Tensor renormalization group calculation for the phase structure of the CP(1) model in the presence of a topological term

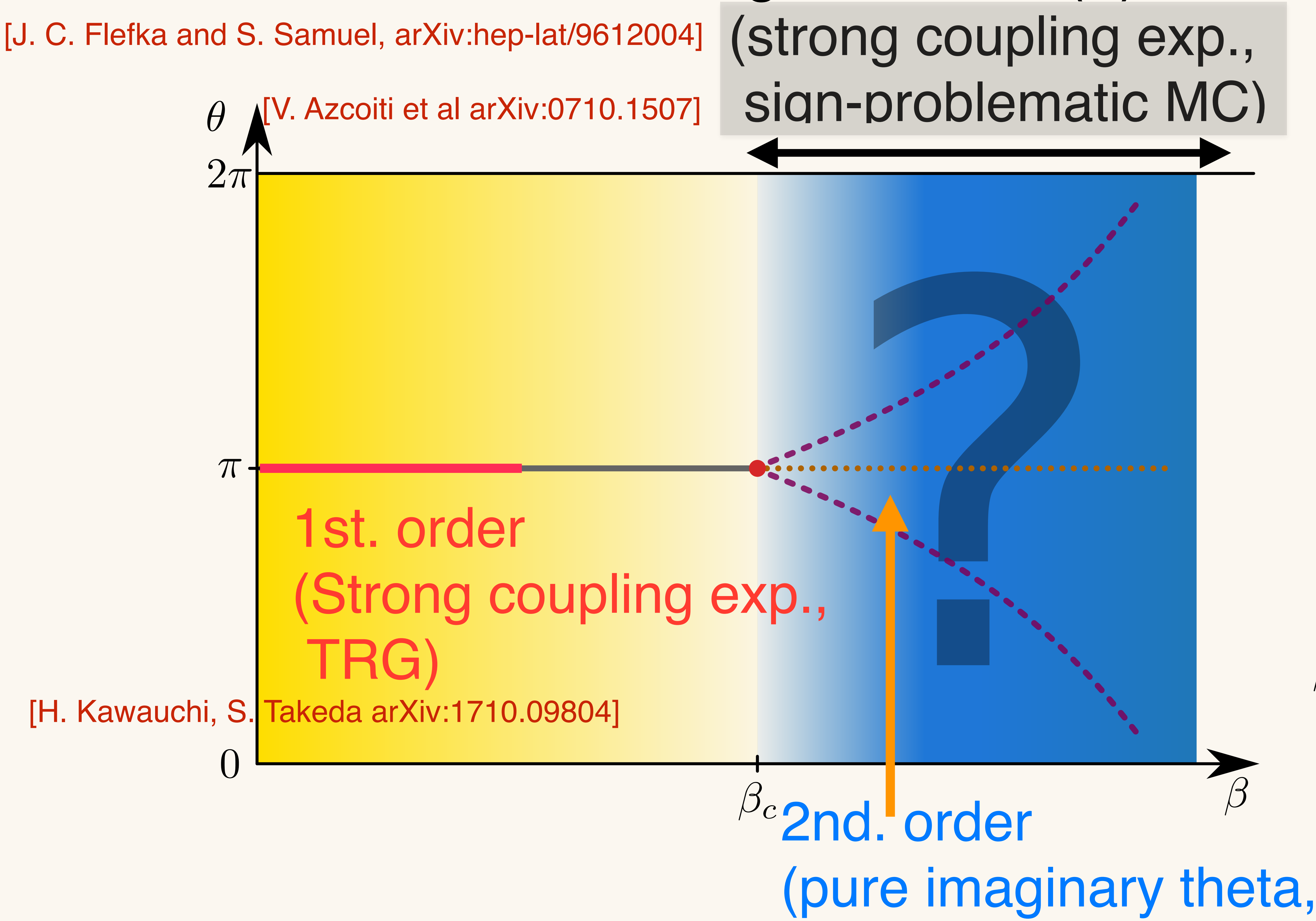
[K.N., L.Funke, K. Jansen et al. arXiv:2107.14220]
[D. Kadoh and K.N. arXiv:1912.02414]

Katsumasa Nakayama (DESY->RIKEN)

2022/11/01@Osaka Univ. with Online

Photo: RIKEN, Japan, 2022

◉ Motivation: Phase diagram of CP(1) model



$$\beta = \frac{1}{g^2}$$

→ Large beta region is not strongly confirmed.

◉ Theoretical aspects of Lattice QFT

◇ What is the Lattice QFT?

Classical field theory on the discrete spacetime

$+$

Path integral quantization

$$Z = \int \mathrm{D}\phi e^{-S_{\mathrm{lattice}}}$$

↓

$$\int \mathrm{d}x \to \sum_x$$

Lattice quantum field theory

(with Monte-Carlo)

◉ Tensor representation of numerical integration

◇ Physically？

1. Generalized transfar matrix (tensor)
2. Spin statistial representation (e.g. Ising model)
3. Hopping term expansion

◇ Advantages and disadvantages

○Sign problem

○Another representation

○Low cost (low dim)

○Partition function

×High cost (high dim)

△Systematic error

→Tensor Renormalization Group

◉ Quantum correspondence

◇ Is there Quantum correspondence?

→Tensor network representation

$$\int D\overline{\psi} D\psi \exp\left[-\sum_x \mathcal{L}(\overline{\psi}_x, \psi_x)\right] \leftrightarrow \sum_{\sigma_i} \exp\left[-\beta\left(J\sum_i \sigma_i\sigma_{i+1} + h\sum_i \sigma_i\right)\right]$$

$$\{\overline{\psi}, \psi\} \leftrightarrow \{\sigma_i\}$$

→ Need to consider path integral measure.

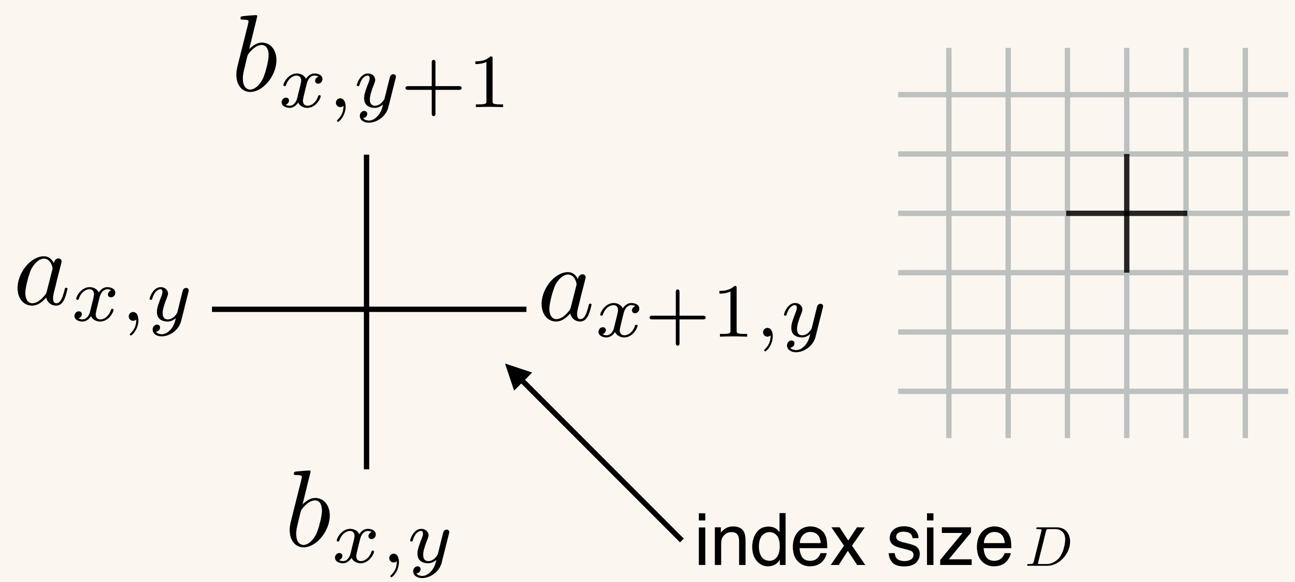
(Lagrangian formalizm)

◉ Tensor representation

- ◇ TRG directly calculate the physical quantity.

$$Z = \sum_{a,b} \prod_{x,y} T_{a_{x,y},a_{x+1,y},b_{x,y},b_{x,y+1}}$$

$$T_{a_{x,y},a_{x+1,y},b_{x,y},b_{x,y+1}} =$$



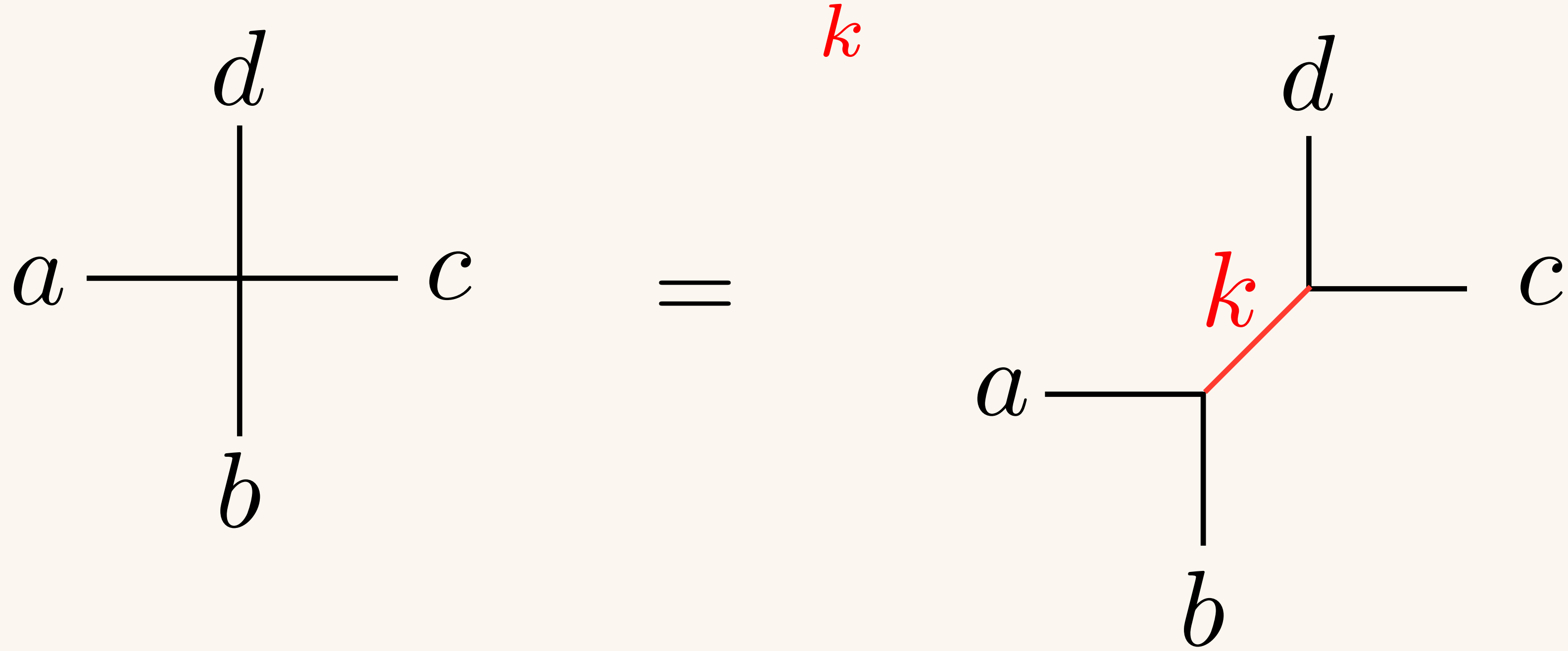
- ◇ No sign-problem. (Using truncation, without sampling)

↓

We need to discuss the systematic error from truncation.

◉ Singular Value Decomposition (SVD)

$$T_{abcd} = \sum_{k}^{D} A_{ab}{}^{k} \lambda^{k} B_{cd}{}^{k}$$



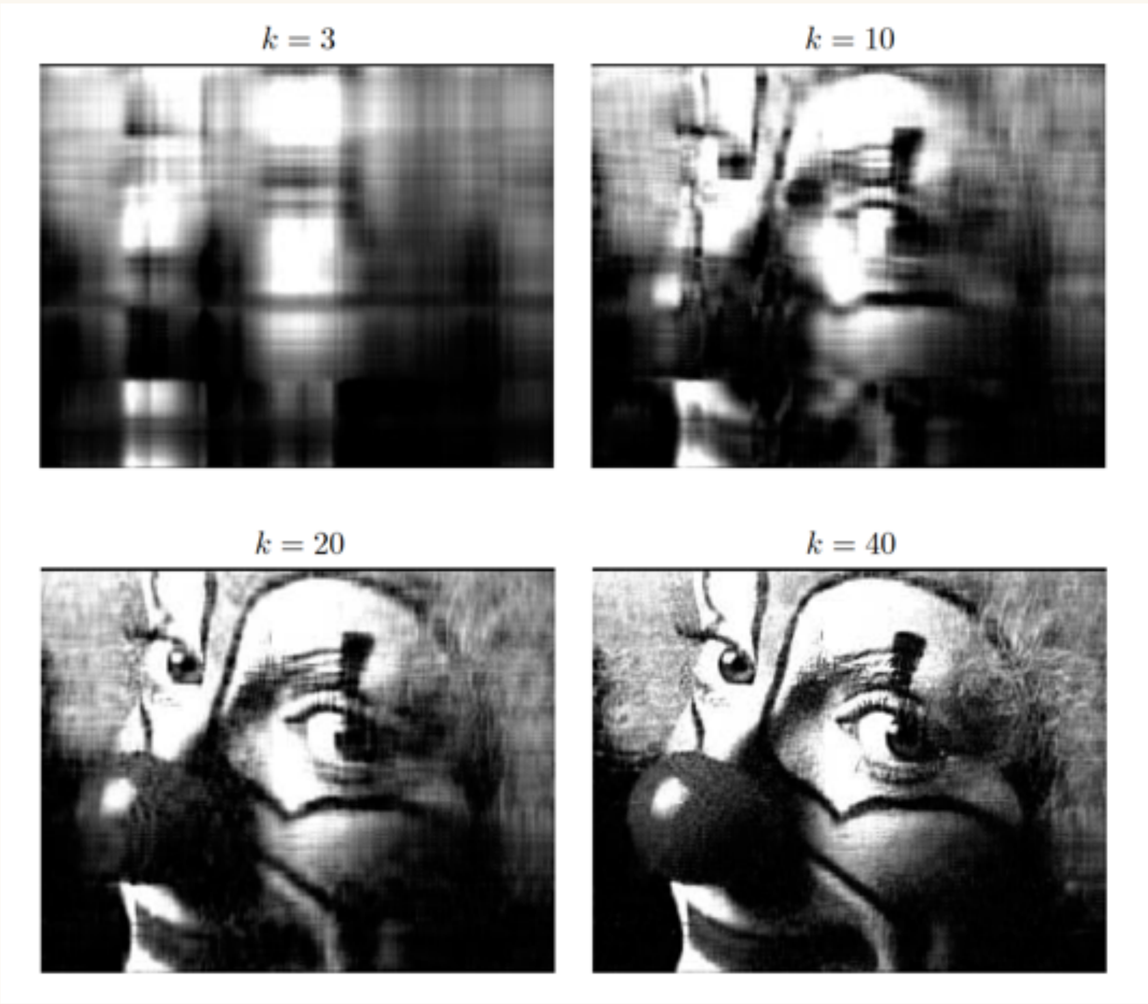
◇ Larger singular values λ have much "information" of T

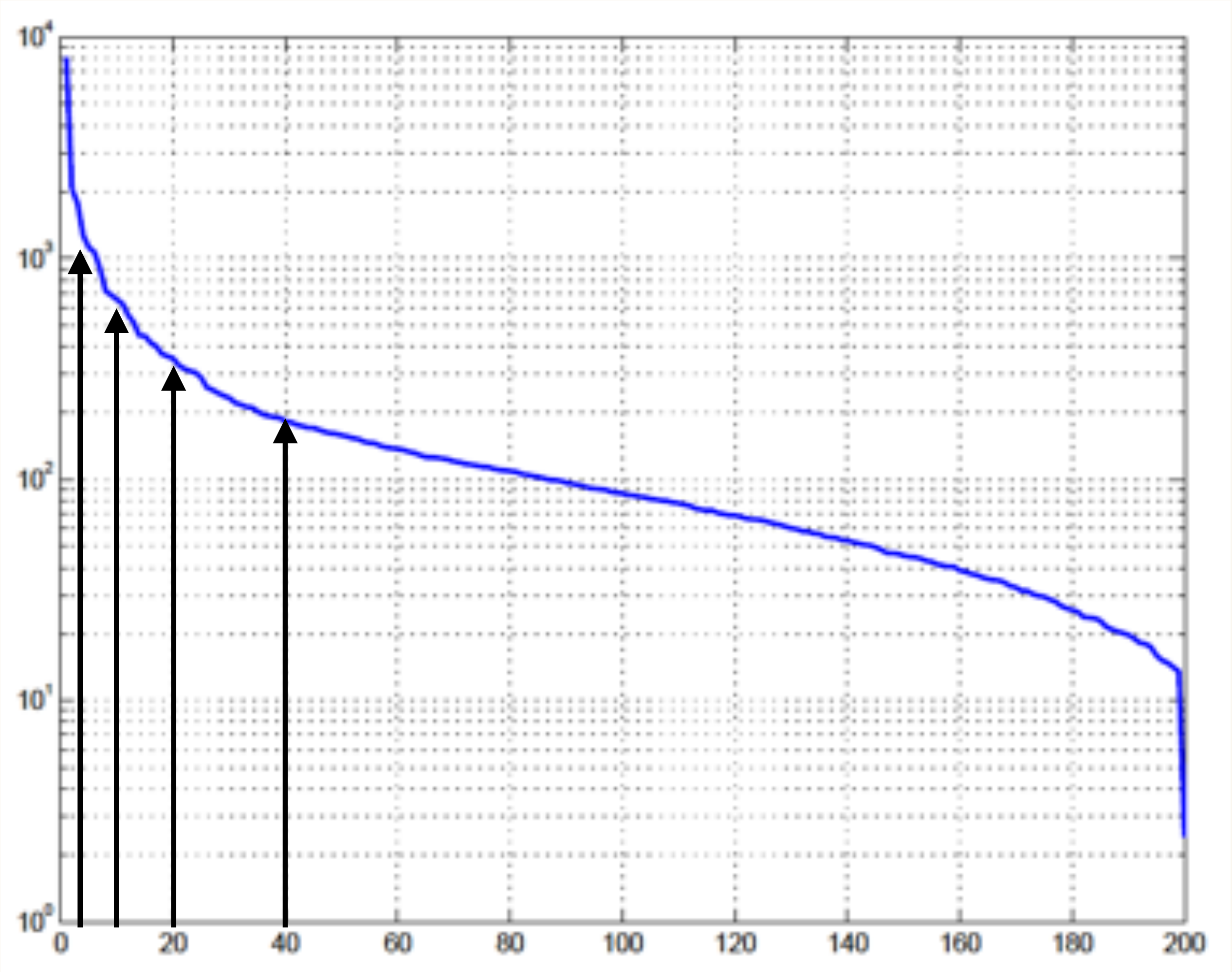
→ (Frobenius norm)

→ We can approximate the matrix by the cutoff of index k

$$\dim(k) = \dim(a)\dim(b) \to D$$

◉ SVD for a coarse graining (e.g. Image)





[http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT]

◉ Hilbert space and many body system

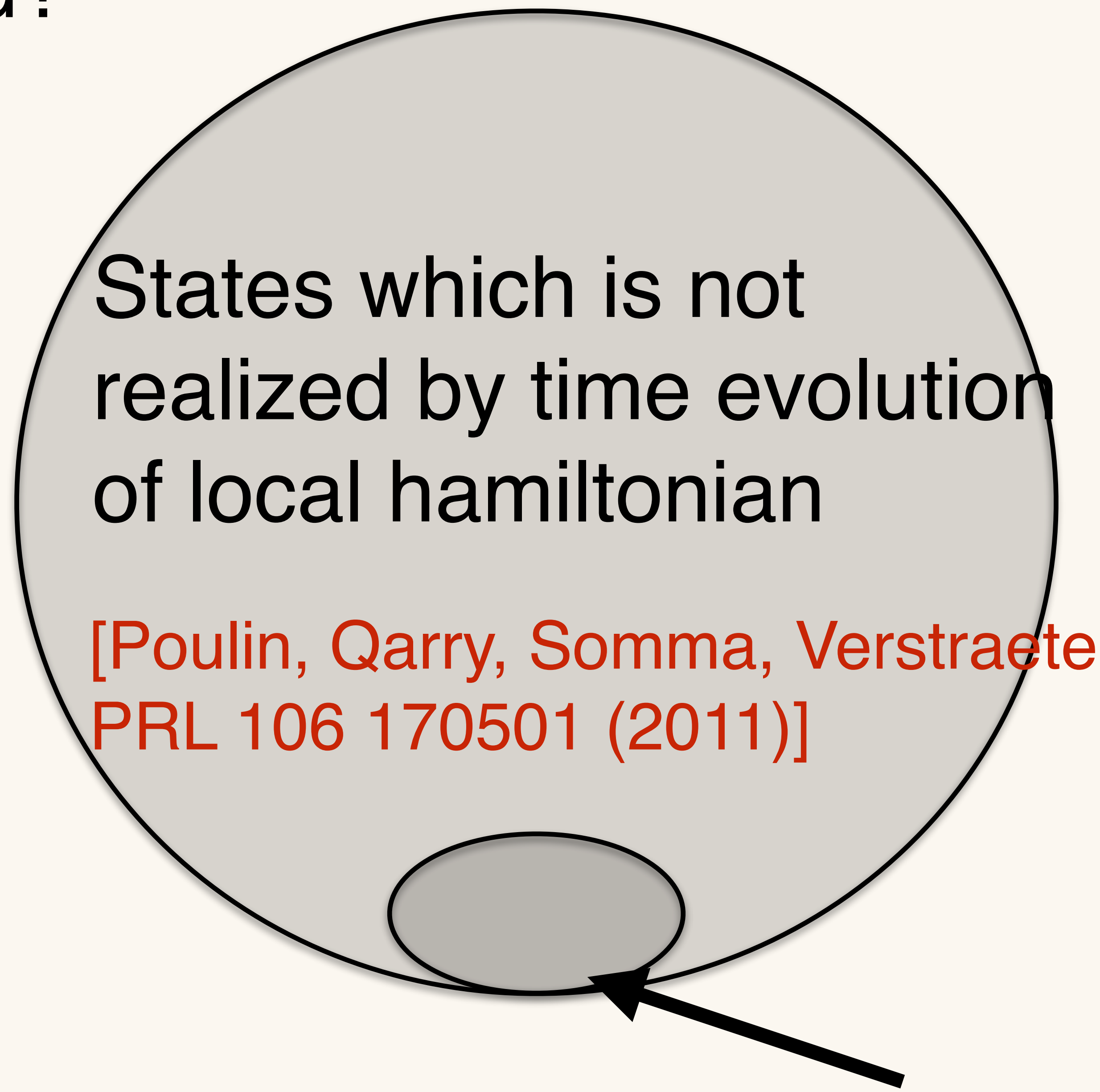
→ Exponentially large (Hilbert) space approximated by polynomially large space.

◇ Is this approximation valid?

→ Approximation relates to area law.

↓

Many one dimensional system follows area law.



approximated space by Tensor network

Numerical costs for simple TRG

◇ How fast?

SVD: $$T_{abcd} = \sum_k A^k_{ac} B^k_{bd}$$ $O(D^6)$

Contraction: $$T^{(n+1)}_{klmn} = \sum_{a,b} A^k_{a_{xy+1} b_{xy+1}} B^l_{a_{xy} b_{x-1y+1}} C^m_{a_{xy} b_{xy+1}} D^n_{a_{xy+1} b_{x-1y+1}}$$ $O(D^6)$

$$2^{2V} \to O((\log V) \times D^6)$$

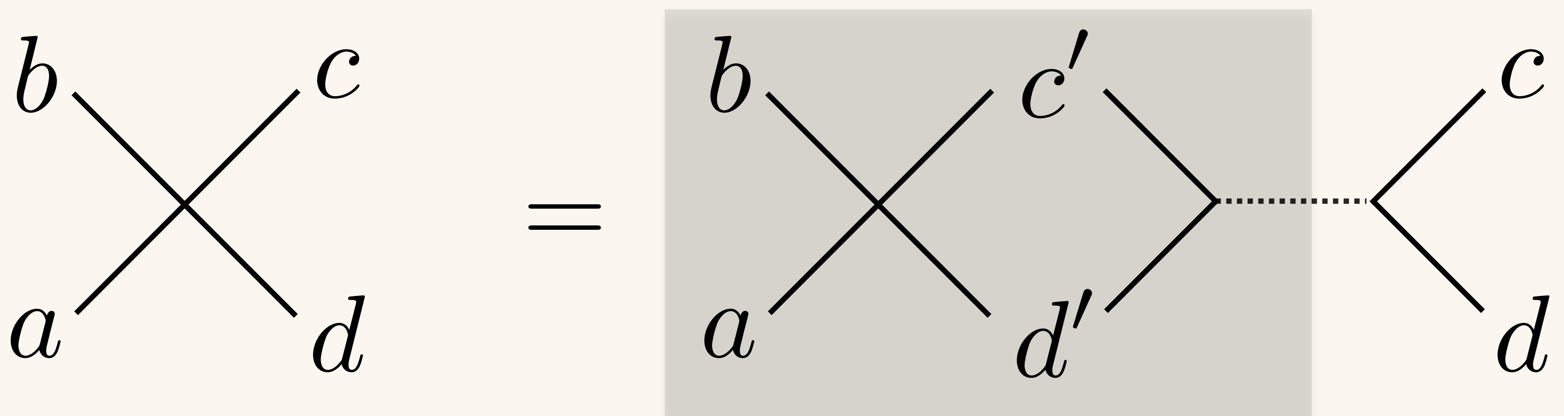
◇ Large D is still difficult.
(Higher dimension, complicated system…)

→ We need more sophisticated algorithm.

Randomized TRG (RTRG)

◇ Approximation by (orthogonal) gaussian noize matrix

$$Y^T Y = \delta \quad Y \equiv T^t O$$



$$T_{abcd} = T_{abc'd'} Y_{c'd'y} Y_{cdy}$$

partially SVD

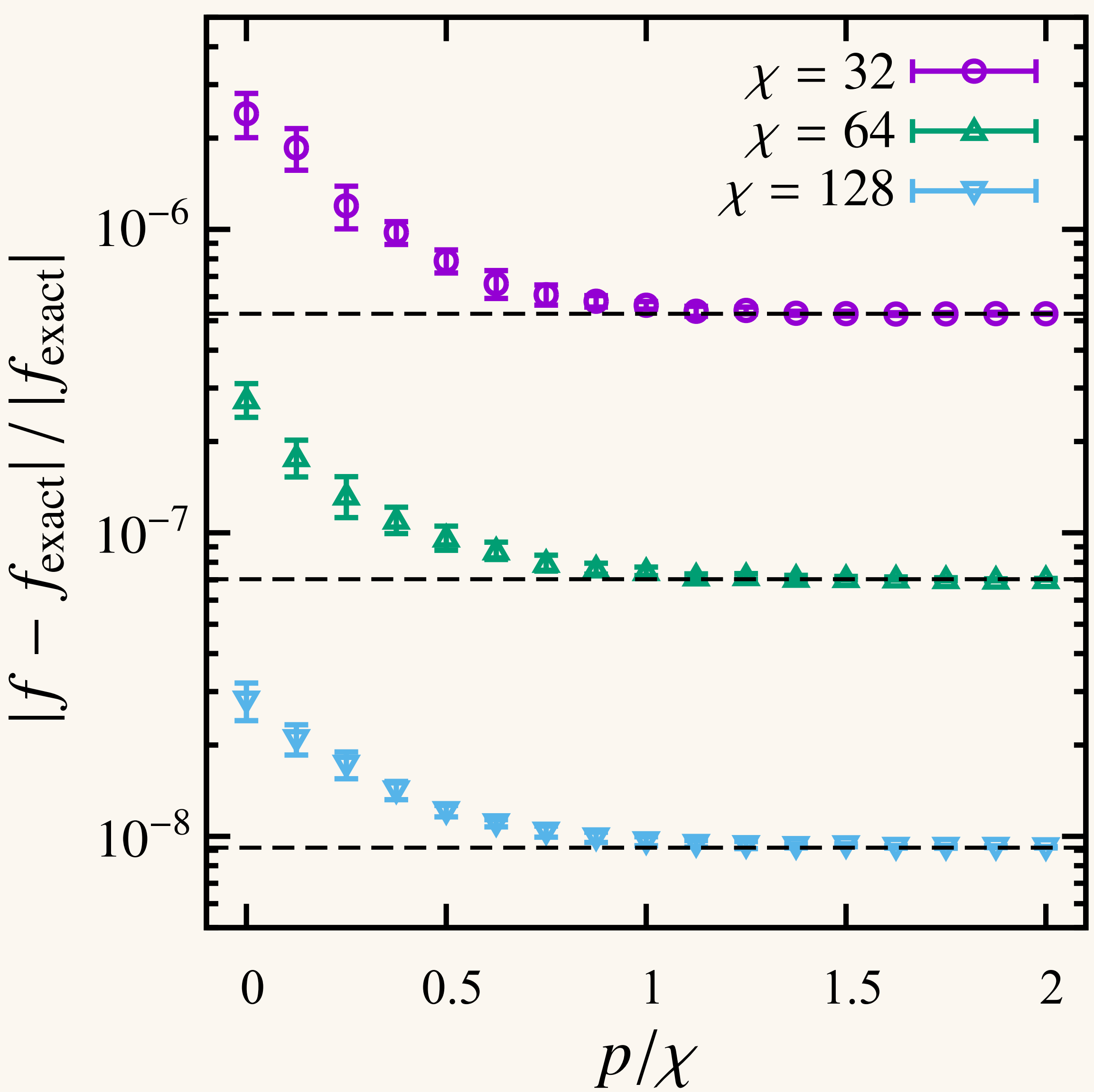
$$[TY]_{aby} = \sum_k A^k_{ab} B^k_y \qquad O(D^5)$$

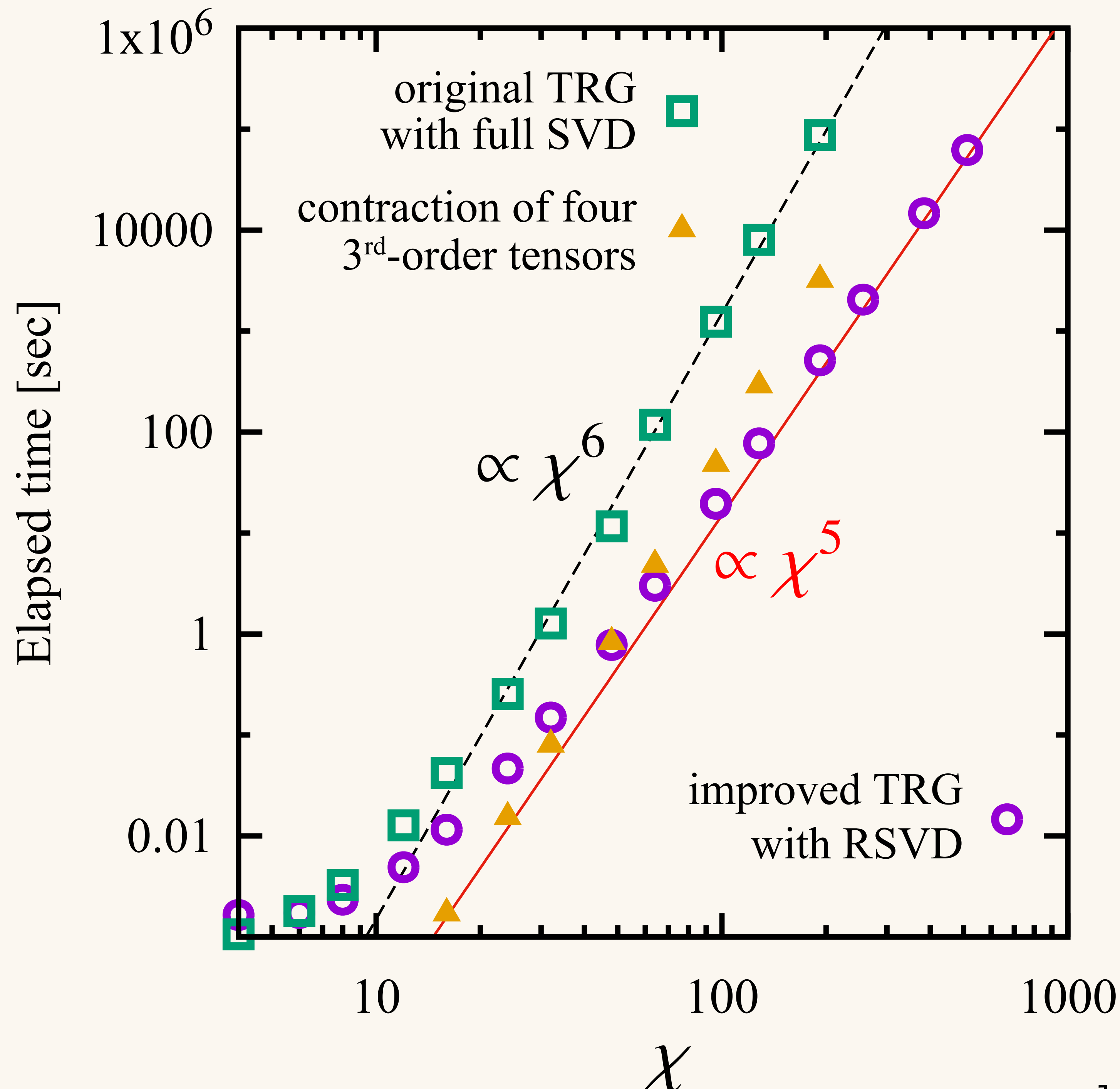
$$T_{abcd} = \sum_k A^k_{ab} [BY]_{cd}$$

[J.T.N. Halko and P.G. Martinsson arXiv:0909.4061]

◉ Numerical costs for randomized TRG

◇ Cost reduced: $O(D^6) \to O(D^5)$

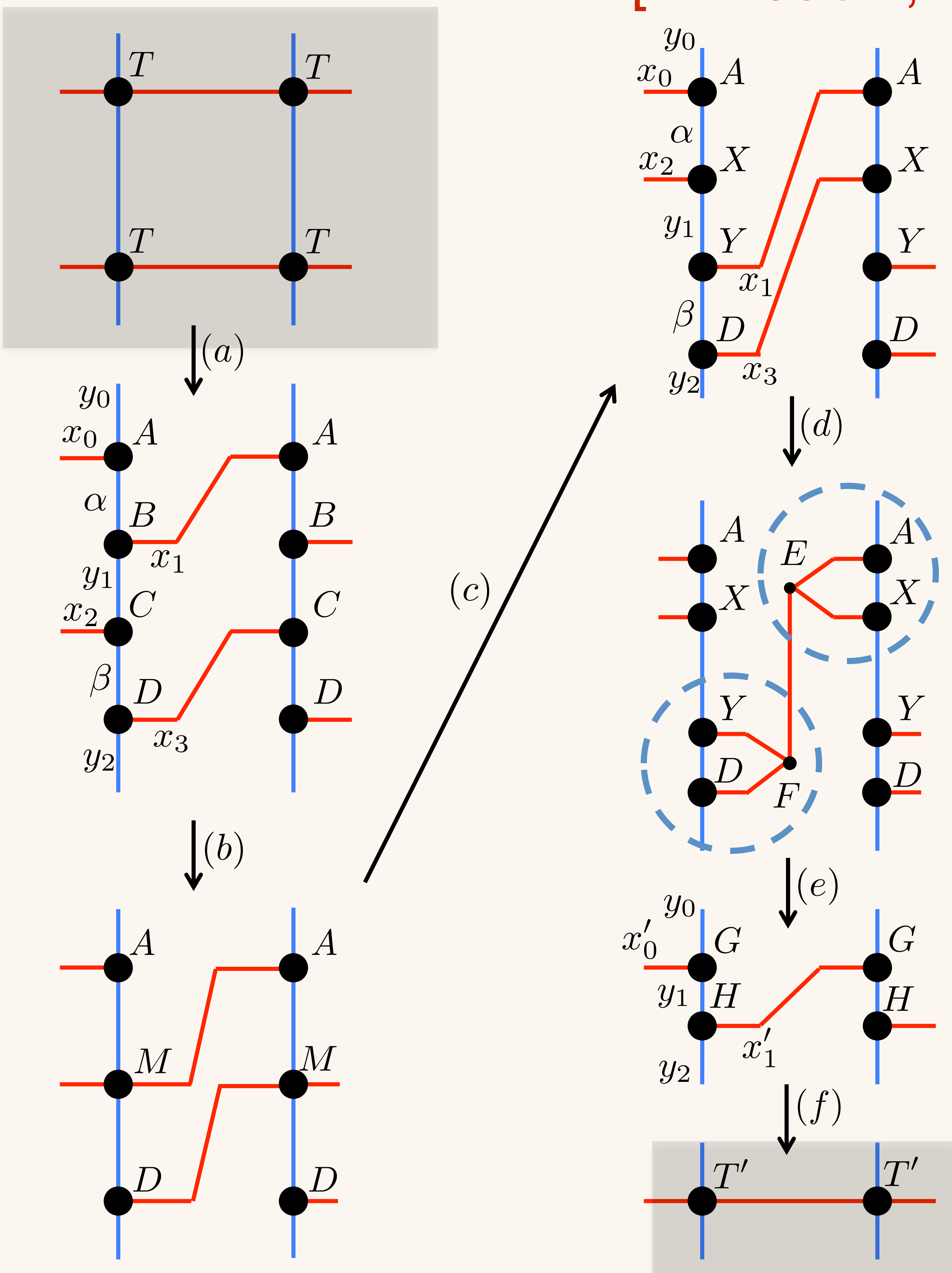




[S. Morita, R. Igarashi et al. arXiv:1712.01458]

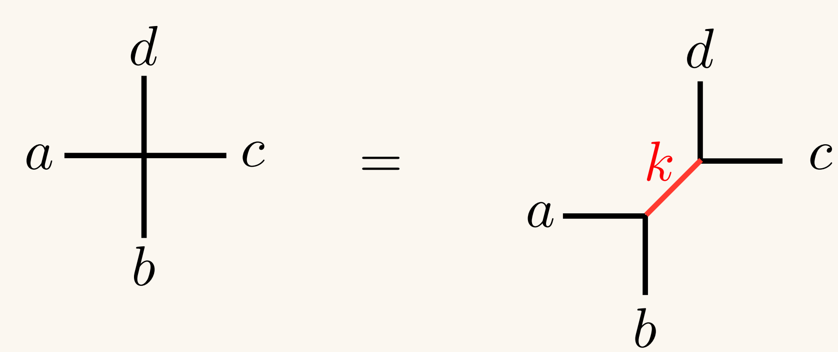
Anisotropic TRG (ATRG)

[D. Adachi, T. Okubo, and S. Todo. arXiv:1906.02007]



◇ Auxiliary SVD before SVD
(In order to reduce indeces)

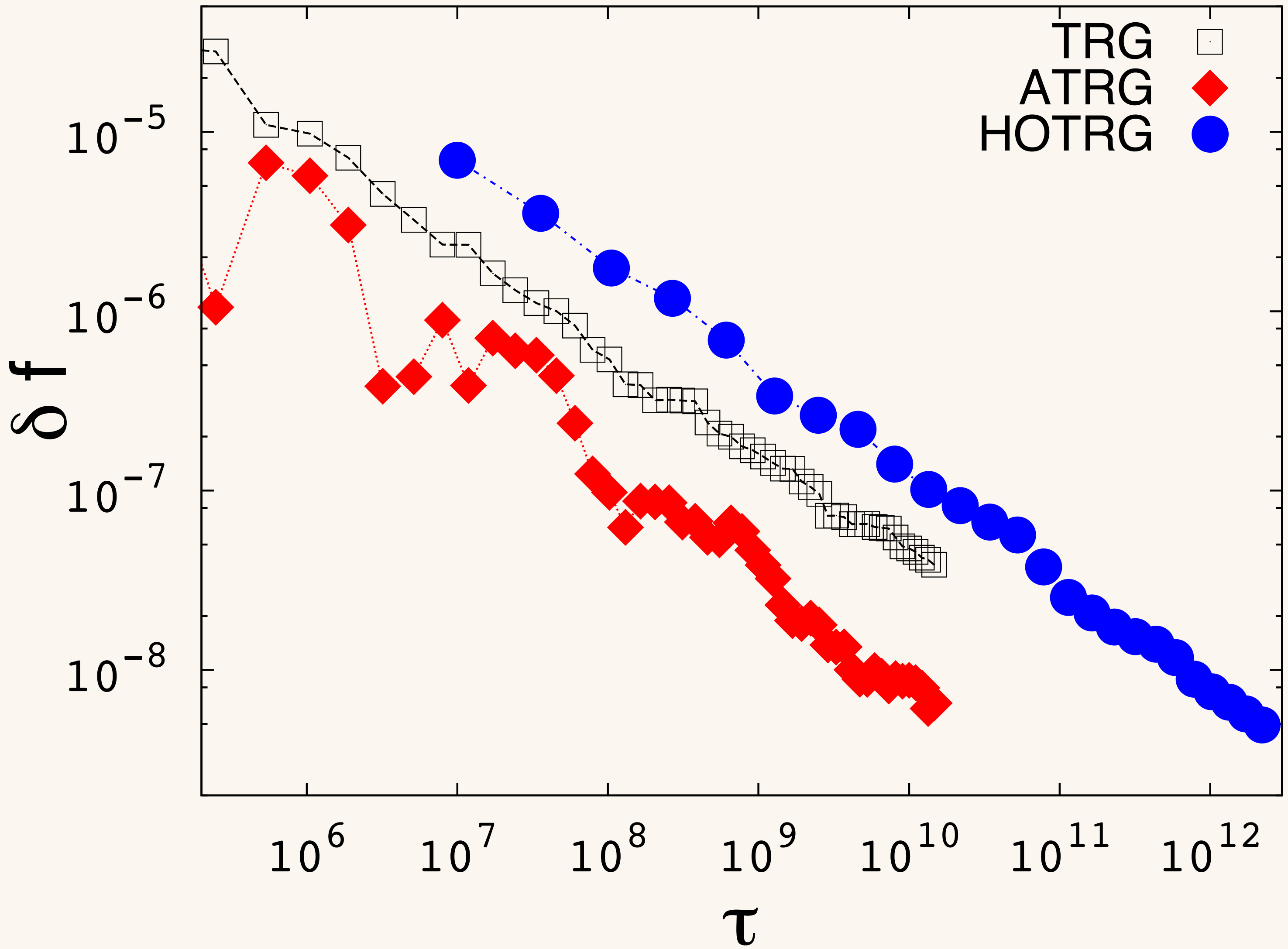
$$O(D^{4\mathrm{dim}-1}) \to O(D^{2\mathrm{dim}+1})$$



◉ Numerical costs for ATRG

[D. Adachi, T. Okubo, and S. Todo. arXiv:1906.02007]

◇ NOTE: This is part of the calculation time.



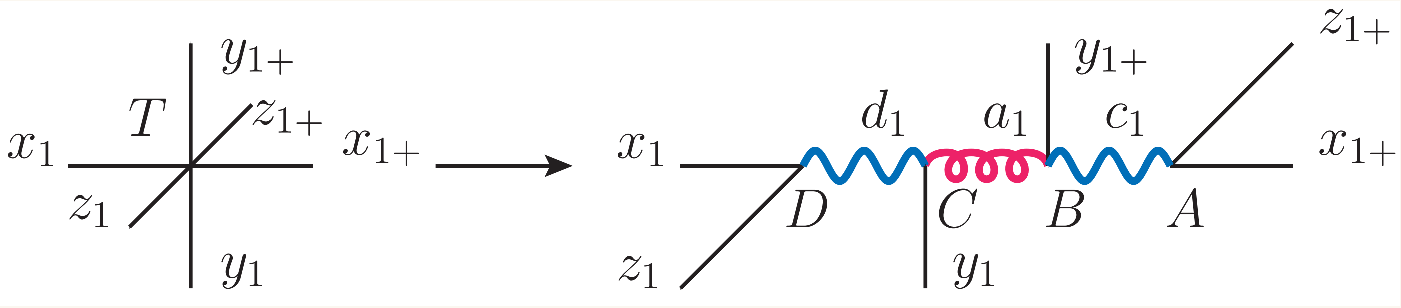
$$\tau = \begin{cases} D^5 & \text{for TRG and ATRG} \\ D^7 & \text{for HOTRG} \end{cases}$$

ATRG, while it is the contraction in HOTRG. In practice, the partial SVD takes much longer time than the contraction, even when their computation costs are in the same order. Thus, the actual performance difference between ATRG and HOTRG is smaller than FIG. 6, though ATRG becomes more and more advantageous for larger

◉ Triad RG

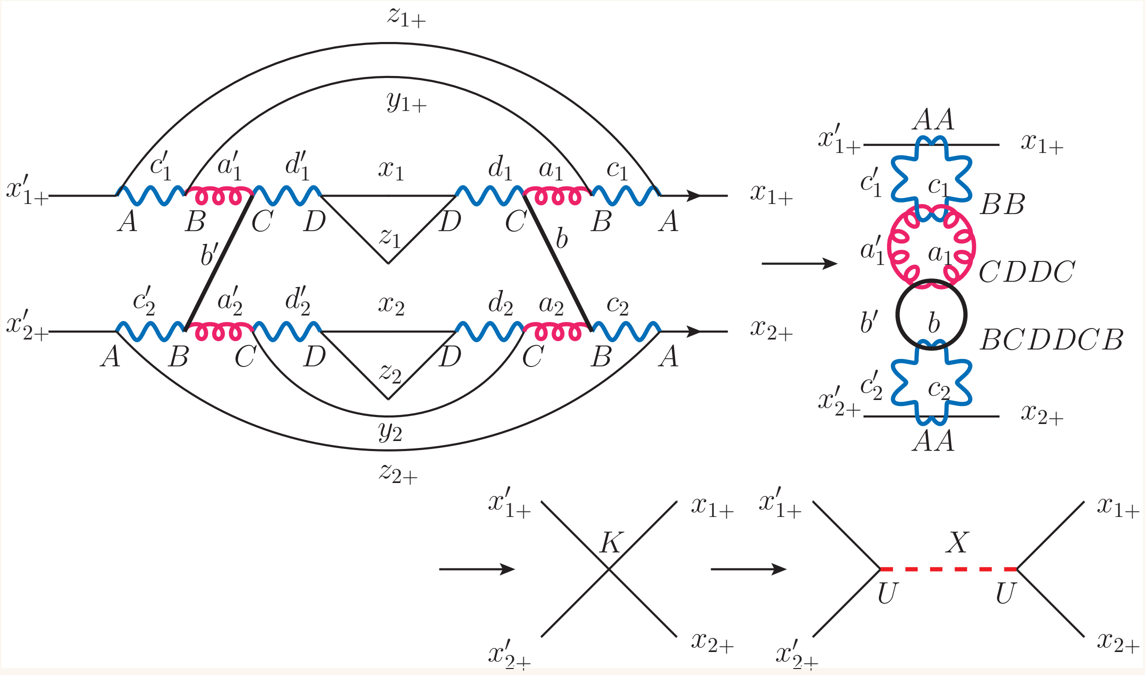
[D. Kadoh and K.N. arXiv:1912.02414]

◇ Using the Triad (Rank-3) tensor as a fundamental tensor

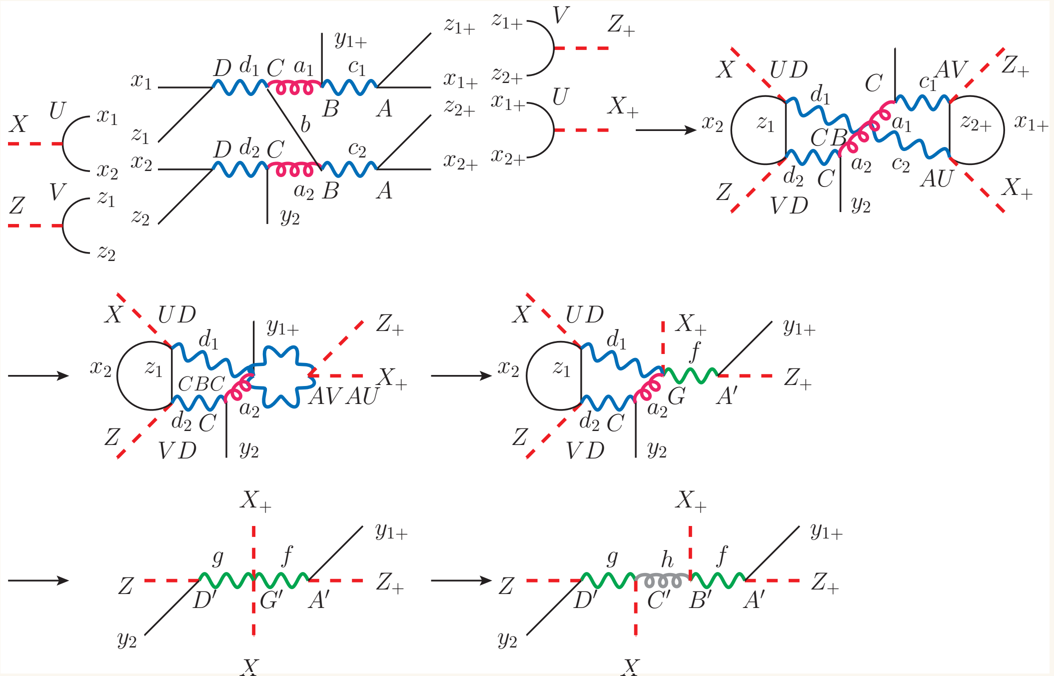


→ We apply HOTRG-like procedure to Triad tensor rep.

◇ projection operator U



◇ contraction part



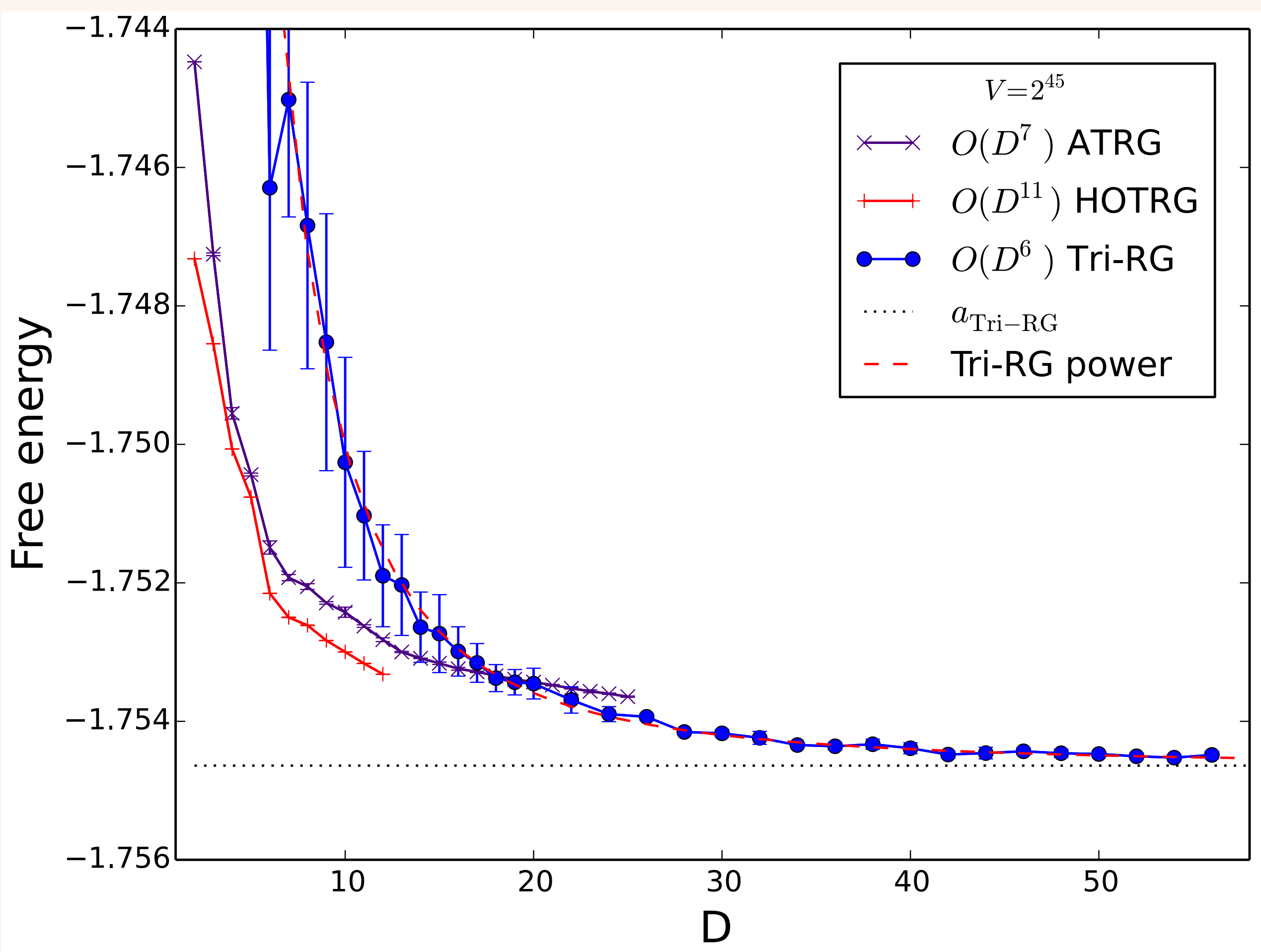
◉ Triad RG

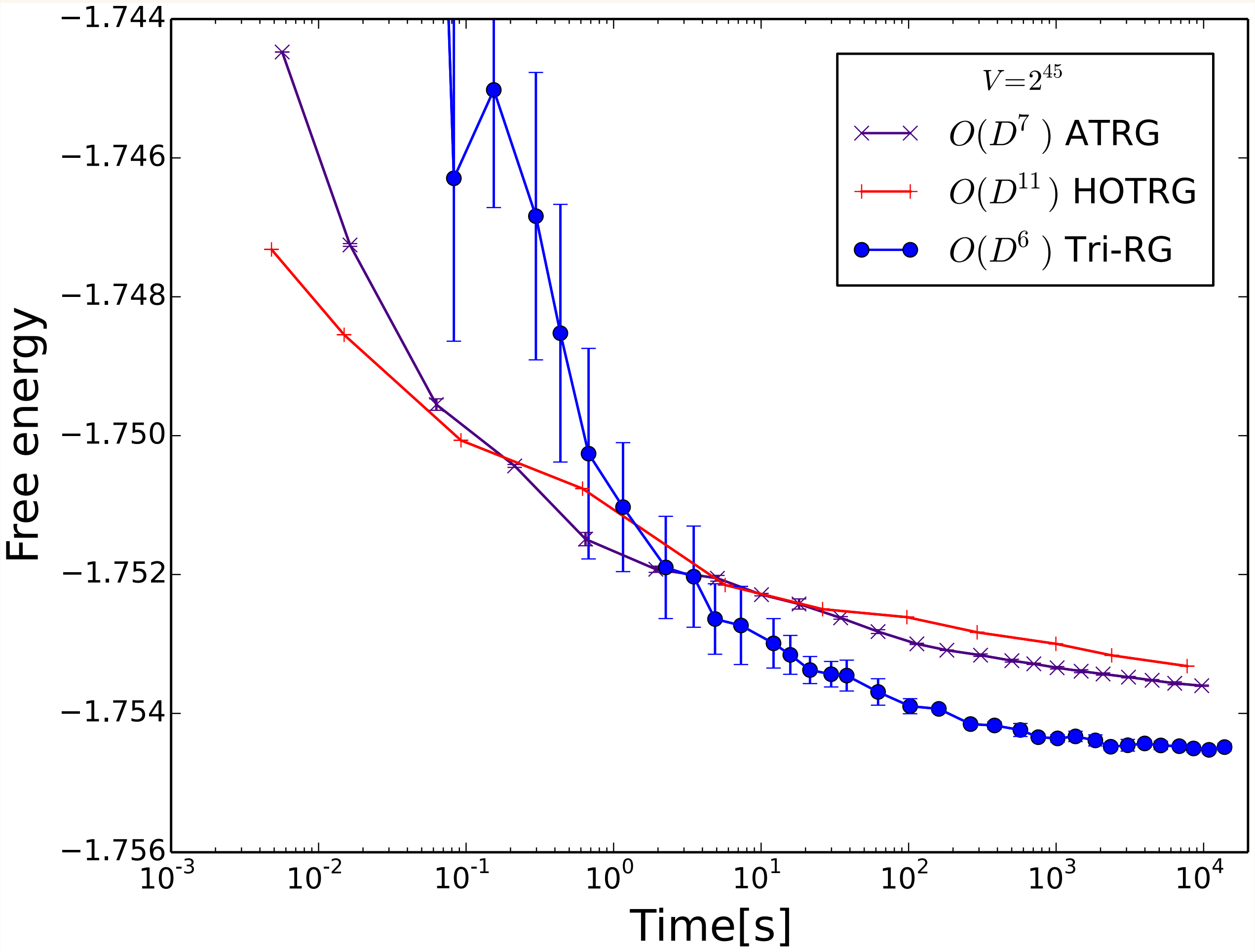
[D. Kadoh and K.N. arXiv:1912.02414]

◇ Triad rep. reduces the cost without loss of precision

$$O(D^{4\text{dim}-1}) \to O(D^{\text{dim}+3})$$

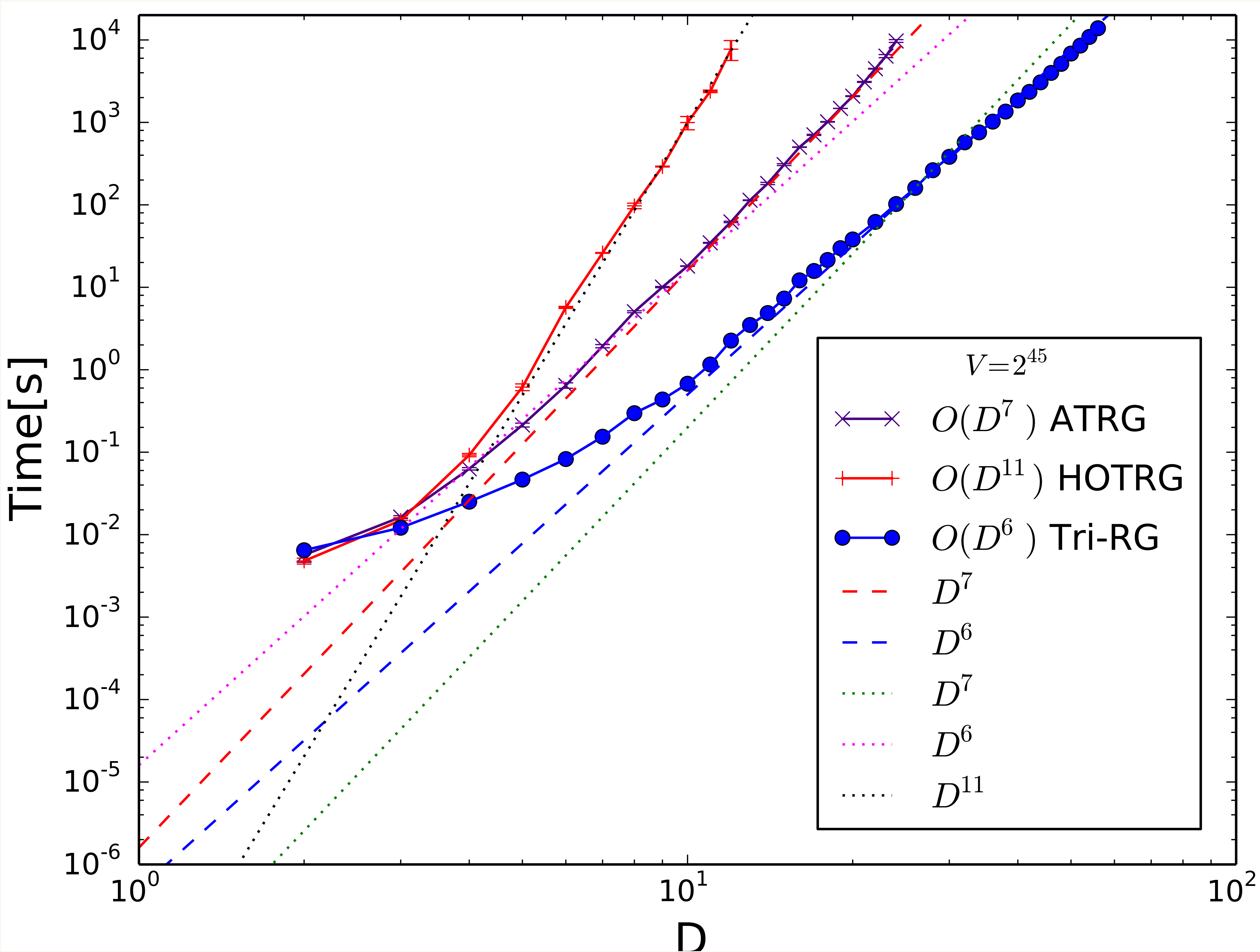
◇ 3-dim Ising



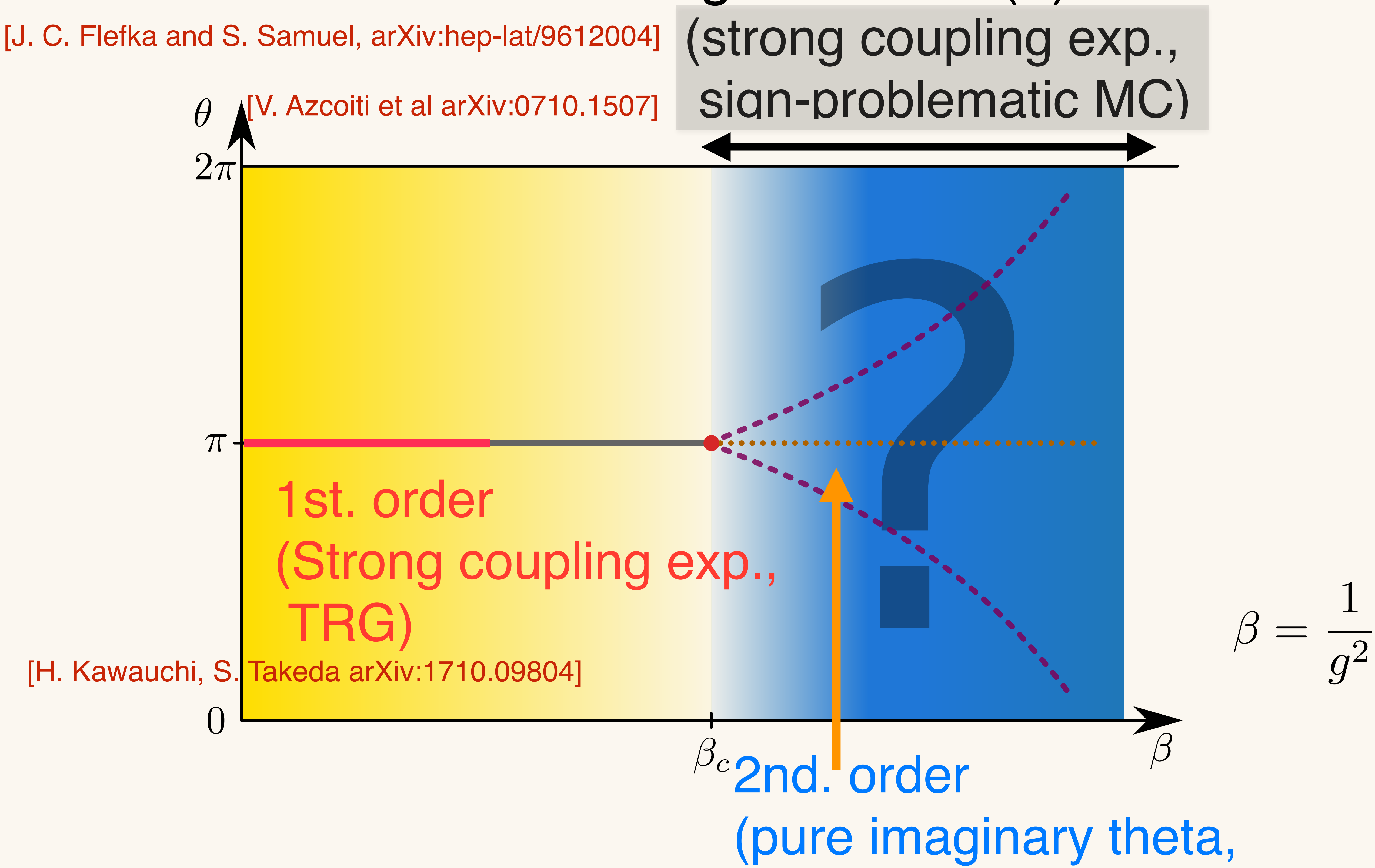


Triad RG

[D. Kadoh and K.N. arXiv:1912.02414]



◉ Motivation: Phase diagram of CP(1) model



→ Large beta region is not strongly confirmed.

$\theta = 0$ case

From Lagrangian to the tensor

$$e^{-S_{\theta=0}} = \exp\left[-2\beta\sum_{x,\mu}\left[z_x^* z_{x+\hat{\mu}}U_\mu + z_x z_{x+\hat{\mu}}^* U_\mu^\dagger\right]\right]$$

z : complex scaler field
U_μ: link variable

◇ Using expansion by orthogonal function $f_{l,m}$

$$\int dz f_{l,m}(z',z) f_{l',m'}^*(z'',z) = \frac{1}{d_{l,m}}\delta_{l,l'}\delta_{m,m'} f_{l,m}(z',z)$$

$l + m \leq k_{\max}$

$I_n(x)$: Modified Bessel

$$e^{-S_{\theta=0}} = \prod_{x,\mu}\frac{1}{2\beta}\sum_{l,m=0}^{\infty} I_{l+m+1}(4\beta) d_{l,m} f_{l,m}(z_x, z_{z+\hat{\mu}})$$

→We need to truncate the index l, m.

From Lagrangian (action) to the tensor

$$e^{-S_{\theta=0}} = \prod_{x,\mu} \frac{1}{2\beta} \sum_{l,m=0}^{\infty} I_{l+m+1}(4\beta) d_{l,m} f_{l,m}(z_x, z_{z+\hat{\mu}})$$

◇ Partition function

$$Z = \int \prod_x dz_x e^{-S_{\theta=0}}$$

→ Integrate out z_x remaining l, m indeces

$$(\{z_x\} \rightarrow \{l, m\})$$

$$Z = \sum_{l,m,a} \prod_x T^{(x)}_{[l_s,m_s,a_s],[l_t,m_t,a_t],[l_u,m_u,a_v],[l_v,m_v,a_s],} = \sum_{l,m,a} \prod_x T^{(x)}_{stuv}$$

→ We introduces truncation for the indices $l + m \leq k_{\max}$.

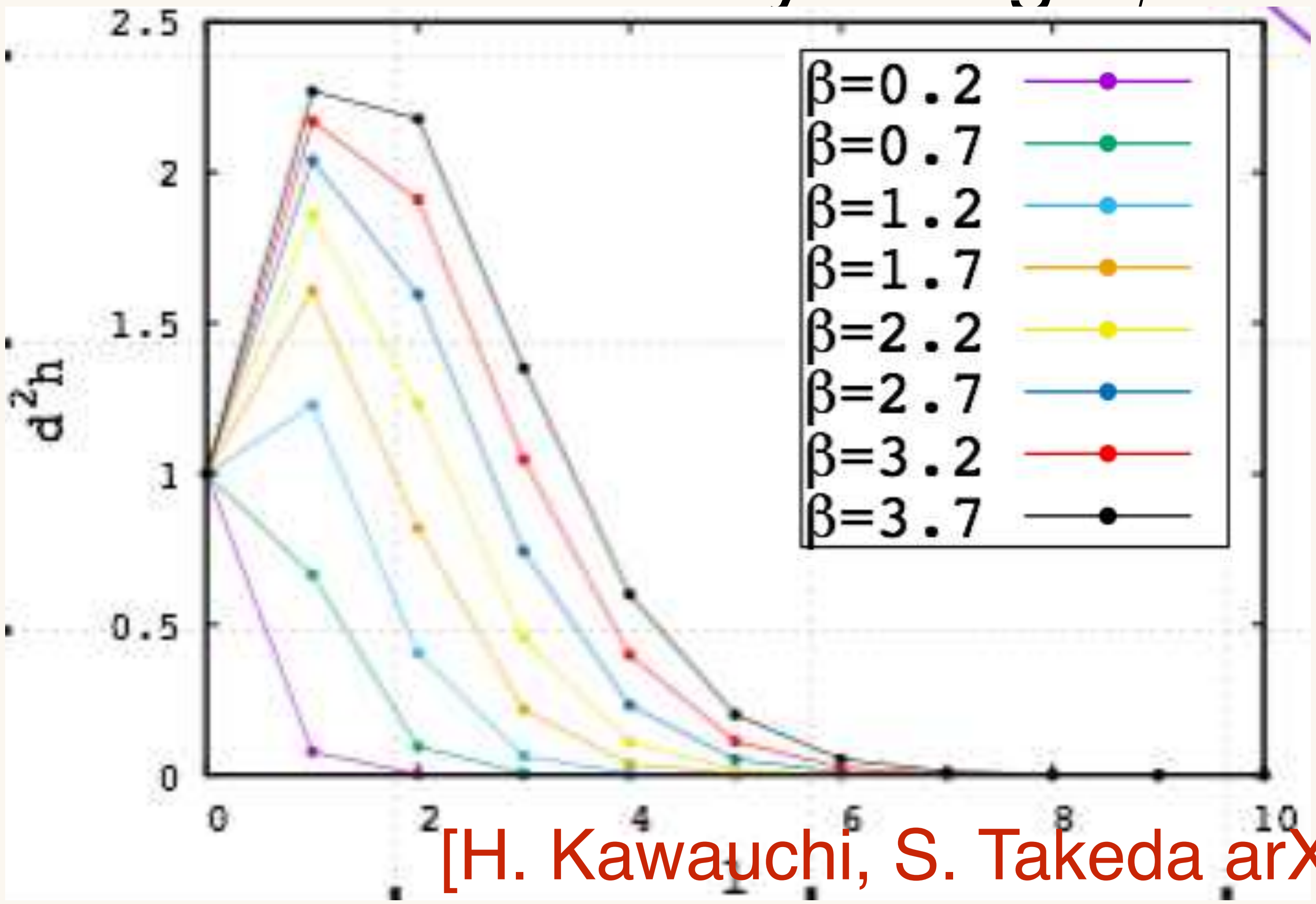
◉ Precision and truncaton

◇ Index size

$$\frac{4^{l_{\max}+1}-1}{3}$$

→ We have to cutoff $l_{\max} \leq 2 \quad (21) \text{ or } 3 \quad (85)$

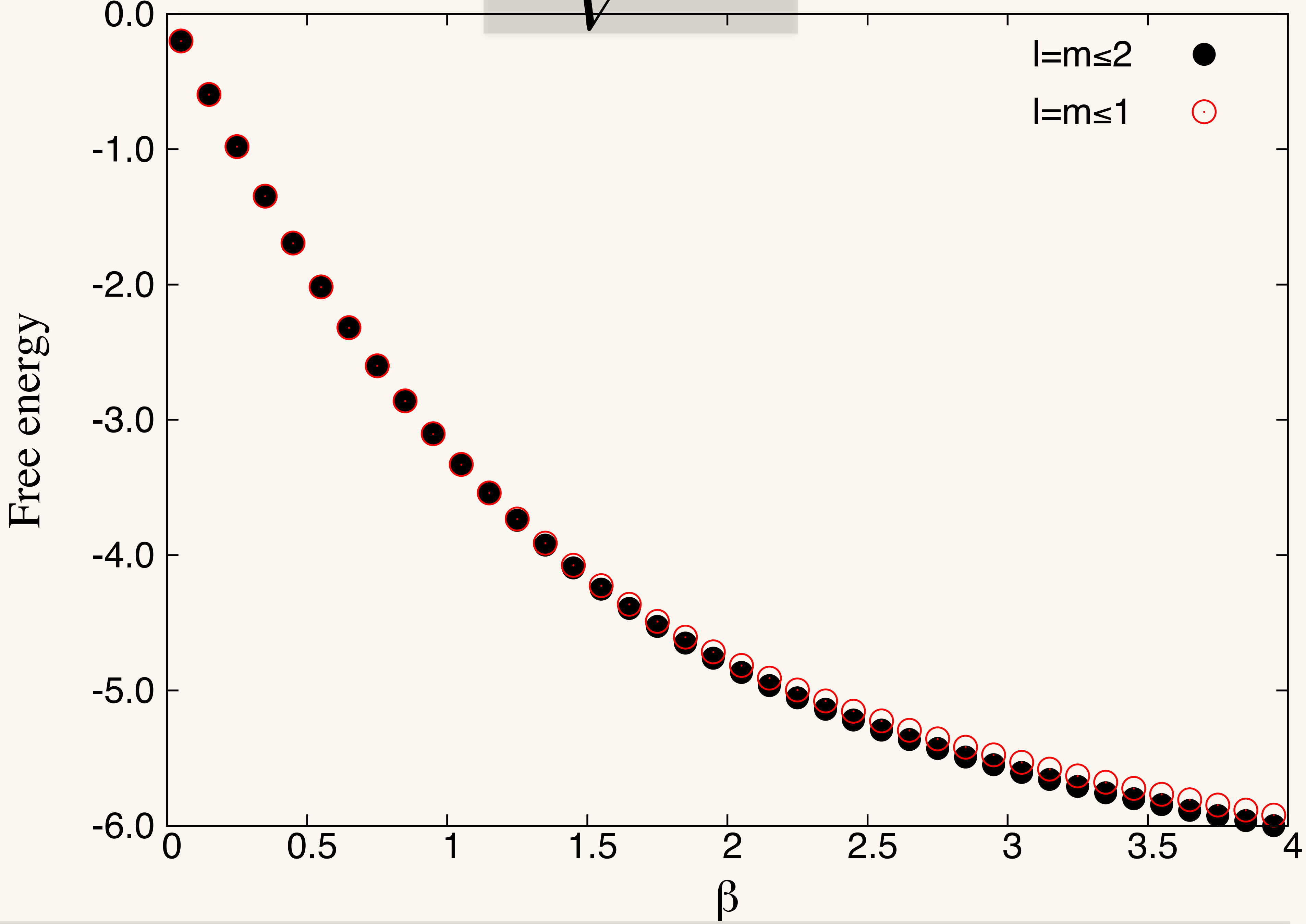
→ This truncation cause difficulty at high β region



[H. Kawauchi, S. Takeda arXiv:1603.09455]

Free energy of CP(1) model with $\theta = 0$

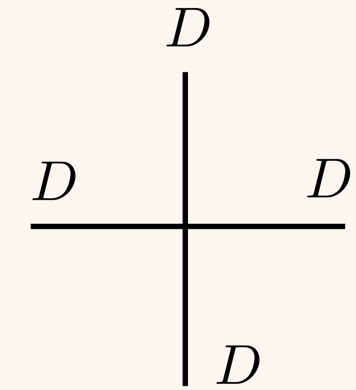
$$-\frac{1}{V}\ln Z$$

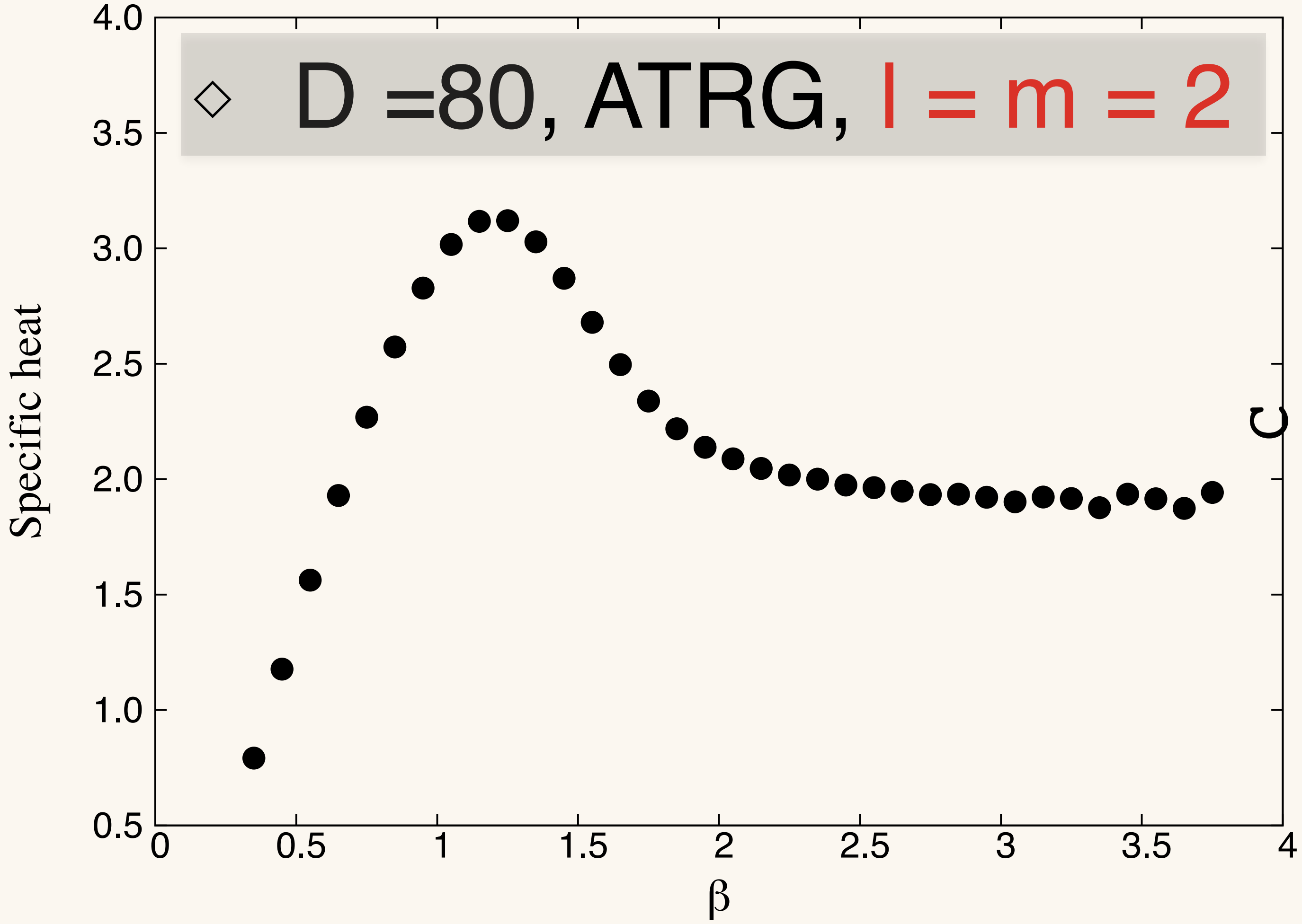


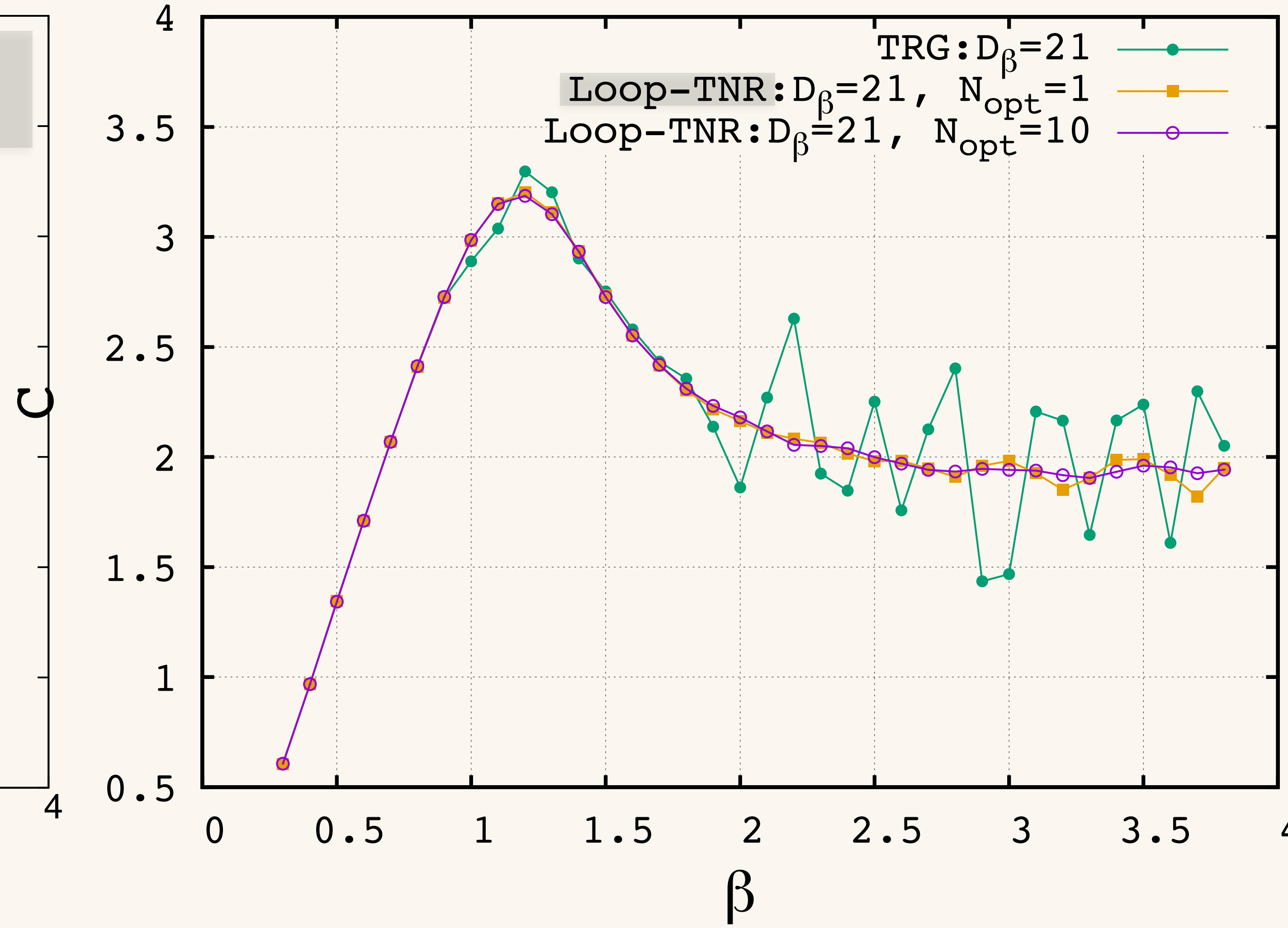
→ Truncation of l, m are small, especially in $\beta < 1.4$.

● Specific heat of CP(1) model with $\theta = 0$.

$$\frac{\beta^2}{V}\frac{\partial^2}{\partial \beta^2}\ln Z$$

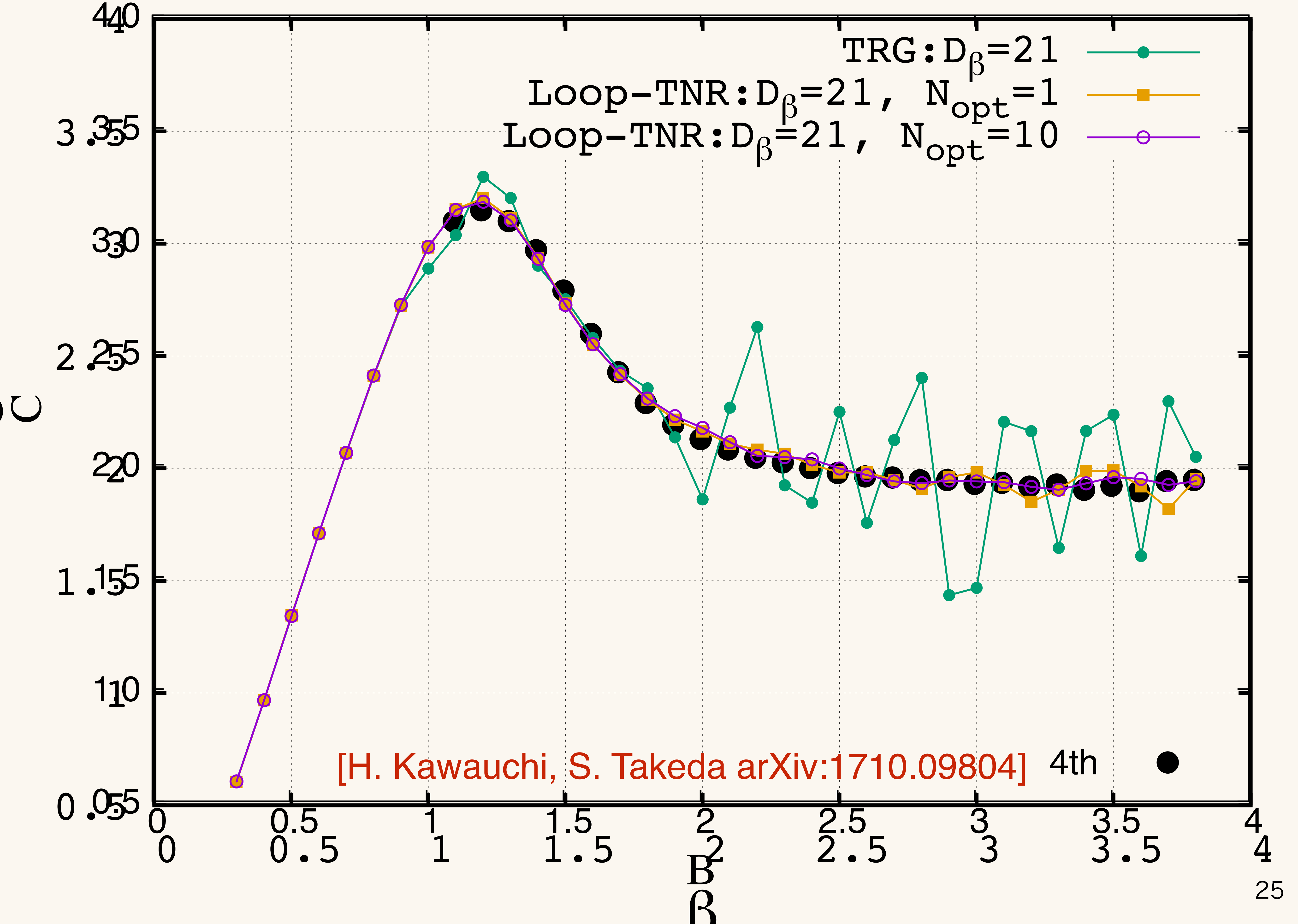


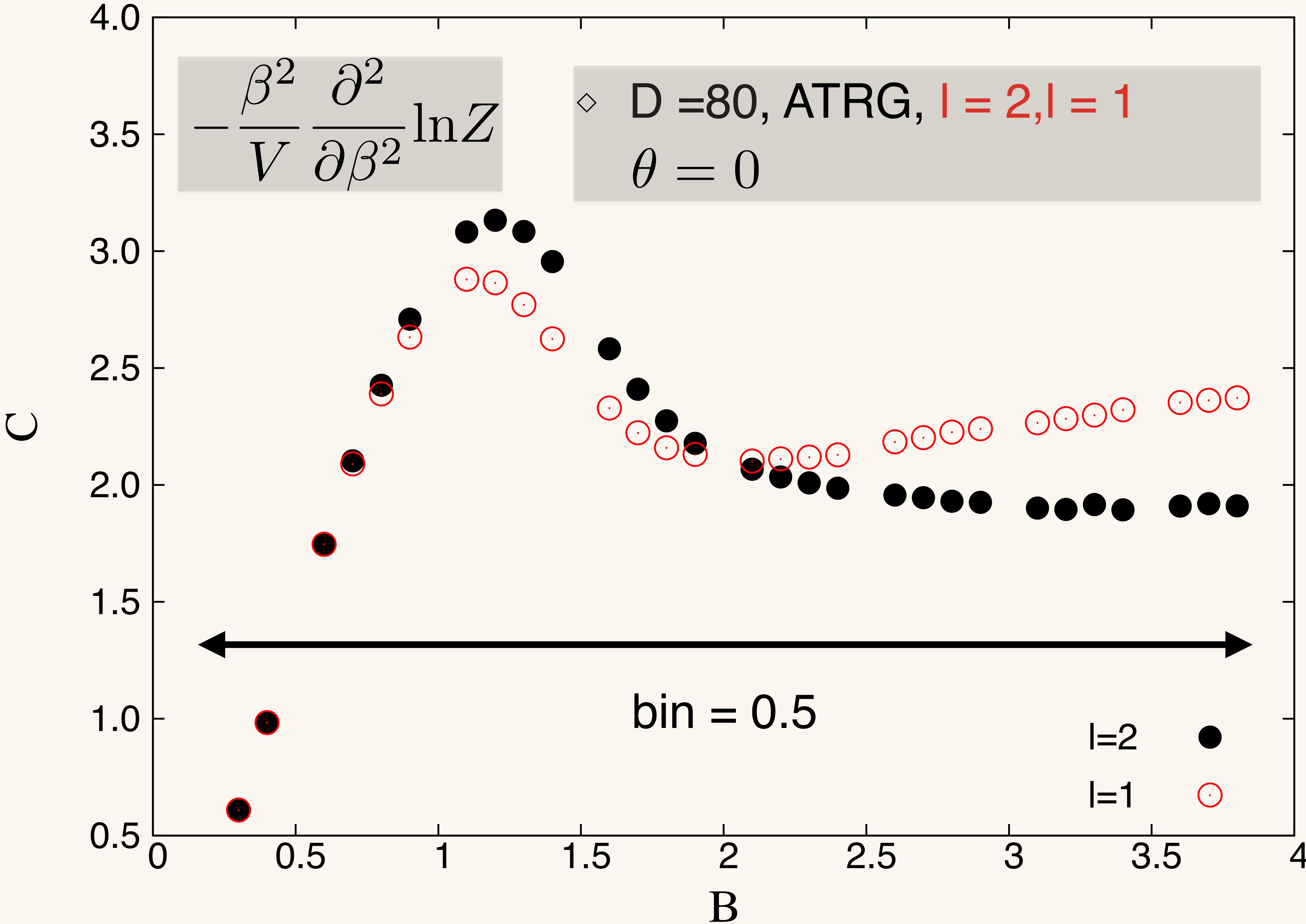




[H. Kawauchi, S. Takeda arXiv:1710.09804]
[ATRG: D. Adachi et al 1906.02007]

→ Sufficiently large bond size D produces reliable results.


TRG:D_β=21
Loop-TNR:D_β=21, N_{opt}=1
Loop-TNR:D_β=21, N_{opt}=10
[H. Kawauchi, S. Takeda arXiv:1710.09804]
4th
C
β
0.5
1
1.5
2
2.5
3
3.5
4
0
0.5
1
1.5
2
2.5
3
3.5
4



$-\frac{\beta^2}{V}\frac{\partial^2}{\partial\beta^2}\ln Z$
◇ D =80, ATRG, l = 2,l = 1
$\theta = 0$
bin = 0.5
l=2
l=1
C
B
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

$\theta \neq 0$ case

Character expansion of the theta term

◇ Theta term is also character expanded.

$$e^{i\frac{\theta}{2\pi}q_p} = \sum_{n_p\in\mathbb{Z}} e^{in_p q_p} \frac{2\sin(\pi n_p + \theta/2)}{\theta + 2\pi n_p}$$

$$q_p = A_{x,1} - A_{x+\hat{1}-\hat{2},2} - A_{x-\hat{2},1} + A_{x-\hat{2},2} \bmod 2\pi$$

↓

We need to truncate the index n_p.

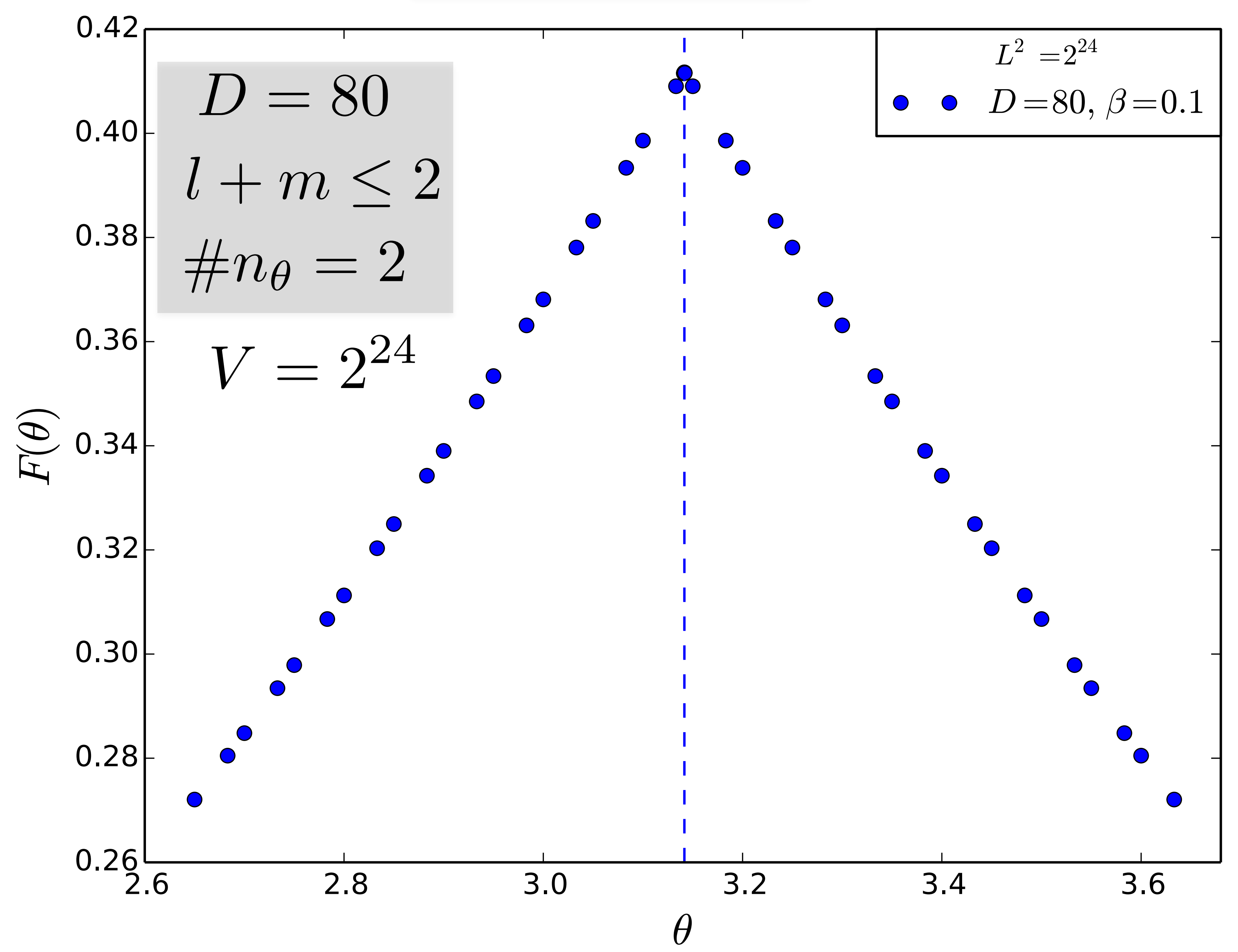
→ The order of the truncation error is $O(1/n_p)$.

◉ Free energy with theta term

[BTRG: D. Adachi et al. 2011.01679]

We use bond-weighted TRG method.

◇ beta = 0.1

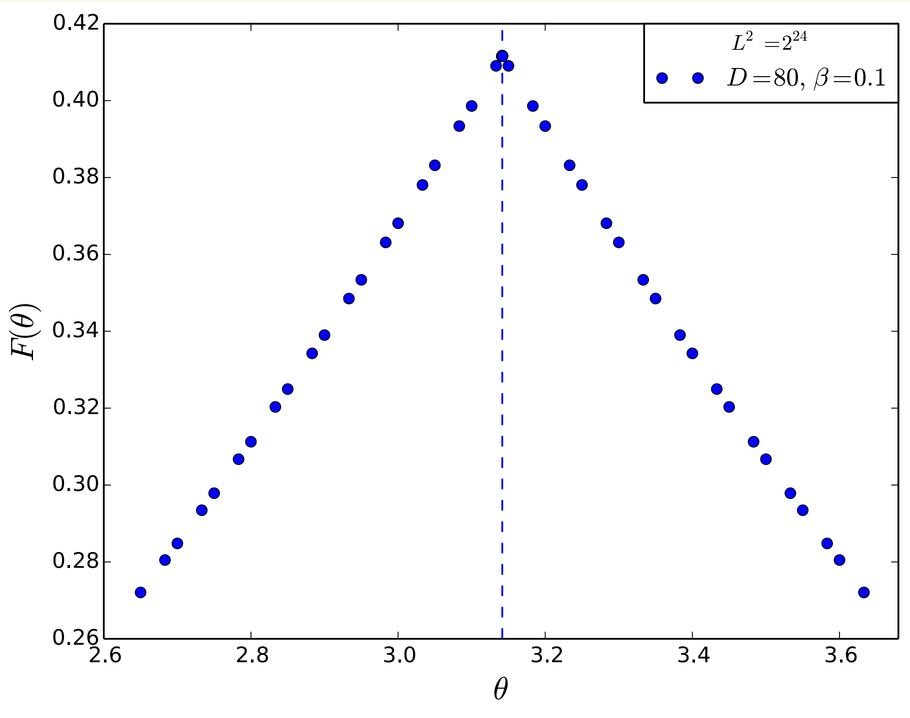


→ Clear kink structure imply the first-order transition

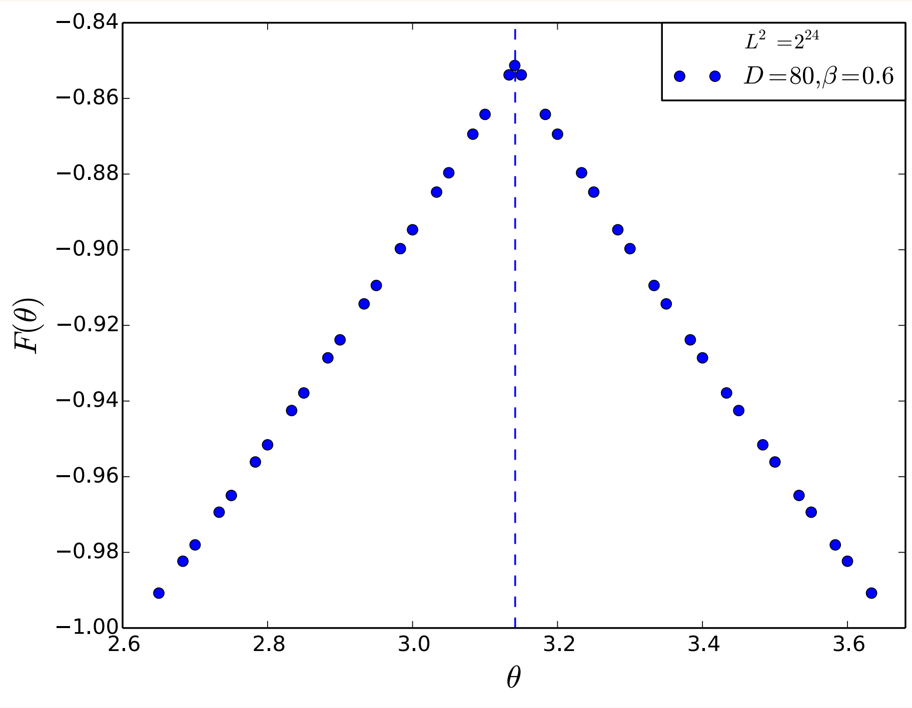
Free energy with theta term

$$D = 80$$
$$l + m \leq 2$$
$$\# n_\theta = 2$$
$$V = 2^{24}$$

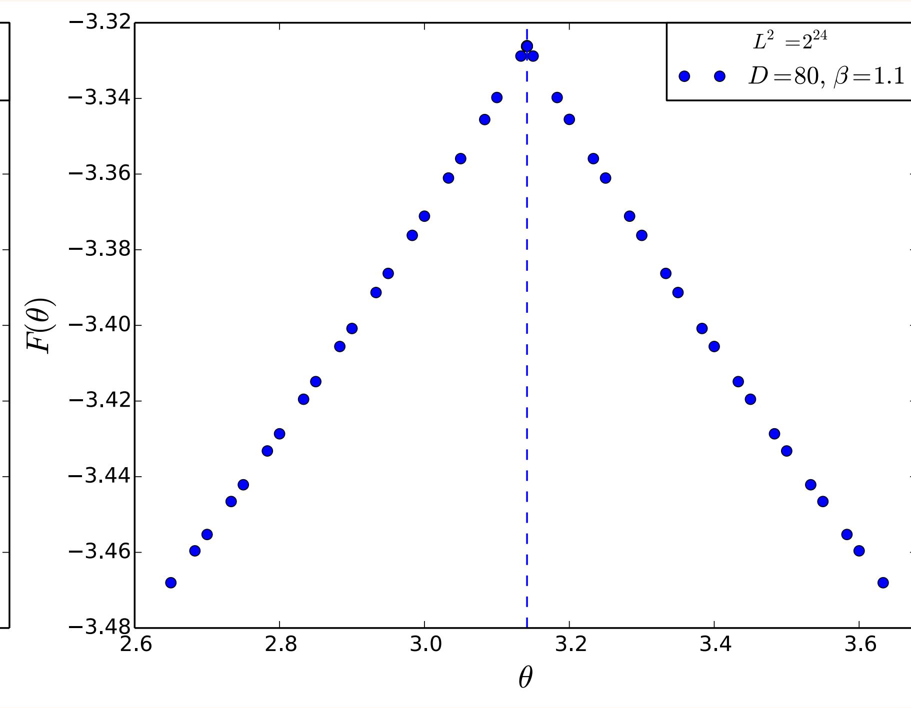
◇ beta = 0.1



◇ beta = 0.6



◇ beta = 1.1



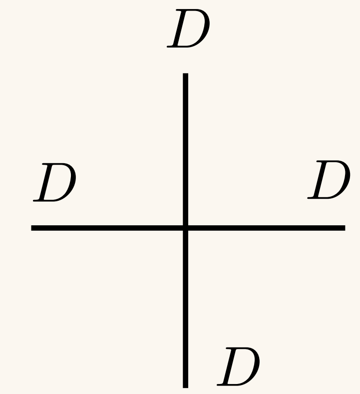
→ Clear kink structure imply the first-order transition

↓

◇ We should estimate the systematic error.

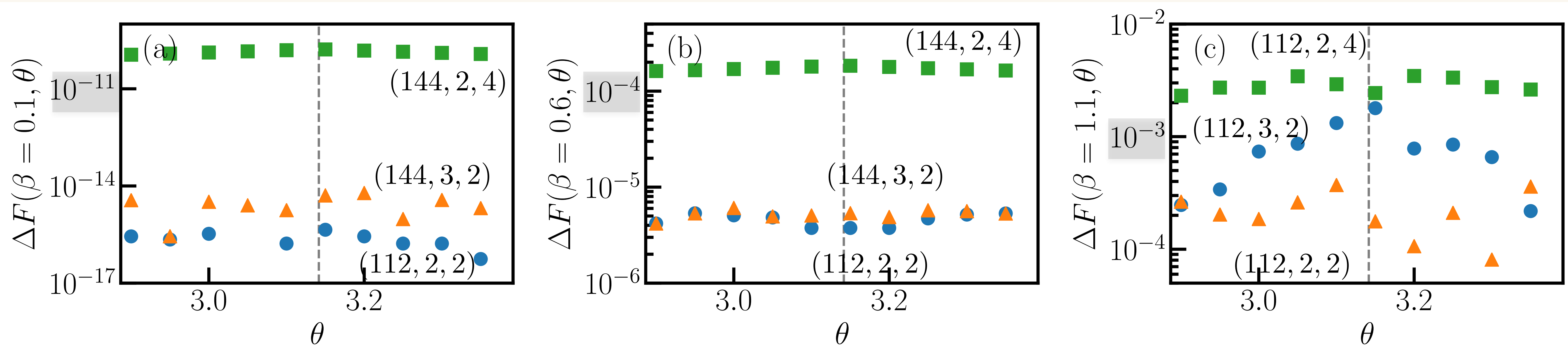
◉ Systematic error

$$l + m \leq k_{\max}$$



$$\Delta F(\beta,\theta) = |F(\beta,\theta)_{(D,k_{\max},\#n_\theta)} - F(\beta,\theta)_{(80,2,2)}|$$

◇ beta = 0.1 ◇ beta = 0.6 ◇ beta = 1.1



◇ SVD bond size truncation: $|F(\beta,\theta)_{(112,2,2)} - F(\beta,\theta)_{(80,2,2)}|$

◇ #Charactor expansion term: $|F(\beta,\theta)_{(144,3,2)} - F(\beta,\theta)_{(80,2,2)}|$

◇ #topological term, $\#n_\theta$: $|F(\beta,\theta)_{(144,2,4)} - F(\beta,\theta)_{(80,2,2)}|$

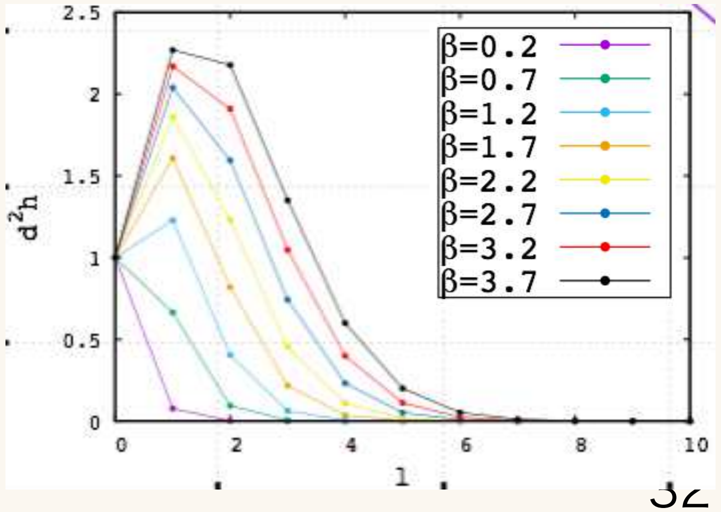
◉ Phase diagram from previous studies

◇ What’s the difference?

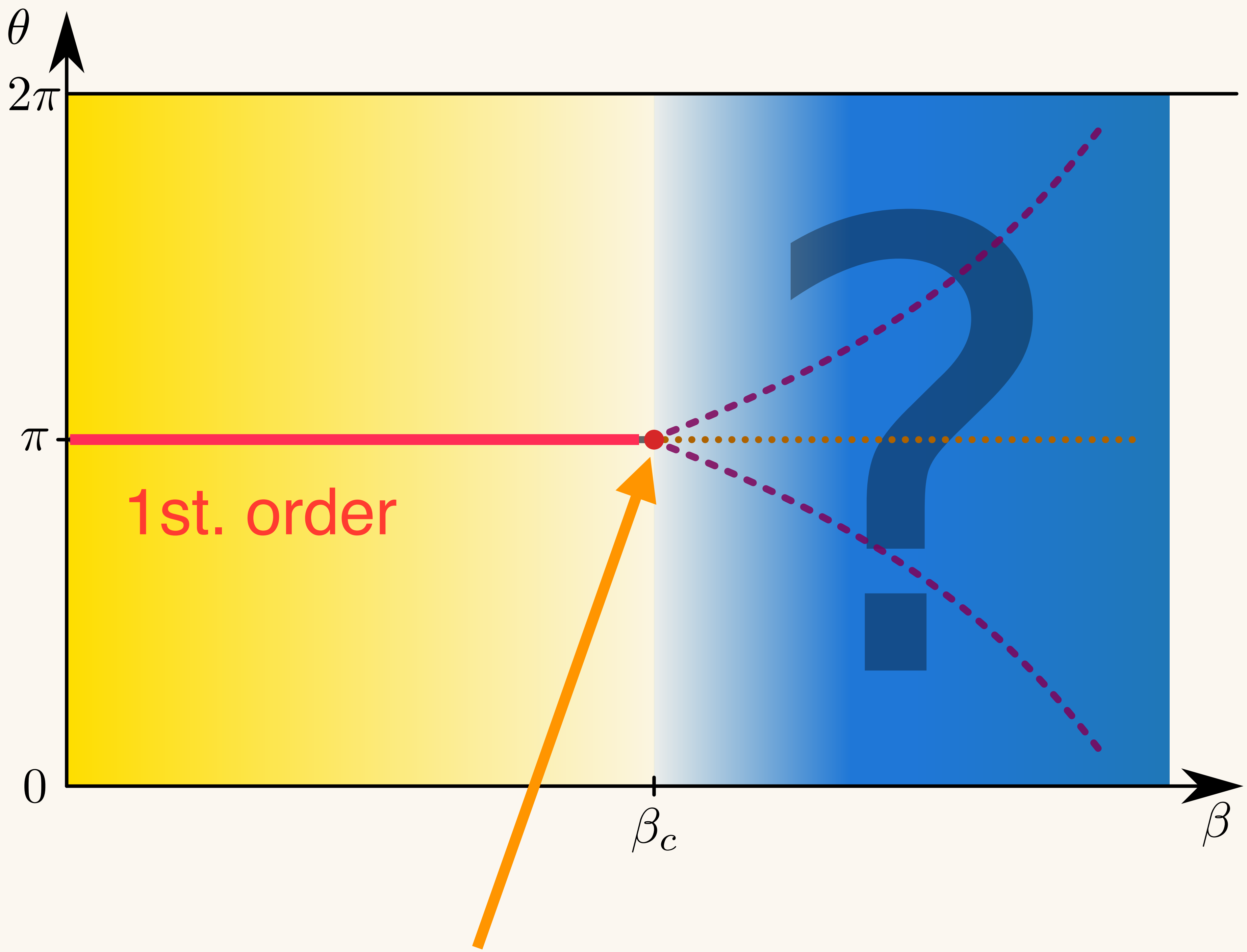
- Character-like expansion is exactly same.
- We introduce additional truncation in theta term.
- We do NOT use additional strong coupling expansion.

→ The truncation of the character-like expansion is GRADUALLY severe in larger beta, but no explicit $\beta << 1$ limitation as strong coupling expansion,

Character-like expansion used properties of Bessel function.



Motivation: Phase diagram of CP(1) model



→ In our analysis, $1.1 \leq \beta_c$ is strongly favored.

◉ Summary

- ◇ We calculate free energy of CP(1) model to study the phase diagrams.
- ◇ From $\theta = 0$ calculation, sufficiently large bond size D is needed for precise calculation.
- ◇ Up to $\beta = 1.1$, our calculation shows first order transition at $\theta = \pi$, without any bifurcation.
- ◇ We estimate the systematic errors, and it is only $O(10^{-2\sim3})$,