# Method of images in defect conformal field theories

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Based on arxiv2205.05370 [hep-th] with T. Nishioka and S. Shimamori

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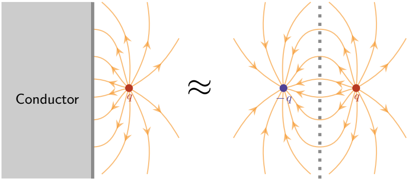
# Quick overview of this paper

Method of images

– a mathematical tool for solving differential equations, in which the domain of the sought function is extended by the addition of its mirror image with respect to a symmetry hyperplane (by Wikipedia)




- In this work, we find that the image method works in what is called **defect conformal field theories**.
- Our prescription provides an efficient way to calculate correlation functions in defect conformal field theories.

# Outline

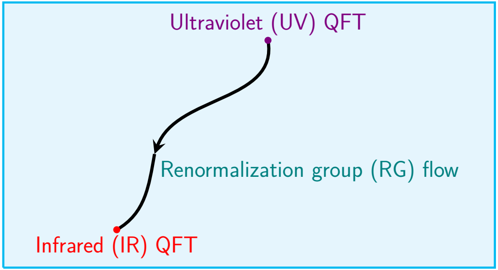
# Table of Contents

# Renormalization group flow

- In local Quantum Field Theory (QFT), integrals that appear in the calculation of physical observables (e.g., cross section, decay rate) are often divergent.
- To tame the UV divergences, we introduce energy cut off: $\Lambda$.
- Because physical coupling constants depend highly on the energy scale $\Lambda$, QFT bahaves differently at different energy scales.

# Renormalization group flow

- Renormalization Group (RG) allows us to understand how a physical system changes as seen at different scale.
- The change in energy scale by integrating out high frequency modes (coarse graining procedure) induces RG flow on the theory space spanned by coupling constants.




- Question: How QFT behaves right on the Infrared fixed point?

# Renormalization group flow

Three possibilities near the Infrared fixed point:

A. a theory with a mass gap (e.g., non-abelian gauge theories).
$\rightarrow$ After integrate out all massive particles, the theory has no dynamical degrees of freedom and is described by topological quantum field theory.

B. a theory with free massless particles in the IR fixed point (e.g., QED)

C. a scale invariant theory with a continuous mass spectrum (e.g., Banks-Zaks fixed point in $4d$ YM)

Under scale transformation $x^\mu \mapsto \lambda x^\mu$, the rest mass changes continuously $m \mapsto \lambda^{-1} m$.

$\rightarrow$ The mass spectrum of scale invariant theories is either continuous or all masses are zero.

In this talk, I focus on scale invariant theories (type C).

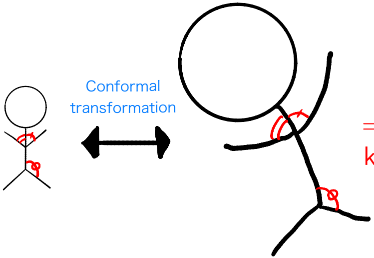
# Analytic continuation to Euclidean

- QFT in Euclidean signature describes statistical mechanics.
- Moreover scale invariant Euclidean QFT describes critical phenomena in statistical mechanics (continuous mass spectrum implies vanishment of typical length scales).
- The Osterwalder-Schrader reconstruction theorem states the equivalence of Euclidean and Lorentzian field theories.
- In what follows, we only focus on QFT in Euclidean signature with the metric $g_{\mu\nu} = \mathrm{diag}(1, \cdots, 1)$.

# scale and conformal incariance

- In relativistic field theories, scale invariance often enhances to conformal invariance. Conformal invariance is the invariance under conformal transformations that locally look like the combination of translation, rotation and scale transformation.

$$x^\mu \mapsto x'^\mu \qquad \frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\, L^\mu{}_\nu \qquad L^\mu{}_\nu \in \mathrm{SO}(d)\ .$$




conformal invariance
= invariance under transformations that keep the angles between any two curves

Conformally invariant theory is called Conformal Field Theory (CFT).

# Conformal transformations

- Conformal transformations in $d \geq 3$ dimensions are finite-dimensional and are composed of
  - translations $\mathbf{P}_\mu : \; x^\mu \mapsto x^\mu + a^\mu$.
  - rotations $\mathbf{M}_{\mu\nu} : \; x^\mu \mapsto L^\mu_{\ \nu}\, x^\nu \;, \qquad L^\mu_{\ \nu} \in \mathrm{SO}(d)$.
  - dilatations $\mathbf{D} : \; x^\mu \mapsto \lambda\, x^\mu$.
  - special conformal transformations $\mathbf{K}_\mu : \; x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$.

  I denoted corresponding generators in **bold** font.

# Conformal transformations

- We can embed these generators into the following $(d+2)\times(d+2)$ antisymmetric matrix $\mathbf{J}_{MN}$ with $M, N = -1, 0, 1, \cdots d$:

$$\mathbf{J}_{MN} = \begin{array}{c} {\scriptstyle M\backslash \; N\rightarrow} \\ {\scriptstyle\downarrow} \\ -1 \\ 0 \\ \mu \end{array} \begin{pmatrix} \scriptstyle -1 & \scriptstyle 0 & \scriptstyle \nu \\ 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_\nu - \mathbf{K}_\nu) \\ -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_\nu + \mathbf{K}_\nu) \\ -\frac{1}{2}(\mathbf{P}_\mu - \mathbf{K}_\mu) & -\frac{1}{2}(\mathbf{P}_\mu + \mathbf{K}_\mu) & \mathbf{M}_{\mu\nu} \end{pmatrix} .$$

- They are subject to the commutation relations of SO$(1, d+1)$:

$$[\mathbf{J}_{KL}, \mathbf{J}_{MN}] = \eta_{LM}\mathbf{J}_{KN} - \eta_{KM}\mathbf{J}_{LN} + \eta_{KN}\mathbf{J}_{LM} - \eta_{LN}\mathbf{J}_{KM} \ ,$$

  with the metric $\eta_{MN} = \text{diag}(-1, 1, 1, \cdots, 1)$.
- In this sense, the conformal group is isomorphic to SO$(1, d+1)$.

# Consequences of conformal symmetry

Due to scale invariance, (interacting) CFTs have a continuous mass spectrum. Therefore,

- We cannot define the S-matrix in CFTs because we do not have isolated one-particle states. And the only physical observables are the correlation functions $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\mathsf{CFT}}$.
- Particle picture breaks down in CFT. It would be nicer to characterize a local field by its response to the scale transformation, rather than its rest mass.

$$\mathcal{O}_\Delta(x) \mapsto \mathcal{O}_\Delta(\lambda x) = \lambda^\Delta \, \mathcal{O}_\Delta(x) \qquad \Delta: \text{ conformal dimension}$$

From now on, I would like to focus on the properties of scalar local fields and their correlation functions.

# Consequences of conformal symmetry

Under conformal transformations, a scalar local field $\mathcal{O}_\Delta$ with conformal dimension $\Delta$ transforms as

- translations + rotation : $x^\mu \mapsto x'^\mu = L^\mu_{\ \nu}\, x^\nu + a^\mu$.

$$\mathcal{O}_\Delta(x) \mapsto \mathcal{O}_\Delta(x')$$

- dilatations : $x^\mu \mapsto x'^\mu = \lambda\, x^\mu$.

$$\mathcal{O}_\Delta(x) \mapsto \lambda^\Delta\, \mathcal{O}_\Delta(x') \ .$$

- special conformal transformations : $x^\mu \mapsto x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$.

$$\mathcal{O}_\Delta(x) \mapsto \gamma^{-\Delta}(x)\, \mathcal{O}_\Delta(x') \qquad \gamma(x) = 1 - 2b\cdot x + b^2 x^2 \ .$$

# Consequences of conformal symmetry

The correlation function of the local scalar fields $\langle \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle_{\mathsf{CFT}}$ must transform covariantly under these transformation laws (last slide).

- translations + rotation : $x_\alpha^\mu \mapsto x_i'^\mu = L^\mu{}_\nu \, x_\alpha^\nu + a^\mu$.

$$\langle \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle_{\mathsf{CFT}} = \langle \mathcal{O}_{\Delta_1}(x_1') \cdots \mathcal{O}_{\Delta_n}(x_n') \rangle_{\mathsf{CFT}} \ .$$

- dilatations : $x_\alpha^\mu \mapsto x_i'^\mu = \lambda \, x_\alpha^\mu$.

$$\langle \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle_{\mathsf{CFT}} = \left( \prod_{\alpha=1}^{n} \lambda^{\Delta_\alpha} \right) \langle \mathcal{O}_{\Delta_1}(x_1') \cdots \mathcal{O}_{\Delta_n}(x_n') \rangle_{\mathsf{CFT}} \ .$$

- special conformal transformations : $x_\alpha^\mu \mapsto x_i'^\mu = \frac{x_\alpha^\mu - b^\mu x_\alpha^2}{1 - 2b \cdot x_\alpha + b^2 x_\alpha^2}$.

$$\langle \mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n) \rangle_{\mathsf{CFT}} = \left[ \prod_{i=1}^{n} \gamma^{-\Delta_i}(x_\alpha) \right] \langle \mathcal{O}_{\Delta_1}(x_1') \cdots \mathcal{O}_{\Delta_n}(x_n') \rangle_{\mathsf{CFT}} \ ,$$

$$\gamma(x_\alpha) = 1 - 2b \cdot x_\alpha + b^2 x_\alpha^2 \ .$$

These requirements impose strong constraints on the correlation functions.

# Consequences of conformal symmetry

- As I will explain later

$$\langle \mathcal{O}_\Delta(x)\rangle_{\text{CFT}} = \begin{cases} \text{const.} & \text{if } \mathcal{O}_\Delta(x) \text{ is the identity operator} \\ 0 & \text{otherwise} \end{cases} ,$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1-x_2|^{2\Delta_1}} ,$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\text{CFT}} = \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}} .$$

- Two-point functions are orthogonal and three-point functions are fixed up to some coefficients. These observations are true even for spinning correlators.

# One-point functions in CFT

- Translational and rotational invariance:

$$\langle \mathcal{O}_\Delta(x) \rangle_{\text{CFT}} = \langle \mathcal{O}_\Delta(0) \rangle_{\text{CFT}} = \text{const.} \qquad .$$

- Covariance under scale transformation:

$$\langle \mathcal{O}_\Delta(x) \rangle_{\text{CFT}} = \lambda^\Delta \langle \mathcal{O}_\Delta(\lambda x) \rangle_{\text{CFT}} \ .$$

- From these two requirements, the one-point function vanishes unless $\Delta = 0$. $\mathcal{O}_{\Delta=0}$ is nothing but the identity operator. Hence,

$$\langle \mathcal{O}_\Delta(x) \rangle_{\text{CFT}} = \begin{cases} \text{const.} & \text{if } \mathcal{O}_\Delta(x) \text{ is the identity operator: } (\Delta = 0) \\ 0 & \text{otherwise} \end{cases} \qquad .$$

# Two-point functions in CFT

- Translational and rotational invariance:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \langle \mathcal{O}_{\Delta_1}(x_1 - x_2)\mathcal{O}_{\Delta_2}(0)\rangle_{\text{CFT}} = f(|x_1 - x_2|) \ .$$

- Covariance under scale transformation:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = \lambda^{\Delta_1+\Delta_2} \langle \mathcal{O}_{\Delta_1}(\lambda x_1)\mathcal{O}_{\Delta_2}(\lambda x_2)\rangle_{\text{CFT}} \ ,$$

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\text{CFT}} = f(|x_1 - x_2|) \propto \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \ .$$

# Two-point functions in CFT

- Translation + rotation + scale transformation:

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\mathrm{CFT}} = f(|x_1 - x_2|) \propto \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \ .$$

- Covariance under special conformal transformations with

$$\gamma(x_\alpha) = 1 - 2b\cdot x + b^2 x^2 \ , \qquad x_i'^{\mu} = \frac{x_\alpha^\mu - b^\mu x_\alpha^2}{1 - 2b\cdot x_\alpha + b^2 x_\alpha^2} \ .$$

Using $|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}(x_1)\gamma_2^{1/2}(x_2)}$, we find

$$\frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}(x_1)\gamma_2^{\Delta_2}(x_2)\,|x_1' - x_2'|^{\Delta_1+\Delta_2}}$$

$$= \frac{\gamma_1^{\frac{\Delta_1+\Delta_2}{2}}(x_1)\gamma_2^{\frac{\Delta_1+\Delta_2}{2}}(x_2)}{\gamma_1^{\Delta_1}(x_1)\gamma_2^{\Delta_2}(x_2)\,|x_1 - x_2|^{\Delta_1+\Delta_2}} \ ,$$

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\mathrm{CFT}} = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \ .$$

The coefficient can be set to unity via field redefinition.

# Three-point functions in CFT

Likewise we can fix three-point functions.

- Translation + rotation + scale transformation:

$$\longrightarrow \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\mathsf{CFT}}$$
$$= \sum_{a+b+c=\Delta_1+\Delta_2+\Delta_3} \frac{C(a,b,c)}{|x_1-x_2|^a|x_2-x_3|^b|x_3-x_1|^c} \, .$$

- Covariance under special conformal transformations completely fixes the form

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_{\mathsf{CFT}}$$
$$= \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}} \, .$$

# Four-point functions in CFT

- Four-point functions in CFT are not determined just by conformal symmetry, because we can make conformal invariants called cross ratios:

$$u = \frac{|x_1 - x_2|^2 |x_3 - x_4|^2}{|x_1 - x_3|^2 |x_2 - x_4|^2} \,, \qquad v = \frac{|x_1 - x_4|^2 |x_2 - x_3|^2}{|x_1 - x_3|^2 |x_2 - x_4|^2} \,.$$

- However, we can evaluate four-point functions via Operator Product Expansion (OPE),

$$\mathcal{O}_\alpha(x_\alpha)\mathcal{O}_\beta(x_\beta) \sim \sum_\sigma \frac{c_{\alpha\beta\sigma}}{|x_\alpha - x_\beta|^{\Delta_\alpha + \Delta_\beta - \Delta_\sigma}} \mathcal{O}_\sigma(x_\beta) + \cdots \,.$$

  The orthogonality of two-point functions allows us to fix OPEs up to any order in $|x_\alpha - x_\beta|$ so as to reproduce three-point functions.
- Similarly, we can calculate $n$-point functions once we know operator spectrum and three-point coupling constants $\{\Delta_\alpha(J_\alpha), c_{\alpha\beta\sigma}\}$.
- Thus, many people are working on the business to derive the CFT data $\{\Delta_\alpha(J_\alpha), c_{\alpha\beta\sigma}\}$. (e.g., conformal bootstrap)

# Summary of CFT

- QFTs acquires scale invariance on the RG fixed point and described by conformal field theory.
- Euclidean conformal field theory describes critical phenomena in statistical mechanics.
- Lower-point scalar correlation functions in CFTs are:

$$\langle \mathcal{O}_\Delta(x) \rangle_{\text{CFT}} = \begin{cases} \text{const.} & \text{if } \mathcal{O}_\Delta(x) \text{ is the identity operator} \\ 0 & \text{otherwise} \end{cases} ,$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2) \rangle_{\text{CFT}} = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}} ,$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3) \rangle_{\text{CFT}} = \frac{c_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4) \rangle_{\text{CFT}} = \text{some function of cross ratios} .$$

# Table of Contents

# Motivation to consider extended object

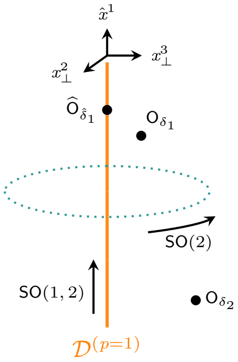
- So far, we have considered conformal field theories, that describe QFT at Infrared fixed point (Lorentzian) and critical phenomena in statistical physics (Euclidean), and investigated correlation functions of local operators.
- However, we can think of extended objects with various dimensions $p$ in QFT, such as:
  - Wilson and 't Hooft line operators ($p = 1$)
  - surface operators ($p = 2$)
  - domain walls, interfaces, boundaries ($p = d - 1$)
- In statistical mechanics (Euclidean QFT), impurities, boundaries and containers can be seen as defects. We should take into account the effect of these objects when we explore critical phenomena in the laboratory.

# Defect conformal field theory

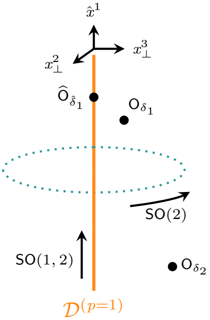
- Defect conformal field theory (DCFT) is a framework to study scale invariant phenomena in the presence of an extended object (defect).
- In DCFT, we approximate the shape of defect to a plain.
  This approximation is reasonable because Infrared theories (critical phenomena) are insensitive to the microscopic structure.
- From now on, I will investigate the lower-point correlation functions in the presence of a defect.

# Defect conformal field theory

- Let us consider a $p$-dimensional planer defect $\mathcal{D}^{(p)}$.
- Without loss of generality, we can place $\mathcal{D}^{(p)}$ at $x^\mu = 0$ for $\mu = p+1, \cdots d$ and decompose the $d$-dimensional coordinates $x^\mu$ into parallel and transverse directions to the defect as $x^\mu = (\hat{x}^a, x_\perp^i)$ with $a = 1, \cdots, p$ and $i = p+1, \cdots, d$:



# Defect conformal field theory




- In the presence of the defect, conformal symmetry $\text{SO}(1, d+1)$ breaks down to $\text{SO}(1, p+1) \times \text{SO}(d-p)$
  - $\text{SO}(1, p+1)$ : conformal group parallel to the defect.
  - $\text{SO}(d-p)$ : transverse rotational group.

- There are two types of operators: *bulk* and *defect* operators. More specifically, we here would like to consider
  - $\text{O}_\delta(x)$ : bulk scalar operator with conformal dimension $\delta$.
  - $\widehat{\text{O}}_{\hat{\delta}}(y)$ : defect scalar operator with conformal dimension $\hat{\delta}$ and $y$ is coordinates on the defect $y = (\hat{y}^a, y_\perp^i = 0)$.

# Correlation functions in DCFT

- We denote correlation functions in DCFT by:

$$\langle \mathsf{O}_\delta(x_1)\cdots\widehat{\mathsf{O}}_{\hat\delta}(y_1)\cdots\rangle_{\mathsf{DCFT}} \equiv \frac{\langle \mathcal{D}^{(p)}\mathsf{O}_\delta(x_1)\cdots\widehat{\mathsf{O}}_{\hat\delta}(y_1)\cdots\rangle_{\mathsf{CFT}}}{\langle\mathcal{D}^{(p)}\rangle_{\mathsf{CFT}}}\,. \qquad (1)$$

  They must transform covariantly under defect conformal group $\mathrm{SO}(1,p+1)\times\mathrm{SO}(d-p)$ that keeps the shape of the defect invariant:

  - parallel translations $\mathbf{P}_a:\ \hat{x}^a\mapsto\hat{x}^a+\hat{c}^a$.
  - parallel rotations $\mathbf{M}_{ab}:\ \hat{x}^a\mapsto L^a{}_b\,\hat{x}^b\ ,\qquad L^a{}_b\in\mathrm{SO}(p)$.
  - parallel special conformal transformations $\mathbf{K}_a:\ \hat{x}^a\mapsto\frac{\hat{x}^a-\hat{b}^a x^2}{\hat\gamma(x)}$ with $\hat\gamma(x)=1-2\hat{b}\cdot\hat{x}+\hat{b}^2x^2$.
  - dilatations $\mathbf{D}:\ x^\mu\mapsto\lambda\,x^\mu$.
  - transverse rotations $\mathbf{M}_{ij}:\ x^i_\perp\mapsto L^i{}_j\,x^j_\perp\ ,\qquad L^i{}_j\in\mathrm{SO}(d-p)$.

# bulk one-point functions in DCFT

- The symmetry breaking patter in the presence of a $p$-dimensional defect is described below

$$
\mathbf{J}_{MN} = \begin{array}{c|cccc} M\backslash N\rightarrow & -1 & 0 & b & j \\ \hline -1 & 0 & \mathbf{D} & \frac{1}{2}(\mathbf{P}_b - \mathbf{K}_b) & \cancel{\frac{1}{2}(\mathbf{P}_j - \mathbf{K}_j)} \\ 0 & -\mathbf{D} & 0 & \frac{1}{2}(\mathbf{P}_b + \mathbf{K}_b) & \cancel{\frac{1}{2}(\mathbf{P}_j + \mathbf{K}_j)} \\ a & -\frac{1}{2}(\mathbf{P}_a - \mathbf{K}_a) & -\frac{1}{2}(\mathbf{P}_a + \mathbf{K}_a) & \mathbf{M}_{ab} & \cancel{\mathbf{M}_{aj}} \\ i & \cancel{-\frac{1}{2}(\mathbf{P}_i - \mathbf{K}_i)} & \cancel{-\frac{1}{2}(\mathbf{P}_i + \mathbf{K}_i)} & \cancel{\mathbf{M}_{ib}} & \mathbf{M}_{ij} \end{array} ,
$$

- Obviously, correlation functions consisting only of defect local operators $\langle \widehat{\mathrm{O}}_{\hat{\delta}}(y_1) \cdots \rangle_{\mathrm{DCFT}}$ are just CFT correlators on the defect. So I only consider DCFT correlators including one or more bulk local operators.

# bulk one-point functions in DCFT

- Let us first focus on a bulk one-point function $\langle \mathsf{O}_\delta(x)\rangle_{\mathsf{DCFT}}$. From parallel translations and transverse rotations:

$$\langle \mathsf{O}_\delta(x)\rangle_{\mathsf{DCFT}} = \langle \mathsf{O}_\delta(\hat{x}^a = 0, x_\perp^i)\rangle_{\mathsf{DCFT}} = f(|x_\perp|) \ .$$

- The covariance under scale transformation uniquely fixed the form:

$$\langle \mathsf{O}_\delta(x)\rangle_{\mathsf{DCFT}} = \frac{a_{\mathsf{O}}}{|x_\perp|^\delta} \ .$$

- The above expression is also consistent with other defect conformal symmetries.
- Bulk one-point functions in DCFT are non-vanishing and behave similarly to the two-point function in CFT: $\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle_{\mathsf{CFT}} = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}$.

# bulk-to-defect two-point functions in DCFT

- Consider bulk-to-defect two-point function. From parallel translations and transverse rotations:

$$\langle \mathsf{O}_\delta(x)\widehat{\mathsf{O}}_{\hat\delta}(y)\rangle_{\mathsf{DCFT}} = \langle \mathsf{O}_\delta(\hat{x}^a - \hat{y}^a, x_\perp^i)\widehat{\mathsf{O}}_{\hat\delta}(0)\rangle_{\mathsf{DCFT}} = f(|\hat{x}-\hat{y}|, |x_\perp|) \ .$$

- Covariance under scale transformation fixes

$$\langle \mathsf{O}_\delta(x)\widehat{\mathsf{O}}_{\hat\delta}(y)\rangle_{\mathsf{DCFT}} = \sum_{m+n=\delta+\hat\delta} \frac{C(m,n)}{|\hat{x}-\hat{y}|^n |x_\perp|^m} \ .$$

- From the covariance under parallel special conformal transformations, we end up with:

$$\langle \mathsf{O}_\delta(x)\widehat{\mathsf{O}}_{\hat\delta}(y)\rangle_{\mathsf{DCFT}} = \frac{b_{\mathsf{O}\widehat{\mathsf{O}}}}{|x_\perp|^{\delta-\hat\delta}\left(|\hat{x}-\hat{y}|^2 + |x_\perp|^2\right)^{\hat\delta}} \ .$$

- As clear from the way the correlator is constrained, the bulk-to-defect two-point function in DCFT looks like a three-point function in CFT.

# higher-point functions in DCFT

- Bulk two-point function

$$\langle \mathrm{O}_{\delta_1}(x_1)\mathrm{O}_{\delta_2}(x_2)\rangle_{\mathrm{DCFT}} ,$$

and bulk-defect-defect three-point function

$$\langle \mathrm{O}_{\delta}(x)\widehat{\mathrm{O}}_{\hat{\delta}_1}(y_1)\widehat{\mathrm{O}}_{\hat{\delta}_2}(y_2)\rangle_{\mathrm{DCFT}} ,$$

cannot be determined solely by defect conformal symmetry, as they depend on defect conformal invariants.
$\rightarrow$ Bulk two-point functions and bulk-defect-defect three-point functions in DCFT seem like four-point functions in CFT.

# Our observations

- Similarity between DCFT correlators and CFT correlators:

| DCFT side | | CFT side |
|---|---|---|
| defect $n$-pt functions | $\Longleftrightarrow$ | $n$-pt function |
| bulk $1$-pt functions | $\Longleftrightarrow$ | $2$-pt functions |
| bulk-to-defect $2$-pt functions | $\Longleftrightarrow$ | $3$-pt functions |
| bulk $2$-pt functions<br>bulk-defect-defect $2$-pt func | $\Longleftrightarrow$ | $4$-pt functions |

- Naive question:
  Is there some correspondance between DCFT correlator with $n$ bulk local scalar and $m$ defect local scalar and $(2n+m)$-point function in CFT?

# Table of Contents

# Cardy's doubling trick in two-dimensional BCFT

- In two-dimensional boundary conformal field theories, such correspondence is established by Cardy and often called *method of images* or *doubling trick*.

$$\begin{aligned}&\langle \mathcal{O}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\cdots\mathcal{O}_{h_n,\bar{h}_n}(z_n,\bar{z}_n)\rangle_{\text{BCFT}_2}\\ &\approx \langle \mathcal{O}_{h_1}(z'_1)\cdots\mathcal{O}_{h_n}(z'_n)\mathcal{O}_{\bar{h}_1}(z'_{n+1})\cdots\mathcal{O}_{\bar{h}_n}(z'_{2n})\rangle_{\text{chiral CFT}_2}\\ &z'_\alpha = z_\alpha \ , \qquad z'_{\alpha+n} = \bar{z}_\alpha \qquad \alpha = 1,\cdots,n \ .\end{aligned}$$

An $n$-point BCFT correlator defined on the upper half plain $\mathrm{Im}\,(z) \geq 0$ is kinematically equivalent (=satisfy same constraint imposed by symmetry) to the $2n$-point correlator in chiral CFT on the full complex plain.

# What we have established in recent paper

- We all know Cardy's *method of images* or *doubling trick* in two-dimensional BCFT.
- For some reason, there was no generalization to higher dimensional DCFT.
- In our recent paper [arxiv2205.05370] we generalize the Cardy's method to higher-dimensional DCFTs, by comparing infinitesimal conformal transformation laws on the both sides (Ward identities).
- To be honest, I still cannot understand why what we did have not been pointed out by somebody else.

# What we have established in recent paper

The correlation function of $n$ bulk scalars $\mathrm{O}_{\delta_\alpha}, \alpha = 1, \cdots, n$ and $m$ defect local scalars $\widehat{\mathrm{O}}_{\hat{\delta}_{\hat{\alpha}}}, \hat{\alpha} = 1, \cdots, m$ in DCFT is equivalent to the correlation function of $n$-pairs of local scalars $\mathcal{O}_{\delta_\alpha/2}$ and $m$ local scalars $\mathcal{O}_{\hat{\delta}_{\hat{\alpha}}}$ in an ancillary CFT by the following relation:

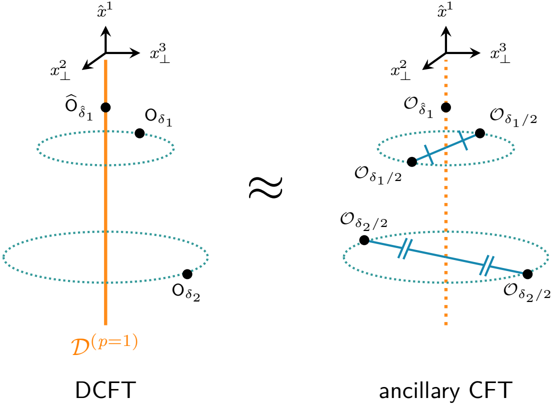
$$\left\langle \prod_{\alpha=1}^{n} \mathrm{O}_{\delta_\alpha}(x_\alpha) \prod_{\hat{\alpha}=1}^{m} \widehat{\mathrm{O}}_{\hat{\delta}_{\hat{\alpha}}}(y_{\hat{\alpha}}) \right\rangle_{\mathrm{DCFT}} \approx \left\langle \prod_{\alpha=1}^{n} \left[\mathcal{O}_{\delta_\alpha/2}(x_\alpha)\, \mathcal{O}_{\delta_\alpha/2}(\bar{x}_\alpha)\right] \prod_{\hat{\alpha}=1}^{m} \mathcal{O}_{\hat{\delta}_{\hat{\alpha}}}(y_{\hat{\alpha}}) \right\rangle_{\mathrm{CFT}} ,$$

where $\approx$ means that both sides satisfy the same differential equations dictated by conformal symmetry.

- $\bar{x}$ : the anti-podal point of $x$ along the transverse direction to the boundary/defect.
- $y = (\hat{y}^a, y^i_\perp = 0)$ : the coordinate on the defect.

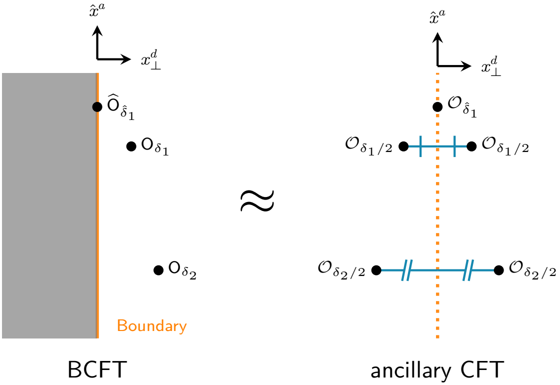
# What we have established in recent paper

- In three-dimensional spacetime with a line defect the bulk-bulk-defect three-point function in DCFT [Left] behaves in the same way as the five-point function in ancillary CFT [Right].



# What we have established in recent paper

- The method of images for BCFT with two bulk and one boundary operators.



# Some working examples of our prescription

- As we saw before, the one-point function of a bulk local operator in DCFT is

$$\langle \mathsf{O}_\delta(x) \rangle_{\mathsf{DCFT}} = \frac{a_{\mathsf{O}}}{|x_\perp|^\delta} \ .$$

On the other hand, the two-point function of bulk operators inserted at $x^\mu = (\hat{x}^a, x_\perp^i)$ and the anti-podal point $\bar{x}^\mu = (\hat{x}^a, -x_\perp^i)$ is

$$\langle \mathcal{O}_{\delta/2}(x)\ \mathcal{O}_{\delta/2}(\bar{x}) \rangle_{\mathsf{CFT}} = \frac{1}{|x-\bar{x}|^\delta} = \frac{1}{2^\delta\, |x_\perp|^\delta} \ ,$$

They are same up to a coefficient.

# Some working examples of our prescription

- The two-point function of a bulk operator at $x^\mu$ and a defect local operator at $y^\mu = (\hat{y}^a, 0)$ reads:

$$\langle \, \mathsf{O}_\delta(x) \, \widehat{\mathsf{O}}_{\hat{\delta}}(y) \, \rangle_{\mathsf{DCFT}} = \frac{b_{\mathsf{O}\widehat{\mathsf{O}}}}{|x_\perp|^{\delta-\hat{\delta}} \left( |\hat{x} - \hat{y}|^2 + |x_\perp|^2 \right)^{\hat{\delta}}} \, .$$

On the CFT side, we have:

$$\begin{aligned} \langle \, \mathcal{O}_{\delta/2}(x) \, \mathcal{O}_{\delta/2}(\bar{x}) \, \mathcal{O}_{\hat{\delta}}(y) \, \rangle_{\mathsf{CFT}} &= \frac{1}{|x-y|^{\hat{\delta}} \, |y-\bar{x}|^{\hat{\delta}} \, |x-\bar{x}|^{\delta-\hat{\delta}}} \\ &= \frac{1}{2^{\delta-\hat{\delta}} \, |x_\perp|^{\delta-\hat{\delta}} \left( |\hat{x}-\hat{y}|^2 + |x_\perp|^2 \right)^{\hat{\delta}}} \, . \end{aligned}$$

which reproduces the bulk-to-defect two-point function as expected.

# Conclusion and outlook

- We generalize the Cardy's method to higher-dimensional DCFTs and provide an efficient way to calculate DCFT correlators in terms of CFT ones.
- Our prescription is purely kinematical. It is still unknown whether the correspondence presented in this paper can be promoted to a complete dictionary that matches dynamical data between DCFT and CFT.
- Nevertheless, once such a dictionary is established, our method should be a powerful tool to study DCFTs.