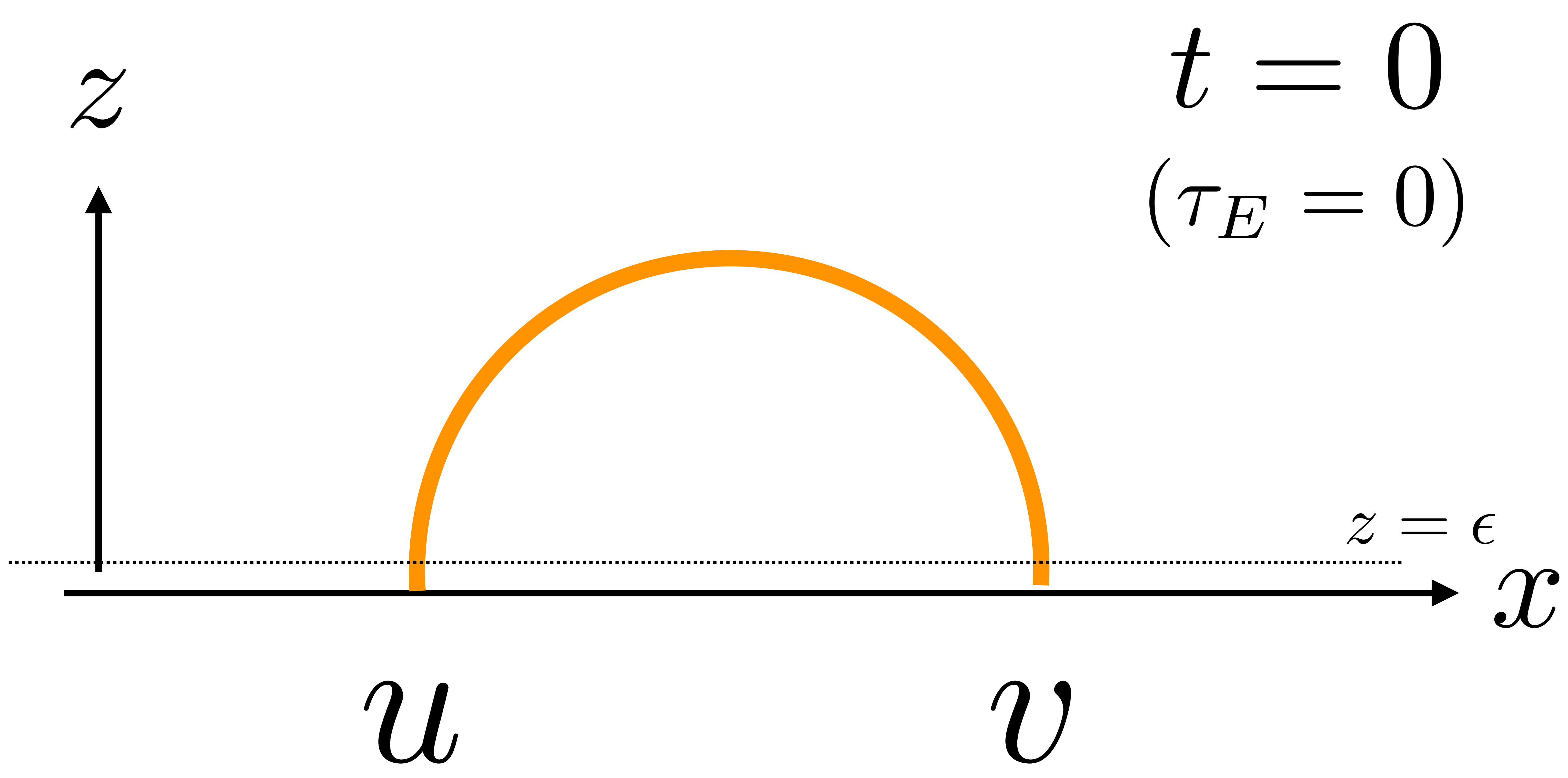
e.g.) EE for Vacuum in CFT ↔ Min. Area for Empty AdS

We have discussed so far

We will see in the next slide

Holographic Entanglement Entropy

Ryu-Takayanagi '06



Poincare AdS

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2}$$

$$\ell_{\mathrm{AdS}_3} \equiv 1$$

Geodesics: semi-circle

$$z^2 + \left(x - \frac{u+v}{2}\right)^2 = \left(\frac{v-u}{2}\right)^2$$

$$\Rightarrow \frac{A}{4G_N} = \frac{1}{2G_N}\log\frac{|u-v|}{\epsilon} = \frac{c}{3}\log\frac{|u-v|}{\epsilon} \qquad \left(c = \frac{3}{2G_N}\right)$$

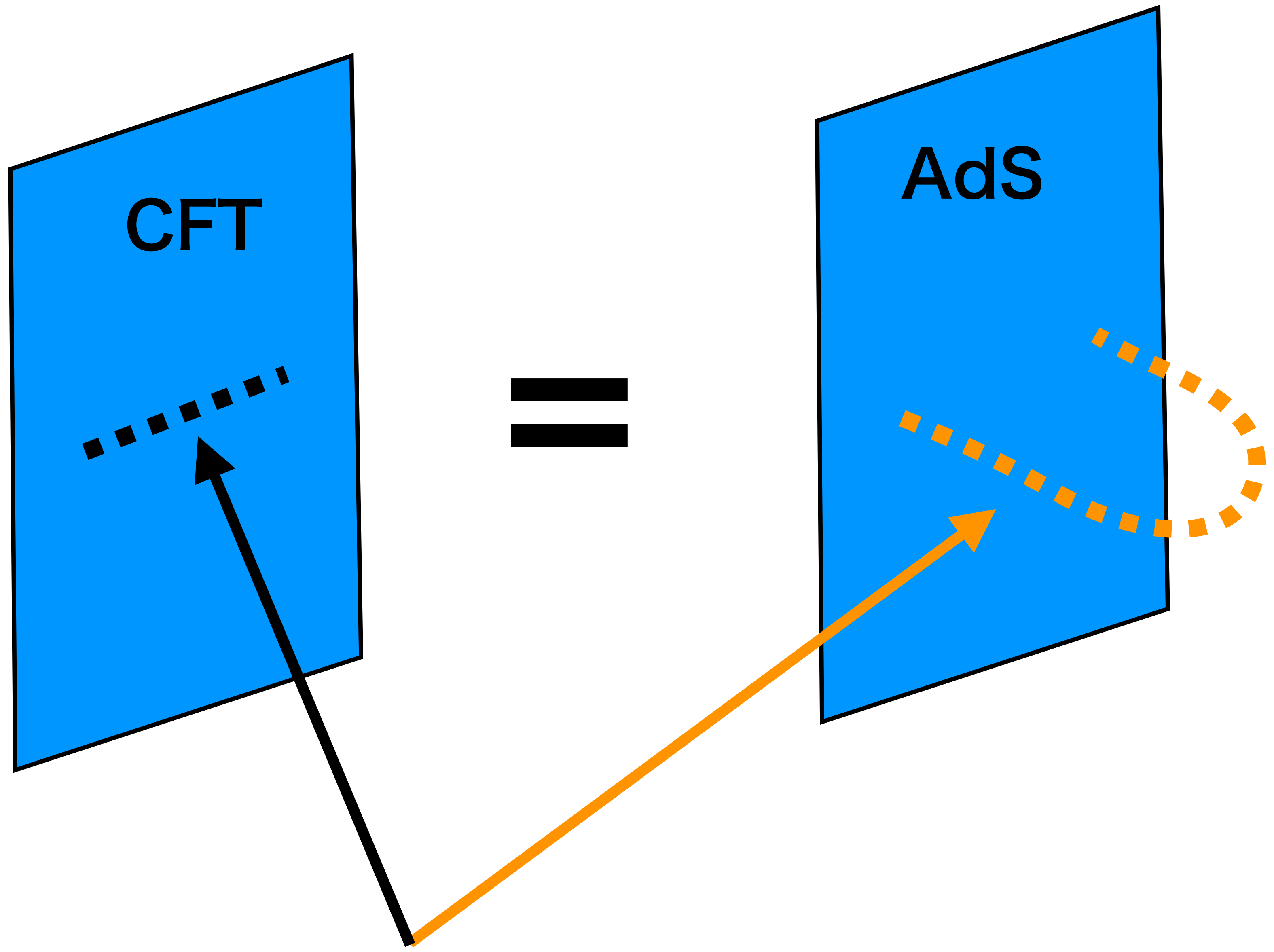
Brown-Henneaux '86

Thanks to the reflection symmetry, the minimal area surface settles down to a canonical time slice!

More general argument

Lewkowycz-Maldacena'13

Just use the AdS/CFT for "partition function"



$$Z^{(n)}_{\mathrm{CFT}_d} \equiv \mathrm{Tr}(\rho_A)^n$$

$$\|$$

$$Z^{(n)}_{\mathrm{AdS}_{d+1}} \overset{\text{large-N}}{\sim} e^{-S^{(n)}} \overset{n\to 1}{\sim} e^{-\boxed{\frac{n-1}{4G_n}A}}$$

Singular parts (contribute to on-shell action)

➡ On-shell action indeed picks up the minimal area!

How holographic states are special?

A Famous Example: Monogamy relation Hayden-Headrick-Maloney '11

$$I(A:BC) \geq I(A:B) + I(A:C)$$

where $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$

A counterexample of this monogamy relation: 2d free scalar

It can be regarded as **a constraint on states with semi-classical gravity dual**

A series of strong conditions from HPE

$$S^{(n)}(\mathcal{T}_A^{\psi|\varphi}) \geq 0, \;\; (n > 0)$$

$\because$) related to the **area** of geometrical objects Dong '16

$$S(\mathcal{T}_A^{\psi|\varphi}) + S(\mathcal{T}_B^{\psi|\varphi}) \geq S(\mathcal{T}_{AB}^{\psi|\varphi})$$

$\because$) follows from **minimization of area**

$$S(\mathcal{T}_{AB}^{\psi|\varphi}) + S(\mathcal{T}_{BC}^{\psi|\varphi}) \overset{?}{\geq} S(\mathcal{T}_{ABC}^{\psi|\varphi}) + S(\mathcal{T}_B^{\psi|\varphi})$$

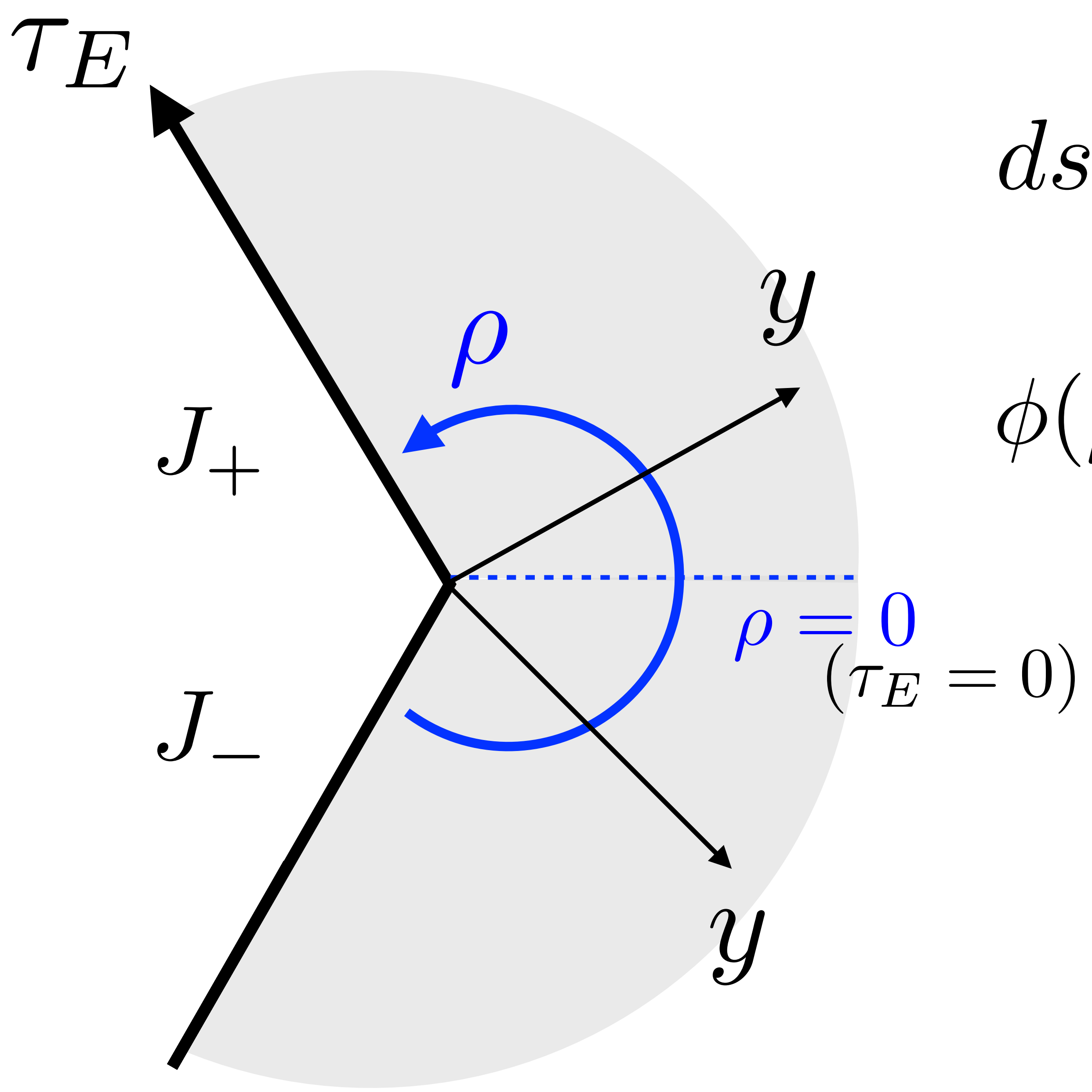
Not proven/disproven yet. **Any counterexamples?**

(By assuming a fixed time slice)

(Note: von-Neumann entropy always satisfies all of them)

Example: Janus AdS$_3$/CFT$_2$

the solution in Einstein + a massless scalar (dilaton)   Bak-Gutperle-Hirano '07



$$ds^2 = d\rho^2 + f(\rho)\frac{dx^2 + dy^2}{y^2}$$

$$\phi(\rho) = \phi_0 + \frac{1}{\sqrt{2}}\log\left(\frac{1+\sqrt{1-2\gamma^2}+\sqrt{2}\gamma\tanh\rho}{1+\sqrt{1-2\gamma^2}-\sqrt{2}\gamma\tanh\rho}\right)$$

$$\underset{\rho\to\pm\infty}{\longrightarrow} J_\pm$$

where $f(\rho) = \frac{1}{2}\left(1+\sqrt{1-2\gamma^2}\cosh(2\rho)\right)$

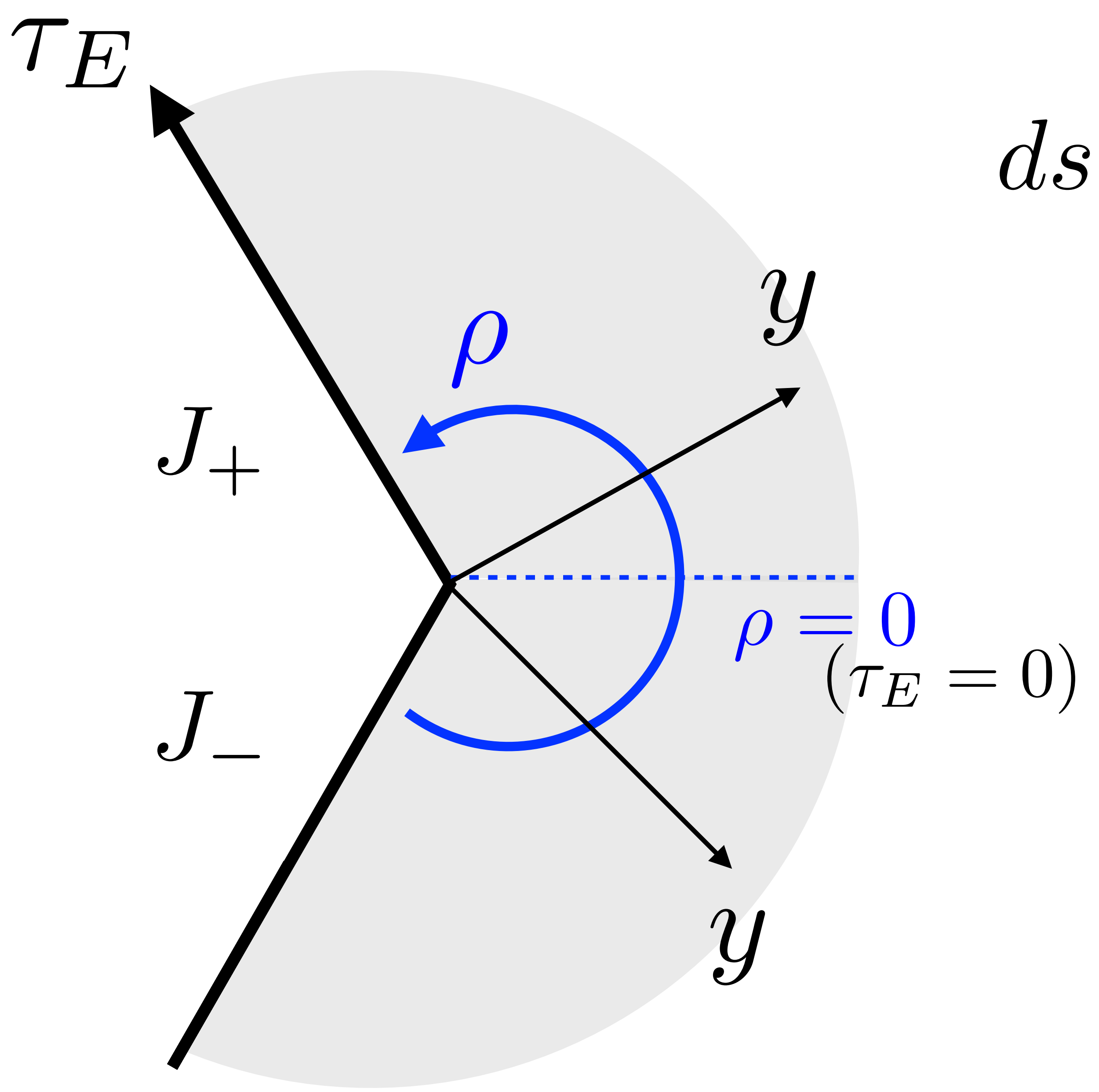
Final/Initial states $|\psi_\pm\rangle$ on boundary

are given by exactly marginal deformation:

$$S = S_{\mathrm{CFT}} + J_\pm \int_{\pm\tau_E>0} d^2x\, \mathcal{O}(x)$$

Example: Janus AdS_3/CFT_2

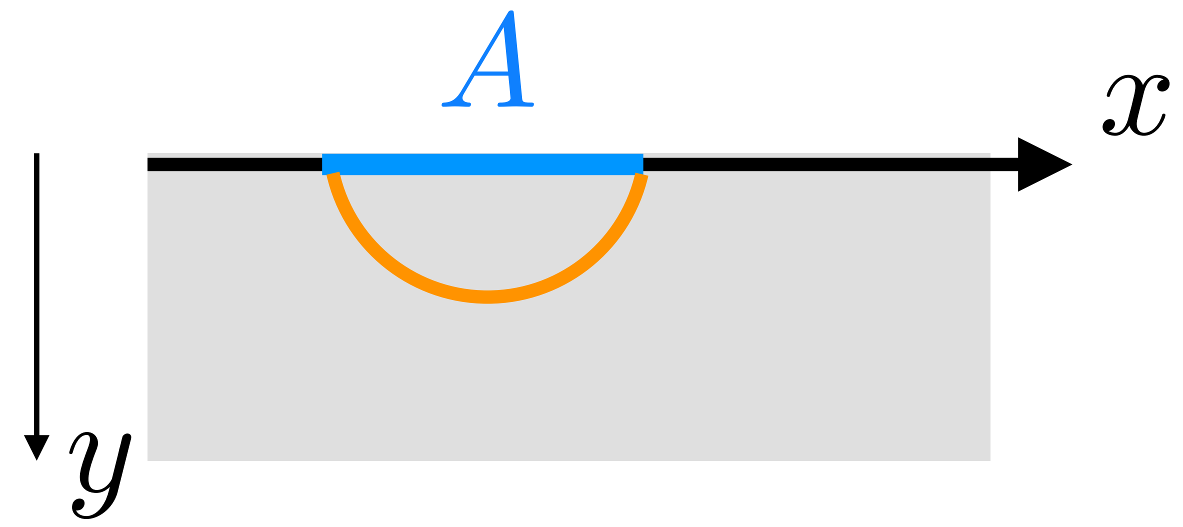
the solution in Einstein + a massless scalar (dilaton)   Bak-Gutperle-Hirano '07



$$ds^2 = d\rho^2 + f(\rho)\frac{dx^2 + dy^2}{y^2}$$

the minimal geodesics (anchored on ∂A):

$$\rho = 0,$$
$$x^2 + y^2 = \ell^2$$



where

$$A = \{(x, \tau_E)\,|\, x \in [-\ell/2, \ell/2],\ \tau_E = 0\}$$

The overall factor $f(0) < 1$ effectively **reduces "the AdS radius"**

and **the cutoff scale in y-direction** (x-direction is the ordinary one)

→ **The Holographic Pseudo Entropy also reduces (natural!)**

Mixed states generalizations

Need mixed states generalization (and holography)

e.g.) $\rho_{AB} = q\,|\Psi\rangle\langle\Psi| + (1-q)\sigma_A \otimes \sigma_B$

"EPR pair" (entangled)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle - |1_A 0_B\rangle)$$

$$\sigma = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1|$$

Not entangled

$$\longrightarrow \quad S(\rho_A) = \log 2$$

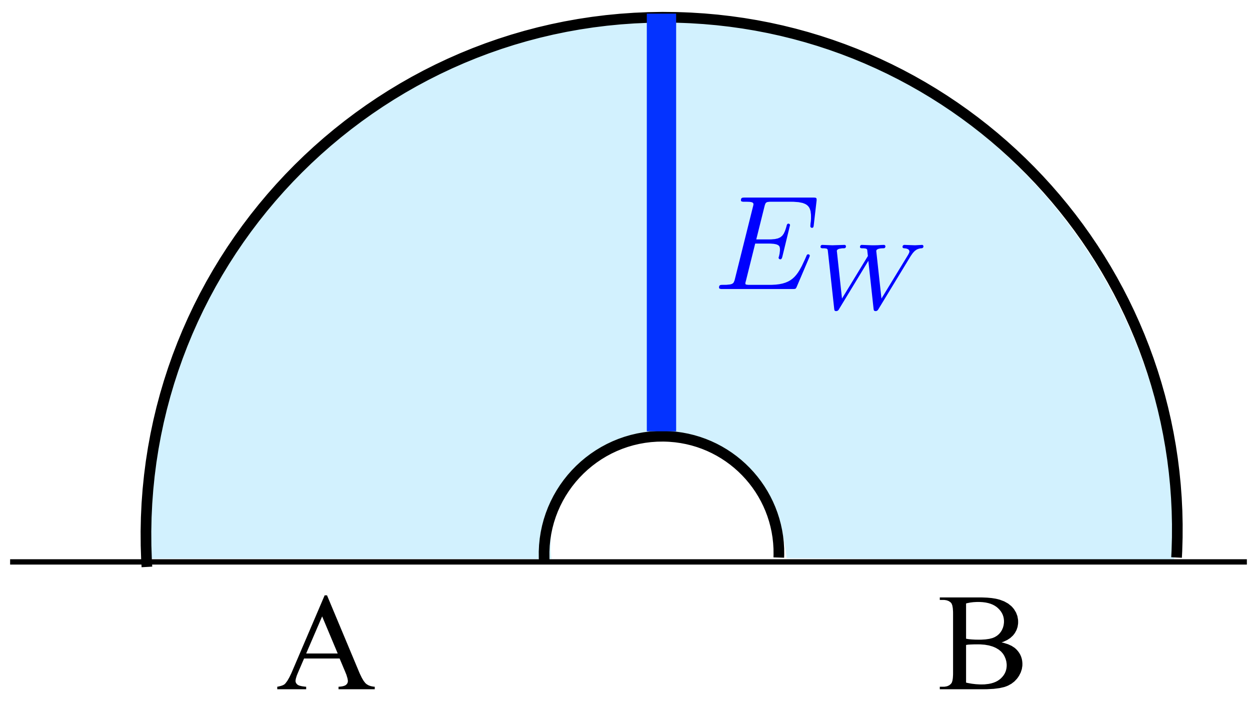
Advertisement: I wrote a brief review in "日本物理学会誌"

For the detail, please take a look it!

最近の研究から

混合状態に対する量子情報量とホログラフィー原理

Entanglement Wedge Cross Section (EWCS)



Entanglement wedge (blue shaded region)

bulk subregion dual to
reduced density matrix (mixed state)

Czech-Karczmarek-Nogueira-VanRaamsdonk, Wall '12,
Headrick, Hubeny-Lawrence-Rangamani '14

EWCS: minimal cross section of the entanglement wedge

Candidates:

Entanglement of Purification Umemoto-Takayanagi '17

Odd Entropy KT '18

Logarithmic Negativity Kulder-Flam—Ryu '18

Reflected Entropy Dutta-Faulkner '19

Q. Boundary dual of EWCS in Euclidian time-dependent background ?

Generalizations to Mixed States ?

Can evaluate correlations for a given "mixed state" $\mathcal{T}_{AB}^{\psi|\varphi}$?

One way to answer it:

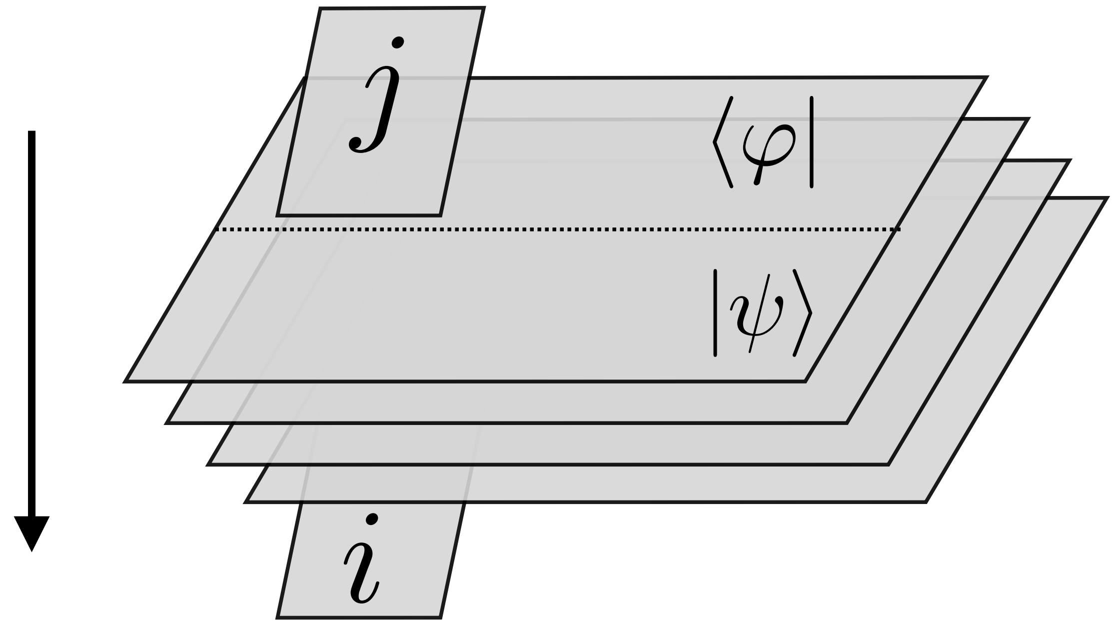
$$\hat{X} = \sum_{ij} X_{ij} \left| i_A \right\rangle \left\langle j_A \right|$$

$$\Rightarrow \quad |\hat{X}_{AA^*}\rangle = \sum_{ij} X_{ij} \left| i_A \right\rangle \left| j^*_{A^*} \right\rangle$$

(for a physical state, it is just the similar procedure in purification)

Have a nice path-integral realization:

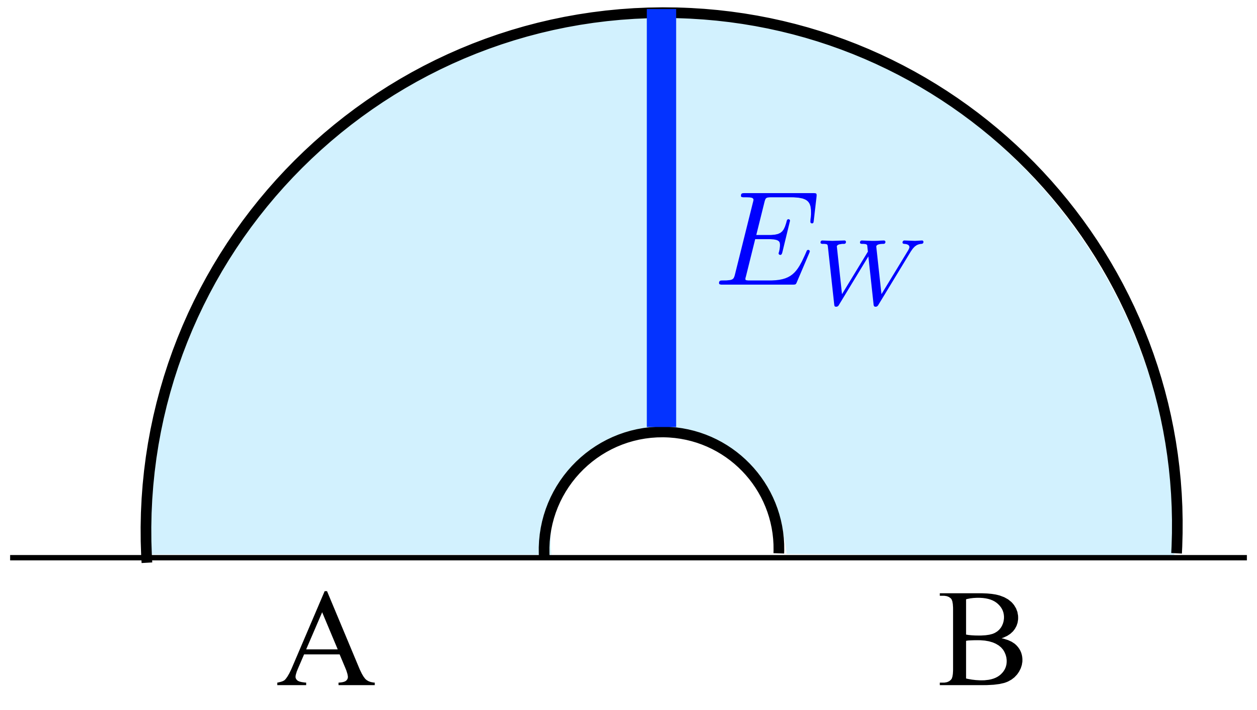
If $X = \mathcal{T}_A^{\varphi|\psi}$, $\quad \left[(\mathcal{T}_A^{\psi|\varphi})^{\frac{m}{2}} \right]_{ij} \quad = \quad \frac{m}{2}$



m: an even integer

"Pseudo Reflected Entropy"

$$S_R(\mathcal{T}_{AB}^{\psi|\varphi}) \equiv S(\mathcal{T}_{AA^*}^{X|X^\dagger}) = 2E_W(\mathcal{T}_{AB}^{\psi|\varphi})$$



where $\mathcal{T}_{ABA^*B^*}^{X|X^\dagger} = \dfrac{|X\rangle\langle X^\dagger|}{\langle X|X^\dagger\rangle}$ with $X = \lim_{m\to 1}(\mathcal{T}_{AB}^{\psi|\varphi})^{\frac{m}{2}}$

$$\hat{X} = \sum_{ij} X_{ij}\,|i_A\rangle\langle j_A|$$

$$\Rightarrow \quad |\hat{X}_{AA^*}\rangle = \sum_{ij} X_{ij}\,|i_A\rangle\,|j^*_{A^*}\rangle$$

(for a physical state, it is just the similar procedure in purification)

- We can also introduce "pseudo odd entropy" *in progress*