Emergence of time from unitary equivalent

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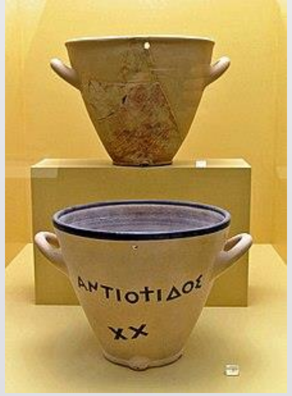
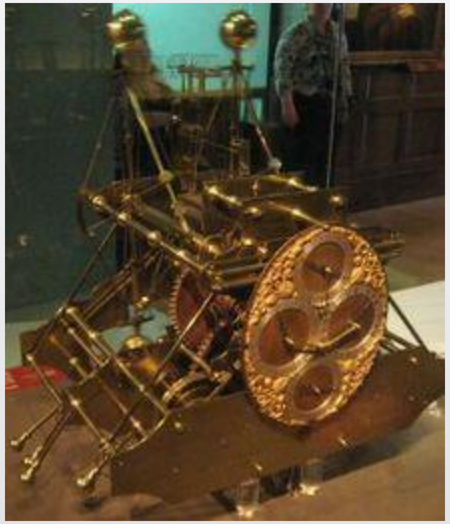
with Chen-Te Ma (APCTP)

arXiv:2204.06366

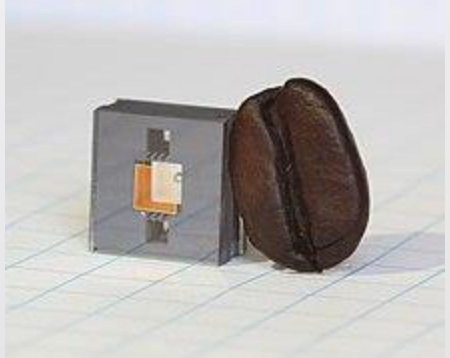
Measurement of Time

Ancient





18th Century



Modern

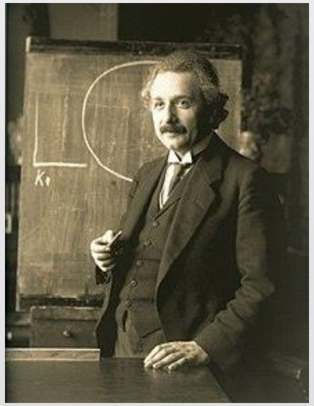
Pictures from Wikipedia

What is Time?

- Newtonian Mechanics
  - Absolute

Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies: and which is vulgarly taken for immovable space ... Absolute motion is the translation of a body from one absolute place into another: and relative motion, the translation from one relative place into another ...

- Relativity
  - Relative



Time is relative; its only worth depends upon what we do as it is passing.

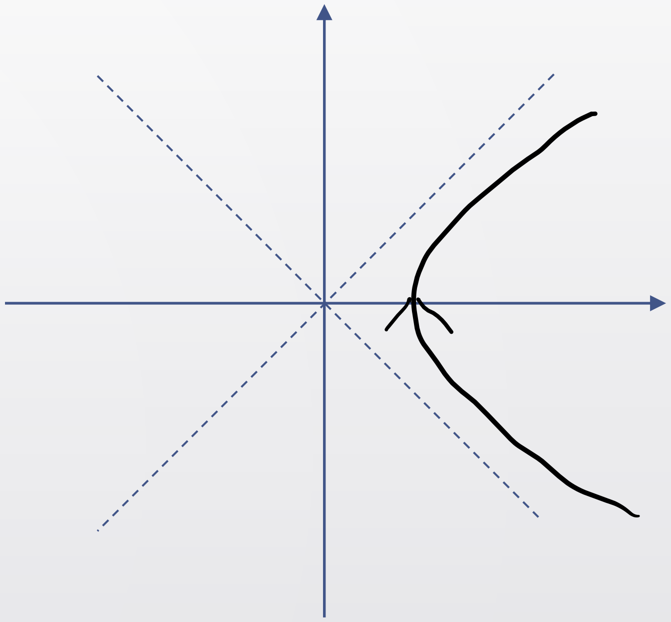
Pictures from Wikipedia

Thermodynamics – Many body

- Time as emergent
- Entropy $\partial S \geq 0 \rightarrow$ arrow of time
- Boltzmann H-theorem [1872]
- Hydro EFT

P. Glorioso, H. Liu, arXiv: 1612.07705

Thermal time hypothesis



- AQFT
  - Algebra of observables
- Modular (Tomita) operator generates a “flow”
- E.g. Rindler spacetime
  - Modular operator $\rho = e^{-2\pi K}$
  - Modular flow $\alpha_s(\hat{a}) = \rho^{is}\hat{a}\rho^{-is}$ $\leftrightarrow$ “Proper time evolution” $\hat{a}(\tau) = e^{iaK\tau}\hat{a}e^{-iaK\tau}$
  - Modular flow as time flow $s = \frac{\tau}{\beta}$ $\leftrightarrow$ Unruh temperature $\beta = \frac{2\pi}{a}$

A Connes, C Rovelli, Class. Quant. Grav.11:2899-2918,1994
J.J. Bisognano, E.H. Wichmann, J. Math. Phys. 16, 985 (1975)

Idea

- Quantum chaos diagnostics
  - Fermionic system
- Entangled state → Unitary Equivalence
  - Subsystem Hamiltonian ↔ Modular Hamiltonian
- Time flow ↔ Modular flow
  - Example of thermal time hypothesis

Classical Chaos

- Sensitivity to initial condition

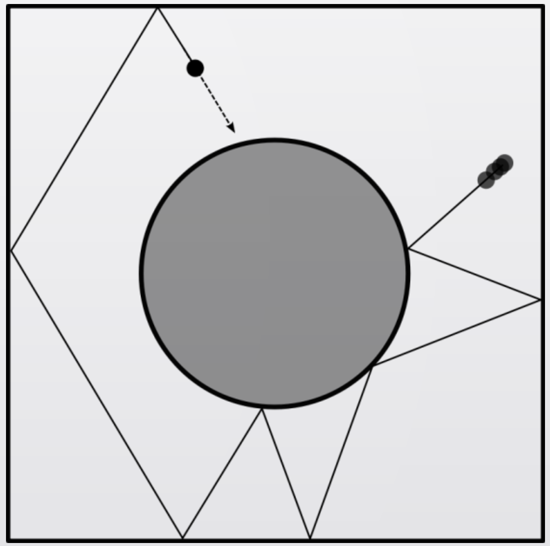
$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda t}$$

- Topological mixing
- Dense periodic orbits

Classical Chaos & RMT

- Sinai billiard
  - Yakov Sinai (1963)
    - Ergodic
  - Chaotic

- Quantum Sinai billiard
  - Spectrum
  - RMT



Y. G. Sinai, Soviet Math. Doklady 4 pp. 1818-1822 (1963)
K. Hashimoto, K. Murata, R. Yoshii JHEP 1710 (2017) 138

https://en.wikipedia.org/wiki/Dynamical_billiards

Quantum Chaos

- Quantum origin of classical chaos?
- Diagnostics
  - Out-of-time ordered correlators
  - Spectral form factors
  - Loschmidt Echo
- (Also : level spacing distribution)

OTOC-1

- Classical

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda t}$$

- Quantum (mechanics)

$$\langle [\hat{x}(t), \hat{p}(0)]^2 \rangle = i^2 \left\langle \left(\frac{\delta x(t)}{\delta x(0)} \right)^2 \right\rangle \sim e^{2\lambda t}$$

OTOC-2

- General

$$C_T = -\left\langle \left[\widehat{W}(t), \widehat{U}(0)\right]^2 \right\rangle_\beta \sim e^{2\lambda t}$$

- Maximum bound

$$\lambda \leq \frac{2\pi k_B}{\beta\hbar}$$

J. Maldacena, S.H. Shenker, D. Stanford JHEP (2016) 08 106

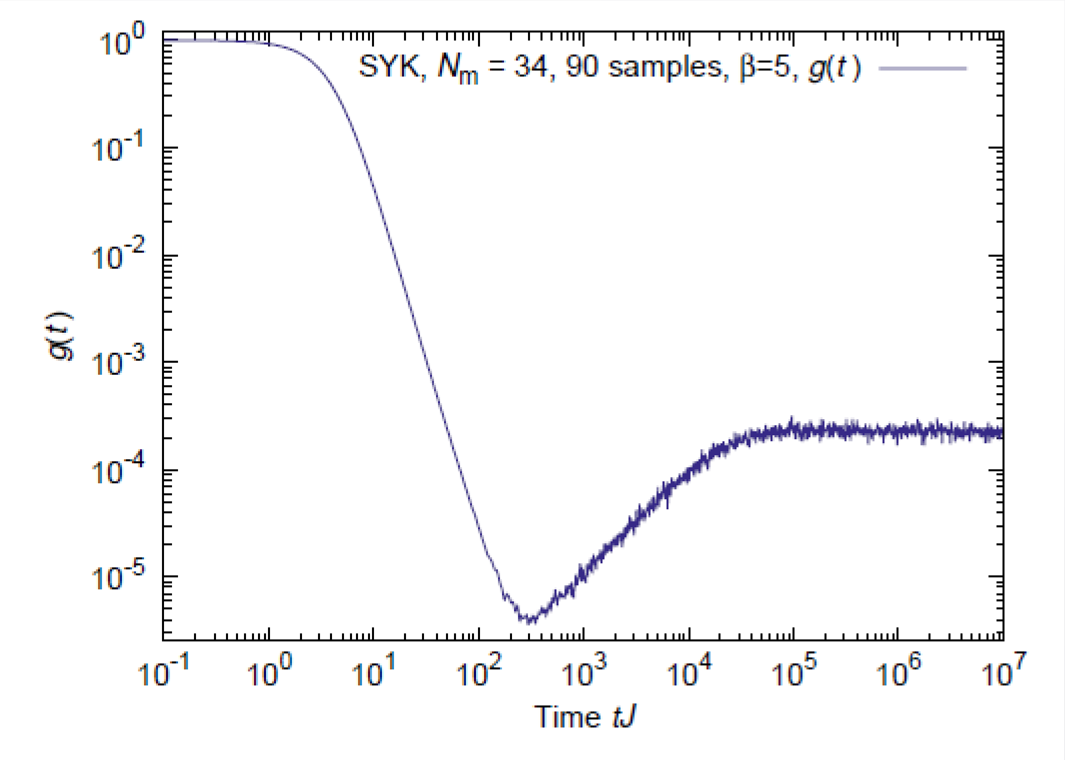
Spectral form factor

- Correlation between eigenvalues

$$g(t) = \frac{|Z(\beta,t)|^2}{|Z(\beta,0)|^2}$$

$$Z(\beta,t) = \mathrm{Tr}\left(e^{(-\beta+it)H}\right)$$

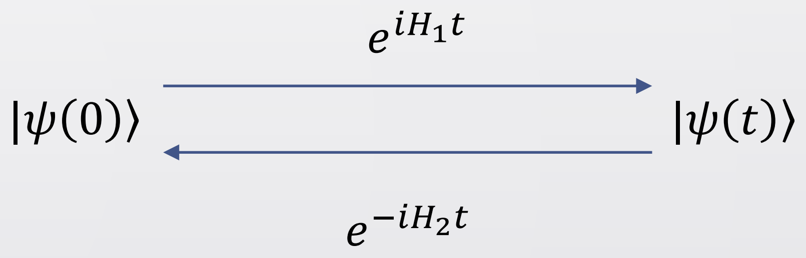
- E.g. SYK model



J.S. Cotler et.al JHEP (2017) 1705: 118.

Loschmidt echo

- Imperfect reverse process
- $M(t) = \left|\langle\psi|e^{iH_2 t}\, e^{-iH_1 t}|\psi\rangle\right|^2$
- Perturbation on Hamiltonian (not state)
  - Unitary evolution



Modular Chaos

- $H_{mod} = -\log \Delta_\psi$
  - Δ_ψ modular operator
- Bound

$$\left\| e^{-iH_{mod}s} e^{i(H_{mod}+\epsilon\delta H_{mod})s} \right\| \leq e^{2\pi s}$$

"Modular Loschmidt Echo"

- $\lambda_{mod} = 2\pi$
- Modular flow parameter - s

J.D. Boer, L. Lamprou, JHEP 06 (2020) 24

Quantum chaos diagnostics

$$\text{SFF} \leftrightarrow \text{OTOC} \leftrightarrow \text{LE}$$

- Bosonic system
- Average over all operators
  - QM : Heisenberg group

$$\text{OTOC} \leftrightarrow \text{SFF}$$

  - Haar integral : N sites lattice system

$$\text{OTOC} \leftrightarrow \text{LE}$$

R.d.M. Koch et.al, Phys.Lett.B 795 (2019) 183-187
A. Bhattacharyya et.al, Eur.Phys.J.C 82 (2022)

Example : Heisenberg group

- Group element

$$\hat{U}(q_1, q_2) = \exp(q_1\hat{x} + q_2\hat{p})$$

- OTOC

$$\frac{1}{2\pi}\int dq_1 dq_2 \langle x|\hat{U}e^{-iHt}\hat{U}^{\dagger}e^{iHt}|x\rangle = \left|\langle e^{iHt}\rangle\right|^2$$

Fermionic system?

- No $\hat{x}$ and $\hat{p}$
- Fermionic coherent state

$$|\psi\rangle = \left(1 - \frac{1}{2}\psi^*\psi\right)|0\rangle - \psi|1\rangle$$

- Displacement operator (c.f. Bosonic case)

$$\hat{D} = \exp\left(\hat{c}^\dagger\psi - \psi^*\hat{c}\right)$$

Fermionic "$\hat{x}$" and "$\hat{p}$"

- Coherent state
  - "$\hat{x}$" operator

$$\hat{c}|\psi\rangle = \psi|\psi\rangle$$

  - "$\hat{p}$" operator (translation) $\hat{t} = \partial_{\psi^*}$ $\hat{t}^\dagger = \partial_\psi$
    - $\hat{\mathrm{T}}_\gamma = \exp(\gamma^* \hat{t} - \hat{t}^\dagger \gamma)$

$$\hat{T}_\gamma|\psi\rangle = |\psi + \gamma\rangle \qquad \langle\psi|\hat{T}_\gamma^\dagger = \langle\psi + \gamma|$$

Fermionic operator

- Inspired by Heisenberg group

$$\hat{U}_{\alpha,\gamma}(0) = \hat{S}_{\alpha}(0)\hat{T}_{\gamma}(0)$$

$$\hat{S}_{\alpha}(0) = e^{i\alpha^{*}\hat{c}}e^{i\hat{c}^{\dagger}\alpha} \qquad \hat{T}_{\gamma}(0) = e^{\gamma^{*}\hat{t}-\hat{t}^{\dagger}\gamma}$$

$$\hat{U}(q_1, q_2) = \exp(q_1\hat{x} + q_2\hat{p})$$

Fermionic OTOC ↔ SFF

$$\int_{\psi,\alpha,\gamma} \langle\psi|e^{-\frac{\beta}{2}\hat{H}}\hat{U}(t)e^{-\frac{\beta}{2}\hat{H}}\hat{U}^{\dagger}(0)|\psi\rangle = \left\langle e^{\left(-\frac{\beta}{2}+it\right)\hat{H}}\right\rangle\left\langle e^{\left(-\frac{\beta}{2}-it\right)\hat{H}}\right\rangle$$

- Can be generalized to higher point OTOC

$$\int_{\psi}\int_{\{\alpha_j,\gamma_j\}} \langle\psi|\hat{U}_1^0\rho_\beta^{\frac{1}{2k}}\hat{U}_2^t\rho_\beta^{\frac{1}{2k}}\hat{U}_3^0\cdots\hat{U}_{2k-2}^t\rho_\beta^{\frac{1}{2k}}\hat{U}_{2k-1}^0\rho_\beta^{\frac{1}{2k}}\hat{W}_{2k}(t)\rho_\beta^{\frac{1}{2k}}|\psi\rangle \longleftrightarrow |\langle e^{(-\frac{\beta}{2k}+it)\hat{H}}\rangle|^{2k}$$

$$\hat{W}_{2k-1}(t) \equiv \left(\hat{U}_1(t)\hat{U}_2(t)\cdots\hat{U}_{2k-1}(t)\right)^{\dagger}$$

$$\int_{\alpha,\gamma} \langle\psi|\hat{U}(0)e^{i\hat{H}t}\hat{U}^{\dagger}(0)|\psi_1\rangle = \langle\psi|\psi_1\rangle\langle e^{i\hat{H}t}\rangle$$

Fermionic OTOC ↔ LE

- Slightly different set-up
- Simplest case : Two-site coherent states

$$|\boldsymbol{\psi}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$\int_{\boldsymbol{\psi}} \int_{U_1,U_2} \langle \boldsymbol{\psi}| \hat{U}_1^\dagger(t) \hat{U}_2^\dagger(0) \hat{U}_1(t) \hat{U}_2(0) |\boldsymbol{\psi}\rangle = \frac{1}{N_1^2} \sum_{P_1,P_1'} \left| \int_{\psi_2} \langle \psi_2 | e^{i(H_2+P_1)t} \, e^{-i(H_2+P_1')t} |\psi_2\rangle \right|^2$$

Assumptions

$$\int_{\boldsymbol{\psi}} \int_{U_1, U_2} \langle \boldsymbol{\psi} | \hat{U}_1^\dagger(t) \hat{U}_2^\dagger(0) \hat{U}_1(t) \hat{U}_2(0) | \boldsymbol{\psi} \rangle = \frac{1}{N_1^2} \sum_{P_1, P_1'} \left| \int_{\psi_2} \langle \psi_2 | e^{i(H_2 + P_1)t} \, e^{-i(H_2 + P_1')t} | \psi_2 \rangle \right|^2$$

- Total Hamiltonian with random interactions

$$H = H_1 \otimes I_2 - I_1 \otimes H_2 + H_1' \otimes H_2'$$

- Similarly, can be generalized to higher point
  - Need to assume weak inter-site coupling

Interpretation

- Consider site 1 and site 2 as "purification"
  - Site 1 and site 2 entangled
  - TFD-like state with perturbation

- Recall
  - $|TFD\rangle = \sum_n e^{-\frac{\beta}{2}E_n}|n\rangle \otimes U|n\rangle$
  - $H_{tot} = H_1 \otimes I - I \otimes H_2$ with $H_1 = H_2 = H$

Unitary equivalence

$$|TFD\rangle = \sum_n e^{-\frac{\beta}{2}E_n}|n\rangle \otimes U|n\rangle$$

- Relates modular H_{mod} and subsystem H

$$H_{mod} = \beta U H U^\dagger$$

- Note : Perturbation on H leads to an unique (upto U) perturbation on H_{mod}

D.L. Jaffreis, L. Lamprou, JHEP 03 (2022) 084

OTOC and Modular LE

- Using $H_{mod} = \beta U H U^{\dagger}$ and $t = \frac{s}{\beta}$

$$\int_{\boldsymbol{\psi}} \int_{U_1,U_2} \langle \boldsymbol{\psi} | \widehat{U}_1^{\dagger}(t) \widehat{U}_2^{\dagger}(0) \widehat{U}_1(t) \widehat{U}_2(0) | \boldsymbol{\psi} \rangle \sim \left| \int_{\psi_2} \langle \psi_2 | e^{i(H_{mod}+\delta H_{mod})s} \, e^{-i(H_{mod}+\delta H'_{mod})s} | \psi_2 \rangle \right|^2$$

- L.H.S. Total Hamiltonian with time t
- R.H.S. Modular Hamiltonian with modular flow s

Chaos and Modular Chaos

OTOC	⟷	LE	⟷ (UE / TTH)	Modular LE
$\lambda = \frac{2\pi}{\beta}$	⟷			$\lambda = 2\pi$
Chaos				Modular Chaos
t	⟷			s
Time flow				Modular flow

Summary

- Fermionic system
  - SSF ↔ OTOC ↔ LE
- Chaos + Unitary equivalence + Thermal Time Hypothesis
  - Chaos ↔ Modular Chaos
  - Reverse : Modular flow ↔ Time flow