Lattice fermions as spectral graphs
-Toward a new theorem-

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1. Review on Naive & Wilson fermions

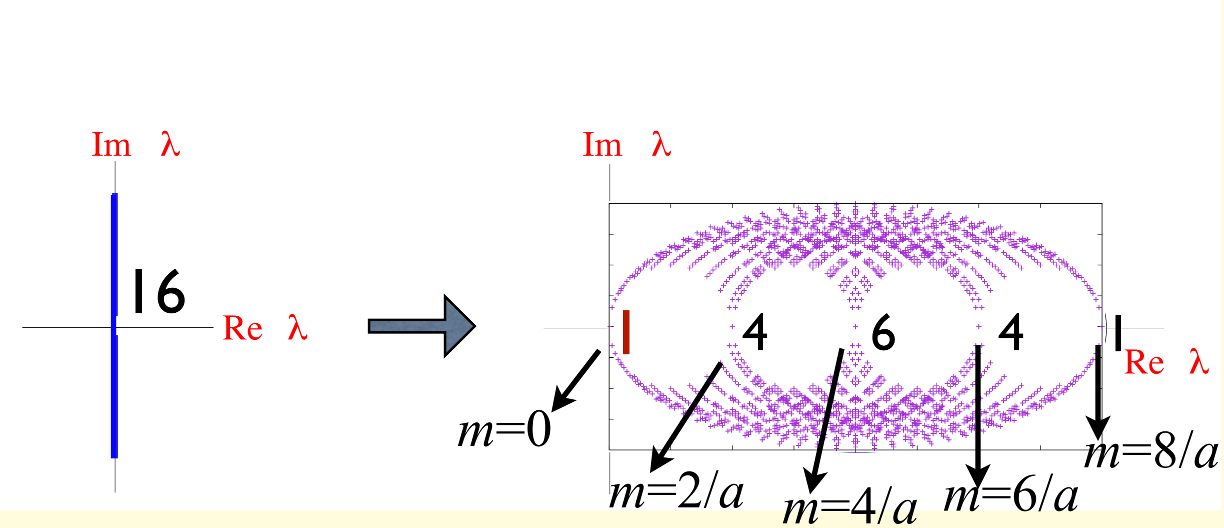
Wilson fermion : species-splitting mass fermion

Lattice fermion action with species-splitting term $\sum_{n,\mu} \frac{a^5}{2} \bar{\psi}_n (2\psi_n - \psi_{n+\mu} - \psi_{n-\mu})$

$$\Rightarrow \quad D_W(p) = \frac{1}{a} \sum_{\mu} [i\gamma_\mu \sin ap_\mu \; + \; \underline{(1 - \cos ap_\mu)}]$$

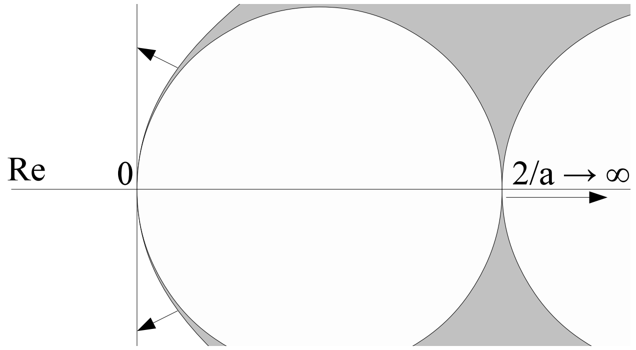
Physical $(0,0,0,0)$: $D_W(p) = i\gamma_\mu p_\mu \; + \; O(a)$

Doubler$(\pi/a,0,0,0)$: $D_W(p) = i\gamma_\mu p_\mu \; + \; \frac{2}{a} \; + \; O(a)$



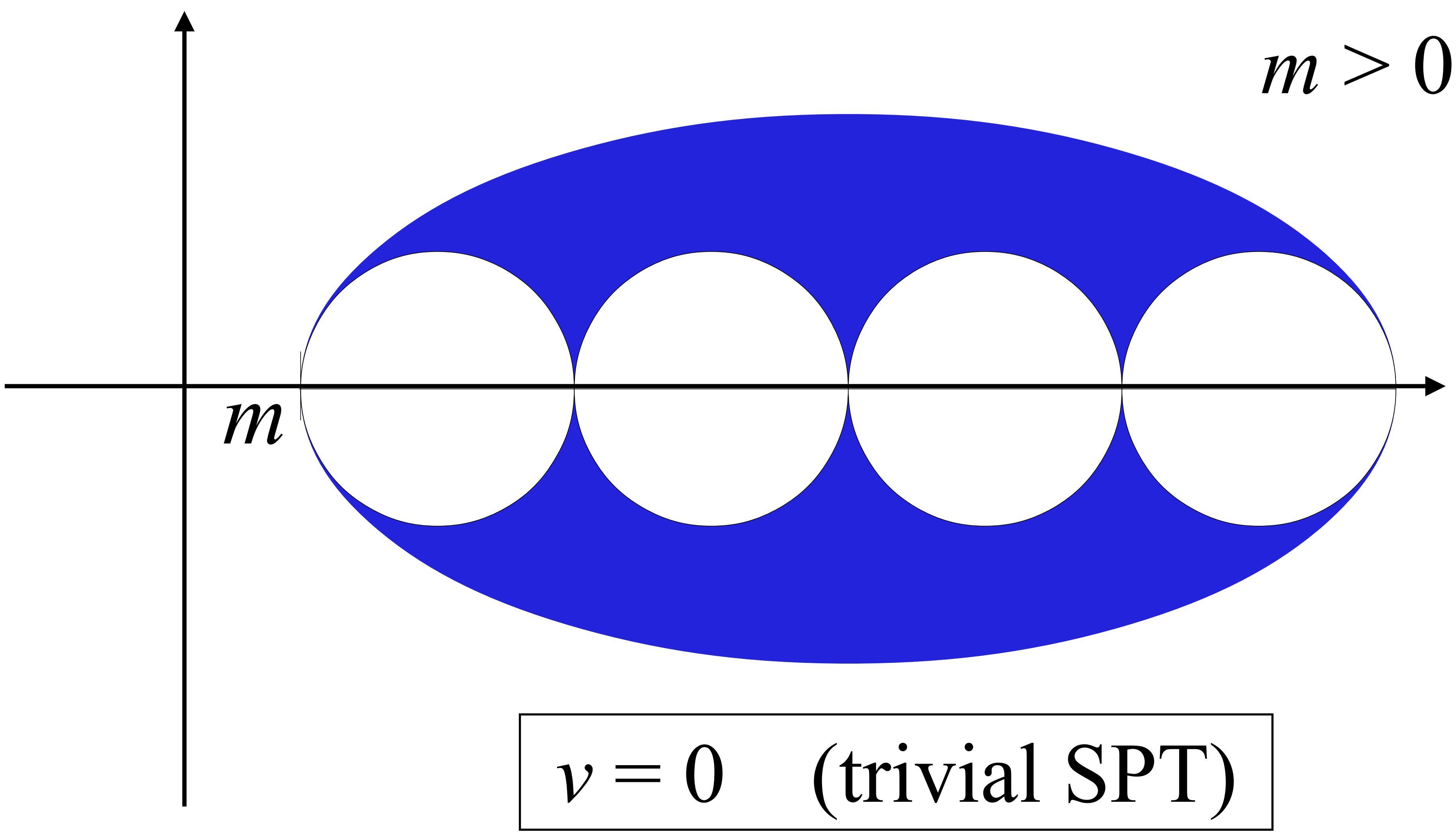
Only one flavor is massless,
while others have $O(1/a)$ mass.

- 15 species are decoupled → doubler-less
- $1/a$ additive mass renormalization → Fine-tune
- Domain-wall & Overlap fermions → costs



Wilson fermion as U(1) SPT phases

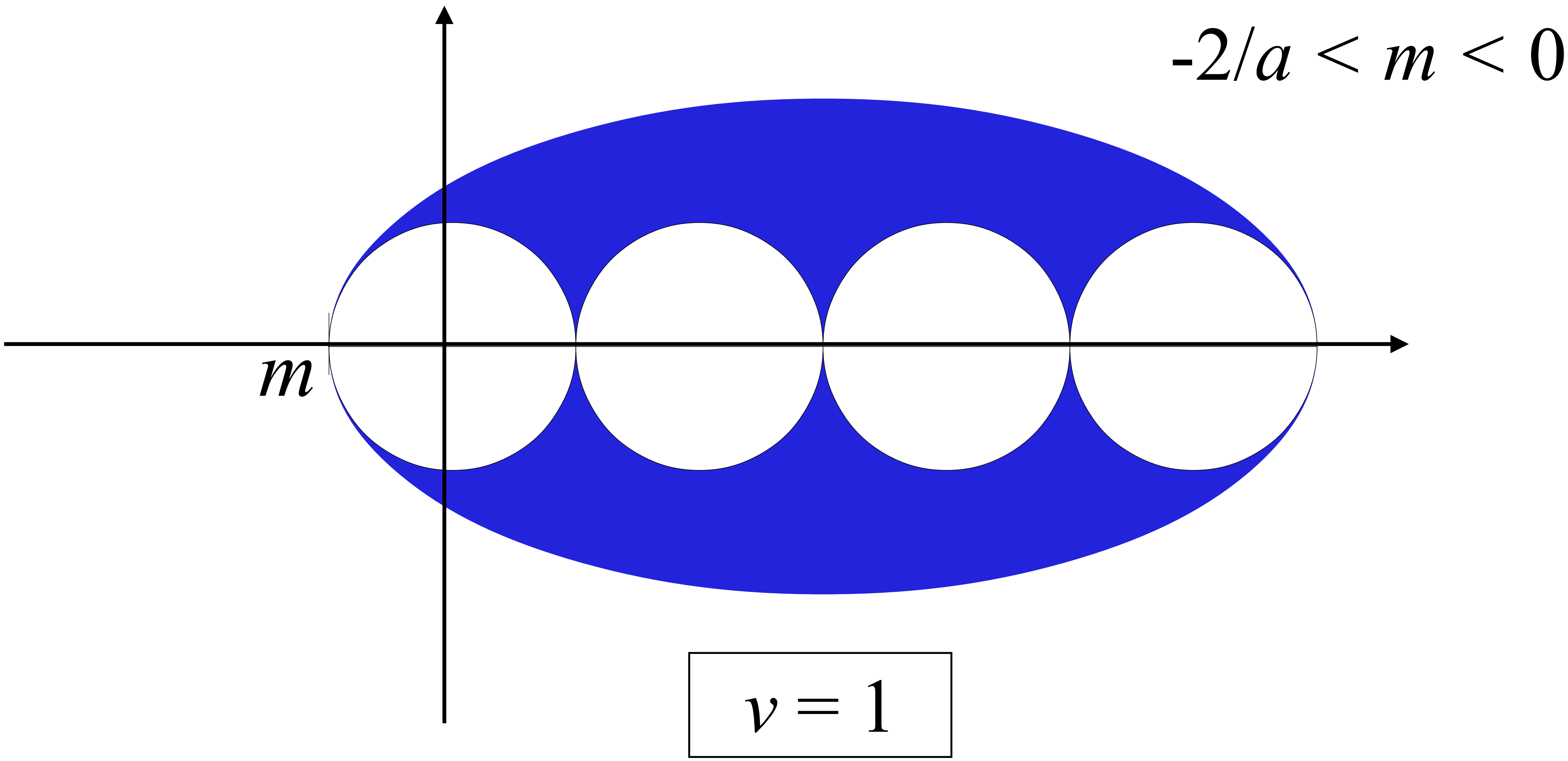
Topological # of SPT ~ index of modes with negative mass



These indices reflect topology of Berry connection for free fermion, while gauge field topology plays the same role in gauged theory.

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass

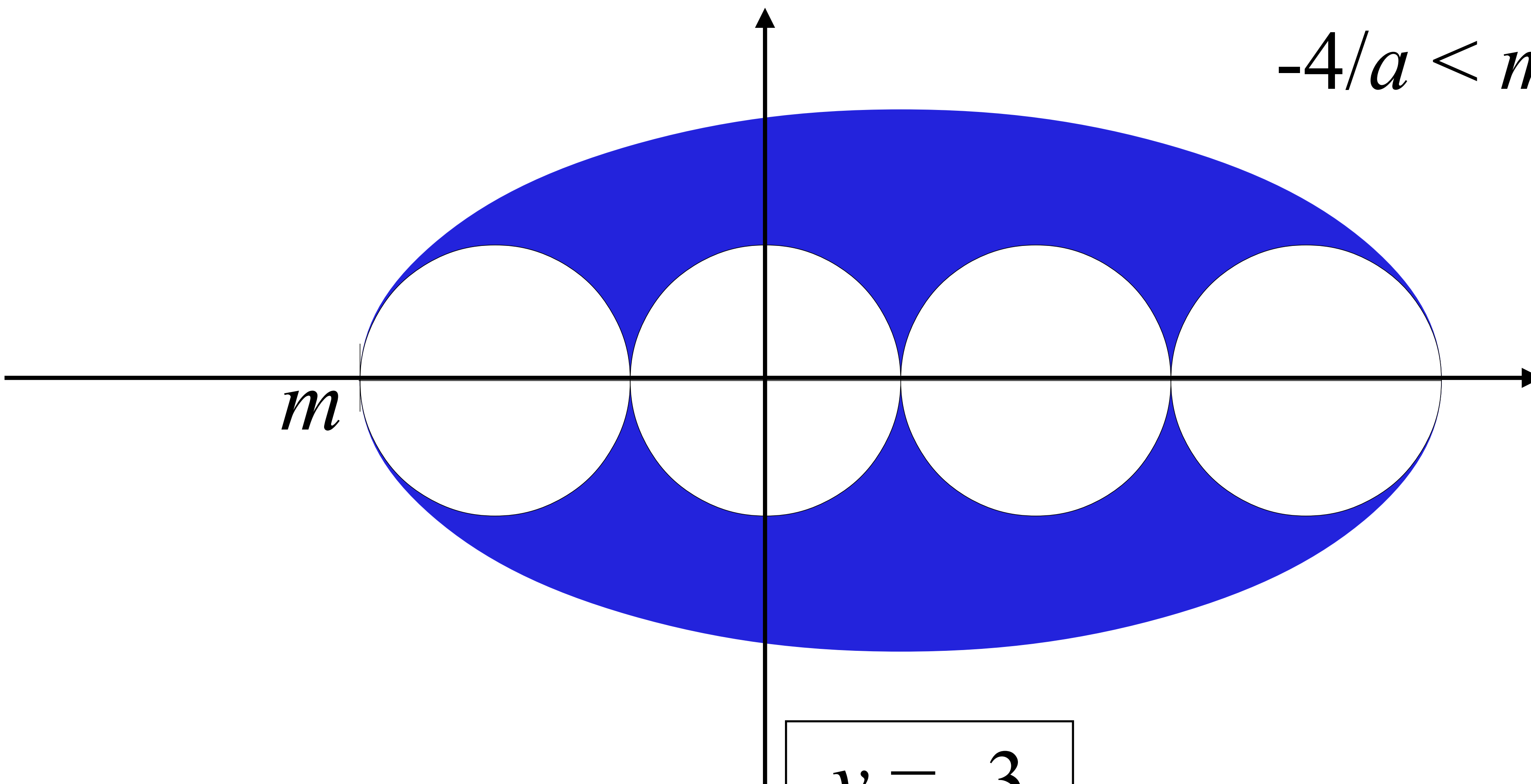


Domain-wall fermion : gapless mode emerging at boundary between ν=0 and ν=1 SPTs, where 't Hooft anomaly cancels.

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass

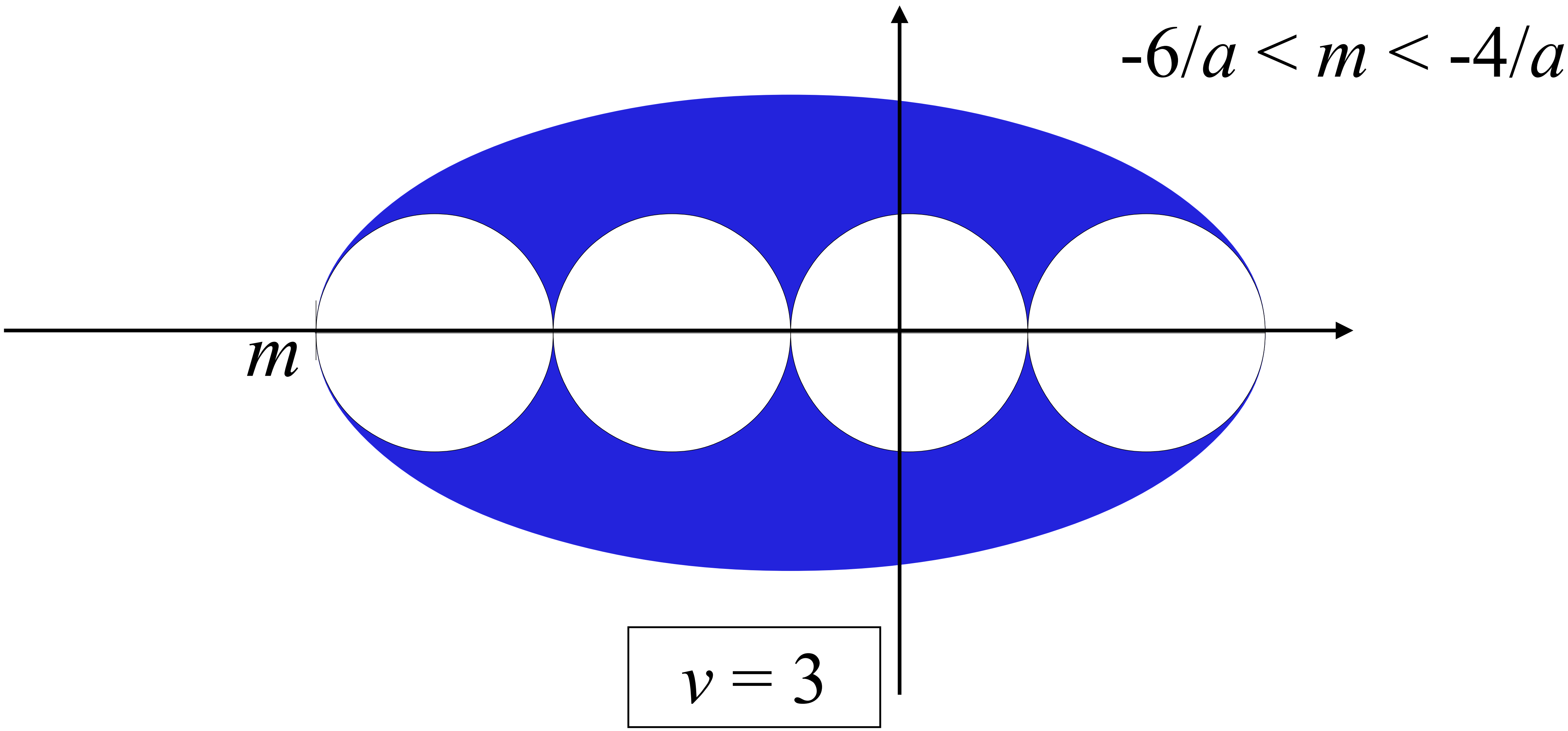
$-4/a < m < -2/a$



$\nu = -3$

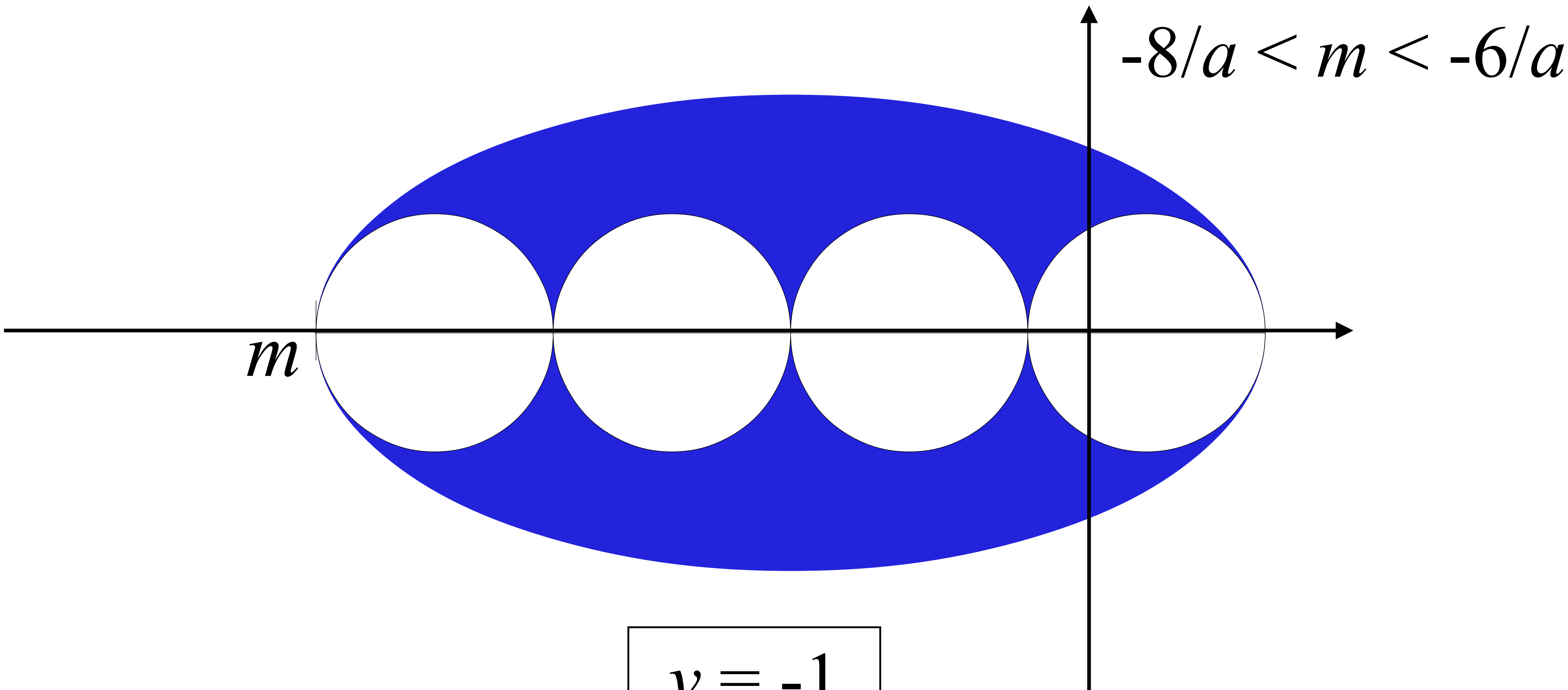
Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass



Wilson fermion as U(1) SPT phases

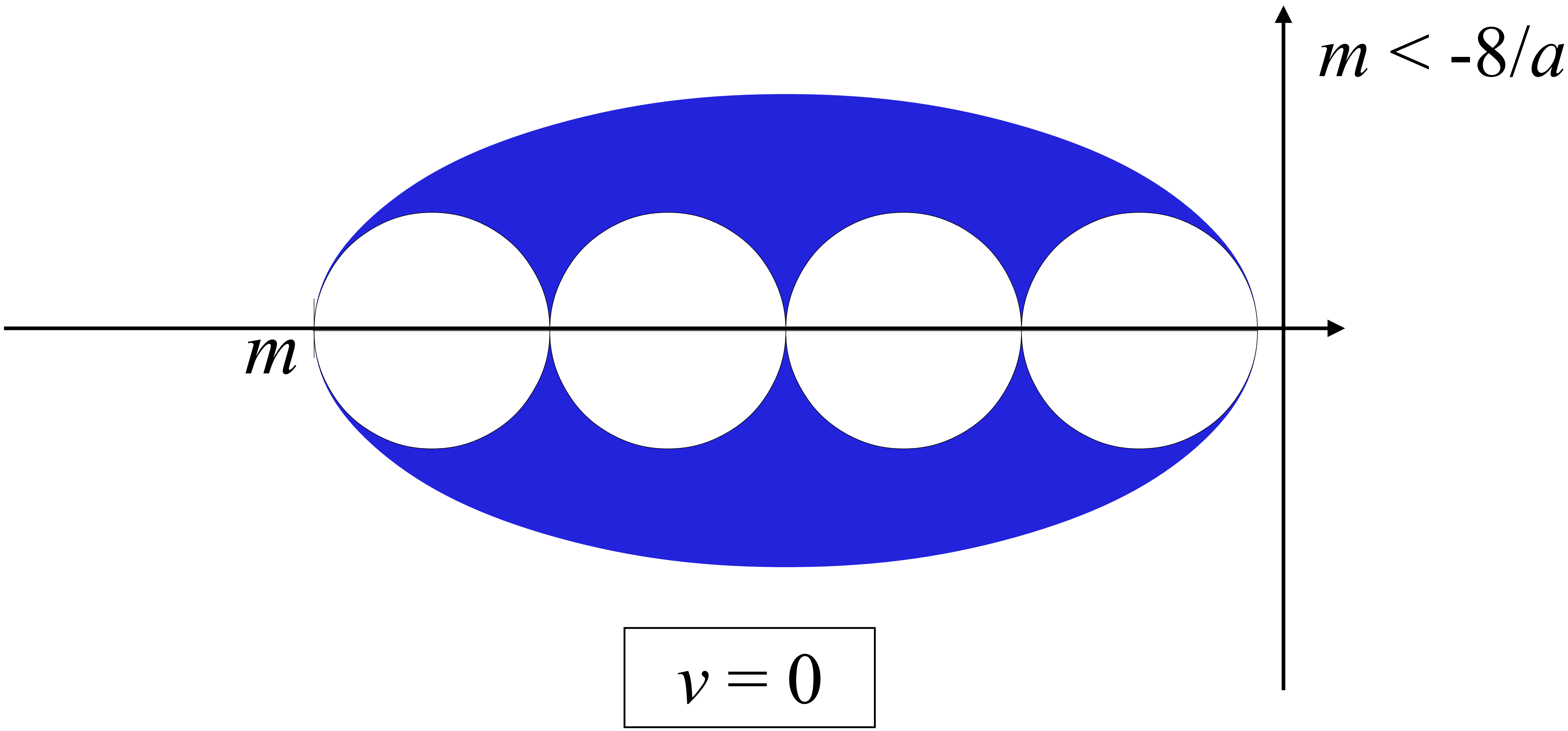
Topological # of SPT ~ index of modes with negative mass



$\nu = -1$

Wilson fermion as U(1) SPT phases

Topological # of SPT ~ index of modes with negative mass



Symmetry-protected topological phase

- G-Symmetry Protected Topological phase (SPT) Wen, et.al., (13)

1. Unique ground state with trivial gap as long as G is unbroken
2. Gap should be closed when moving to another SPT
3. Massless modes at boundary btwn two different SPTs
4. 't Hooft anomaly cancelled btwn bulk & boundary with gauged G

All 't Hooft anomalies are (expected to be) classified by SPTs.

Kapustin (14), Witten (15), Yonekura (16), Yonekura, Witten (19)

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

m < 0

$$Z = e^{-2\pi i \eta}$$

(η: APS η-invariant $\equiv \sum_i \mathrm{sgn}[\lambda_i]$)

m > 0

$$Z = 1$$

2-dim chiral fermions Z_{bndry}

Symmetry-protected topological phase

ex.) (2+1)-dim free massive Dirac fermion = U(1) SPT = IQHS

m < 0

$$Z = e^{-2\pi i\eta}$$

APS index theorem
cf.)Fukaya, Onogi, Yamaguchi,et.al.(17-19)

$$= e^{\frac{i}{4\pi}\int AdA}$$

m > 0

$$Z = 1$$

2-dim chiral fermions Z_{bndry}

$$Z_{\mathrm{total}} = Z_{\mathrm{bulk}} \cdot Z_{\mathrm{bndry}} \longrightarrow Z_{\mathrm{bulk}} e^{\frac{i}{4\pi}\int F} \cdot Z_{\mathrm{bndry}} e^{-\frac{i}{4\pi}\int F} = Z_{\mathrm{total}}$$

’t Hooft anomaly is cancelled between bulk and boundary

2. Lattice fermions as spectral graphs

Lattice fermion as spectral graph

Yumoto, TM (21)
cf.) Ohta, Sakai (20)
Ohta, Matsuura (21)

Definition 1. ***A graph G is a pair G = (V, E).*** *V is a set of vertices and E is a set of edges.*

Definition 2. ***A directed graph is a pair (V, E) of sets of vertices and edges together with two maps* init *: E → V and* ter *: E → V.*** *The two maps are assigned to every edge e_{ij} with an initial vertex* $\mathrm{init}(e_{ij}) = v_i \in V$ *and a terminal vertex* $\mathrm{ter}(e_{ij}) = v_j \in V$. *If* $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$, *the edge e_{ij} is called a loop.*

Definition 3. ***A weighted graph has a value (weight) for each edge in a graph.***

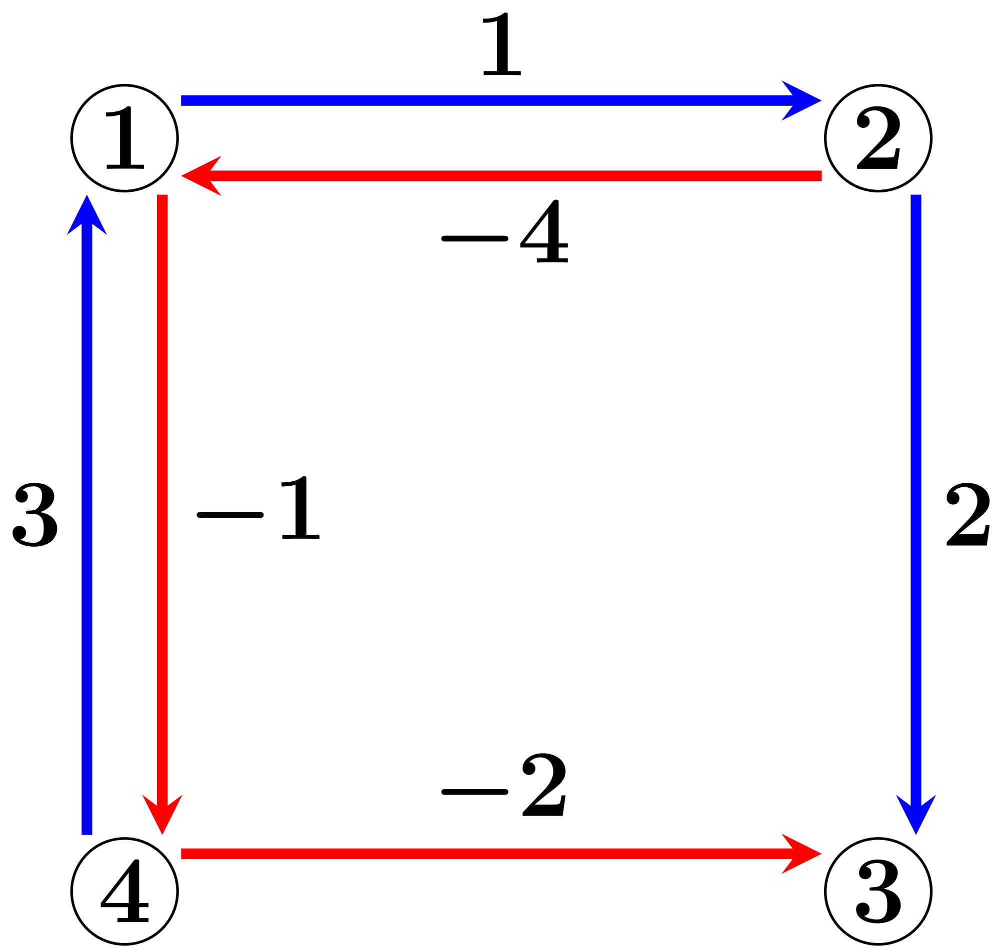
Definition 4. ***A adjacency matrix A of a graph is the |V| × |V| matrix given by***

$$A_{ij} = \begin{cases} w_{ij} & \textit{if there is a edge from } i \textit{ to } j \\ 0 & \textit{otherwise} \end{cases},$$

where w_{ij} is the weight of an edge from i to j.

Lattice fermion as spectral graph

Yumoto, TM (21)



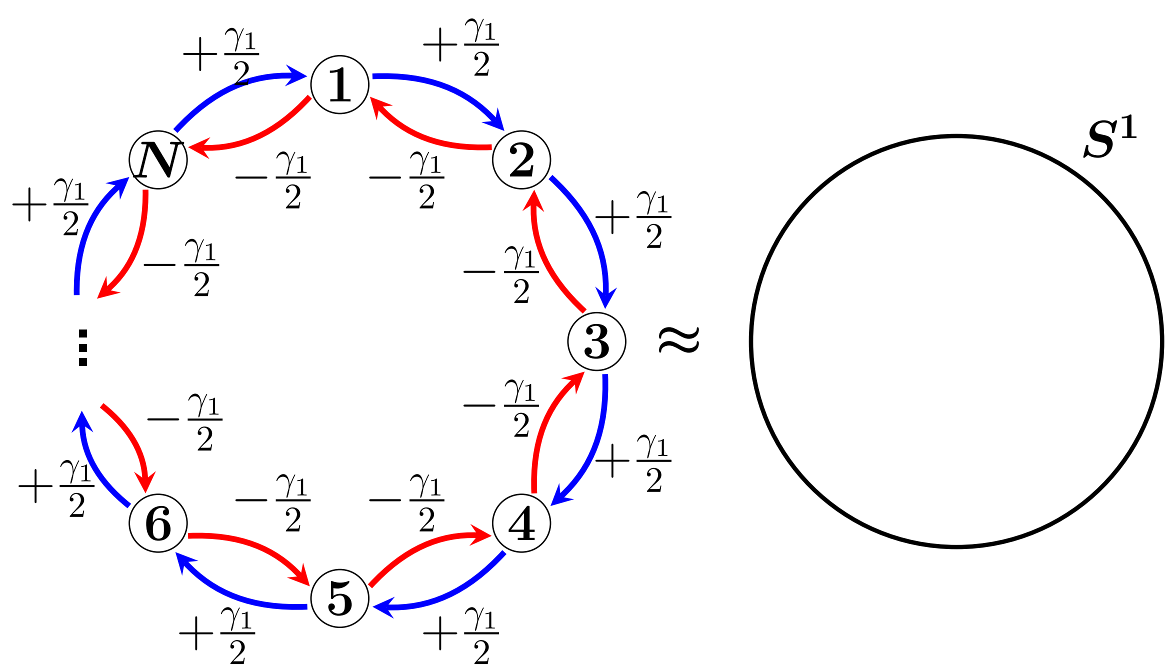
$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \end{pmatrix}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

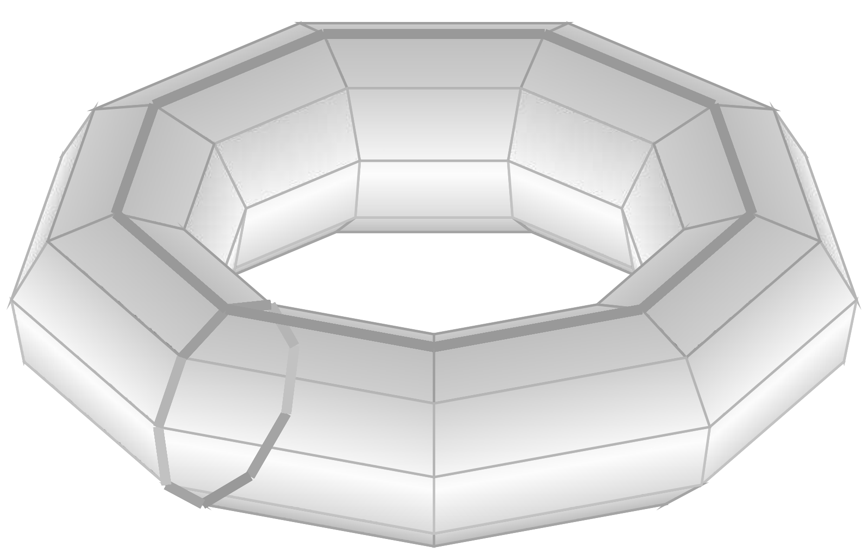
- Naive fermion

1D



$$\mathcal{D}^{1\mathrm{d}} = P_N \otimes \gamma_1 .$$

2D

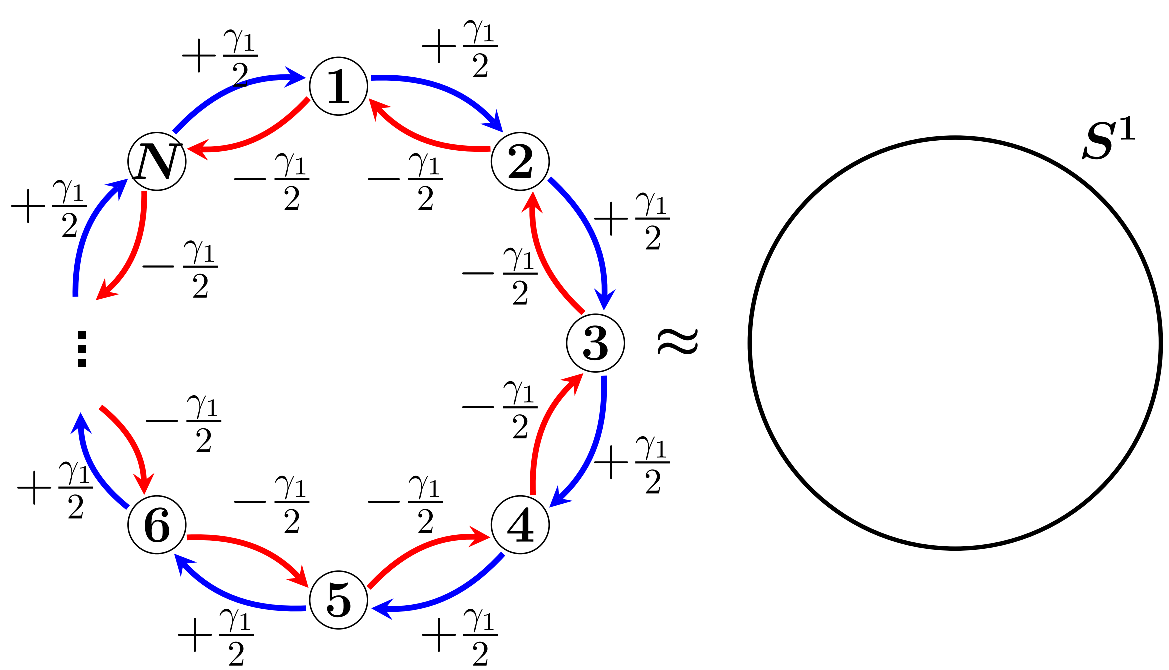


$$D^{2\mathrm{d}} = \mathbf{1}_N \otimes P_N \otimes \gamma_1 + P_N \otimes \mathbf{1}_N \otimes \gamma_2$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- 4D Naive fermion



$\times 4$

$$\begin{aligned}\mathcal{D} = & \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ & + P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4\end{aligned}$$

$$P_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Diagonalization

$$P_N X = i\,\mathrm{Diag}\left[0,\ \sin\frac{2\pi}{N},\ \sin\frac{4\pi}{N},\ \cdots,\ \sin\frac{2(N-1)\pi}{N}\right] X \equiv \Lambda_{P_N} X.$$

$$\Rightarrow \quad \begin{aligned} \mathcal{U}^\dagger \mathcal{D} \mathcal{U} = & \ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \gamma_1 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ & + \Lambda_{P_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4 \end{aligned} \qquad \mathcal{U} = \bigotimes_{\mu=1}^4 X \otimes \mathbf{1}_4$$

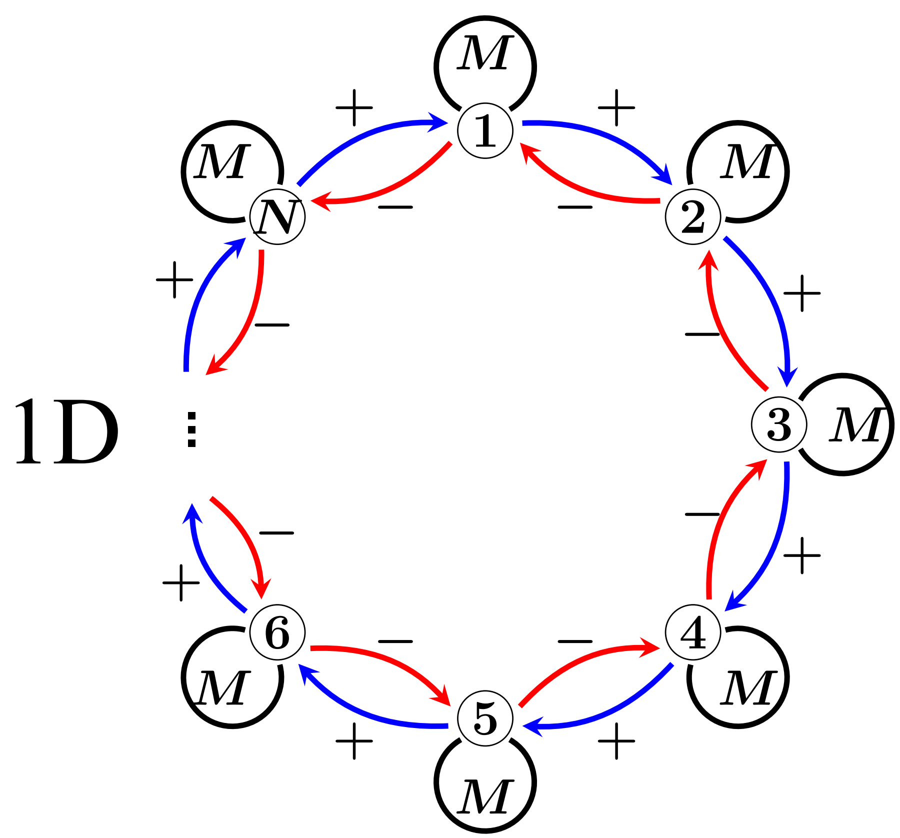
$$\Rightarrow \quad \sum_{\mu=1}^4 \sin\left(\frac{2\pi(k_\mu - 1)}{N}\right)\gamma_\mu = \mathbf{0}. \quad \Rightarrow$$

16 zero modes
=
16 species

Lattice fermion as spectral graph

Yumoto, TM (21)

◆ Wilson fermion



$$D_W^{1\mathrm{d}} = P_N \otimes \gamma_1 + m\mathbf{1}_N \otimes \mathbf{1}_4 + rM_W \otimes \mathbf{1}_4$$

$$M_W = \mathbf{1}_N - (E + E^\dagger)/2$$

$$D_W^{2\mathrm{d}} = \mathbf{1}_N \otimes P_N \otimes \gamma_1 + P_N \otimes \mathbf{1}_N \otimes \gamma_2$$

$$+ m\left(\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4\right) + r\left(\mathbf{1}_N \otimes M_W + M_W \otimes \mathbf{1}_N\right) \otimes \mathbf{1}_4$$

$\pm(\gamma_1 \mp r\mathbf{1}_4)/2$

$M = m + r$

$$\sin\left(\frac{2\pi(k_\mu - 1)}{N}\right) = 0$$

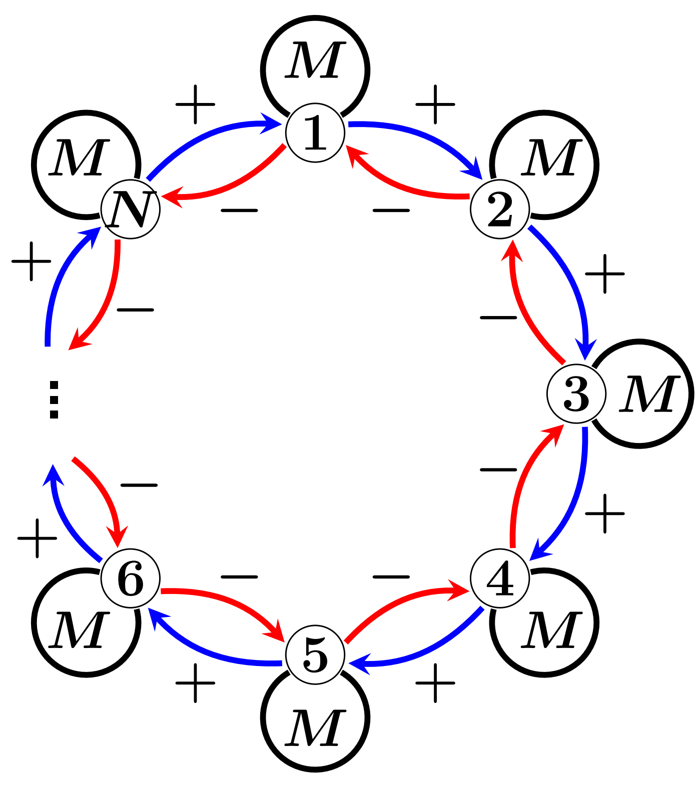
$$m + 4r - r\sum_\mu \cos\left(\frac{2\pi(k_\mu - 1)}{N}\right) = 0$$

single zero mode
=
one species

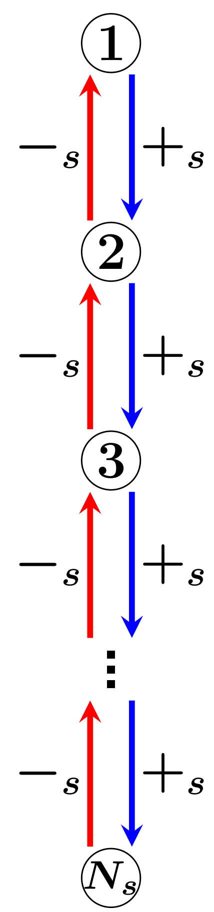
Lattice fermion as spectral graph

Yumoto, TM (21)

- Domain-wall fermion



$\square$



$$S_{DW} = \sum_{s,t} \bar{\psi}_s \left[\mathcal{D}_{DW} + \mathcal{M}_{DW}\right]_{st} \psi_t$$

$$\begin{aligned}[\mathcal{D}_{DW}]_{st} =& \delta_{st} \cdot \big(\mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 + \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ &+ \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 + P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4 \big)\end{aligned}$$

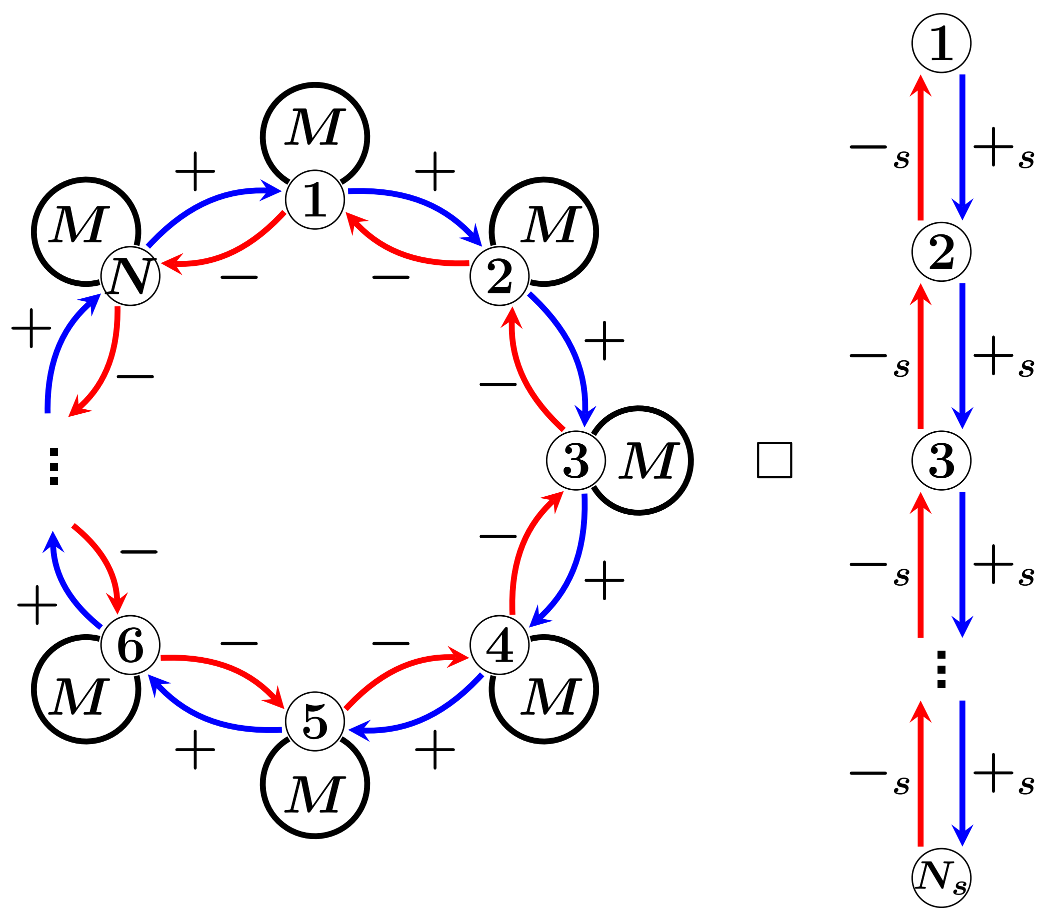
$$\begin{aligned}[\mathcal{M}_{DW}]_{st} =& \frac{1}{2}\Delta^{(-)}_{st} \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_5 \\ &+ \delta_{st}\big(-M_0 \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \\ &+ \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_4 + \mathbf{1}_N \otimes \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \\ &+ \mathbf{1}_N \otimes M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 + M_W \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4 \big) \\ &+ \frac{1}{2}\left(2\delta_{st} - \Delta^{(+)}_{st}\right) \cdot \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_4\end{aligned}$$

$$\Delta^{(\pm)}_{st} = \sum_{i=1}^{N-1} \delta_{si}\delta_{i+1\,t} \pm \sum_{l=2}^{N} \delta_{sl}\delta_{l-1\,t}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Domain-wall fermion



$$S_{DW} = \sum_{s,t} \bar{\psi}_s \left[\mathcal{D}_{DW} + \mathcal{M}_{DW}\right]_{st} \psi_t$$

the solutions of k_μ	the range of M_0	the number of zero eigenvalues
any $k_\mu = 1$	$0 < M_0 < 2$	1
one $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$2 < M_0 < 4$	4
two $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$4 < M_0 < 6$	6
three $k_\mu = 1 + N/2$ otherwise $k_\mu = 1$	$6 < M_0 < 8$	4
any $k_\mu = 1 + N/2$	$8 < M_0 < 10$	1

# of zero modes depends on the range of mass parameter

Lattice fermion as spectral graph

Yumoto, TM (21)

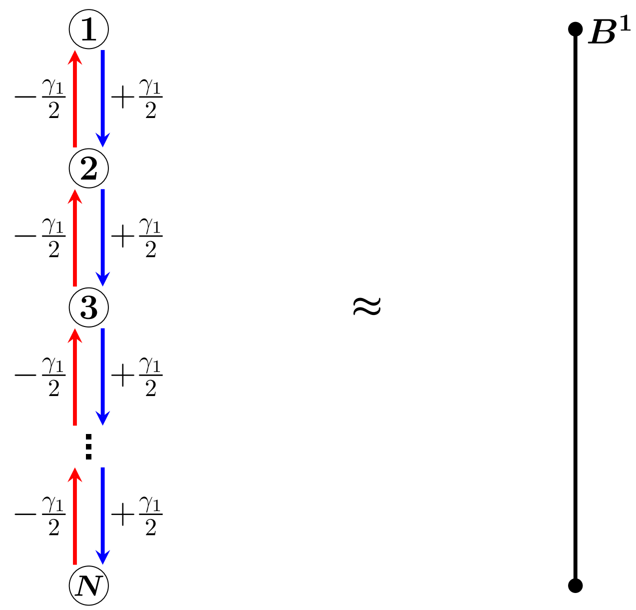
- Naive fermion on hyperball

$$\mathcal{D}_{B^1} = Q_N \otimes \gamma_1$$

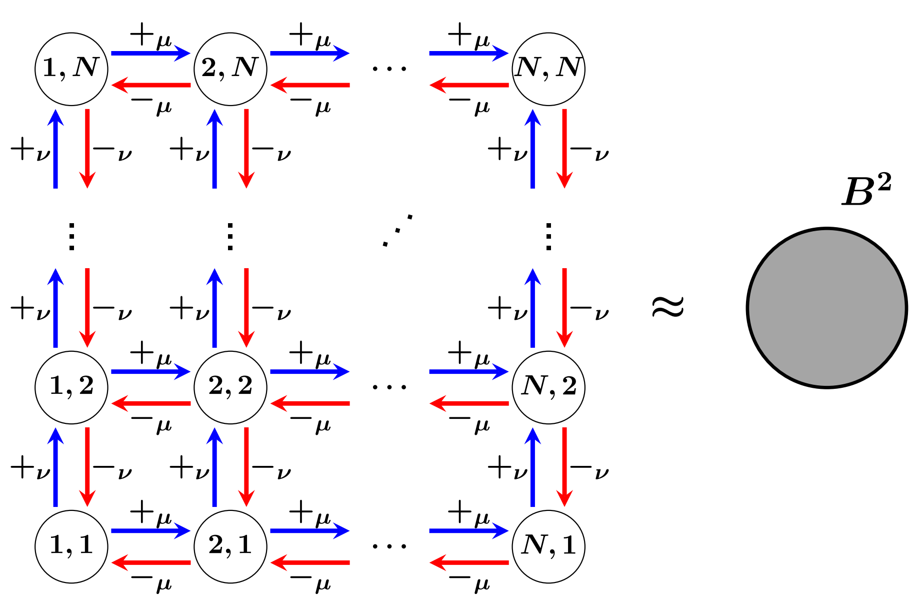
$$\mathcal{D}_{B^2} = \mathbf{1}_N \otimes Q_N \otimes \gamma_1 + Q_N \otimes \mathbf{1}_N \otimes \gamma_2$$

$$\begin{aligned}\mathcal{D}_{B^4} = \ & \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes Q_N \otimes \gamma_1 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes Q_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes Q_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ & + Q_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4.\end{aligned}$$

1D



2D



$$Q_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Lattice fermion as spectral graph

Yumoto, TM (21)

- Naive fermion on 4D hyperball

$$Q_N Y = i\mathrm{Diag}\left[\cos\left(\frac{\pi}{N+1}\right),\ \cos\left(\frac{2\pi}{N+1}\right),\ \cdots,\ \cos\left(\frac{N\pi}{N+1}\right)\right] \equiv \Lambda_{Q_N} X$$

$$\begin{aligned}\mathcal{V}^\dagger \mathcal{D}_{B^4} \mathcal{V} = {} & \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \gamma_1 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ & + \Lambda_{Q_N} \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4\end{aligned}$$

$$\mathcal{V} \equiv \bigotimes_{\mu=1}^4 Y \otimes \mathbf{1}_4$$

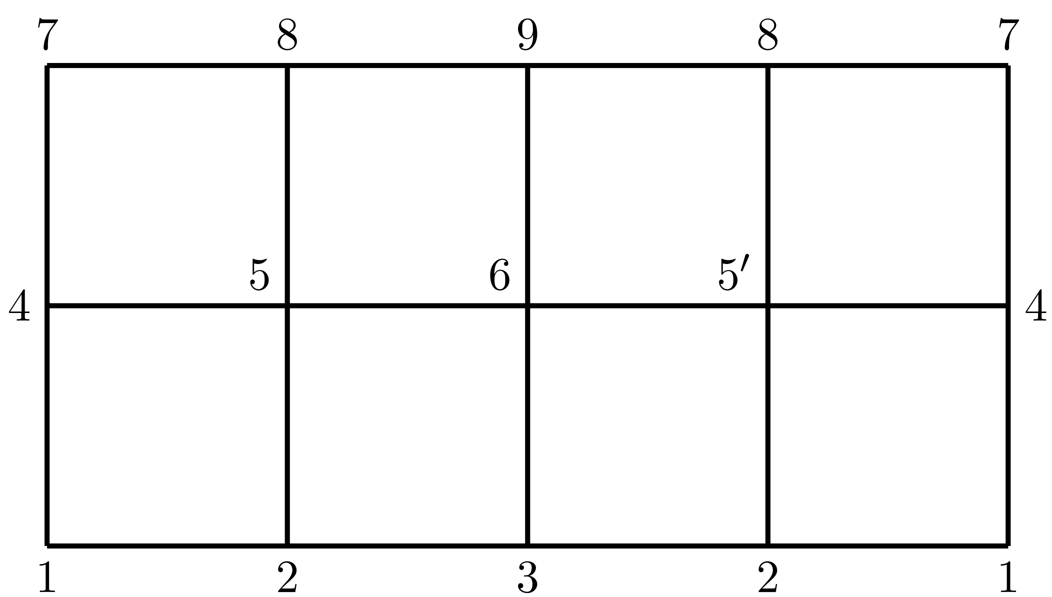
$$\sum_\mu \cos\left(\frac{\lambda_\mu \pi}{N+1}\right)\gamma_\mu = 0$$

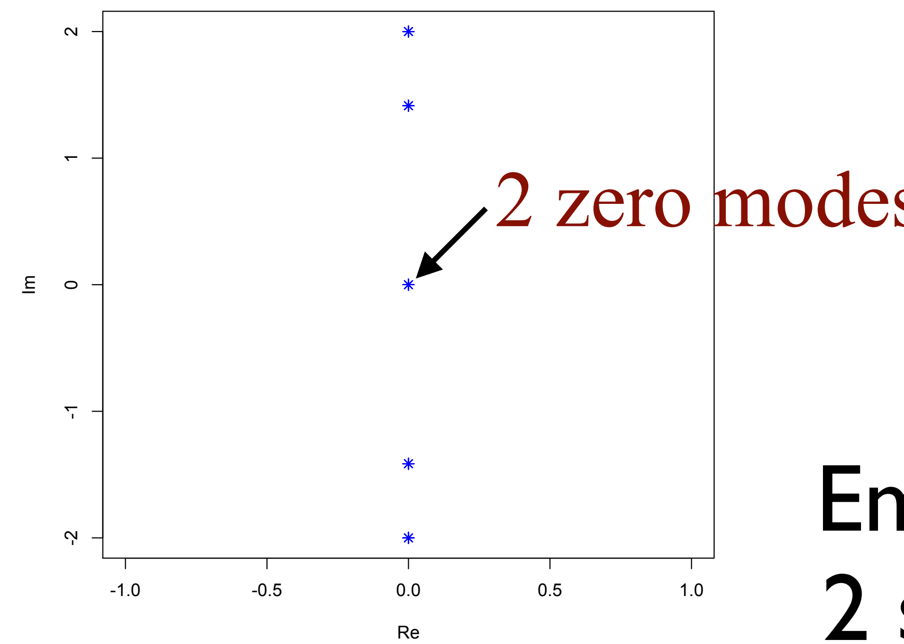
1 zero mode in bulk
=
1 species in bulk

Lattice fermion as spectral graph

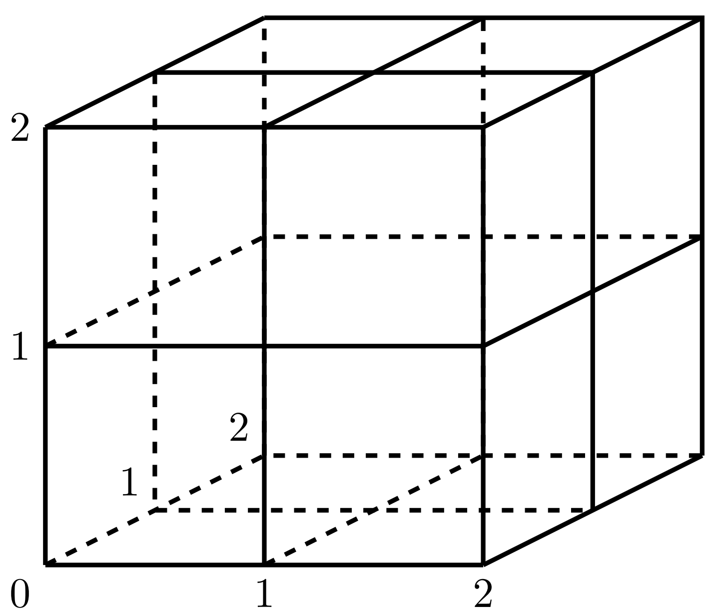
Yumoto, TM (21)

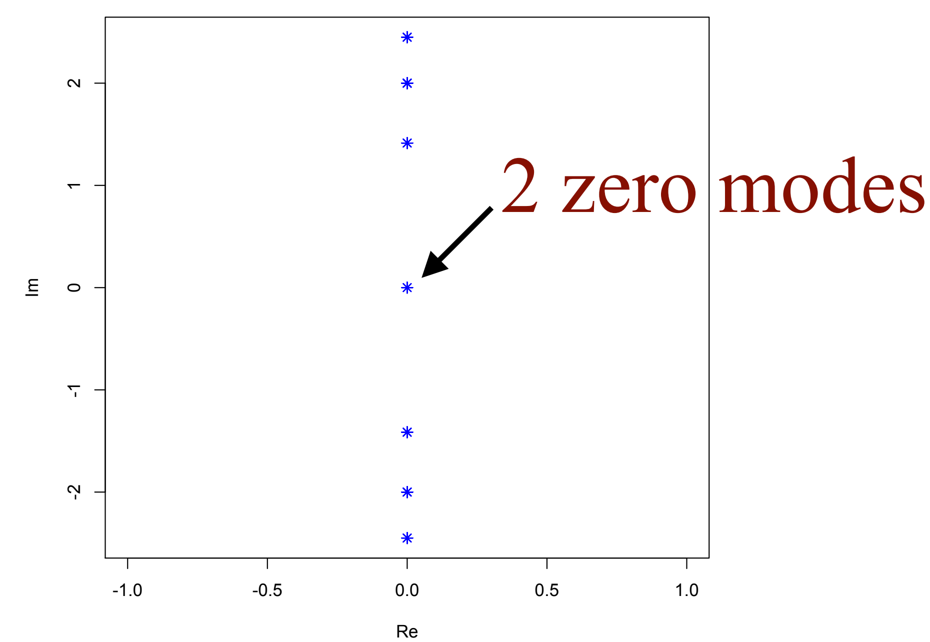
- Naive fermion on sphere





Empirically
2 species !





Lattice fermion as spectral graph

Yumoto, TM (21)

Lattice field theory	➡	Spectral graph theory
Lattice fermion	➡	Directed and Weighted spectral graph
# of Fermion species	➡	Nullity of spectral matrix

We can study lattice field theory in terms of SGT.

3. New conjecture on fermion doubling

Nielsen-Ninomiya's no-go theorem is just no-go theorem.

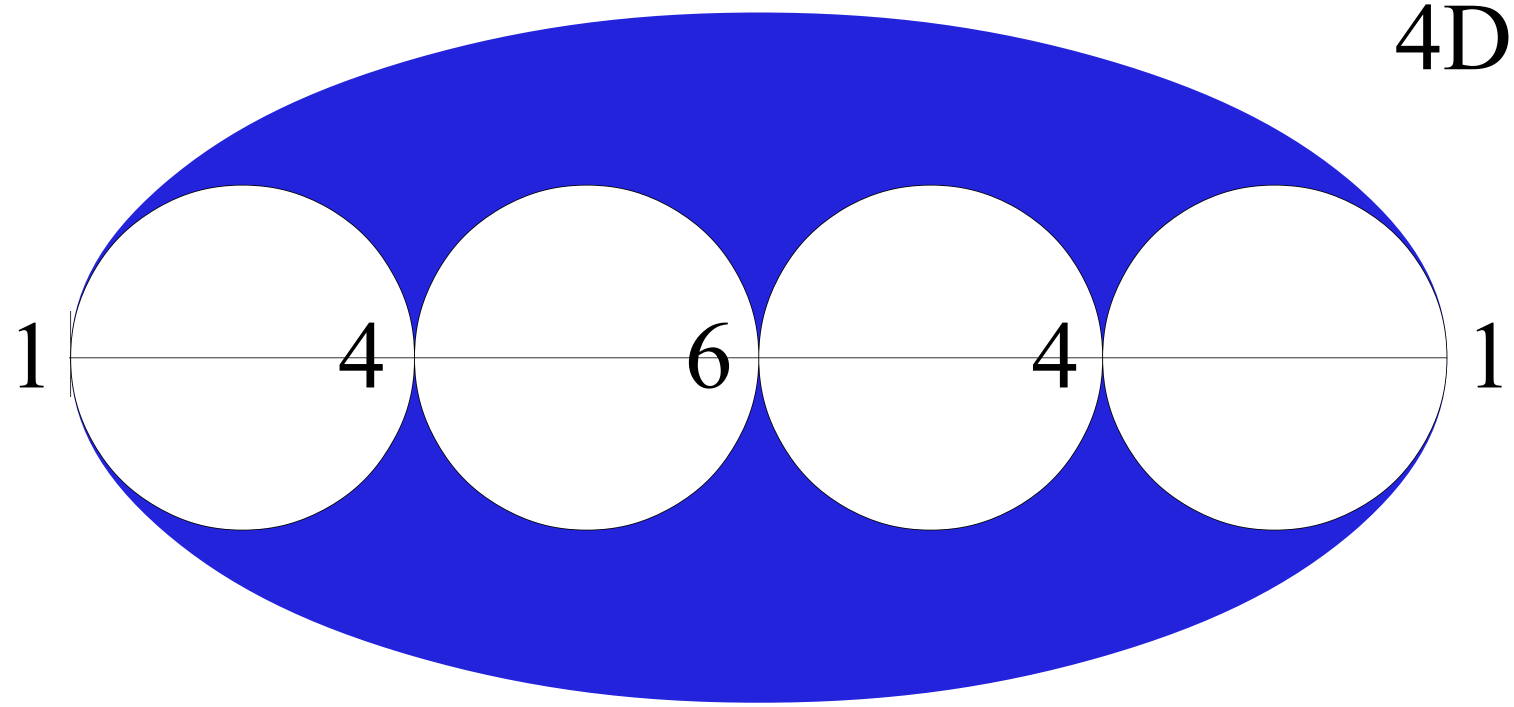
It never tells us how many fermion species emerge given a lattice fermion formulation.

Is there a theorem which informs us of # of species?

Reconsider Naive and Wilson

Yumoto, TM (22)

4D Wilson



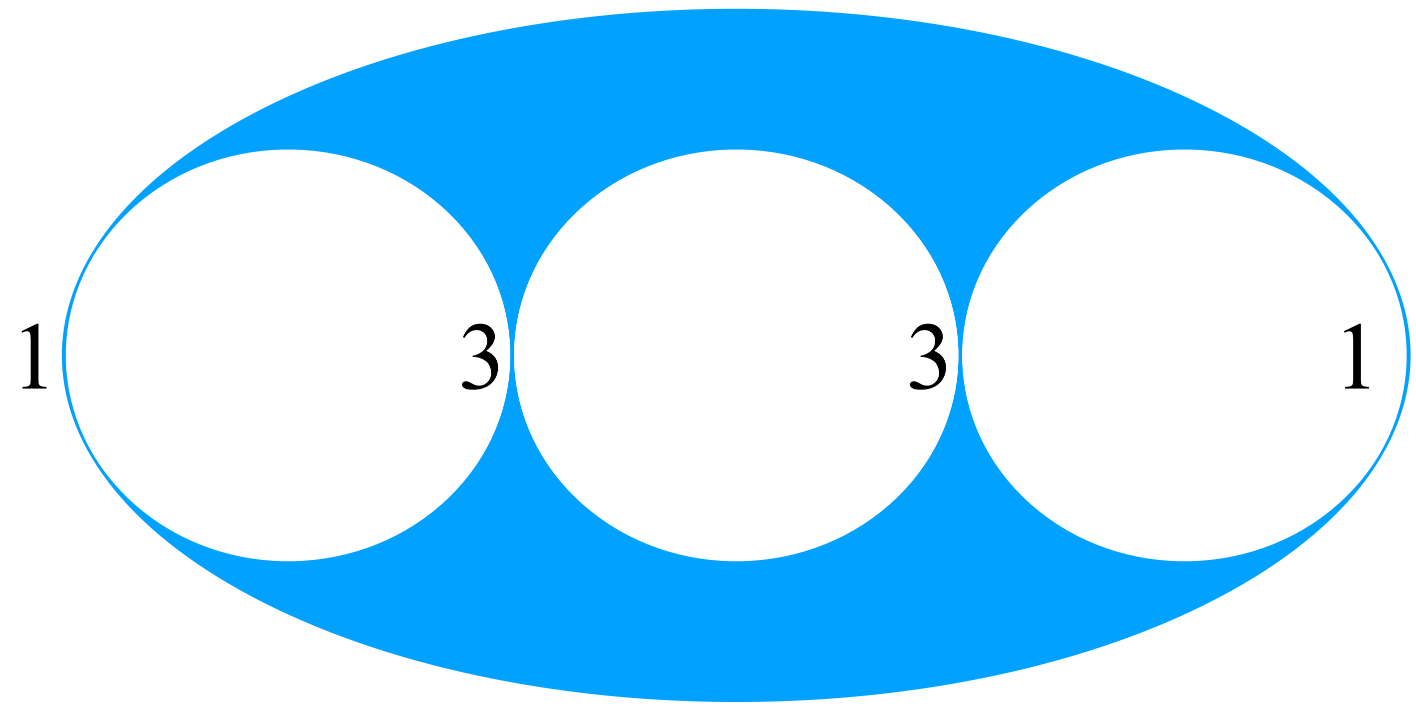
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$(0,\pi,0,0)$	$(\pi,0,\pi,0)$	$(\pi,\pi,0,\pi)$	
	$(0,0,\pi,0)$	$(\pi,0,0,\pi)$	$(\pi,0,\pi,\pi)$	
	$(0,0,0,\pi)$	$(0,\pi,0,\pi)$	$(0,\pi,\pi,\pi)$	
		$(0,0,\pi,\pi)$		
		$(0,\pi,\pi,0)$		

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

3D Wilson



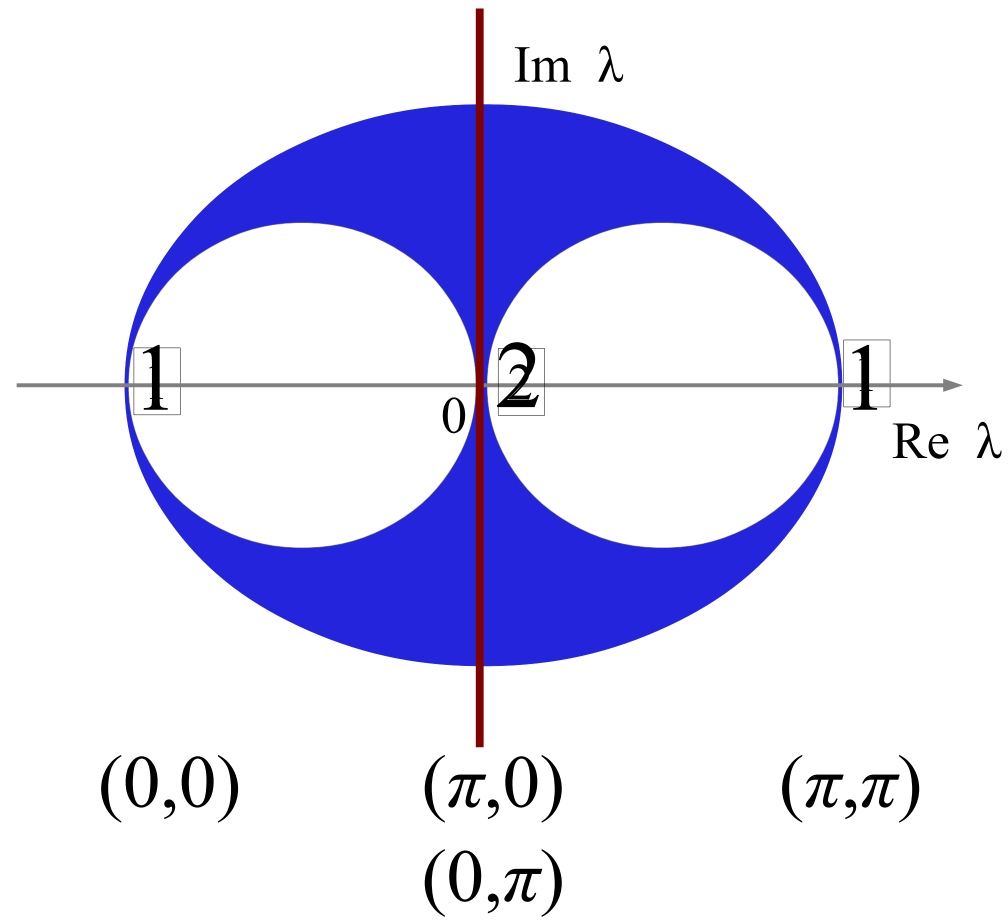
$(0,0,0)$	$(\pi,0,0)$	$(\pi,\pi,0)$	(π,π,π)
	$(0,\pi,0)$	$(\pi,0,\pi)$	
	$(0,0,\pi)$	$(0,\pi,\pi)$	

What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

2D Wilson

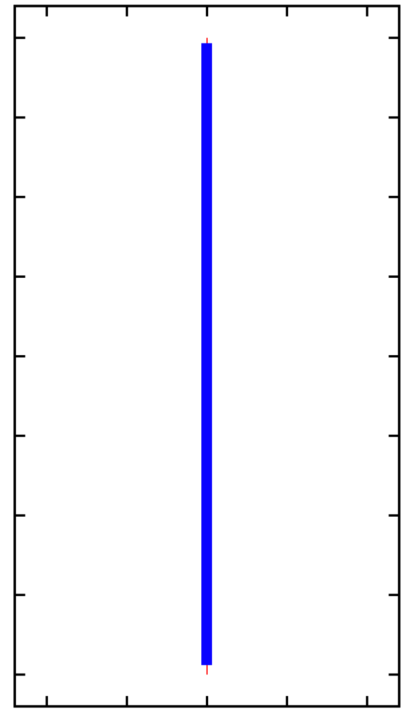


What is the meaning of the numbers?

Reconsider Naive and Wilson

Yumoto, TM (22)

4D Naive



16

1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$\vdots$	$\vdots$	$\vdots$	

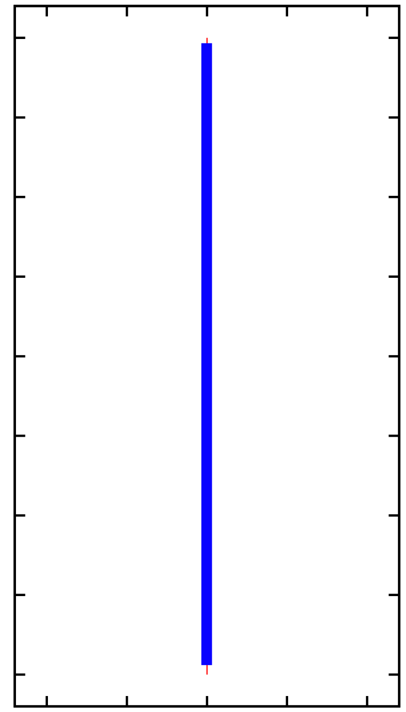
What is the meaning of the numbers?

The reason why $p=\pi$ becomes zero of Dirac operator is "periodicity"

Reconsider Naive and Wilson

Yumoto, TM (22)

4D Naive



16

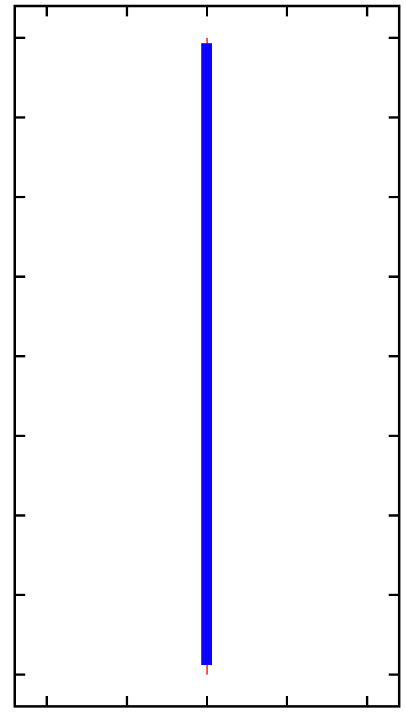
1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)

What is the meaning of the numbers?

It means these numbers are related to certain topological invariants

Reconsider Naive and Wilson

Yumoto, TM (22)



4D Naive

16

1	4	6	4	1
$(0,0,0,0)$	$(\pi,0,0,0)$	$(\pi,\pi,0,0)$	$(\pi,\pi,\pi,0)$	(π,π,π,π)
	$\vdots$	$\vdots$	$\vdots$	
# of 0d holes	# of 1d holes	# of 2d holes	# of 3d holes	# of 4d holes

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **4D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 4 \qquad \beta_2(M) = 6 \qquad \beta_3(M) = 4 \qquad \beta_4(M) = 1$$

$\Downarrow$

Sum of Betti numbers is 16 $\longrightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

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n-th Betti number is a rank of n-th homology group

- **3D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 3 \qquad \beta_2(M) = 3 \qquad \beta_3(M) = 1$$

$\Downarrow$

Sum of Betti numbers is 8 $\rightarrow$ # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **2D torus**

$$\beta_0(M) = 1 \qquad \beta_1(M) = 2 \qquad \beta_2(M) = 1$$

$$\Downarrow$$

Sum of Betti numbers is 4 → # of naive fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **D-dim hyperball**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 0 \quad \ldots..$$

$$\Downarrow$$

Sum of Betti numbers is 1 $\rightarrow$ # of bulk fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **D-dim hyperball**

$$\beta_0(M) = 1$$

$$Q_N = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Sum of Betti numbers is 1 $\longrightarrow$ # of bulk fermion species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- $\mathbf{T^4 \times R^1}$

$\beta_0(M) = 1 \quad \beta_1(M) = 4 \quad \beta_2(M) = 6 \quad \beta_3(M) = 4 \quad \beta_4(M) = 1 \quad \beta_5(M) = 0$

Sum of Betti numbers is 16 $\rightarrow$ maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- $\mathbf{T^2 \times R^2}$

$\beta_0(M) = 1 \quad \beta_1(M) = 2 \quad \beta_2(M) = 1 \quad \beta_3(M) = 0 \quad \beta_4(M) = 0$

Sum of Betti numbers is 4 $\rightarrow$ maximal # of species !

Topological invariants

Yumoto, TM (22)

- Topological invariant

Betti number is an indicator how many n-dimensional holes the space has.

$$\beta_n(M) \quad = \quad \text{rank of } H_n(M) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$$

n-th Betti number is a rank of n-th homology group

- **2D Spheres**

$$\beta_0(M) = 1 \quad \beta_1(M) = 0 \quad \beta_2(M) = 1$$

Kamata, Matsuura, TM, Ohta (16)
Yumoto, TM (21)

Sum of Betti numbers is 2 $\longrightarrow$ # of fermion species !

Topological invariants

Yumoto, TM (22)

	sum of $\beta_n(M)$	maximal # of species
1D torus	1+1	2
2D torus	1+2+1	4
3D torus	1+3+3+1	8
4D torus	1+4+6+4+1	16
T^D	$(1+1)^D$	2^D
Hyperball	1+0+0+….	1 for bulk
Sphere	1+0+0+…+1	2
$T^D \times R^d$	$2^D + 0$	2^D

Conjecture on fermion species

Yumoto, TM (22)

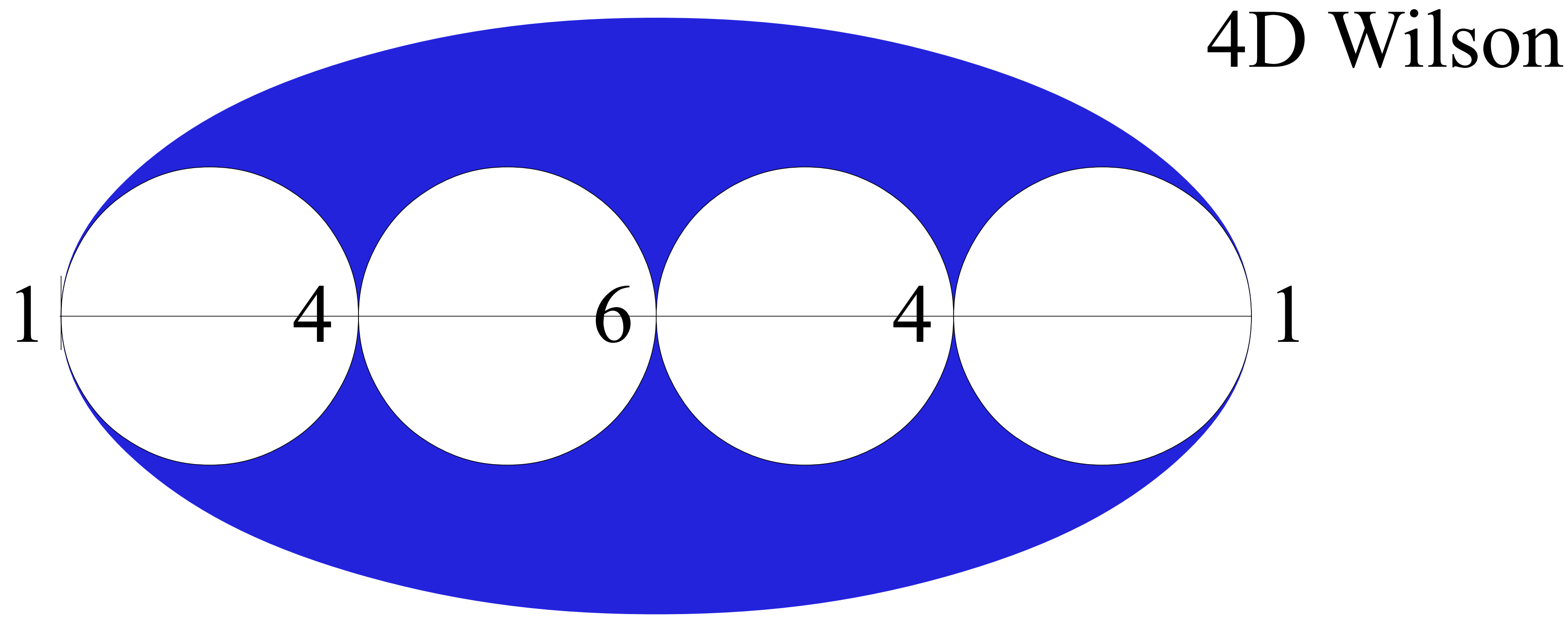
- **Conjecture**

A sum of Betti numbers of background space is
a maximal number of fermion species
when the fermion is defined on the discretized space.

How can we prove it?

Definition of maximal # of species

Yumoto, TM (22)



maximal number of fermion species

=

number of modes on real axis

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

↓

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4D Torus and Hyperball.

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*)$$

↓

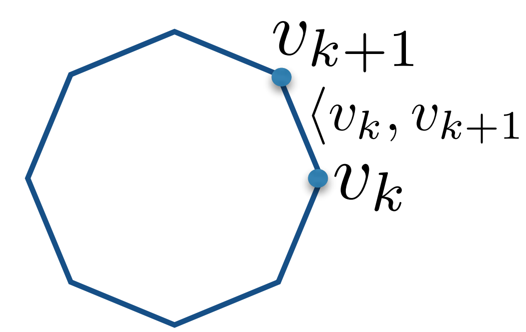
Classify necessary conditions and complete proof.

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- 1D torus lattice fermion as Chain complex



$$\mathcal{C}^1(L_N^{(\mathrm{p})}) \equiv \left\{ \sum_{k=1}^{N} a_{k,k+1} \langle v_k, v_{k+1} \rangle \;\middle|\; a_{k,k+1} \in \mathbb{Z} \right\}$$
$$= \{ a_{1,2} \langle v_1, v_2 \rangle + a_{2,3} \langle v_2, v_3 \rangle + \cdots + a_{N,1} \langle v_N, v_1 \rangle \mid a_{1,2}, a_{2,3}, \cdots, a_{N,1} \in \mathbb{Z} \}$$

$L_N^{(\mathrm{p})}$: simplical complex $\rightarrow$ graph (1D lattice)

$\langle v_k, v_{k+1} \rangle$: 1-simplices of complex $L_N^{(\mathrm{p})}$ $\rightarrow$ edges (links)

v_k : boundaries of simplices $\rightarrow$ vertices (lattice points)

$a_{k,k+1}$: coefficients of simplices (should be abelian ring)

Sketch of proof

Yumoto, TM (22)

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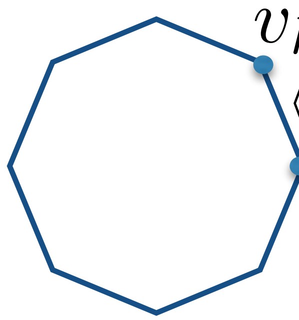


$$\mathcal{C}^1(L_N^{(\mathrm{p})}) \equiv \left\{ \sum_{k=1}^{N} a_{k,k+1} \langle v_k, v_{k+1} \rangle \,\middle|\, a_{k,k+1} \in \mathbb{Z} \right\}$$
$$= \{ a_{1,2} \langle v_1, v_2 \rangle + a_{2,3} \langle v_2, v_3 \rangle + \cdots + a_{N,1} \langle v_N, v_1 \rangle \mid a_{1,2}, a_{2,3}, \cdots, a_{N,1} \in \mathbb{Z} \}$$

$L_N^{(\mathrm{p})}$: simplical complex $\rightarrow$ graph (1D lattice)

$\langle v_k, v_{k+1} \rangle$: 1-simplices of complex $L_N^{(\mathrm{p})}$ $\rightarrow$ edges (links)

$\boxed{v_k \text{ : boundaries of simplices } \rightarrow \text{ vertices (lattice points)}}$

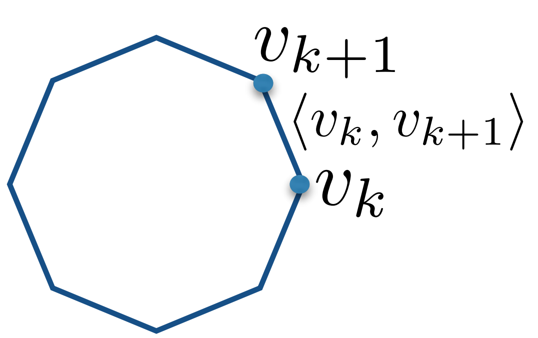
$a_{k,k+1}$: coefficients of simplices (should be abelian ring)

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- 1D torus lattice fermion as Chain complex



$$\mathcal{C}^1(L_N^{(\mathrm{p})}) \equiv \left\{ \sum_{k=1}^{N} a_{k,k+1} \langle v_k, v_{k+1} \rangle \;\middle|\; a_{k,k+1} \in \mathbb{Z} \right\}$$
$$= \{ a_{1,2} \langle v_1, v_2 \rangle + a_{2,3} \langle v_2, v_3 \rangle + \cdots + a_{N,1} \langle v_N, v_1 \rangle \mid a_{1,2}, a_{2,3}, \cdots, a_{N,1} \in \mathbb{Z} \}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \qquad \cdots\cdots\cdots \qquad v_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}$$

represented as linearly independent vectors

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$\beta_1(M) \;=\; \text{rank of } H_1(M)\text{=Ker}\partial_1$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

Cycle group Ker∂_1 $\quad c_1 \in \mathcal{C}^1(L_N^{(\mathrm{p})})$

$$\begin{aligned}
\partial_1 c_1 &= \sum_{k=1}^{N} a_{k,k+1} \partial_1 \langle v_k, v_{k+1} \rangle = \sum_{k=1}^{N} a_{k,k+1} \left(v_k - v_{k+1} \right) \\
&= a_{1,2} \left(v_1 - v_2 \right) + a_{2,3} \left(v_2 - v_3 \right) + \cdots + a_{N,1} \left(v_N - v_1 \right) \\
&= \left(a_{12} - a_{N1} \right) v_1 + \left(a_{23} - a_{12} \right) v_2 + \cdots + \left(a_{N,1} - a_{N-1,N} \right) v_N = 0
\end{aligned}$$

$$\Downarrow \qquad a = a_{1,2} = a_{2,3} = \ldots = a_{N-1,N}$$

$$c'_1 = a \sum_{k=1}^{N} \langle v_k, v_{k+1} \rangle \qquad c'_1 \in \mathrm{Ker}\,\partial_1$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

Cycle group Ker∂_1 $\quad c_1 \in \mathcal{C}^1(L_N^{(\mathrm{p})})$

$$
\begin{aligned}
\partial_1 c_1 &= \sum_{k=1}^{N} a_{k,k+1} \partial_1 \langle v_k, v_{k+1} \rangle = \sum_{k=1}^{N} a_{k,k+1} (v_k - v_{k+1}) \\
&= a_{1,2}(v_1 - v_2) + a_{2,3}(v_2 - v_3) + \cdots + a_{N,1}(v_N - v_1) \\
&= (a_{12} - a_{N1}) v_1 + (a_{23} - a_{12}) v_2 + \cdots + (a_{N,1} - a_{N-1,N}) v_N = 0
\end{aligned}
$$

$$\Downarrow \qquad a = a_{1,2} = a_{2,3} = \ldots = a_{N-1,N}$$

$$H_1(L_N^{(\mathrm{p})}) = \mathrm{Ker}\,\partial_1 = \{a(\langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \cdots + \langle v_N, v_1 \rangle) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

Cycle group Ker∂_1 $\quad c_1 \in \mathcal{C}^1(L_N^{(\mathrm{p})})$

$$\partial_1 c_1 = \sum_{k=1}^{N} a_{k,k+1} \partial_1 \langle v_k, v_{k+1} \rangle = \sum_{k=1}^{N} a_{k,k+1} \left(v_k - v_{k+1} \right)$$
$$= a_{1,2} \left(v_1 - v_2 \right) + a_{2,3} \left(v_2 - v_3 \right) + \cdots + a_{N,1} \left(v_N - v_1 \right)$$
$$= \left(a_{12} - a_{N1} \right) v_1 + \left(a_{23} - a_{12} \right) v_2 + \cdots + \left(a_{N,1} - a_{N-1,N} \right) v_N = 0$$

$\Downarrow$ $\quad a = a_{1,2} = a_{2,3} = \ldots = a_{N-1,N}$

$$\boxed{\beta_1(L_N^{(\mathrm{p})}) = 1}$$

Sketch of proof

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$c_1' = a \sum_{k=1}^{N} \langle v_k, v_{k+1} \rangle \qquad c_1' \in \mathrm{Ker}\, \partial_1$$

$$\Rightarrow \quad \partial_1 c_1' = a \textstyle\sum_{k=1}^{N} \partial_1 \langle v_k, v_{k+1} \rangle = a \sum_{k=1}^{N} (v_k - v_{k+1}) = 0$$

$$\Rightarrow \quad \begin{aligned} \partial_1 c_1' &= a\,(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1) \\ &= a\,(-v_2 + v_N + v_1 - v_3 + \cdots + v_{N-1} - v_1) \end{aligned}$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$c_1' = a \sum_{k=1}^{N} \langle v_k, v_{k+1} \rangle \qquad c_1' \in \mathrm{Ker}\, \partial_1$$

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$$\Rightarrow \quad \begin{aligned} \partial_1 c_1' &= a \left(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1 \right) \\ &= a \left(\underline{-v_2 + v_N} + \underline{v_1 - v_3} + \cdots + \underline{v_{N-1} - v_1} \right) = 0 \end{aligned}$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

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$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \cdots\cdots \quad \begin{pmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$\partial_1 c'_1 = a\left(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1\right)$$
$$= a\left(\underline{-v_2 + v_N} + \underline{v_1 - v_3} + \cdots + \underline{v_{N-1} - v_1}\right) = 0$$

$$\underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}}_{w_1} \quad \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}}_{w_2} \quad \cdots\cdots \quad \underbrace{\begin{pmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}}_{w_N}$$

$$\mathcal{D}_{1\mathrm{D}} = \frac{1}{2}\left(w_1, w_2 \cdots w_N\right)$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove β_1=1 is equivalent to degeneracy of Dirac matrix

$$c_1' = a \sum_{k=1}^{N} \langle v_k, v_{k+1} \rangle \qquad c_1' \in \mathrm{Ker}\,\partial_1$$

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$$\begin{aligned} \Rightarrow \quad \partial_1 c_1' &= a\,(v_2 - v_1 + v_2 - v_3 + \cdots + v_N - v_1) \\ &= a\,(-v_2 + v_N + v_1 - v_3 + \cdots + v_{N-1} - v_1) \\ &= w_1 + w_2 + \cdots + w_N = 0 \end{aligned}$$

$\Rightarrow$ degeneracy (nullity) of Dirac matrix

↓

zero mode (fermion species)

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

- prove $\beta_0=1$ is equivalent to degeneracy of Dirac matrix

$$\beta_0(M) \;=\; \text{rank of } H_0(M) = \ker\partial_0/\mathrm{Im}\partial_1$$

$$\Downarrow$$

$$w_1 - w_2 + w_3 - w_4 \cdots + w_{N-1} - w_N = 0 \quad \Rightarrow$$

degeneracy (nullity)
of Dirac matrix
↓
zero mode
(fermion species)

Sketch of proof

Yumoto, TM (22)

Prove each of Betti numbers (β_0=1 and β_1=1) is equivalent to each of nullity of the Dirac matrix on 1D torus or 1D ball by regarding lattice fermion as chain complex.

↓

By use of Künneth theorem, elevate the above argument to higher dimensional space such as 4D Torus and Hyperball.

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*)$$

↓

Classify necessary conditions and complete proof.

Summary

- Lattice fermions are interpreted as spectral graphs. It means we can study them in terms of topology of graphs.

- New conjecture on fermion doubling is proposed: The maximal # of species is the sum of Betti numbers.

- The proof is based on use of chain complex and Kunneth's theorem.