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Bi-local Holography of the SYK Model

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Motivations

• Simple and Solvable examples of AdS/CFT are needed for better understanding of holography itself and quantum gravity problems.

• Based on earlier Sachdev-Ye model [Sachdev & Ye '93], Sachdev-Ye-Kitaev (SYK) model was proposed as a simpler version of AdS/CFT, which is a quantum mechanical many-body system based on $N(\gg 1)$ fermionic sites. [Kitaev '15]

• From the maximally chaotic behavior of the model in large N, dual gravity theory was conjectured to be AdS_2 black hole theory. [Kitaev '15]

• In order to understand the dual gravity theory of the SYK model, in this talk I will explore some aspects of the SYK model, in the large N limit.

Outline

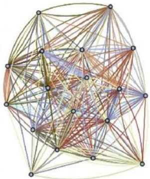
1. SYK Model
2. Quadratic Fluctuations
3. Schwarzian Action
4. Partially Disorder-Averaged SYK
5. Summaries

1. SYK Model

- SYK model [Kitaev '15] consists of Majorana fermions on N sites ($N \gg 1$):

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l\,, \qquad \text{with} \qquad \{\chi_i\,,\, \chi_j\} = \delta_{ij}$$

- J_{ijkl} are all-to-all & random; distributions are Gaussian:



$$P(J_{ijkl}) \propto \exp\left(-\frac{N^3 J_{ijkl}^2}{12 J^2}\right)$$

and the disordered average is defined by

$$\langle \mathcal{O} \rangle_J \equiv \int \prod_{i,j,k,l}^{N} dJ_{ijkl}\, P(J_{ijkl})\, \mathcal{O}$$

- The model is known to be self-averaging at large N. [Sachdev & Ye '93] Namely the quenched disorder (i.e. averaging over collerators) = the annealed disorder (i.e. averaging over partition function).

Collective theory

- The Large N theory is represented through a bi-local collective field:

$$\Psi(\tau_1, \tau_2) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i(\tau_1)\chi_i(\tau_2)$$

The corresponding path-integral is [Jevicki, K.S. & Yoon '16]

$$\langle Z \rangle_J = \int \prod_{\tau_1, \tau_2} \mathcal{D}\Psi(\tau_1, \tau_2)\, e^{-S_{\text{col}}[\Psi]}$$

where S_{col} is the collective action (generalized to q-point interaction):

$$S_{\text{col}}[\Psi] = \frac{N}{2} \int d\tau \left[\partial_\tau \Psi(\tau, \tau')\right]_{\tau'=\tau} + \frac{N}{2} \operatorname{Tr} \log \Psi - \frac{J^2 N}{2q} \int d\tau_1 d\tau_2 \left[\Psi(\tau_1, \tau_2)\right]^q$$

- Another formalism of the effective action by G and Σ is equivalent after integrating out Σ. Then $G = \Psi$.

Saddle-point

- In strongly coupling limit $J|\tau_{12}| \gg 1$ (critical IR fixed point), one can drop the Kinetic term:

$$S_{\mathsf{c}} = \frac{N}{2}\mathrm{Tr}(\log\Psi) - \frac{J^2 N}{2q}\int d\tau_1 d\tau_2 \, [\Psi(\tau_1,\tau_2)]^q$$

- Saddle-Point Equation:

$$0 = \frac{\delta S_{\mathsf{c}}}{\delta\Psi(\tau_1,\tau_2)} = \frac{N}{2}\Big[\Psi^{-1}(\tau_2,\tau_1) - J^2\,\Psi^{q-1}(\tau_1,\tau_2)\Big]$$

- Critical Saddle-Point Solution [Kitaev '15]:

$$\Psi_0(\tau_1,\tau_2) \propto \frac{1}{J^{\frac{2}{q}}}\,\frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{\frac{2}{q}}}$$

with $\tau_{12} \equiv \tau_1 - \tau_2$.

Conformal symmetry

• The critical action

$$S_{\mathsf{c}} = \frac{N}{2}\mathsf{Tr}(\log\Psi) - \frac{J^2N}{2q}\int d\tau_1 d\tau_2\,[\Psi(\tau_1,\tau_2)]^q$$

exhibits the conformal reparametrization symmetry [Kitaev '15]: $\tau \to f(\tau)$ with

$$\Psi(\tau_1,\tau_2) \to \left|f'(\tau_1)f'(\tau_2)\right|^{\frac{1}{q}}\,\Psi(f(\tau_1),f(\tau_2))$$

• $f(\tau)$ is the dynamical symmetry mode, whose effective action is given by a Schwarzian derivative. [Kitaev '15] [Maldacena & Stanford '16] [Jevicki & K.S. '16]

• In dual gravity theory, it corresponds to a dynamical boundary time (or boundary graviton) [Jensen '16] [Maldacena, Stanford & Yang '16] [Engelsöy, Mertens & Verlinde '16]

Finite-temperature Ψ_0

- The critical action

$$S_{\mathsf{c}} = \frac{N}{2}\mathrm{Tr}(\log\Psi) - \frac{J^2 N}{2q}\int d\tau_1 d\tau_2\,[\Psi(\tau_1,\tau_2)]^q$$

is invariant under $\tau \to f(\tau)$ and

$$\Psi(\tau_1,\tau_2) \to \left|f'(\tau_1)f'(\tau_2)\right|^{\frac{1}{q}}\,\Psi(f(\tau_1),f(\tau_2))$$

with an arbitrary (monotonically increasing) function $f(\tau)$.

- Finite-temperature solution can be obtained by $f(\tau)=\frac{\beta}{\pi}\tan(\frac{\pi\tau}{\beta})$ as

$$\Psi_{0,\beta}(\tau_1,\tau_2) \propto \left[\frac{\pi}{J\beta\sin(\pi\tau_{12}/\beta)}\right]^{\frac{2}{q}}\mathrm{sgn}(\tau_{12})$$

2. Quadratic Fluctuations

- Expansion around the critical IR background Ψ_0 as

$$\Psi(\tau_1,\tau_2) \,=\, \Psi_0(\tau_1,\tau_2) \,+\, \frac{1}{\sqrt{N}}\,\eta(\tau_1,\tau_2)$$

the collective action S_{col} leads to the systematic $1/N$ expansion:

$$S_{\text{col}} \,=\, N\,S_{(0)} \,+\, S_{(2)} \,+\, \frac{1}{\sqrt{N}}\,S_{(3)} \,+\, \cdots$$

where the bi-local Quadratic Action:

$$S_{(2)} \,=\, -\,\frac{1}{2}\int\prod_{i=1}^{4} d\tau_i\;\eta(\tau_1,\tau_2)\,\mathcal{K}(\tau_1,\tau_2;\tau_3,\tau_4)\,\eta(\tau_3,\tau_4)$$

- Quadratic kernel $\mathcal{K}$ must a function of the bi-local $SL(2,\mathcal{R})$ Casimir

$$\begin{aligned} C_{1+2} \,&=\, \big(\hat{D}_1+\hat{D}_2\big)^2 \,-\, \frac{1}{2}\big(\hat{P}_1+\hat{P}_2\big)\big(\hat{K}_1+\hat{K}_2\big) \,-\, \frac{1}{2}\big(\hat{K}_1+\hat{K}_2\big)\big(\hat{P}_1+\hat{P}_2\big) \\ &=\, -\,(\tau_1-\tau_2)^2\,\partial_1\partial_2 \end{aligned}$$

Diagonalizing the kernel

- The eigenfunctions of the bi-local $SL(2,\mathcal{R})$ Casimir C_{1+2} are given by conformal three-point function $\left\langle \mathcal{O}_h(\tau_0)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\right\rangle$ with a unit OPE coefficient.

- It is more useful to Fourier transform from τ_0 to ω by

$$\left\langle \widetilde{\mathcal{O}_h}(\omega)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\right\rangle \equiv \int d\tau_0\, e^{i\omega\tau_0}\left\langle \mathcal{O}_h(\tau_0)\,\mathcal{O}_\Delta(\tau_1)\,\mathcal{O}_\Delta(\tau_2)\right\rangle$$
$$\propto \frac{\operatorname{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta-\frac{1}{2}}}\; e^{i\omega(\frac{\tau_1+\tau_2}{2})}\, Z_\nu(|\tfrac{\omega\tau_{12}}{2}|)$$

where $h=\nu+1/2$ and

$$Z_\nu(x) \;=\; J_\nu(x) \,+\, \xi_\nu\, J_{-\nu}(x)\,, \qquad \xi_\nu \;=\; \frac{\tan(\pi\nu/2)+1}{\tan(\pi\nu/2)-1}$$

- The complete set of ν can be fixed from the representation theory of $SL(2,\mathcal{R})$ [Kitaev '17]. The discrete modes $\nu = 2n+3/2$ $(n=0,1,2,\cdots)$ and the continuous modes $\nu = ir$ $(0<r<\infty)$.

Bi-local propagator

- Therefore, the bi-local propagator ($\mathcal{D} = \mathcal{K}^{-1}$) is

$$\mathcal{D}(\tau_1, \tau_2; \tau_1', \tau_2') \propto J^{-1} \left[\int_0^\infty dr \int_{-\infty}^\infty dw \, \frac{r}{\sinh(\pi r)} \frac{u^*_{ir,w}(\tau_1, \tau_2)\, u_{ir,w}(\tau_1', \tau_2')}{\widetilde{g}(ir) - 1} + \sum_{n=0}^\infty \int_{-\infty}^\infty dw \, \frac{\nu\, u^*_{\nu,w}(\tau_1, \tau_2)\, u_{\nu,w}(\tau_1', \tau_2')}{\widetilde{g}(\nu) - 1} \Bigg|_{\nu = \frac{3}{2} + 2n} \right]$$

where the eigenvalue (for $q = 4$) is

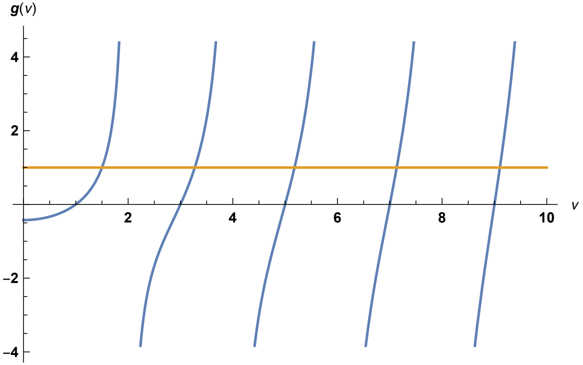
$$\widetilde{g}(\nu) \equiv -\frac{2\nu}{3} \cot\left(\frac{\pi\nu}{2}\right)$$

- The zero mode $\nu = \frac{3}{2}$, ($n = 0$) gives a divergence since

$$\widetilde{g}\left(\tfrac{3}{2}\right) = 1$$

Poles

- Poles are determined by $$\widetilde{g}(\nu) = -\frac{2\nu}{3}\cot\left(\frac{\pi\nu}{2}\right) = 1$$



$$\nu_0 = 3/2$$
$$\nu_1 = 3.274...$$
$$\nu_2 = 5.179...$$
$$\nu_3 = \cdots$$

2D masses m_{2d} are obtained by $\nu = \sqrt{\frac{1}{4} + m_{2d}^2}$.

3. Schwarzian Action

• The effective action associated with the zero mode was conjectured to be given by the Schwarzian derivative [Kitaev '15].

$$S[f] \propto \int d\tau \, \mathrm{Sch}(f(\tau), \tau)\,, \qquad \mathrm{Sch}(f(\tau), \tau) \equiv \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2$$

• This was confirmed at the quadratic level together with symmetry discussions [Maldacena & Stanford '16]; $f(\tau) = \tau + \epsilon(\tau)$

$$S[f] \propto \int d\tau \left[(\epsilon'')^2 - \left(\tfrac{2\pi}{\beta}\right)^2 (\epsilon')^2 \right]$$

• We demonstrate how to derive Schwarzian action for all order, up to a numerical constant, (which needs to be fixed by a numerical evaluation).
[Jevicki, K.S. & Yoon '16], [Jevicki & K.S. '16]

IR breaking

- The kinetic term breaks the conformal symmetry:

$$S_{\mathsf{col}}[\Psi] = \frac{N}{2}\int d\tau_1 \left[\partial_1 \Psi(\tau_1,\tau_2)\right]_{\tau_2=\tau_1} + S_{\mathsf{c}}$$

- Hence, the (leading order in $1/J$) effective action is given by this form:

$$S[f] = \frac{N}{2}\int d\tau_1 \left[\partial_1 \Psi_{0,f}(\tau_1,\tau_2)\right]_{\tau_2=\tau_1}$$

- Now the question is how to evaluate this action.

$q = 2$ **model**

• $q = 2$ case is simple, [Jevicki, K.S. & Yoon '16]:

$$\Psi_{0,f}(\tau_1, \tau_2) = -\frac{1}{\pi J}\left(\frac{\sqrt{f'(\tau_1)f'(\tau_2)}}{|f(\tau_1) - f(\tau_2)|}\right) = -\frac{1}{\pi J}\left(\frac{1}{|\tau_{12}|} + \frac{|\tau_{12}|}{12}\mathrm{Sch}(f(\tau_2),\ t_2) + \cdots\right)$$

where we expanded in the $\tau_1 \to \tau_2$ limit.

• Eliminate the diverging contribution from the first term, this leads to

$$S[f] = -\frac{N}{24\pi J}\int d\tau\ \mathrm{Sch}(f(\tau), \tau)$$

• For $q > 2$ models, more complicated regularization is needed. Nevertheless, still one can derive the Schwarzian action for all order, up to a numerical coefficient.

$1/J$ saddle-point correction

- We expand the exact saddle-point solution by $1/J$ as:

$$\Psi_{\text{cl}}(\tau_1,\tau_2) \;=\; \Psi_0(\tau_1,\tau_2) \,+\, \Psi_1(\tau_1,\tau_2) \,+\, \cdots$$

- The saddle-point equation of Ψ_1 is given by

$$\int d\tau_3 d\tau_4 \, \mathcal{K}(\tau_1,\tau_2;\tau_3,\tau_4)\Psi_1(\tau_3,\tau_4) \;=\; \partial_1\delta(\tau_{12})$$

where $\mathcal{K}(\tau_1,\tau_2;\tau_3,\tau_4) = \Psi_0^{-1}(\tau_{13})\Psi_0^{-1}(\tau_{24}) + (q-1)J^2\delta(\tau_{13})\delta(\tau_{24})\Psi_0^{q-2}(\tau_{12})$.

- Using an ansatz for Ψ_1 $(s>0)$:

$$\Psi_1^{(s)}(\tau_1,\tau_2) \;=\; B_1\,\frac{\text{sgn}(\tau_{12})}{(J|\tau_{12}|)^{\frac{2}{q}+2s}}$$

it turned out that $s = 1/2$ is the correct answer.

s-regularization

• Using the Ψ_1 equation, we rewrite

$$S[f] = -\frac{N}{2}\lim_{s\to 1/2}\int d\tau_1 d\tau_2 d\tau_3 d\tau_4\, \Psi_{0,f}(\tau_1,\tau_2)\mathcal{K}(\tau_1,\tau_2;\tau_3,\tau_4)\Psi_1^{(s)}(\tau_3,\tau_4)$$

• Here, we define our regularization by exchanging the order of the integrations and $s \to 1/2$ limit.

• After evaluating the integrals and limit, we obtain [Jevicki & K.S. '16]

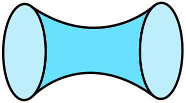
$$S[f] = -\frac{\gamma B_1 N}{2\pi J}\int d\tau\, \mathrm{Sch}(f(\tau),\tau)$$

with a q-dependent function γ.

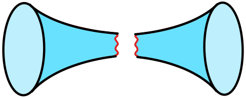
• A similar regularization method replacing the delta function source by a polynomial type source was used in [Kitaev & Suh '17].

4. Partially Disorder-Averaged SYK

• Two set of SYK model with the common random coupling (with disorder average) has wormhole saddles [Saad, Shenker & Stanford '18].



• However, the SYK model with a fixed-coupling constant (without disorder average) has the so called "half-wormhole" saddles [Saad, Shenker, Stanford & Yao '21].



• This discovery led us to study more about the fixed-coupling SYK model and the transition from the ordinary totally disorder-averaged model.

Partial disorder-averaging

• Now we introduce a partial disorder-averaging by deforming the probability distribution of the random coupling as

$$P(J_{i_1\cdots i_q}) = \exp\left[-\frac{N^{q-1}}{2(q-1)!}\sum_{i_1<\cdots<i_q}^{N}\left(\frac{J_{i_1\cdots i_q}^2}{J^2}+\frac{(J_{i_1\cdots i_q}-J_{i_1\cdots i_q}^{(0)})^2}{\sigma^2}\right)\right]$$

with

$$\left\langle\mathcal{O}\right\rangle_J \equiv \mathcal{N}_\sigma^{-1}\int\prod_{i_1<\cdots<i_q}^{N} dJ_{i_1\cdots i_q}\,P(J_{i_1\cdots i_q})\,\mathcal{O}$$

• $\sigma=\infty$ corresponds to the total disorder-averaging coincides with the ordinary SYK model, while $\sigma=0$ corresponds to totally fixing the coupling constant $J_{i_1\cdots i_q}=J_{i_1\cdots i_q}^{(0)}$.

Partial disorder-averaged partition function

- Taking the partial disorder-averaging of the partition function, we obtain

$$\big\langle Z \big\rangle_J = \int D\chi_i \, e^{-S_{\text{eff}}[\chi]}$$

with

$$S_{\text{eff}}[\chi] = \frac{1}{2}\int d\tau \sum_{i=1}^{N} \chi_i \partial_\tau \chi_i - \frac{\widetilde{J}^2}{2qN^{q-1}} \int d\tau_1 d\tau_2 \left(\sum_{i=1}^{N} \chi_i(\tau_1)\chi_i(\tau_2) \right)^q - \frac{i^{\frac{q}{2}}\widetilde{J}^2}{\sigma^2} \int d\tau \sum_{i_1<\cdots<i_q}^{N} J^{(0)}_{i_1\cdots i_q} \chi_{i_1}\cdots\chi_{i_q}$$

where we introduced

$$\frac{1}{\widetilde{J}^2} \equiv \frac{1}{J^2} + \frac{1}{\sigma^2}$$

- This coupling behaves as $\widetilde{J} \to J$ $(\sigma \to \infty)$ and $\widetilde{J} \to \sigma$ $(\sigma \to 0)$.

Decomposition of external coupling

- Let us now consider a specific form of the external coupling $J^{(0)}_{i_1\cdots i_q}$, such that it's decomposed by anti-symmetric vector-like variables θ_i as

$$J^{(0)}_{i_1\cdots i_q} = \frac{(q-1)!J_0}{i^{\frac{q}{2}}N^{q-1}}\,\theta_{i_1}\cdots\theta_{i_q}\,, \qquad \text{with} \qquad \{\theta_i,\theta_j\} = \delta_{ij}\,,$$

- After using this decomposition, we have the effective action

$$S_{\text{eff}}[\chi] = \frac{1}{2}\int d\tau\sum_{i=1}^{N}\chi_i\partial_\tau\chi_i - \frac{\widetilde{J}^2}{2qN^{q-1}}\int d\tau_1 d\tau_2\left(\sum_{i=1}^{N}\chi_i(\tau_1)\chi_i(\tau_2)\right)^q - \frac{J_\sigma}{qN^{q-1}}\int d\tau\left(\sum_{i=1}^{N}\theta_i\chi_i(\tau)\right)^q$$

where

$$J_\sigma \equiv \frac{\widetilde{J}^2 J_0}{\sigma^2}\,, \qquad J_\sigma \to 0\ (\sigma\to\infty)\,, \qquad J_\sigma \to J_0\ (\sigma\to 0)$$

Local collective fields

- Now it is useful to employ the following Hubbard–Stratonovich trick:

$$
\begin{aligned}
1 = & \int \prod_{\tau_1,\tau_2} DG(\tau_1,\tau_2) \int \prod_{\tau} DG_\sigma(\tau) \\
& \times \delta\left(G(\tau_1,\tau_2) - \frac{1}{N}\sum_{i=1}^{N}\chi_i(\tau_1)\chi_i(\tau_2)\right)\delta\left(G_\sigma(\tau) - \frac{1}{N}\sum_{i=1}^{N}\theta_i\chi_i(\tau)\right)
\end{aligned}
$$

- By inserting the above identity to the partially disorder-averaged partition function and performing the Gaussian integral for χ_i, we obtain

$$
\left\langle Z \right\rangle_J = \mathcal{N}_\sigma^{-1} \int DG D\Sigma DG_\sigma D\Sigma_\sigma \, e^{-S_{\text{eff}}[G,\Sigma,G_\sigma,\Sigma_\sigma]}
$$

New effective action

- The new effective action is now

$$
\begin{aligned}
S_{\text{eff}}[G,\Sigma,G_\sigma,\Sigma_\sigma] \;=\; & -\frac{N}{2}\mathrm{Tr}\log(-\Sigma) - \frac{N}{2}\int d\tau \left[\partial_\tau G(\tau,\tau')\right]_{\tau'=\tau} \\
& + \frac{N}{2}\int d\tau_1 d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{\widetilde{J}^2}{q}G(\tau_1,\tau_2)^q\right) \\
& + N\int d\tau \left(\Sigma_\sigma(\tau)G_\sigma(\tau) - \frac{J_\sigma}{q}G_\sigma(\tau)^q\right) \\
& - \frac{N}{4}\int d\tau_1 d\tau_2\, \Sigma_\sigma(\tau_1)\Sigma^{-1}(\tau_1,\tau_2)\Sigma_\sigma(\tau_2)
\end{aligned}
$$

- The last term is the interaction term between the bi-local sector and the local sector.

As "half" of the bi-local fields

- If we consider the combination of $(G_\sigma(\tau_1)G_\sigma(\tau_2))^q$, this is written as

$$\Big(G_\sigma(\tau_1)G_\sigma(\tau_2)\Big)^q = \frac{i^q}{\big((q-1)!\big)^2 J_0^2 N^2} \sum_{i_1,\cdots,i_q}^{N} J^{(0)}_{i_1\cdots i_q}\, \chi_{i_1}(\tau_1)\cdots\chi_{i_q}(\tau_1) \times \sum_{j_1,\cdots,j_q}^{N} J^{(0)}_{j_1\cdots j_q}\, \chi_{j_1}(\tau_2)\cdots\chi_{j_q}(\tau_2)$$

- If we take the total disorder-averaging of $J^{(0)}_{i_1\cdots i_q}$ for this quantity, we find

$$\Big\langle G_\sigma^q(\tau_1)G_\sigma^q(\tau_2)\Big\rangle_{J_0} = \frac{i^q \widetilde{J}^2}{(q-1)!J_0^2 N}\, G(\tau_1,\tau_2)^q$$

5. Summaries

• The SYK model might be very useful to understand holography and quantum gravity better.

• The zero mode sector of the model dominates the low energy limit of the model, which is described by the Schwarzian theory, that is dual to JT gravity in AdS_2.

• However, the model predicts an infinitely many matter contributions coming from the non-zero mode sector.

• A partially disorder-averaged SYK model is also helpful to understand Euclidean wormhole physics, and possible for condensed matter physics as well. [Goto, KS & Ugajin '21]

• The SYK model might also be useful to understand de-Sitter quantum gravity [Susskind '21]

Thank you!