$T\bar{T}$ Deformation of Stress-Tensor Correlators from Random Geometry

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Shinji Hirano (Wits) and MS, 2003.06300
Tatsuki Nakajima (Nagoya), SH, MS, 2012.03972

What is $T\bar{T}$?

In 2D QFT, the $T\bar{T}$ operator is defined by

$$\mathcal{O}_{T\bar{T}} \equiv T\bar{T} - \Theta^2$$

$$T = T_{zz}\,,\quad \bar{T} = T_{\bar{z}\bar{z}}\,,\ \Theta = T_{z\bar{z}}$$

$$z = x^1 + ix^2$$

Various other expressions:

$$\mathcal{O}_{T\bar{T}} = \frac{1}{8}\left(T^{ij}T_{ij} - T^i_iT^j_j\right)$$

$$= -\frac{1}{4}\det T^i{}_j = -\frac{1}{8}\epsilon^{ik}\epsilon^{jl}T_{ij}T_{kl}$$

What is $T\bar{T}$?

[Zamolodchikov 2004]

Nice properties:

$$\lim_{z\to z'}[T(z)\bar{T}(z') - \Theta(z)\Theta(z')] = \mathcal{O}_{T\bar{T}}(z') + (\text{derivatives})$$

$\langle \mathcal{O}_{T\bar{T}} \rangle = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle^2$ ← Not just for $|0\rangle$ but for $|n\rangle$

$T\bar{T}$ deformation

$$S_{\text{deformed}} = S_{\text{CFT}} + \mu \int \mathcal{O}_{T\bar{T}}$$ **(roughly)**

- Non-renormalizable ($[\mu] = \text{mass}^{-2} = \text{length}^2$).
- Still, surprisingly predictable
  - Energy spectrum of a theory on a circle of radius R

non-unitary

$$E_n(R,\mu) = \frac{\pi R}{\mu}\left(1 - \sqrt{1 - \frac{2C_n}{\pi R^2}\mu}\right), \quad C_n = \frac{\Delta_n + \bar{\Delta}_n - c/12}{R}$$

[Smirnov-Zamolodchikov '16]
[Cavaglia, Negro, Szecsenyi, Tateo '16]
Cf. [Haruna, Ishii, Kawai, Sakai, Yoshida '20]

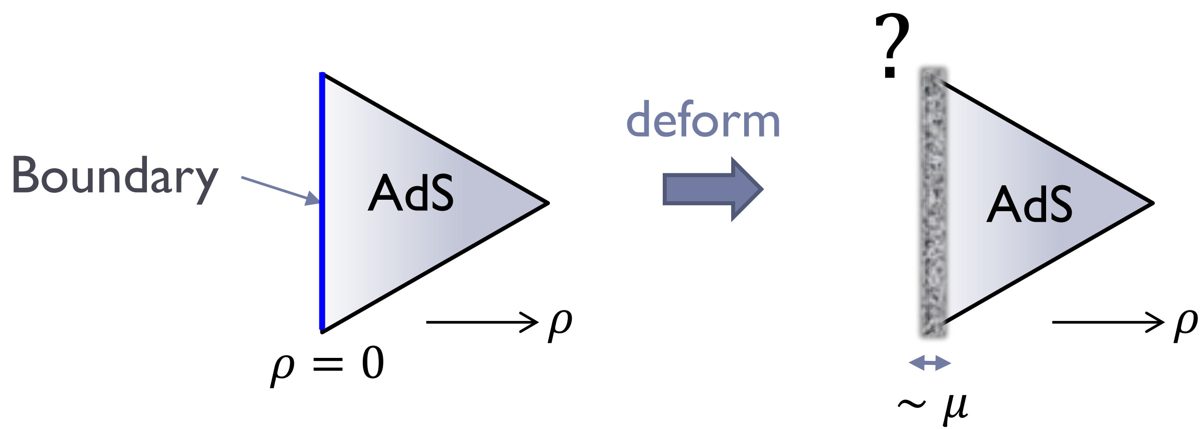
  - Integrable
  - Thermodynamics ($\mu < 0$: Hagedorn)
  - Deformed theory with $(g, T) \sim$ undeformed theory with (g', T')

Holography

- $T\bar{T}$: irrelevant in IR, relevant in UV

  → Non-normalizable in AdS

  → Change AdS asymptotics?



"Undo decoupling"?

Holography: moving field theory into bulk

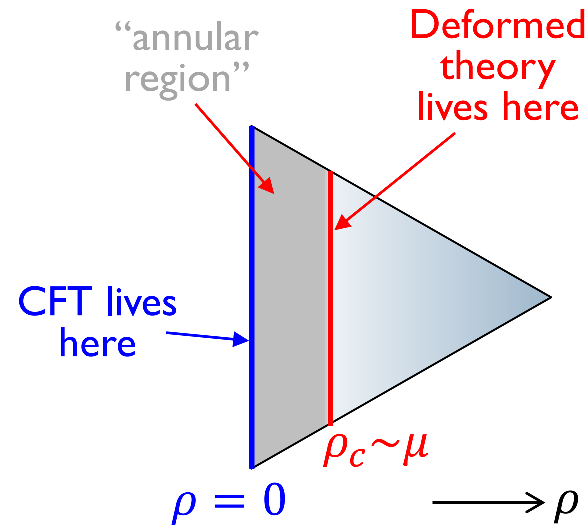
The deformed theory corresponds to placing the field theory at $\rho_c \sim \mu$ of the bulk

(ρ: radial coordinate in Fefferman-Graham coordinates)

- $T\bar{T}$ deformation makes the bndy cond be naturally given at ρ_c
- CFT living at $\rho = 0$ is "equivalent" to deformed theory at $\rho_c \sim \mu$

$$S_{\mathrm{CFT}} = S_{\mathrm{deformed}} + S_{\mathrm{annular}}$$

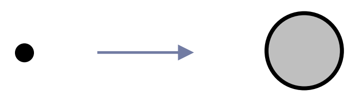
- Valid only in pure gravity without matter (so far)



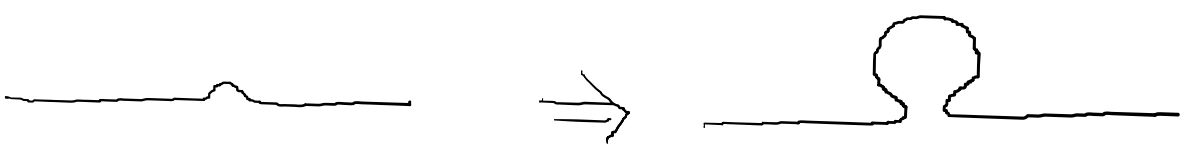
[McGough, Mezei, Verlinde '16]
[Guica-Monten '19]
[Caputa-Datta-Jiang-Kraus '20]

$T\bar{T}$ and width of particles [Cardy-Doyon '20] [Jiang '20]

Deformation makes particles have "width" μ



- $T\bar{T}$ deformation dynamically changes the metric
- $T\bar{T}$ deforming free scalars → Nambu-Goto with tension $-\mu$
  [Cavaglia, Negro, Szecsenyi, Tateo '16]



A lot more to explore about $T\bar{T}$ deformation!

“Random geometry” by Cardy [Cardy ’18]

Idea: rewrite $T\bar{T}$ using Hubbard-Stratonovich transformation

$$\exp\left[-\mu \int T^2\right] \sim \int [dh] \exp\left[\frac{1}{\mu}\int h^2 - \underbrace{\int hT}_{\text{deformation of backgnd geometry}}\right]$$

(because $T^{ij} \sim \frac{\delta S}{\delta g_{ij}}$)

This talk:

- Apply this to compute $T\bar{T}$-deformed T correlators
  - Previous results: 2pt func at $\mathcal{O}(\mu^2)$, 3pt func at $\mathcal{O}(\mu)$
- Develop a new method to compute T-correlators
  - We computed 4pt func at $\mathcal{O}(\mu)$
- Result is applicable to any deformed CFT

What we do:

- Compute $\mathcal{O}(\mu)$ correction to Liouville-Polyakov anomaly action

$$S_0[g] = \frac{c}{96\pi} \int d^2x \sqrt{g}\, R\, \Box^{-1} R$$

$$\equiv S_{\mu=0}[g] \quad \rightarrow \quad S_{\mu}[g]$$

- Vary backgnd metric $g \rightarrow g + h$ to compute T-correlators (explicit computations at $\mathcal{O}(\mu)$)

Plan

1. Introduction $\checkmark$
2. $T\bar{T}$ deformation as random geometries
3. $T\bar{T}$-deformed Liouville action
4. Stress tensor correlators (technical!)
5. $T\bar{T}$-deformed OPEs
6. Discussions

$T\overline{T}$ deformation as random geometries

$T\bar{T}$ deformation

- $T\bar{T}$-deformed theory is defined incrementally by:

$$S[\mu+\delta\mu]-S[\mu]=\frac{\delta\mu}{\pi^2}\int d^2x\sqrt{g}\,\mathcal{O}_{T\bar{T}}\equiv\delta S$$

$$\mathcal{O}_{T\bar{T}}\equiv T\bar{T}-\Theta^2=-\frac{1}{8}\epsilon_{ik}\epsilon_{jl}T^{ij}T^{kl}$$

T_{ij} : stress-energy tensor of the μ-deformed theory

* Our convention for T_{ij}: $\delta_g S=\frac{1}{4\pi}\int T^{ij}\delta g_{ij}$

- $T\bar{T}$-deformed theory with finite μ is obtained by iteration
- We will often suppress $d^2x\sqrt{g}$

$T\bar{T}$ = random geometries

- HS transformation:

$$e^{-\delta S} = e^{\frac{\delta\mu}{8\pi^2}\int d^2x\sqrt{g}\epsilon_{ik}\epsilon_{jl}T^{ij}T^{kl}}$$

$$\propto \underbrace{\int [dh]\exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}}_{\text{Gaussian integral over } h} - \frac{1}{4\pi}\underbrace{\int d^2x\sqrt{g}\,h_{ij}T^{ij}}_{\text{Change in backgnd metric } g\to g+h}\right]$$

Saddle point: $h^*_{ij} = -\frac{\delta\mu}{\pi}\,\epsilon_{ik}\epsilon_{jl}T^{kl} = \mathcal{O}(\delta\mu)$

⇨ "Master formula"

$$\boxed{\langle\ldots\rangle_{\mu+\delta\mu,g} = \mathcal{N}\int [dh]\,\exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right]\langle\ldots\rangle_{\mu,g+h}}$$

"Master formula"

$$\langle \ldots \rangle_{\mu+\delta\mu,g} = \mathcal{N} \int [dh] \, \exp\left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g}\, \epsilon^{ik} \epsilon^{jl} h_{ij} h_{kl} \right] \langle \ldots \rangle_{\mu,g+h}$$

Can be used to reproduce various results [Hirano+MS 2003.06300]

- Energy spectrum

$$E_n(R,\mu) = \frac{\pi R}{\mu}\left(1 - \sqrt{1 - \frac{2C_n}{\pi R^2}\mu}\right), \qquad C_n = \frac{\Delta_n + \bar{\Delta}_n - c/12}{R}$$

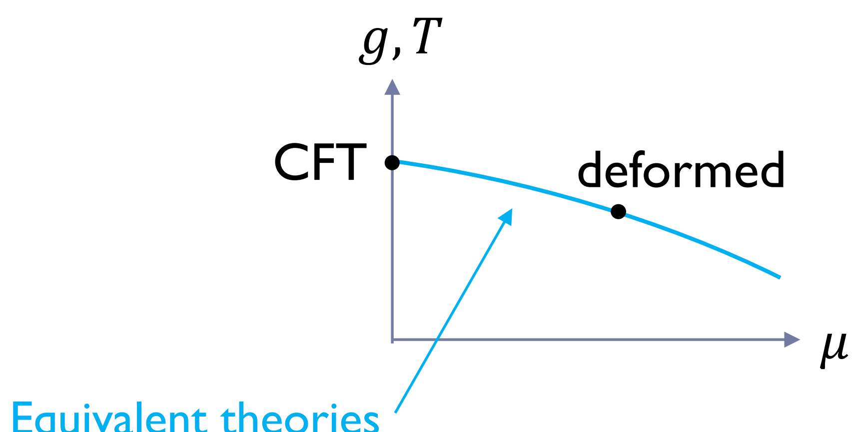
- Cylinder partition function
- Matter correlation function

$$\boxed{\langle \dots \rangle_{\mu+\delta\mu,g} = \mathcal{N} \int [dh] \, \exp \left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g} \, \epsilon^{ik} \epsilon^{jl} h_{ij} h_{kl} \right] \langle \dots \rangle_{\mu,g+h}}$$

In saddle-pt approximation,

$T\bar{T}$-deformed theory with g, T	=	Undeformed theory with g', T'

(up to the HS action)



Cf. bulk picture

Parametrizing h_{ij}

In 2D, any $h_{ij} = \underbrace{\text{diff}}_{x \to x + \alpha} + \underbrace{\text{Weyl}}_{ds^2 \to e^{2\Phi} ds^2}$

$$\Rightarrow \quad h_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + 2 g_{ij} \Phi$$

Useful to write $\Phi = \phi - \frac{1}{2} \nabla_k \alpha^k$

$\Rightarrow$ Another form of the master formula

$$\langle \ldots \rangle_{\mu+\delta\mu,g} = \mathcal{N} \int [d\alpha][d\phi] \, \exp\left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g} \left(\alpha_i \left(\Box_{\rm v} + \frac{R}{2} \right) \alpha^i + 4\phi^2 \right) \right] \langle \ldots \rangle_{\mu,g+h}.$$

Summary so far

- $T\bar{T}$ = random geometries

$$h_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + 2 g_{ij} \left(\phi - \frac{1}{2} \nabla \cdot \alpha \right)$$

- "Master formula"

$$\langle \ldots \rangle_{\mu+\delta\mu, g} = \mathcal{N} \int [d\alpha][d\phi] \, \exp\left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g} \left(\alpha_i \left(\Box_{\mathrm{v}} + \frac{R}{2} \right) \alpha^i + 4\phi^2 \right) \right] \langle \ldots \rangle_{\mu, g+h} .$$

$T\overline{T}$ deformed Liouville action

Liouville-Polyakov action (1)

- Dependence of CFT_2 on backgnd metric $g_{ij}(x)$ is determined by conformal anomaly [Polyakov '81]:

$$Z_0[g] = e^{-S_0[g]} \underline{Z_0[\delta]}$$

$= 1$ for flat $\mathbb{R}^2$

$$S_0[g] = \frac{c}{96\pi} \int d^2x \sqrt{g}\, R\, \Box^{-1} R$$

Liouville (-Polyakov) action

$\Downarrow$ Conformal gauge $ds^2 = e^{2\Omega} dz\, d\bar{z}$

$$\boxed{Z_0[e^{2\Omega}\delta] = e^{-S_0[e^{2\Omega}\delta]}, \qquad S_0[e^{2\Omega}\delta] = -\frac{c}{24\pi} \int d^2x\, \delta^{ij} \partial_i \Omega\, \partial_j \Omega}$$

conformal-gauge Liouville action

Cf. For general fiducial metric $\hat{g}$,

$$S_0[e^{2\Omega}\hat{g}] = -\frac{c}{24\pi} \int d^2x \sqrt{\hat{g}} \left(\hat{g}^{ij} \partial_i \Omega \partial_j \Omega + \hat{R}\Omega \right) + S_0[\hat{g}]$$

But note that we are not doing Liouville field theory

Liouville-Polyakov action (2)

$$S_0[g] = \frac{c}{96\pi} \int d^2x \sqrt{g}\, R\, \Box^{-1} R$$
$$S_0[e^{2\Omega}\delta] = -\frac{c}{24\pi} \int d^2x\, \delta^{ij} \partial_i \Omega\, \partial_j \Omega$$

- Valid for *any* CFT_2
- We can compute correlators $\langle TT \dots \rangle$ by varying $g \to g + h$ and taking derivatives of S_0 with respect to h
- The conformal form $S_0[e^{2\Omega}\delta]$ contains the same information as $S_0[g]$

→ Can also use $S_0[e^{2\Omega}\delta]$ to compute $\langle TT \dots \rangle$

(We will come back to this point later)

$T\bar{T}$-deforming Liouville action (1)

- Let's consider how Liouville action $S_0[g]$ is $T\bar{T}$-deformed (at $\mathcal{O}(\delta\mu)$)

$$e^{-(S_0[g]+\delta S[g])} \sim \int [d\alpha][d\phi] e^{-\frac{1}{8\delta\mu}\int \epsilon\epsilon hh - S_0[g+h]}$$

From the deformed action $\delta S[g]$, we can compute any $\langle TT \ldots \rangle$ for *any* $T\bar{T}$-deformed CFT

$$\Downarrow$$

Need to evaluate $S_0[g+h]$

Possible approaches:

- We could expand $S_0[g+h]$ in h
- But we take a different approach

$T\bar{T}$-deforming Liouville action (2)

Our approach:

- In 2D, we can bring $g+h$ back to original metric via diff, up to Weyl rescaling (we have already discussed this):

$$\begin{cases} \left(g_{ij}(x)+h_{ij}(x)\right)dx^i dx^j = e^{2\Psi(\tilde{x})} g_{ij}(\tilde{x}) d\tilde{x}^i d\tilde{x}^j \\ \tilde{x}^i = x^i + A^i(x) \end{cases}$$

For some $A^i(x), \Psi(\tilde{x})$

This is possible even for <u>finite</u> $h_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + 2 g_{ij} \Phi$

$$A^i(x) = \alpha^i(x) + A^i_{(2)}(x) + A^i_{(3)}(x) + \cdots\,,$$
$$\Psi(\tilde{x}) = \Phi(\tilde{x}) + \Psi_{(2)}(\tilde{x}) + \Psi_{(3)}(\tilde{x}) + \cdots\,.$$

$A_{(2)}, \Psi_{(2)}$, ... are complicated (nonlocal)

$\rightarrow \quad S_0[g(x)+h(x)] = S_0[e^{2\Psi(\tilde{x})} g(\tilde{x})] \quad \rightarrow$ easy to evaluate

$T\bar{T}$-deforming Liouville action (3)

For $e^{2\Psi(\tilde{x})} g_{ij}(\tilde{x}) \equiv g'_{ij}(\tilde{x})$,

$$\sqrt{g'(\tilde{x})} = e^{2\Psi(\tilde{x})}\sqrt{g(\tilde{x})}, \qquad R_{g'}(\tilde{x}) = e^{-2\Psi(\tilde{x})}(R_g(\tilde{x}) - 2\tilde{\Box}_g\Psi(\tilde{x})),$$
$$\tilde{\Box}_{g'} = e^{-2\Psi(\tilde{x})}\tilde{\Box}_g, \qquad \tilde{\Box}_{g'}^{-1} = \tilde{\Box}_g^{-1} e^{2\Psi(\tilde{x})},$$

$$\begin{aligned} S_0[g(x)+h(x)] &= S_0[g'(\tilde{x})] \\ &= \frac{c}{96\pi}\int d^2\tilde{x}\sqrt{g'(\tilde{x})}\, R_{g'}(\tilde{x})\, \tilde{\Box}_{g'}^{-1} R_{g'}(\tilde{x}) \\ &= \frac{c}{96\pi}\int d^2\tilde{x}\sqrt{g(\tilde{x})}\, \left(R_g(\tilde{x}) - 2\tilde{\Box}_g\Psi(\tilde{x})\right) \tilde{\Box}_g^{-1} \left(R_g(\tilde{x}) - 2\tilde{\Box}_g\Psi(\tilde{x})\right) \\ &= \frac{c}{96\pi}\int d^2x\sqrt{g(x)}\, (R_g(x) - 2\Box_g\Psi(x))\, \Box_g^{-1}\, (R_g(x) - 2\Box_g\Psi(x)) \\ &= \frac{c}{96\pi}\int d^2x\sqrt{g}\left(R\Box^{-1}R - 4R\Psi + 4\Psi\Box\Psi\right), \end{aligned}$$

→ Using the master formula and carrying out Gaussian integration, we can get δS

$T\bar{T}$-deforming Liouville action (4)

$$\delta S[g] = \delta S_{\text{saddle}}[g] + \delta S_{\text{fluct}}[g]$$

$$\boxed{\delta S_{\text{saddle}}[g] = -\left(\frac{c}{48\pi}\right)^2 \delta\mu \int d^2x\sqrt{g}\,R\left(1 - \nabla^k \frac{1}{\Box_{\text{v}} + R/2}\nabla_k\right) R}$$

- Exact at $\mathcal{O}(\delta\mu)$
- Nonlocal (expected of $T\bar{T}$-deformed theory, but so was original LP action...)
- δS_{fluct} is fluctuation term coming from Gaussian integral.
  - Contains contribution from $\Psi_{(2)}$. Very complicated and divergent. Depends on measure.
  - Vanishes after regularization (this can be shown using conformal pert theory)
  - Non-vanishing at $\mathcal{O}(\delta\mu^2)$ [Kraus-Liu-Marolf '18]
  - Can be dropped at large c ($\delta S_{\text{saddle}} \sim c^{n+1}\delta\mu^n$), which is relevant for holography

$T\bar{T}$-deforming Liouville action (5)

δS is very simple in conformal gauge:

$$\boxed{\delta S_{\text{saddle}}[g] = \frac{c^2\,\delta\mu}{72\pi^2}\int d^2z\, e^{-2\Omega}\Big[-2(\partial\Omega)(\bar{\partial}\Omega)(\partial\bar{\partial}\Omega) + (\partial\Omega)^2(\bar{\partial}\Omega)^2\Big]}$$

- "Local" in Ω but it's really nonlocal (just like the CFT Liouville action)
- We will use this to compute T correlators

Higher order

- The same procedure gives a differential equation to determine the effective action $S_\mu[e^{2\Omega}\delta]$ at finite deformation μ:

$$\frac{\partial}{\partial\mu}S_\mu = \frac{1}{16}\int d^2z\, \frac{\delta S_\mu}{\delta\Omega}\, e^{-2\Omega}\left(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial e^{-2\Omega} + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial e^{-2\Omega} - 2\right)\frac{\delta S_\mu}{\delta\Omega}$$

- We ignored fluctuation (i.e., it's valid at large c)
- Recursive relation:

$$\mathbf{S}_{n+1} = \frac{1}{16}\sum_{k=0}^{n}\binom{n}{k}\int d^2z\,\frac{\delta\mathbf{S}_{n-k}}{\delta\Omega}\, e^{-2\Omega}\left(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial e^{-2\Omega} + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial e^{-2\Omega} - 2\right)\frac{\delta\mathbf{S}_k}{\delta\Omega}$$

where $S_\mu = \sum_n \frac{\mu^n}{n!}\mathbf{S}_n$

- No longer "local" in Ω at $\mathcal{O}(\mu^2)$

$$\frac{\delta\mathbf{S}_1}{\delta\omega} = -\frac{c^2}{18\pi^2}e^{-2\omega}\left[\left(\partial^2\omega - (\partial\omega)^2\right)\left(\bar\partial^2\omega - (\bar\partial\omega)^2\right) - (\partial\bar\partial\omega)^2\right]$$

Summary so far

Polyakov-Liouville action

$$S_0[g] = \frac{c}{96\pi}\int d^2x\sqrt{g}\,R\,\Box^{-1}R$$

$$S_0[e^{2\Omega}\delta] = -\frac{c}{24\pi}\int d^2x\,\delta^{ij}\partial_i\Omega\,\partial_j\Omega$$

$$\Downarrow$$

Deformed action at $\mathcal{O}(\delta\mu)$

$$\boxed{\begin{aligned}\delta S_{\text{saddle}}[g] &= -\left(\frac{c}{48\pi}\right)^2\delta\mu\int d^2x\sqrt{g}\,R\left(1-\nabla^k\frac{1}{\Box_{\text{v}}+R/2}\nabla_k\right)R\\ \delta S_{\text{saddle}}[g] &= \frac{c^2\,\delta\mu}{72\pi^2}\int d^2z\,e^{-2\Omega}\left[-2(\partial\Omega)(\bar\partial\Omega)(\partial\bar\partial\Omega)+(\partial\Omega)^2(\bar\partial\Omega)^2\right]\end{aligned}}$$

Was useful: $$h_{ij} = \underbrace{\nabla_i\alpha_j+\nabla_j\alpha_i}_{\text{diff}} + \underbrace{2g_{ij}\Phi}_{\text{Weyl}}$$

Stress-tensor correlators (technical!)

Getting T correlators (1)

- We can get $\langle TT \dots \rangle$ by varying $g \to g + h$, expanding $Z[g+h] = e^{-S_{\text{eff}}[g+h]}$ in h, and reading off coefficients.

$$\underline{S_{\text{eff}}[g+h] - S_{\text{eff}}[g]} = \frac{1}{4\pi}\int d^2x\sqrt{g}\, h_{ij}\langle T^{ij}\rangle_{g,\text{c}} - \frac{1}{2(4\pi)^2}\iint d^2x\sqrt{g}\, d^2x'\sqrt{g'}\, h_{ij}\, h'_{kl}\langle T^{ij}T'^{kl}\rangle_{g,\text{c}} + \cdots$$

We know this.
CFT: Liouville action S_0
deformed: δS we just computed

$$\langle \dots \rangle_{g,\text{c}} \equiv \langle \dots \rangle_{g,\text{connected part}} / \langle 1 \rangle_g$$

Getting T correlators (2)

- Flat background: $g=\delta,\ ds^2 = dz d\bar{z}.$ $\quad h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i + 2 g_{ij}\Phi$

$$\Rightarrow \quad h_{zz} = 2\partial\alpha, \quad h_{\bar{z}\bar{z}} = 2\bar{\partial}\bar{\alpha}, \quad h_{z\bar{z}} = \phi, \qquad \Phi = \phi - (\bar{\partial}\alpha + \partial\bar{\alpha})$$

$$\partial \equiv \partial_z, \quad \bar{\partial} \equiv \partial_{\bar{z}}, \qquad \alpha \equiv \alpha_z, \quad \bar{\alpha} \equiv \alpha_{\bar{z}}$$

$$S_{\text{eff}}[\delta + h] = \frac{2}{\pi}\int d^2x \left\langle \partial\alpha\,\bar{T} + \bar{\partial}\bar{\alpha}\,T + \phi\,\Theta \right\rangle_{\text{c}}$$
$$- \frac{1}{2}\left(\frac{2}{\pi}\right)^2 \iint d^2x\, d^2x' \left\langle (\partial\alpha\,\bar{T} + \bar{\partial}\bar{\alpha}\,T + \phi\,\Theta)(\partial'\alpha'\,\bar{T}' + \bar{\partial}'\bar{\alpha}'\,T' + \phi'\,\Theta') \right\rangle_{\text{c}} + \cdots$$

Coeff of $\partial\alpha$ $\Longleftrightarrow$ $\bar{T} = T_{\bar{z}\bar{z}}$

Coeff of $\bar{\partial}\bar{\alpha}$ $\Longleftrightarrow$ $T = T_{zz}$

Coeff of ϕ $\Longleftrightarrow$ $\Theta = T_{z\bar{z}}$

e.g.

$$S_{eff}[\delta + h] \supset \iint \frac{\bar{\partial}\bar{\alpha}(x)\bar{\partial}\bar{\alpha}(x')}{(z-z')^4}$$

$$\rightarrow \quad \langle T T' \rangle \sim \frac{1}{(z-z')^4}$$

Using conformal-gauge action

- Can use conformal-gauge action $S_{\text{eff}}[e^{2\Omega}\delta]$ instead of $S_{\text{eff}}[g]$ to compute T correlators

1. Start with flat backgnd, $ds^2 = dz\,d\bar{z}$
2. Vary $\delta \to \delta + h$. $ds'^2 = dz\,d\bar{z} + 2\left(\partial\alpha\,dz^2 + \bar{\partial}\bar{\alpha}\,d\bar{z}^2 + \phi\,dz\,d\bar{z}\right)$.
   (Here α, ϕ are finite.)
3. Find diff $x \to \tilde{x} = x + A(x)$ that brings metric into conformal gauge:
   $$ds'^2 = e^{2\Psi(\tilde{z},\bar{\tilde{z}})} d\tilde{z}\,d\bar{\tilde{z}}$$
4. Compute $S_{\text{eff}}[e^{2\Psi}\delta] = S_{\text{eff}}[\delta + h]$
5. Read off correlator from expansion

Similar to what we did when we computed deformed Liouville action. But here we need A, Ψ to higher order to compute higher correlator $\langle TTT \ldots \rangle$.

Check: the case of CFT

To check that this method works, let's apply it to some CFT correlators.

Conformal-gauge undeformed Liouville action:

$$S_{\text{eff}} = S_0\left[e^{2\Omega}\delta\right] = -\frac{c}{12\pi}\int d^2z\,\partial\Omega\,\bar{\partial}\Omega$$

Let's reproduce the known expressions

2pt: $\langle T_1 T_2 \rangle = \dfrac{c}{2z_{12}^4}$

3pt: $\langle T_1 T_2 T_3 \rangle = \dfrac{c}{z_{12}^2 z_{13}^2 z_{23}^2}$

CFT 2pt func (1)

To see how it goes, let's carry out this procedure for CFT.

$$\tilde{z} = z + A^z_{(1)} + A^z_{(2)} + \cdots, \qquad A^z_{(1)} = \alpha^z$$

$$\Psi = \Psi_{(1)} + \Psi_{(2)} + \cdots, \qquad \Psi_{(1)} = \Phi = \phi - (\partial\bar{\alpha} + \bar{\partial}\alpha)$$

Varied action:

$$S_0\left[e^{2\Psi(\tilde{z},\bar{\tilde{z}})}\delta\right] = -\frac{c}{12\pi}\int d^2\tilde{z}\,\partial_{\tilde{z}}\Psi\,\partial_{\bar{\tilde{z}}}\Psi$$

$$= -\frac{c}{12\pi}\int d^2z\,\partial\Psi\,\bar{\partial}\Psi + (\text{higher})$$

$$= -\frac{c}{12\pi}\int d^2z\,\partial\left(\phi - (\partial\underline{\bar{\alpha}} + \bar{\partial}\alpha)\right)\bar{\partial}\left(\phi - (\partial\underline{\bar{\alpha}} + \bar{\partial}\alpha)\right) + (\text{higher})$$

Want $\langle TT \rangle$ → Extract coeff of $\bar{\partial}\bar{\alpha}\,\bar{\partial}\bar{\alpha}$

CFT 2pt func (2)

$$S_0\left[e^{2\Psi(\tilde{z},\bar{\tilde{z}})}\delta\right] \supset -\frac{c}{12\pi}\int d^2z\,\partial^2\bar{\alpha}\,\partial\bar{\partial}\bar{\alpha} \;=\frac{c}{12\pi}\int d^2z\,\partial^3\bar{\alpha}\,\bar{\partial}\bar{\alpha}$$

$$=\frac{c}{12\pi}\int d^2z\,\partial^3\frac{1}{\bar{\partial}}\bar{\partial}\bar{\alpha}\,\bar{\partial}\bar{\alpha} \qquad \text{(“created” } \bar{\partial}\bar{\alpha}\text{)}$$

Here

$$\bar{\partial}\frac{1}{z}=2\pi\delta^2(z) \;\Rightarrow\; \frac{1}{\bar{\partial}}=\frac{1}{2\pi}\int\frac{d^2z'}{z-z'} \qquad \partial^3\frac{1}{\bar{\partial}}=\frac{-3}{\pi}\int\frac{d^2z'}{(z-z')^4}$$

Therefore

$$(\text{above})=\frac{-c}{4\pi}\int d^2z\,\frac{\bar{\partial}\bar{\alpha}(z)\bar{\partial}\bar{\alpha}(z')}{(z-z')^4} \;\Rightarrow\; \langle TT'\rangle=\frac{c}{2(z-z')^4} \quad \checkmark$$

Also,

$$\langle\Theta(z,\bar{z})\Theta(0)\rangle=-\frac{\pi c}{6}\partial\bar{\partial}\delta^2(z)\,, \qquad \left\langle T(z,\bar{z})\bar{T}(0)\right\rangle=-\frac{\pi c}{6}\partial\bar{\partial}\delta^2(z)\,,$$

$$\langle\Theta(z,\bar{z})T(0)\rangle=\frac{\pi c}{6}\partial^2\delta^2(z)\,, \qquad \left\langle\Theta(z,\bar{z})\bar{T}(0)\right\rangle=\frac{\pi c}{6}\bar{\partial}^2\delta^2(z)\,.$$

Cf. $\Theta=-\frac{1}{48}R$

CFT 3pt func

Conformal-gauge action $S_0[e^{2\Omega}\delta]$ is quadratic. How can we get $\langle TTT\rangle$?

1. Need to rewrite $\tilde{z}, \bar{\tilde{z}}$ in $S_0[e^{2\Psi(\tilde{z},\bar{\tilde{z}})}\delta]$ in terms of $z, \bar{z}$
2. $\Psi = \underbrace{\Psi_{(1)}}_{\mathcal{O}(h)} + \underbrace{\Psi_{(2)}}_{\mathcal{O}(h^2)} + \cdots$

$x + A(x)$

$$\Rightarrow \quad S_0\left[e^{2\Psi(\tilde{z},\bar{\tilde{z}})}\delta\right] = -\frac{c}{12\pi}\int d^2\tilde{z}\ \ \partial_{\tilde{z}}\Psi(\tilde{x})\ \partial_{\bar{\tilde{z}}}\Psi(\tilde{x})$$

$$\frac{\partial(\tilde{z},\bar{\tilde{z}})}{\partial(z,\bar{z})}d^2z \qquad \frac{\partial z}{\partial\tilde{z}}\partial + \frac{\partial\bar{z}}{\partial\tilde{z}}\bar{\partial} \qquad \frac{\partial z}{\partial\bar{\tilde{z}}}\partial + \frac{\partial\bar{z}}{\partial\bar{\tilde{z}}}\bar{\partial}$$

$$= \frac{c}{6\pi}\int d^2z(\bar{\partial}\bar{\alpha})\left[\partial^2(\partial\bar{\alpha})^2 - \partial^2(\bar{\alpha}\partial^2\bar{\alpha}) - \partial^3(\bar{\alpha}\partial\bar{\alpha}) - \partial^2(\partial\bar{\alpha})^2 + \partial^3\bar{A}_{(2)} + \cdots\right]$$

- By similar manipulations, can check $\langle T_1T_2T_3\rangle = \dfrac{c}{z_{12}^2z_{13}^2z_{23}^2}$
- All contributions are needed to reproduce the correct result

Explicit forms of second-order terms:

$$A_{(1)} = \alpha, \qquad \bar{A}_{(1)} = \bar{\alpha}, \qquad \Psi_{(1)} = \Phi = \phi - (\partial\bar{\alpha} + \bar{\partial}\alpha),$$

$$A_{(2)} = -\frac{2}{\partial}\big((\phi - \bar{\partial}\alpha)\partial\alpha\big), \qquad \bar{A}_{(2)} = -\frac{2}{\bar{\partial}}\big((\phi - \partial\bar{\alpha})\bar{\partial}\bar{\alpha}\big),$$

$$\Psi_{(2)} = -\phi^2 - 2(\alpha\bar{\partial}\phi + \bar{\alpha}\partial\phi) + (\bar{\partial}\alpha)^2 + 2\alpha\bar{\partial}^2\alpha + (\partial\bar{\alpha})^2 + 2\bar{\alpha}\partial^2\bar{\alpha}$$
$$+ 2\alpha\partial\bar{\partial}\bar{\alpha} + 2\bar{\alpha}\partial\bar{\partial}\alpha - 2\partial\alpha\bar{\partial}\bar{\alpha} + 2\frac{\partial}{\bar{\partial}}\big((\phi - \partial\bar{\alpha})\bar{\partial}\bar{\alpha}\big) + 2\frac{\bar{\partial}}{\partial}\big((\phi - \bar{\partial}\alpha)\partial\alpha\big).$$

$$A_{(n)} \equiv A_{(n)z}, \qquad \bar{A}_{(n)} \equiv A_{(n)\bar{z}}.$$

Deformed T-correlators (1)

We just reproduced CFT correlators.

Now consider deformed ones.

$$\delta S[e^{2\Omega}\delta] = \frac{c^2\,\delta\mu}{72\pi^2}\int d^2z\, \underbrace{e^{-2\Omega}\left[-2(\partial\Omega)(\bar\partial\Omega)(\partial\bar\partial\Omega) + (\partial\Omega)^2(\bar\partial\Omega)^2\right]}_{= -2(\partial\Omega)(\bar\partial\Omega)(\partial\bar\partial\Omega) + \mathcal{O}(\Omega^4)}$$

What's known in the literature at $\mathcal{O}(\delta\mu)$:

- 3pt functions (based on conformal pert theory [Kraus-Liu-Marolf '18], and random geom and WT id [Aharony-Vaknin '18])
- No 4pt functions

Deformed 3pt functions

Deformed 3pt func is just like CFT 2pt func;
Simply set $\Omega \to \Psi \sim \Phi = \phi - (\partial\bar{\alpha} + \bar{\partial}\alpha)$ and also $\tilde{x} \sim x$

$$\delta S[e^{2\Omega}\delta] \supset -\frac{c^2\delta\mu}{36\pi^2}\int d^2z\, \partial\bar{\partial}\Phi\, \partial\Phi\, \bar{\partial}\Phi \qquad\qquad \Phi = \phi - (\partial\bar{\alpha} + \bar{\partial}\alpha)$$

➡ Straightforward to read off

$$\langle \Theta(z_1) T(z_2) \bar{T}(z_3) \rangle = -\frac{c^2\delta\mu}{4\pi}\frac{1}{z_{12}^4 \bar{z}_{13}^4}$$

$$\langle T(z_1) \bar{T}(z_2) \bar{T}(z_3) \rangle = -\frac{c^2\delta\mu}{3\pi}\frac{1}{z_{12}^3 \bar{z}_{23}^5} + (z_2 \leftrightarrow z_3)$$

Reproduces known results

Deformed 4pt functions

Deformed 4pt func is similar to CFT 3pt func.

We have to take into account $\tilde{x} = x + A(x)$ and correction $\Psi_{(2)}$.
Also, δS has quartic terms $\mathcal{O}(\Omega^4)$ as well.

$$\langle T(z_1)T(z_2)\bar{T}(z_3)\Theta(z_4)\rangle = -\frac{c^2\delta\mu}{2\pi}\frac{1}{z_{41}^2 z_{42}^2 z_{12}^2 \bar{z}_{34}^4}$$

$$\langle T(z_1)T(z_2)T(z_3)\bar{T}(z_4)\rangle = \frac{c^2\delta\mu}{6\pi}\left[\frac{1}{z_{12}^2 z_{13}^3 z_{23}^2} + \frac{1}{z_{12}^3 z_{13}^2 z_{23}^2}\right]\frac{1}{\bar{z}_{14}^3} + \mathrm{perm}(z_1, z_2, z_3)$$

$$\langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\rangle = \frac{2c^2\delta\mu}{\pi z_{12}^5 \bar{z}_{34}^5}\left[\frac{z_{12}}{z_{31}} + \frac{\bar{z}_{34}}{\bar{z}_{13}} + 2\ln|z_{13}|^2\right] + (z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4)$$

All contributions are needed for final results

Deformed OPEs

Deformed OPEs?

$$\langle \Theta(z_1) T(z_2) \bar{T}(z_3) \rangle = -\frac{c^2 \delta\mu}{4\pi} \frac{1}{z_{12}^4 \bar{z}_{13}^4}$$

$$\langle T(z_1) \bar{T}(z_2) \bar{T}(z_3) \rangle = -\frac{c^2 \delta\mu}{3\pi} \frac{1}{z_{12}^3 \bar{z}_{23}^5} + (z_2 \leftrightarrow z_3)$$

$$\langle T(z_1) T(z_2) \bar{T}(z_3) \Theta(z_4) \rangle = -\frac{c^2 \delta\mu}{2\pi} \frac{1}{z_{41}^2 z_{42}^2 z_{12}^2 \bar{z}_{34}^4}$$

$$\langle T(z_1) T(z_2) T(z_3) \bar{T}(z_4) \rangle = \frac{c^2 \delta\mu}{6\pi} \left[\frac{1}{z_{12}^2 z_{13}^3 z_{23}^2} + \frac{1}{z_{12}^3 z_{13}^2 z_{23}^2} \right] \frac{1}{\bar{z}_{14}^3} + \mathrm{perm}(z_1, z_2, z_3)$$

$$\langle T(z_1) T(z_2) \bar{T}(z_3) \bar{T}(z_4) \rangle = \frac{2c^2 \delta\mu}{\pi z_{12}^5 \bar{z}_{34}^5} \left[\frac{z_{12}}{z_{31}} + \frac{\bar{z}_{34}}{\bar{z}_{13}} + 2 \ln |z_{13}|^2 \right] + (z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4)$$

Can understand these from deformed OPEs?

OPEs from 3pt funcs (1)

$$\left\langle \Theta(z_1)T(z_2)\bar{T}(z_3)\right\rangle = -\frac{c^2\delta\mu}{4\pi}\frac{1}{z_{12}^4\bar{z}_{13}^4}$$

$$\Rightarrow \quad \Theta(z)T(w) \sim -\frac{c\,\delta\mu}{2\pi}\frac{\bar{T}(z)}{(z-w)^4}+\cdots\ , \qquad \Theta(z)\bar{T}(w) \sim -\frac{c\,\delta\mu}{2\pi}\frac{T(z)}{(\bar{z}-\bar{w})^4}+\cdots$$

- 3pt func is reproduced
- Consistent with the flow equation $\Theta = -\frac{\delta\mu}{\pi}T\bar{T}$

OPEs from 3pt funcs (2)

More interesting:

$$\langle T(z_1)\bar{T}(z_2)\bar{T}(z_3)\rangle = -\frac{c^2\delta\mu}{3\pi}\frac{1}{z_{12}^3\bar{z}_{23}^5} + (z_2 \leftrightarrow z_3)$$

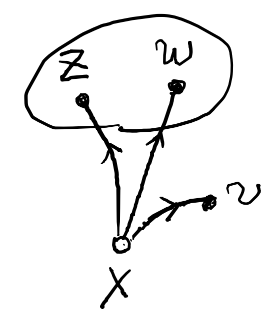
$$\Rightarrow \quad \bar{T}(z)\bar{T}(w) \sim -\frac{c\,\delta\mu}{\pi^2}\frac{1}{(\bar{z}-\bar{w})^5}\int d^2z' \ln(z-z')\bar{\partial}' T(z') + (z \leftrightarrow w) + \cdots$$

$$= -\frac{2c\,\delta\mu}{\pi}\frac{1}{(\bar{z}-\bar{w})^5}\int_X^z dz'\, T(z') + (z \leftrightarrow w)$$

- Can be regarded as coming from field-dependent diff
  [Conti, Negro, Tateo '18,'19] [Cardy '19]

$$\bar{T}(\bar{z}) \to \bar{T}(\bar{z}+\bar{\epsilon}) = \bar{T} + \bar{\partial}\bar{T}\bar{\epsilon}, \qquad \bar{\epsilon} = \frac{\delta\mu}{\pi}\int_X^z dz' T(z')$$

$$\Rightarrow \quad \bar{T}(\bar{z})\bar{T}(\bar{w}) \to \bar{T}(\bar{z})\bar{\partial}\bar{T}(\bar{w})\bar{\epsilon} \sim -\frac{2c}{(\bar{z}-\bar{w})^5}\bar{\epsilon}$$



- Non-local OPE

OPEs from 3pt funcs (3)

Different pair in the same 3pt func:

$$\langle \underline{T(z_1)\bar{T}(z_2)}\bar{T}(z_3)\rangle = -\frac{c^2\delta\mu}{3\pi}\frac{1}{z_{12}^3\bar{z}_{23}^5} + (z_2 \leftrightarrow z_3)$$

$$\Rightarrow \quad T(z)\bar{T}(w) \sim \frac{c\,\delta\mu}{6\pi}\frac{\bar{\partial}\bar{T}(w)}{(z-w)^3} - \frac{c\,\delta\mu}{6\pi}\frac{\partial T(z)}{(\bar{z}-\bar{w})^3} + \cdots$$

- Again, comes from field-dependent diff

$$\bar{T}(\bar{z}) \to \bar{T}(\bar{z}+\bar{\epsilon}) = \bar{T} + \bar{\partial}\bar{T}\bar{\epsilon}, \qquad \bar{\epsilon} = \frac{\delta\mu}{\pi}\int_X^z dz' T(z')$$

$$T(z)\bar{T}(\bar{w}) \to T(z)\,\bar{\epsilon}\,\bar{\partial}\bar{T}(\bar{w}) \sim \frac{\delta\mu}{\pi} T(z) \int_X^z dz'\, T(z')\bar{\partial}\bar{T}(\bar{w}) \to \text{(above)}$$

Consistency check with 4pt funcs

Check OPEs derived above using 4pt funcs:

$$\langle T(z_1)T(z_2)\underbrace{\bar{T}(z_3)\bar{T}(z_4)}_{\text{OPE}}\rangle \sim \frac{2}{\bar{z}_{34}^2}\langle T(z_1)T(z_2)\bar{T}(z_4)\rangle + \frac{1}{\bar{z}_{34}}\bar{\partial}_4\langle T(z_1)T(z_2)\bar{T}(z_4)\rangle$$ ← $\bar{T}\bar{T}$ OPE at $\mathcal{O}(\delta\mu^0)$

$$-\frac{2c\,\delta\mu}{\pi\bar{z}_{34}^5}\int_X^{z_3} dz'\langle T(z_1)T(z_2)T(z')\rangle + (z_3 \leftrightarrow z_4)$$ ← $\bar{T}\bar{T}$ OPE at $\mathcal{O}(\delta\mu)$

$$= -\frac{c^2\delta\mu}{3\pi z_{12}^5}\left[\frac{2}{\bar{z}_{41}^3\bar{z}_{34}^2} - \frac{3}{\bar{z}_{41}^4\bar{z}_{34}}\right] + \frac{2c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\left[-\frac{z_{12}z_{34}}{z_{31}z_{41}} + 2\ln\frac{z_{31}}{z_{41}}\right] + (z_1 \leftrightarrow z_2)$$

Agrees with the $z_{34} \to 0$ expansion of the 4pt func

Discussions

Summary:

- Studied the stress-energy sector of $T\bar{T}$-deformed theories using random geometry approach
  - Found $T\bar{T}$-deformed Liouville-Polyakov action exactly at $\mathcal{O}(\delta\mu)$
  - Derived equation to determine deformed action for finite μ
  - Developed technique to compute T-correlators

  Correlators at $\mathcal{O}(\delta\mu)$ is computable also by conformal perturbation theory. Random geometry approach seems to allow straightforward generalization to higher order, at least formally

- Deformed OPEs show sign of non-locality

Future directions:

- Go to higher order in $\delta\mu$
  - Solve the differential equation for $S_\mu[g]$? Large c (dual to classical gravity)?
  - All-order correlators, such as $G_\Theta(|z_{12}|) \equiv \langle \Theta(z_1)\Theta(z_2) \rangle$?
    Expectation: $G_\Theta < 0$ at short distances (negative norm)
    cf. [Haruna-Ishii-Kawai-Skai-Yoshida '20]
- Better understand the fluctuation part S_{fluct}
  - Why does it vanish at $\mathcal{O}(\delta\mu)$, as indicated by conf pert theory?
  - Use vanishing of it at $\mathcal{O}(\delta\mu)$ to regularize S_{fluct}
- More
  - Compute physically interesting quantities
  - Inclusion of matter → modify the $T\bar{T}$ operator?
  - Position-dependent coupling cf. [Chandra et al., 2101.01185]